ON THE CLASSIFICATION OF ANALYTIC DIFFERENTIAL EQUATIONS IN $(\mathbb{C}^2, 0)$

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ABSTRACT. We study the classification of analytic differential equations in $(\mathbb{C}^2, 0)$. We give a complete set of analytic invariants determining the analytic type of the underling foliation. In particular we generalize some previous results about dicritical singularities in a straightforward manner. Such problem has its origins in a conjecture proposed by R. Thom in the middle 1970s.

CONTENTS

1. INTRODUCTION

We study the classification of differential equations of the form $Adx + bdy$ in two variables, by their underlying foliations. Our treatment goes in the same spirit of Sophus Lie invariant-theoretic approach for the integration of linear partial differential equations, as in the Doctoral dissertation

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of G. Halphen ([13]) and the work of H. Poincaré ([23]), both related to linear differential equations in dimension one with order $n \geq 2$. As it is well known, every analytic differential equation in $(\mathbb{C}^2, 0)$ has a non-void set of integral curves through the origin, called *separatrix* set, and denoted by $Sep(\mathcal{F})$. In [3] we use the blow-up method in order to identify a complete set of invariants describing the analytic type of a generic subset of the above differential equations: the *generalized curves*. By definition they have only a finite number of separatrices and no saddle-nodes along their minimal resolutions ([24]). In fact, we introduce a new kind of resolution of foliations (called rectifier resolutions) taking care not only of the topological behavior of the singularities along the "solved" foliation but paying attention to the disposition of the strict transform of $Sep(\mathcal{F})$, say $Sep(\widetilde{\mathcal{F}})$. In fact, if D_j is an irreducible component of the exceptional divisor of $\widetilde{\mathcal{F}}$ and $Sep(\widetilde{\mathcal{F}}_j)$ is the restriction of $Sep(\widetilde{\mathcal{F}})$ to a neighborhood of D_j, then $Sep(\widetilde{\mathcal{F}}_j)$ will be a subset of fibers of a fibration transversal to D_j. Thence we determine the analytic type of each Hopf component (see definition below), say $\widetilde{\mathcal{F}}_j$, of the solved foliation by its set of singular points (up to homographic conjugacy class), the first jets of their singularities and the conjugacy class of the holonomy group in $Diff(\mathbb{C}, 0)$. Indeed, we use classical topological methods of path lifting due to Ehersmann and analytic extension technics for non-degenerated reduced singularities (c.f. [18], [20]). Finally the remaining invariants appear as characteristic classes which arise naturally from this geometric construction.

In order to extend the previous construction for any foliation, we have the following hindrances to overcome:

First consider the case of a non-dicritical foliation with a Hopf component $\widetilde{\mathcal{F}}_j$ defined in the Hopf bundle $\mathbb{H}_j$ (see definition below), having just non-degenerated singularities and some saddle-nodes with strong variety ([7]) siting over the invariant projective line D_j. Although the conjugacy class of the projective holonomy determines the analytic type of each singularity ([18], [20]), we cannot guarantee that it determines the analytic type of the Hopf component of the foliation, even if $Sep(\widetilde{\mathcal{F}}_j)$ is contained in a fibration transversal to D_j. In fact, there is no analytic extension theorem for saddle-nodes as in the non-degenerated case. As we shall see, in order to overcome this obstacle, we have to "isolate" such saddle-nodes (after blowing-up) with just one resonant saddle and then, verify that the projective holonomy of the new Hopf component determine the analytic type of the original saddle-node, by studying its Stokes phenomenon.

Now, suppose that $\widetilde{\mathcal{F}}_j$ has just non-degenerated singularities and at least one saddle-node with central variety ([7]) siting on D_j. In this case the conjugacy class of the projective holonomy does not determines the analytic type of the saddle-node in general ([9], [19]). But exploring the "symmetry" of saddle-nodes with central variety, we can define an equivalence relation (with analytic moduli space) such that the new class of the projective holonomy determines the analytic type of $\widetilde{\mathcal{F}}_j$, since $Sep(\widetilde{\mathcal{F}}_j)$ is contained in fibration transversal to D_j.

On the other hand, we recall that until now, the dicritical singularities — that is with an infinite number of separatrices — are not well understood ([4], [16]). In fact, their **extended resolutions** ([2]) are composed by non-dicritical and dicritical Hopf components (see definition below) satisfying:

i) The non-dicritical components are solved, that is, they have just reduced singularities;
ii) The dicritical components are pair-wise non-intercepting, and have no singular or tangential points in the zero section of its Hopf bundle.

We apply the same reasonings in order to obtain an "extended rectifier resolution". First remark that the zero section of a Hopf bundle corresponding to a dicritical component is transversal to the foliation. It follows that the only invariants provided by it are the corners and the Chern class of its Hopf bundle (we verify this by path lifting). In particular, it follows that any two analytically equivalent dicritical foliations, namely $\mathcal{F}^i$, $i=1,2$, with dicritical components $\widetilde{\mathcal{F}}^i_{j,0}$ intercepting just the non-dicritical Hopf components, say $\widetilde{\mathcal{F}}^i_{j,k}$, $k=1\cdots n_i$, must have the same global conjugacy in $PGL(2,\mathbb{C})$ which does not only carry corners of $\widetilde{\mathcal{F}}^1_{j,0}$ into corners of $\widetilde{\mathcal{F}}^2_{j,0}$ but simultaneous conjugate all the projective holonomies of $\widetilde{\mathcal{F}}^i_{j,k}$ evaluated over the transversal section $D^i_{j,0}$ — the zero section of the Hopf bundle determined by the dicritical component $\widetilde{\mathcal{F}}^i_{j,0}$ (see figure 2). So, we cannot choose the projective holonomies of the non-dicritical Hopf components freely, in the extended rectifier resolution of a germ of dicritical foliation. In other words, the "cellular" objects of the resolution of a dicritical singularity is no more just the Hopf components, but the following ones:

a) The union of a dicritical component together with the non-dicritical components intercepting it, say $\widetilde{\mathcal{F}}_{j,0}\bigcup_{k=1}^{n_j}\widetilde{\mathcal{F}}_{j,k}$;
b) The non-dicritical Hopf components of the foliation non-intercepting any dicritical foliation.

In fact the analytic type of this objects are determined by the projective holonomies of the non-dicritical Hopf components of the foliation in the extended rectifier resolution and by the tree of corners and singularities of each cell. It follows that this objects are suitable for the definition of a good covering for the first cohomology set of the isotropy group of a foliation model as in [3].

2. Basic definitions and notations

A germ of singular foliation, say $(\mathcal{F}:\omega=0)$, in $(\mathbb{C}^2,0)$ of codimension 1 is, roughly, the set of integral curves of a given germ of 1-form $\omega\in\Omega^1(\mathbb{C}^2,0)$ which may be assumed to have just an isolated singularity at the origin. Let $Diff(\mathbb{C}^k,0)$ (respect. $\widehat{Diff}(\mathbb{C}^k,0)$) be the group of germs of analytic (respect. formal) diffeomorphisms of $(\mathbb{C}^k,0)$ fixing the origin and $Diff_{\mathbf{I}}(\mathbb{C}^k,0)\subset Diff(\mathbb{C}^k,0)$ (respect. $\widehat{Diff}_{\mathbf{I}}(\mathbb{C}^k,0)\subset\widehat{Diff}(\mathbb{C}^k,0)$) be the subset of such diffeomorphisms, tangent to the identity. We say that two germs of foliations in $(\mathbb{C}^2,0)$, namely $(\mathcal{F}_j:\omega_j=0)$, $j=1,2$, are analytically conjugated if there is $\Phi\in Diff(\mathbb{C}^2,0)$, such that Φ sends leaves of $\mathcal{F}_1$ into leaves of $\mathcal{F}_2$. We say that $h_1,h_2\in Diff(\mathbb{C},0)$ are analytically conjugated if there is $\phi\in Diff(\mathbb{C},0)$ such that $Ad(\phi)(h_2):=\phi^{-1}\circ h_2\circ\phi=h_1$. We denote by $Iso(\mathcal{F})$ the isotropy group of the germ of foliation $(\mathcal{F}:\omega=0)$ in $(\mathbb{C}^2,0)$, given by

$$Iso(\mathcal{F})=\{\phi\in Diff(\mathbb{C}^2,0):\phi^*\omega\wedge\omega=0\}$$

Further, let us denote the *Hopf bundle of Chern class* $-k$ by $\mathbb{H}(-k):\big(p_{(k)}:\mathcal{H}(-k)\to\mathbb{CP}(1)\big)$, or just by its total space $\mathbb{H}(-k)$. Recall, from the theory of Algebraic Curves, that whenever $\pi:(\widetilde{X},D)\longrightarrow(\mathbb{C}^2,0)$ is a map resulting from the iteration of finite number of blow-ups, with exceptional curve $D=\pi^{-1}(0)$, whose irreducible components are D_j, $j=1\cdots n$, with Chern classes $-k_j$, then a suitable neighborhood of D in $\widetilde{X}$ results from pasting together suitable neighborhoods of the zero sections of $\mathbb{H}(-k_j)$. Thus, for each Hopf bundle component $p:\mathcal{H}_j\to D_j$ of a given resolution, we shall denote by $\widetilde{\mathcal{F}}_j$, the germ of foliation at $D_j\subset\mathcal{H}_j$ induced by the restriction of $\widetilde{\mathcal{F}}$, and call it the $j^{\mathbf{th}}$ **Hopf component** of the resolution. Further, recall that $\widetilde{\mathcal{F}}_j$ is **non-dicritical** if D_j is an invariant set of the foliation, and **dicritical** otherwise. In the former case, the holonomy of $\widetilde{\mathcal{F}}_j$ with respect to a section Σ transversal to $D\simeq\mathbb{CP}(1)$ is called the **projective holonomy** of $\widetilde{\mathcal{F}}_j$, and denoted by $Hol_\Sigma(\mathcal{F}_j,D_j)$, furthermore one says that it is **solved** if it has just *reduced* singularities (c.f. [18]), and **rectified** if $Sep(\widetilde{\mathcal{F}}_j)$ is contained in a fibration transversal to D_j in $(\mathcal{H}_j,D_j)$. In both cases one calls the set of singularities of $\widetilde{\mathcal{F}}_j$ the **first tangent cone** of $\mathcal{F}_j$ ([5]), and, in the later case, the set of tangencies between $\widetilde{\mathcal{F}}_j$ and D_j the **second tangent cone** of $\widetilde{\mathcal{F}}$ ([16]).

Let $\widetilde{\mathcal{F}}^i$, $i = 1, 2$, be two germs of singular non-dicritical foliation at $\mathbb{CP}(1) \subset \mathcal{H}$, and $\varphi \in PGL(2, \mathbb{C})$ be an isomorphism between their sets of singular points $\{t^i_j\}^n_{j=1}$, that is $\varphi(t^1_j) = t^2_j$. Further let $t^1_0 \in \mathbb{CP}(1)$ be a regular point of $\widetilde{\mathcal{F}}^1$, $t^2_0 = \varphi(t^1_0)$, and denote by h^i_γ the holonomy of a path $\gamma \in \pi_1(\mathbb{CP}(1)\backslash\{t^i_j\}^n_{j=1}, t^i_0)$, with respect to sections Σ_i, $i = 1, 2$ transversal to $\mathbb{CP}(1)$. Then recall that the projective holonomies of these foliations have an analytic **conjugation subordinated to** φ if there is $\phi \in Diff(\mathbb{C}, 0)$ such that $Ad(\phi)(h^2_\gamma) = h^1_{\varphi^*\gamma}$, for every $\gamma \in \pi_1(\mathbb{CP}(1)\backslash\{t^1_j\}^n_{j=1}, t^1_0)$. Further we set $Diff(\widetilde{\mathcal{F}}^1, \widetilde{\mathcal{F}}^2) = \{\Phi \in Diff(\mathcal{H}, \mathbb{CP}(1)) : \Phi_*(\widetilde{\mathcal{F}}^1) = \widetilde{\mathcal{F}}^2, \Phi(Sing(\widetilde{\mathcal{F}}^1)) = \varphi(Sing(\widetilde{\mathcal{F}}^1)\}$. In particular we denote the isotropy group of $\widetilde{\mathcal{F}}$ by $Iso(\widetilde{\mathcal{F}}) = Diff(\widetilde{\mathcal{F}}, \widetilde{\mathcal{F}})$.

We say that a resolution of a non-dicritical foliation is simple if its exceptional divisor has only one projective line with three or more singularities (see figure 1), which is called **principal projective line** of the exceptional divisor; its holonomy is called the **projective holonomy** of the foliation. In particular the Hopf component of a simple non-dicritical foliation about the principal project line is called the **principal Hopf component**.

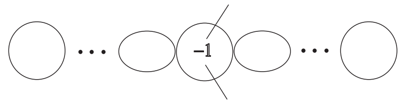


Figure 1: A simple resolution for a non-dicritical foliation

Now recall that a saddle-node is a singularity with a degenerated non-vanishing linear part which presents two invariant curves passing through the origin ([7]): the **strong variety** which is analytic and a formal one, which is called **central variety** when it converges. Their holonomies are called **strong holonomy** in the former case, and **weak holonomy** in the later. Let $\widetilde{\mathcal{F}}$ be a non-dicritical solved foliation defined around the zero section of a Hopf bundle $\mathbb{H} : (p : \mathcal{H} \to D)$, with saddle-nodes along the divisor. Then one says that a saddle-node defined in $(\mathcal{H}, D)$ is **transversal** if its strong variety is transversal to D. Otherwise one say that the saddle-node is **parallel**.

Recall from [8] that every saddle-node $(\mathcal{F} : \omega = 0)$ can be analytic reduced to the form $(\mathcal{F}_{p,A} : \omega_{p,A} = 0)$ where

$$\omega_{p,A}(x, y) = x^{p+1}dy - A(x, y)dx$$

with $A \in \mathcal{O}_2$ and $A(0, y) = y$. That is, there is $\Phi_{p,A} \in Diff(\mathbb{C}^2, 0)$ such that

$$(\Phi_{p,A})^*\omega_{p,A} \wedge \omega = 0,$$

which shall be called an **analytic Dulac reduction**. On the other hand each singularity of the form $(\mathcal{F}_{p,A} : \omega_{p,A} = 0)$ has a formal normal form $(\mathcal{F}_{p,\lambda} : \omega_{p,\lambda} = 0)$ where

$$\omega_{p,\lambda}(x, y) = x^{p+1}dy - y(1 + \lambda x^p)dx$$

In other words, there is $\widehat{N}(\mathcal{F}_{p,A}) \in \widehat{Diff_{\mathbf{I}}}(\mathbb{C}^2, 0)$ given by $\widehat{N}(\mathcal{F}_{p,A})(x, y) = (x, g(x, y))$, with $g(x, y) = y + \sum_{j\geq 1} a_j(y)x^j$, where all a_j converges with the same radius of convergence, such that

$$(\widehat{N}(\mathcal{F}_{p,A}))^*\omega_{p,\lambda} \wedge \omega_{p,A} = 0,$$

which shall be called a **formal Dulac reduction**. Furthermore, for each $U_j = \{(x, y) \in \mathbb{C}^2 : \arg(x) - \frac{(2j+1)}{2p}\pi \in] - \pi/p, \pi/p[, 0 < |y| < r, r \sim 0)$, with $j \in \mathbb{Z}_{2p}$, there is an unique holomorphic transformation of U_j, namely $N_j(\mathcal{F}_{p,A})$, such that $N_j(\mathcal{F}_{p,A}) \underset{1/p}{\longrightarrow} \widehat{N}(\mathcal{F}_{p,A})$, i.e. $\widehat{N}(\mathcal{F}_{p,A})$ is the asymptotic expansion of each $N_j(\mathcal{F})$ (in fact p-summable, c.f. [15],[19]) satisfying

$$(N_j(\mathcal{F}_{p,A}))^*\omega_{p,\lambda} \wedge \omega_{p,A} = 0$$

Now let $\widetilde{U}_j := \Phi^{-1}_{p,A}(U_j)$ and $N_j(\mathcal{F}) := N_j(\mathcal{F}_{p,A}) \circ \Phi_{p,A} : \widetilde{U}_j \longrightarrow U_j$ for $j \in \mathbb{Z}_{2p}$. Then one has that $N_{j+1,j}(\mathcal{F}) := N_{j+1}(\mathcal{F}) \circ N_j(\mathcal{F})^{-1}$ can be identified with $\mathbb{C}$ for $j \equiv 0 \bmod 2$ and with $Diff_{\mathbf{I}}(\mathbb{C}, 0)$

for $j \equiv 1 \bmod 2$. Hence one has that (up to a linear change of coordinates $(x, y) \mapsto (\alpha x, \beta y)$, with $\alpha^p = 1$, and $\beta \in \mathbb{C}^*$) the map

$$\begin{array}{cccc} \Theta: & \Sigma_{p,\lambda} & \longrightarrow & \mathbb{C} \times Diff_{\mathbf{I}}(\mathbb{C}, 0) \times \overset{p \text{ times}}{\cdots} \times \mathbb{C} \times Diff_{\mathbf{I}}(\mathbb{C}, 0) \\ & (\mathcal{F} : \omega = 0) & \longmapsto & (N_{1,0}(\mathcal{F}), N_{2,1}(\mathcal{F}), \cdots, N_{2p-1,2p-2}(\mathcal{F}), N_{2p,2p-1}(\mathcal{F})) \end{array}$$

is a well defined biholomorphism, where $\Sigma_{p,\lambda}$ is the orbit space under analytic equivalence of the set of saddle-nodes with formal normal form $(\mathcal{F}_{p,\lambda} : \omega_{p,\lambda} = 0)$. In particular we have that $(\mathcal{F} : \omega = 0)$ has central variety if and only if $\Theta(\mathcal{F}) = (\mathbf{I}, N_{2,1}(\mathcal{F}), \cdots, \mathbf{I}, N_{2p,2p-1}(\mathcal{F}))$. Note that to determine the analytic type of a saddle-node, it suffices to determine $N_j(\mathcal{F})$ for all $j \in \mathbb{Z}_{2p}$ ([19]). So we say that $(\widetilde{U}_j, N_j(\mathcal{F}))$ is a **sectorial chart** and that $\bigcup_{j \in \mathbb{Z}_{2p}} (\widetilde{U}_j, N_j(\mathcal{F}))$ is a **sectorial atlas** for the saddle-node $(\mathcal{F} : \omega = 0)$.

Now let $\mathcal{F}^1, \mathcal{F}^2 \in \Sigma_{p,\lambda}$ be two saddle-nodes with central variety and formal normal form $\mathcal{F}_{p,\lambda}$. Let $\Sigma_a = (x = a)$, $\widetilde{\Sigma}_a^j := (\Phi_{p,A}^j)^{-1}(\Sigma_a)$, $S_j := (\Phi_{p,A}^j)^{-1}((y = 0))$, and ϕ be a conjugation between $Hol_{\widetilde{\Sigma}_a^1}(\mathcal{F}^1, S_1)$ and $Hol_{\widetilde{\Sigma}_a^2}(\mathcal{F}^2, S_2)$. Then one says that ϕ is p-**periodic** in sectorial charts if

$$(N_j(\mathcal{F}^2)|_{\widetilde{\Sigma}_a^2}) \circ \phi \circ (N_j(\mathcal{F}^1))^{-1}|_{\Sigma_a}(y) = \exp(\frac{2\pi i}{p}) \cdot y$$

Henceforth, we say that a saddle-node with central variety has a q-**symmetric Stokes phenomenon** or has q-**order of symmetry** if q is the greater integer for which the map $R_q(z) = \exp(\frac{2\pi i}{q}) \cdot z$ commutes with $N_{j+1,j}$ for every $j \equiv 1 \bmod 2$.

Let $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{F}}'$ be two analytically equivalent, up to order one (for reduced non-degenerated singularities) and formal type (for saddle-nodes), solved foliations defined in the Hopf bundle $\mathbb{H} : (p : \mathcal{H} \longrightarrow D)$, with $\{t_j\}_{j=1}^n$ and $\{t'_j\}_{j=1}^n$ being transversal saddle-nodes, such that $t'_j = \varphi(t_j)$, where $\varphi \in Diff(D) \approx PGL(2, \mathbb{C})$ is their tree isomorphism, and with $\widetilde{\mathcal{F}}|_{(D,t_j)}$ and $\widetilde{\mathcal{F}}'|_{(D,t'_j)}$ having the same order of symmetry, say q_j. Then one says that a holomorphic conjugation between $Hol_\Sigma(\widetilde{\mathcal{F}}, D)$ and $Hol_{\Sigma'}(\widetilde{\mathcal{F}}', D)$ **commutes with the stokes phenomenon** of $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{F}}'$ if for each pair of equivalent points (t_j, t'_j), ϕ is p_j-periodic in sectorial charts, where p_j is a divisor of $\gcd(\{q_j\}_{j=1}^n)$.

Let $\mathcal{F}$ be a singular foliation in $(\mathbb{C}^2, 0)$ with dicritical component, and $\widetilde{\mathcal{F}}$ its extended resolution, then one says that each non-dicritical Hopf component that does not intercept a dicritical Hopf component is a **non-dicritical cell**. Further, if $\mathcal{F}$ has a dicritical component, then the union of each dicritical Hopf component with the non-dicritical ones intercepting it, is called a **dicritical cell**, and denoted by $\widetilde{\mathcal{C}}_j = \widetilde{\mathcal{F}}_{j,0} \cup_{k=1}^n \widetilde{\mathcal{F}}_{j,k}$ where $\widetilde{\mathcal{F}}_{j,0}$ is the dicritical component of the cell.

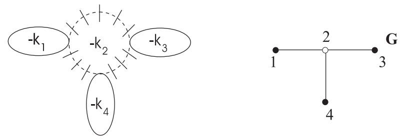


Figure 2: A dicritical cell

Let $\widetilde{\mathcal{C}}^i = \widetilde{\mathcal{F}}_0^i \cup_{k=1}^n \widetilde{\mathcal{F}}_k^i$, $i = 1, 2$, be two germs of dicritical cells, where each $\widetilde{\mathcal{F}}_k^i$ is defined at the zero section of the Hopf bundle $\mathbb{H}_k : (p_k : \mathcal{H}_k \longrightarrow D_k)$. Then one says that $\bigoplus Hol_{D_0}(\widetilde{\mathcal{F}}_k^1, D_k)$ and $\bigoplus Hol_{D_0}(\widetilde{\mathcal{F}}_k^2, D_k)$ have a **Möbius conjugation** if there is $\phi \in Diff(D_0) \simeq PGL(2, \mathbb{C})$ such that $Ad(\phi)(Hol_{D_0}(\widetilde{\mathcal{F}}_k^2, D_k)) = Hol_{D_0}(\widetilde{\mathcal{F}}_k^1, D_k)$. Furthermore, let $\varphi_k \in Diff(D_k) \simeq \mathbb{PGL}(2, \mathbb{C})$ be an isomorphism between $Sing(\widetilde{\mathcal{F}}_k^1)$ and $Sing(\widetilde{\mathcal{F}}_k^2)$ for $k = 1 \cdots n$ and between the corners of $\widetilde{\mathcal{F}}_0^1$ and $\widetilde{\mathcal{F}}_0^2$. Then we say that the Möbius conjugation is **subordinated to** $\varphi := (\varphi_k)_{k=0}^n$ if it is subordinated to each φ_k as a local conjugation, for $k = 1 \cdots n$, and is such that $\phi|_{\text{corners of } \widetilde{\mathcal{F}}_0^1} = \varphi|_{\text{corners of } \widetilde{\mathcal{F}}_0^1}$. Further we set $\mathcal{BH} := \mathcal{H}_0 \cup_{k=1}^n \mathcal{H}_k$ and $BD := D_0 \cup_{k=1}^n D_k$ and define $Diff(\mathcal{BH}, BD)$ to be the set of diffeomorphisms of $\mathcal{BH}$ preserving D_k, for $k = 0 \cdots n$. Moreover we set $Diff_{\widetilde{\mathcal{C}}^1,\widetilde{\mathcal{C}}^2}(\mathcal{BH}, BD) := \{\Phi \in Diff(\mathcal{BH}, BD) : \Phi_*(\widetilde{\mathcal{C}}^1) = \widetilde{\mathcal{C}}^2, \Phi(Sing(\widetilde{\mathcal{F}}^1_k)) = \varphi(Sing(\widetilde{\mathcal{F}}^1_k)\}$. In particular we define the isotropy group of the dicritical cell $\widetilde{\mathcal{C}}$ by $Iso(\widetilde{\mathcal{C}}) := Diff_{\widetilde{\mathcal{C}},\widetilde{\mathcal{C}}}(\mathcal{BH}, BD)$.

Further we shall say that a dicritical cell is **solved** if each of its singularities are reduced. Analogously we shall say that a dicritical cell is **rectified** if each of its non-dicritical component is solved and rectified, and its dicritical component has void first and second tangent cones ([3]). Note that two dicritical cells may intercept, but if it happens, it most be in just one non-dicritical Hopf component of the foliation.

Let us call the larger simply-connected subsets of the resolution space, containing just non-dicritical Hopf components of the foliation, the **non-dicritical branches** of $\widetilde{\mathcal{F}}$. In particular, we shall say that a non-dicritical branch is simple if it has just one projective line with three or more singularities, which shall be called the **principal projective line** of the branch. So we extend the concept of simple resolution foliation in the following way: we say that a resolution of a dicritical foliation is **simple** whenever its has just one dicritical component, and each non-dicritical branch is simple with principal projective line intercepting the dicritical component (see figure 3).

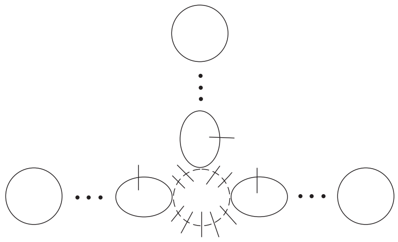


Figure 3: A dicritical foliation with simple extended rectifier resolution

Further, let $\widetilde{\mathcal{F}}$ be a simple non-dicritical branch of a solved and rectified foliation. Then we say that a dicritical singular foliation, say $\widetilde{\mathcal{G}}$, is a **companion fibration** for $\widetilde{\mathcal{F}}$, if $\widetilde{\mathcal{G}}$ satisfies the following properties:

a) $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{G}}$ have the same chain of projective lines (see [3]);
b) $\widetilde{\mathcal{G}}$ is simple and its (only) dicritical component is a transversal fibration (see §5.1) to the principal Hopf component of $\widetilde{\mathcal{F}}$;

We say that a singular foliation $(\mathcal{F} : \omega = 0)$ has a (**extended**) **rectifier resolution**, say $(\widetilde{\mathcal{F}} : \widetilde{\omega} = 0)$ if each of its Hopf components, namely $\widetilde{\mathcal{F}}_j$, satisfy one of the following conditions:

i) $\widetilde{\mathcal{F}}_j$ is non-dicritical, solved, rectified and does not have parallel saddle-nodes;
ii) $\widetilde{\mathcal{F}}_j$ has just two singularities, and in antipodal points of $D_j \approx \mathbb{CP}(1)$: a parallel saddle-node and a (corner) resonant saddle with index -1 with respect to D_j;
iii) $\widetilde{\mathcal{F}}_j$ is dicritical and has void first and second tangent cones.

Notice that similarly to the case of Seidenberg's resolution, we can talk about a minimal rectifier resolution, although this is not unique in general. Two rectifier resolutions, namely $\widetilde{\mathcal{F}}^1$ and $\widetilde{\mathcal{F}}^2$, are **cell-wise equivalent** if they have isomorphic trees of singularities and there is a collection of biholomorphisms Φ_j taking the leaves of $\widetilde{\mathcal{C}}_j^{\,1}$ into the leaves of $\widetilde{\mathcal{C}}_j^{\,2}$. In particular

Notation 1. *Let us denote by* $\sum_\omega^c$ *(respect.* $\sum_\omega^{c,1,F}$*) the set of germs of foliations which has a minimal (extended) rectifier resolution cell-wise equivalent (respect. up to order one for the non-degenerated reduced singularities and formal type for the saddle-nodes) to a minimal rectifier resolution of* $(\mathcal{F} : \omega = 0)$*. Further we denote as* $\sum_{\omega,f}^{c,1,F}$ *the subset of* $\sum_\omega^{c,1,F}$ *whose separatrix set has the same analytic type of the curve* $f^{-1}(0)$*.*

3. Statement of the main results

The above definitions of rectifier resolution is supported by the following

Proposition 1. *Every germ of analytic foliation in* $(\mathbb{C}^2, 0)$ *has an (extended) rectifier resolution.*

Let $(\mathcal{F} : \omega = 0)$ be a germ of foliation in $(\mathbb{C}^2, 0)$, then pick a foliation $\mathcal{F}^{\mathrm{o}}$ cell-wise isomorphic to $\mathcal{F}$, which shall be called a **fixed model** for $\mathcal{F}$, and consider the elements $\Phi_j \in Diff(\widetilde{\mathcal{C}}_j, \widetilde{\mathcal{C}}_j^{\mathrm{o}})$, which shall be called **cell charts** for the j^{th} cell of the fixed model. Then it is straightforward that for each cell $\widetilde{\mathcal{C}}_j$ and each fixed model cell $\widetilde{\mathcal{C}}_j^{\mathrm{o}}$, there exists only one cell chart, up to left composition by an element of $Iso(\widetilde{\mathcal{C}}_j^{\mathrm{o}})$. So, consider the sheaf of non-abelian groups $\Lambda^{\mathrm{o}} := Iso(\widetilde{\mathcal{F}}^{\mathrm{o}})$, then we say that $\mathcal{U} := \cup U_j$ is a good covering for Λ^{o} if U_j are neighborhoods of $D_j \subset \mathcal{H}_j$ or of $BD_j \subset \mathcal{BH}_j$, respectively for non-dicritical or dicritical cells. Therefore, consider the first cohomology set $H^1(\mathcal{U}, \Lambda^{\mathrm{o}})$ associated to the good covering $\mathcal{U}$, and set $H^1(D, \Lambda^{\mathrm{o}})$ as the direct limit of $H^1(\mathcal{U}, \Lambda^{\mathrm{o}})$ for the good coverings for of Λ^{o}, associated to $D = \cup D_j$, the exceptional divisor of a minimal rectifier resolution of $\mathcal{F}$.

Thus we define the map

$$\begin{array}{ccc} \sum_\omega^{\mathrm{c}} & \xrightarrow{\Theta} & H^1(D, \Lambda^{\mathrm{o}}) \\ \mathcal{F} & \mapsto & (\Phi_{i,j}) \end{array}$$

where $\Phi_{i,j} := \Phi_i \circ \Phi_j^{-1}$. Note that Θ does not depend on the fixed models up to cell-wise conjugacy class. Then, as a consequence of path-lifting construction, one has that:

Theorem 1. *Let* $\mathcal{F}, \mathcal{F}'$ *be two germs of singular foliations in* $(\mathbb{C}^2, 0)$, *and* $\widetilde{\mathcal{F}}, \widetilde{\mathcal{F}}'$ *one of their minimal rectifier resolutions respectively. Then* $\mathcal{F}, \mathcal{F}'$ *are analytically equivalent if and only if they satisfy:*

- i) $\mathcal{F}, \mathcal{F}' \in \sum_\omega^{c,1,F}$ *for some* 1*-form* $\omega \in \Omega^1(\mathbb{C}^2, 0)$ *(holomorphic one-forms). In particular we denote by* $\varphi = (\varphi_j) : \Gamma_s \longrightarrow \Gamma_s'$ *the component-wise isomorphism between the trees of singularities of their minimal rectifier resolution.*
- ii) *Two equivalent (up to formal type) transversal saddle-nodes* $t_{i,j}, t'_{i,j} = \varphi_j(t_{i,j})$ *have the same order of symmetry.*
- iii) *For each non-dicritical cell* $\widetilde{\mathcal{F}}_j$ *and* $\widetilde{\mathcal{F}}_j'$ *their projective holonomies have an analytic conjugation, namely* ϕ_j, *subordinated to* φ_j, *and commuting with the Stokes phenomenon of the transversal saddle-nodes of* $\widetilde{\mathcal{F}}_j$ *and* $\widetilde{\mathcal{F}}_j'$.
- iv) *For each dicritical cell* $\widetilde{\mathcal{C}}_j = \widetilde{\mathcal{F}}_{j,0} \cup_{k=1}^{n_j} \widetilde{\mathcal{F}}_{j,k}$ *and* $\widetilde{\mathcal{C}}_j' = \widetilde{\mathcal{F}}_{j,0}' \cup_{k=1}^{n_j} \widetilde{\mathcal{F}}_{j,k}'$, *we have that* $\bigoplus_{k=1}^{n_j} Hol_{D_{j,0}}(\widetilde{\mathcal{F}}_{j,k}, D_{j,k})$ *and* $\bigoplus_{k=1}^{n_j} Hol_{D_{j,0}}(\widetilde{\mathcal{F}}_{j,k}', D_{j,k})$ *have a Möbius conjugation, namely* ϕ_j, *subordinated to* φ, *and commuting with the Stokes phenomenon of the transversal saddle-nodes of* $\widetilde{\mathcal{F}}_{j,k}$ *and* $\widetilde{\mathcal{F}}_{j,k}'$.
- v) $\Theta(\mathcal{F}) = \Theta(\mathcal{F}')$, *for one (and every) fixed model of* $\sum_\omega^c$.

Notice that ii) above answers in an affirmative way the following conjecture of S. M. Voronin (c.f. [10]): *Beyond the resolution tree of the minimal resolution and the projective holonomies there are some further numerical invariants for singular foliations in* $(\mathbb{C}^2, 0)$. In fact, these new invariants are the order of symmetry of each transversal saddle-node.

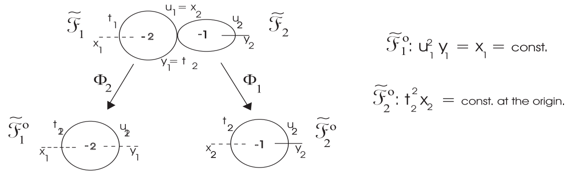


Figure 4: A fixed model for $\sum_{d(y^2+x^4)}^{\mathrm{c},1}$

Recall that in [3], for $f(x,y)=\prod_{j=1}^{d}(y^p-\lambda_j x^q)$, with $1<p<q\in N^*$, $g.c.d.(p,q)=1$, and $\lambda_j\in\mathbb{C}^*$, we classify the set $\sum_{\omega,f}^{\mathrm{c},1}$ (the subset of generalized curves of $\sum_{\omega,f}^{\mathrm{c},1,\mathrm{F}}$) by its projective holonomy, generalizing a result of [6] concerning the subset $\sum_{\omega,y^2+x^n}^{\mathrm{c},1}$. Here we give a slight generalization of this result, including in particular a result of [21] concerning $\sum_{\omega,y^2+x^n}^{\mathrm{c},1,\mathrm{F}}$.

Theorem 2. *Let $f(x,y)=\prod_{j=1}^{d}(y^p-\lambda_j x^q)$, where $1<p<q\in N^*$, $g.c.d.(p,q)=1$, and $\lambda_j\in\mathbb{C}^*$. Then, for every $\mathcal{F}\in\sum_{\omega}^{c,1,F}$ we have that $\mathcal{F}$ has a simple minimal rectifier resolution, and $\Theta(\mathcal{F})$ is trivial in $H^1(D,\Lambda^o)$. In particular if $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{F}}'$ are minimal rectifier resolutions of $\mathcal{F},\mathcal{F}'\in\sum_{\omega}^{c,1,F}$, and $\widetilde{\mathcal{F}}_0,\widetilde{\mathcal{F}}_0'$ their principal projective lines, respectively, then $\mathcal{F}$ and $\mathcal{F}'$ are analytically equivalent if and only if:*

i) *If φ is the isomorphism between the trees of $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{F}}'$ with $t'_{ij}=\varphi(t_{ij})$, then t_{ij} and t'_{ij} have the same order of symmetry, whenever they are transversal saddle-nodes;*
ii) *Their projective holonomies have a conjugation subordinated to φ and commuting with the Stokes phenomenon of the transversal saddle-nodes of $\widetilde{\mathcal{F}}_0$ and $\widetilde{\mathcal{F}}_0'$.*

We would like to stress the importance of the fixed separatrix type in the above theorem, remarking that in general $\sum_{\omega,f}^{\mathrm{c},1,\mathrm{F}}\subsetneq\sum_{\omega}^{\mathrm{c},1,\mathrm{F}}$.

Theorem 3. *Let $\mathcal{F}$ be a germ of dicritical foliation which has a simple minimal rectifier resolution, say $\widetilde{\mathcal{F}}$, whose non-dicritical branches have a companion fibration. Then $\Theta(\mathcal{F})$ is trivial in $H^1(D,\Lambda^o)$. In particular if $\mathcal{F},\mathcal{F}'\in\sum_{\omega}^{c,1,F}$ then they are analytically equivalent if and only if they satisfy:*

i) *If φ is the isomorphism between the trees of $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{F}}'$ with $t'_{ij}=\varphi(t_{ij})$, then $\widetilde{\mathcal{F}}|_{(D_i,t_{ij})}$ and $\widetilde{\mathcal{F}}'|_{(D_i,t'_{ij})}$ have the same order of symmetry, whenever t_{ij} and t'_{ij} are transversal saddle-nodes;*
ii) *For any dicritical cells $\widetilde{\mathcal{C}}_j=\widetilde{\mathcal{F}}_{j,0}\cup_{k=1}^{n}\widetilde{\mathcal{F}}_{j,k}$ and $\widetilde{\mathcal{C}}_j'=\widetilde{\mathcal{F}}'_{j,0}\cup_{k=1}^{n}\widetilde{\mathcal{F}}'_{j,k}$, we have that $\bigoplus_{k=1}^{n}Hol_{D_{j,0}}(\widetilde{\mathcal{F}}_{j,k},D_{j,k})$ and $\bigoplus_{k=1}^{n}Hol_{D_{j,0}}(\widetilde{\mathcal{F}}'_{j,k},D_{j,k})$ have a Möbius conjugation, namely ϕ, subordinated to φ, and commuting with the Stokes phenomenon of the transversal saddle-nodes of $\widetilde{\mathcal{F}}_{j,k}$ and $\widetilde{\mathcal{F}}'_{j,k}$.*

As we shall verify, a particular instance of a foliation in the hypothesis of the above theorem happens when the minimal (Seidenberg) resolution of a dicritical singularity is simple and has just tangencies and the kind of singularities we study in theorem2, in the dicritical component.

We want to point out that theorem 1, is proved in §8, proposition 1 is proved in §7.2, theorem 2 is proved in §9.2, and theorem 3 is proved in §9.3.

In a next work we shall describe more precisely $H^1(D,\Lambda^{\mathrm{o}})$ showing that, roughly speaking, it can be split in the first cohomology group of the coherent sheaf of vector fields tangent to the model $\mathcal{F}^{\mathrm{o}}$ and the first cohomology group of a (in general finitely generated) sheaf of abelian groups generated by the centralizers of the projective holonomy groups of $\widetilde{\mathcal{F}}_j^{\mathrm{o}}$. In particular we will present some examples for which the cohomology is non-trivial.

4. Saddle-nodes and weak holonomy

Now let us understand the relationship between saddle-nodes with central variety and its weak holonomy. First we recall that for a non-degenerated reduced singularity, any conjugation between the holonomies of one of its separatrices can be lift to a (unique) fiber-equivalence between the singularities ([21]). For the saddle-nodes this is even true for the strong holonomy of singularities with fixed formal type but not in general for the weak holonomy ([19]). On the other hand, by the following proposition, there are a few number of such isotropies, thus we can find a "suitable" equivalence class between elements of the weak holonomy of saddle-nodes with a fixed formal type — finer than the conjugacy class — that can be lift to a equivalence class between two saddle-nodes. In order to verify this we need some preliminary results.

Let $\Psi \in Diff(\mathbb{C}^2, 0)$ be an analytic equivalence between the saddle-nodes $(\mathcal{F} : \omega = 0)$ and $(\mathcal{F}' : \omega' = 0)$, i.e. $\Psi^*\omega' \wedge \omega = 0$. Then we have in particular that $\mathcal{F}$ and $\mathcal{F}'$ have the same formal normal form $\mathcal{F}_{p,\lambda}$. Therefore we have the expression of Ψ in sectorial charts given by

$$\Psi_j := ((N_{\mathcal{F}'})_j \circ \Phi_{\mathcal{F}'}) \circ \Psi \circ ((N_{\mathcal{F}})_j \circ \Phi_{\mathcal{F}})^{-1}$$

where $\Phi_{\mathcal{G}} \in Diff(\mathbb{C}^2, 0)$ is the analytic Dulac reduction, $\widehat{N}_{\mathcal{G}}$ is the formal Dulac reduction and $N_{\mathcal{G}} \underset{1/p}{\longrightarrow} \widehat{N}_{\mathcal{G}}$. In particular we have the compatibility equations

$$\Psi_{j+1} = (N_{\mathcal{F}'})_{j+1,j} \circ \Psi_j \circ (N_{\mathcal{F}})^{-1}_{j+1,j} \tag{4.1}$$

in $U_{j,j+1} = U_j \cap U_{j+1}$, where $(N_{\mathcal{G}})_{j+1,j} := (N_{\mathcal{G}})_{j+1} \circ (N_{\mathcal{G}})_j^{-1}$, with $\mathcal{G} \in \{\mathcal{F}, \mathcal{F}'\}$, and $j \in \mathbb{Z}_{2p}$.

Now, let $\Lambda(\mathcal{F}_{p,\lambda})(U) := \{\Psi \in Iso(\mathcal{F}_{p,\lambda})(U) : \Psi(x, y) = (x, g(x, y))\}$, then the sectorial expression for an analytic equivalence is determined by the following:

Proposition 2 ([19]). *Let $U \subset \mathbb{C}$ be a sectorial open set with vertex at the origin, and $\Psi(x, y) = (x, g(x, y)) \in \Lambda(\mathcal{F}_{p,\lambda})(U)$. Then one has that:*

(i) *If $U \subset \mathrm{Re}(x^p) > 0$ then*
$$g(x, y) = a_1 y + a_0 x^\lambda \cdot \exp(-1/px^p).$$
(ii) *If $U \subset \mathrm{Re}(x^p) < 0$ then*
$$g(x, y) = a_1 y + \sum_{n\geq 2} a_n [x^\lambda \cdot \exp(1/px^p)]^{n-1} y^n,$$
where $\sum_{n\geq 2} a_n t^n \in \mathbb{C}\{x\}$.
(iii) *If $U \subset \mathrm{Re}(x^p) = 0$ then*
$$g(x, y) = a_1 y.$$
where $a_n \in \mathbb{C}$, for all $n \geq 0$.

Let $a \in U_j$, then the set $\Delta_j(a, \varepsilon) := \{z \in U_j : |a| - \varepsilon < |z| < |a| + \varepsilon\}$ is called the **sectorial annulus** through a of radii $|a| - \varepsilon$ and $|a| + \varepsilon$.

Lemma 1. *Let $a \in U_j$ and $\Delta_j(a, \varepsilon)$ the sectorial annulus through a of radii $|a| - \varepsilon$ and $|a| + \varepsilon$. Let $\Phi_\varepsilon \in \Lambda(\mathcal{F}_{p,\lambda})(U_j(a, \varepsilon))$ be such that $(\Phi_\varepsilon|_{\Sigma_a})(y) = \exp(\frac{2\pi i}{q}) \cdot y$ then $\Phi_\varepsilon(x, y) = (x, \exp(\frac{2\pi i}{q}) \cdot y)$. That is, if Φ_ε is a q-rotation along the fiber $(x = a)$ it is a q-rotation along all fibers.*

Proof. The proof is obtained in a similar fashion to the proof of the proposition above. In fact, a straightforward calculation from $\Phi_\varepsilon^*(\omega_{p,\lambda}) \wedge \omega_{p,\lambda} = 0$, shows that $\Phi_\varepsilon(x, y) = (x, g(x, y))$, where $g(x, y) = \sum_{n\geq 0} g_n(x) y^n$, with $g_n(x) = a_n[\exp(-\frac{1}{px^p}) \cdot x^\lambda]^{1-n}$. Hence, as $\Phi_\varepsilon(a, y) = (a, e^{\frac{2\pi i}{q}} \cdot y)$, then $a_n = \exp(\frac{2\pi i}{q})$ if $n = 1$ and $a_n = 0$ otherwise. □

Theorem 4. *Let $\mathcal{F}^j$, $j = 1, 2$, be two saddle-nodes with central variety. Then they are analytically equivalent if and only if.*

i) *They have the same formal normal form $\mathcal{F}_{p,\lambda}$;*
ii) *Their Stokes phenomenon have the same order of symmetry, namely q;*
iii) *Their holonomies have a conjugation which is q-periodic in sectorial charts.*

Proof. First let us suppose that $\mathcal{F}^1$ and $\mathcal{F}^2$ are analytically equivalent, then in particular they are formally equivalent and their Stokes phenomenon are equivalent. Thus i) and ii) follows. On the other hand, by proposition 2.(iii) and the compatibility equations (4.1) we have that $\Phi(x,y)=(x,e^{\frac{2\pi i}{q}}\cdot y)$ commutes with the Stokes Phenomenon of $\mathcal{F}^i$. But as $N(\mathcal{F}^i)_{j+1,j}$ is tangent to the identity in each fiber for all $j\in\mathbb{Z}_{2p}$, then a_1 must be a q-root of unity, where q is the order of symmetry of the Stokes Phenomenon of $\mathcal{F}^1$ and $\mathcal{F}^2$.

Now let us verify the converse. Suppose that $\mathcal{F}^1$ and $\mathcal{F}^2$ satisfy i), ii) and iii) above. Then, by path lifting, one can extend the holonomy conjugation to an equivalence, say Ψ, between $\mathcal{F}^1$, and $\mathcal{F}^2$ in a sectorial annulus, say $\widetilde{\Delta}_j(a,\varepsilon)$, so by lemma 1 one has that it can be lift to the whole "sector" $\widetilde{U}_j$, as a q-rotation in sectorial charts. Now, by ii) and the compatibility equations (4.1) one has an equivalence between $\mathcal{F}^1$ and $\mathcal{F}^2$ in $\widetilde{U}_j\cap\widetilde{U}_{j+1}$. Applying once again lemma 1 we can extend it to the whole sector $\widetilde{U}_{j+1}$, and so on. Then the compatibility conditions guarantees a conjugation between $\mathcal{F}^1$, and $\mathcal{F}^2$ in a neighborhood of $(x=0)$, except for itself. But, by the asymptotic behavior of $(N_{\mathcal{F}})_j$ near the origin, we have (in view of (4.1)) that Ψ can be extended continuously to $(x=0)$. Finally applying Riemann's extension theorem ([12]), the result follows. ☐

5. Non-dicritical cells and projective holonomy

5.1. **Non-degenerated reduced singularities.** Now let us recall a result of [3] which determines the analytic type of solved rectified non-dicritical foliations in Hopf bundles.

In order to clarify these ideas we recall the following:

Definition 1. *Let $\mathbb{H}:(p:\mathcal{H}\to\mathbb{CP}(1))$ be the Hopf bundle, $\mathcal{F}$ a germ of singular solved non-dicritical foliation at $\mathbb{CP}(1)$, without saddle nodes. Then we say that a germ of holomorphic map at $\mathbb{CP}(1)$, $f:(\mathcal{H},\mathbb{CP}(1))\to\mathbb{CP}(1)$ is a* ***transversal fibration*** *to $\mathcal{F}$, if it satisfies:*

1. *f is a retraction, that is f is a submersion and $f|_{\mathbb{CP}(1)}=Id|_{\mathbb{CP}(1)}$;*
2. *the fiber $f^{-1}(t_j)$ is a separatrix of $\mathcal{F}$ for each $t_j\in Sing(\mathcal{F})$;*
3. *$f^{-1}(t)$ is transversal to $\mathcal{F}$ for every point $t\in\mathcal{F}\backslash Sing(\mathcal{F})$.*

In fact this is a fundamental tool for the proof of:

Proposition 3 ([3]). *Let $\mathbb{H}:(p:\mathcal{H}\to\mathbb{CP}(1))$ be a Hopf bundle, $\mathcal{F}_i$, $i=1,2$, be two solved and rectified non-dicritical foliations with just non-degenerated singularities in $\mathcal{H}$. Further let $\varphi\in PSL(2,\mathbb{C})$ a conjugation between the trees of singularities of $\mathcal{F}_1$ and $\mathcal{F}_2$. Then $\mathcal{F}_1$ and $\mathcal{F}_2$ are analytic conjugated if and only if their projective holonomies have a conjugation subordinated to φ.*

5.2. **Parallel saddle-nodes.** We describe the invariants which determine the analytic type of solved foliations, with parallel saddle-nodes along the extended resolution of a germ of singularity.

Recall that according to proposition 1, we only have to deal with parallel saddle-nodes appearing in solved non-dicritical Hopf components, say $\widetilde{\mathcal{F}}_j$, which has just two singularities, and in antipodal points of $D_j\approx\mathbb{CP}(1)$: a parallel saddle-node and a (corner) resonant saddle with index -1 with respect to D_j. Hereafter we shall briefly refer to these foliations as of the **type (*)**.

Lemma 2. *Every singularity of a germ of foliation of type (*) comes from the blow-up of a saddle-node.*

Proof. First notice that we may suppose that the foliation of type (*), namely $(\widetilde{\mathcal{F}}:\widetilde{\omega}=0)$, defined in $(\mathcal{H},D)$, is given by

$$\left\{\begin{array}{l}\widetilde{\omega}(t,x)=(\alpha t+\cdots)dx+(\alpha x+\cdots)dt,\\ \widetilde{\omega}(u,y)=(\beta u+\cdots)dy+(\cdots)du,\end{array}\right. \tag{5.1}$$

$\alpha,\beta\in\mathbb{C}^*$ (where the dots denote higher order terms). For, after a blow-down and a holomorphic change of variables, we may suppose that the separatrix of $\mathcal{F}:=\pi_*(\widetilde{\mathcal{F}})$, determined by the resonant saddle, is the (germ of) axis $(y=0)$. Now, suppose that $(\mathcal{F}:\omega=0)$ is given by $\omega(x,y)=A(x,y)dy+B(x,y)dx$. So after a blow-up we obtain that

$$\pi^*\omega(t,x)=\sum_{\nu\geq p}(x^\nu B_\nu(1,t)+tx^\nu A_\nu(1,t))dx+\sum_{\nu\geq p}x^{\nu+1}A_\nu(1,t)dt.$$

where $A(x,y)=\sum_{\nu\geq p}A_\nu(x,y)$, $B(x,y)=\sum_{\nu\geq p}B_\nu(x,y)$, with A_ν,$B_\nu\in\mathbb{C}[x,y]$ homogeneous polynomials of order ν, and $p=\min\{ord(A),ord(B)\}$. Hence

$$\begin{aligned}\widetilde{\omega}(t,x)=\frac{\pi^*\omega(t,x)}{x^p}&=[B_p(1,t)+tA_p(1,t)+x\sum_{\nu\geq p+1}x^{\nu-(p+1)}(B_\nu(1,t)+tA_\nu(1,t))]dx\\&+x\sum_{\nu\geq p}x^{\nu-p}A_\nu(1,t)dt;\\\widetilde{\omega}(u,y)=\frac{\pi^*\omega(u,y)}{y^p}&=[A_p(u,1)+uB_p(u,1)+y\sum_{\nu\geq p+1}y^{\nu-(p+1)}(A_\nu(u,1)+uB_\nu(u,1))]dy\\&+y\sum_{\nu\geq p}y^{\nu-p}B_\nu(u,1)du.\end{aligned}$$

As the only singularity in the chart (t,x) is a resonant one given by (5.1), then one has that $A_p(1,t)=\alpha\operatorname{mod}O(|t|)$ and $B_p(1,t)+tA_p(1,t)=at$, thus

$$B_p(1,t)=0\ \text{ and } A_p(1,t)=\alpha. \tag{5.2}$$

Analogously, the only singularity in the chart (u,y) is a parallel saddle-node (once again given by (5.1)), then $A_p(u,1)+uB_p(u,1)=bu$ and $B_p(u,1)=\gamma u\operatorname{mod}O(|u|)$, and thus

$$B_p(u,1)=0\ \text{ and }\ A_p(u,1)=\beta. \tag{5.3}$$

Now let $A_p(x,y)=\sum_{\nu\geq 0}a_\nu x^\nu y^{p-\nu}$ and $B_p(x,y)=\sum_{\nu\geq 0}b_\nu x^\nu y^{p-\nu}$. Therefore by (5.2) and (5.3) we have that $\alpha=\sum_{\nu=0}^p a_\nu t^{p-\nu}$, $\beta=\sum_{\nu=0}^p a_\nu u^\nu$, and $B_p\equiv 0$. Hence

$$a_\nu=\begin{cases}\alpha & \text{if}\quad \nu=p-1,\\ 0 & \text{if}\quad \nu\neq p-1;\end{cases}\quad\text{and}\quad a_\nu=\begin{cases}\beta & \text{if}\quad \nu=0,\\ 0 & \text{if}\quad \nu\neq 0.\end{cases}$$

Finally we have that $p=1$, $\alpha=\beta$, $A_1(x,y)=\alpha x$, and $B_1(x,y)\equiv 0$; and the result follows. ☐

Now we study the relationship between the formal type and Stokes Phenomenon of the parallel saddle-node of the type (*) foliation, namely $(\widetilde{\mathcal{F}}:\widetilde{\omega}=0)$, and of its blow-down, say $\mathcal{F}=\pi_*(\widetilde{\mathcal{F}})$. First recall that $\widehat{N}(\mathcal{F})$ denotes the formal normalizer of $\mathcal{F}$ and that $N_j(\mathcal{F})$, $j\in\mathbb{Z}_{2p}$ are the sectorial charts of $\mathcal{F}$. Further, let $\widetilde{\widehat{N}(\mathcal{F})}:=\pi^*\widehat{N}(\mathcal{F})$ and $\widetilde{N_j(\mathcal{F})}:=\pi^*N_j(\mathcal{F})$, then

Lemma 3. *Suppose that $\mathcal{F}$ has the same formal type of $(\mathcal{F}_{p,\lambda}:\omega_{p,\lambda}=0)$. Then the parallel saddle-node of $\widetilde{\mathcal{F}}$ has the formal type of $(\widetilde{\mathcal{F}}_{p,\lambda-1}:\omega_{p,\lambda-1}=0)$, and its formal normalizer is given by $\widetilde{\widehat{N}(\mathcal{F})}$, i.e. $\widehat{N}(\widetilde{\mathcal{F}})=\widetilde{\widehat{N}(\mathcal{F})}$.*

Proof. As a formal conjugation lifts under blow-up, then this is just a matter to verify the behavior of $\mathcal{F}_{p,\lambda}$ after a blow-up. In fact,

$$\widetilde{\omega}_{p,\lambda}(u,y)=\frac{\pi^*\omega_{p,\lambda}(uy,y)}{y}=y^{p+1}du-u(1+(\lambda-1)y^p)dy=\omega_{p,\lambda-1}(u,y).$$

☐

Lemma 4. *The sectorial charts of the parallel saddle-node of $\widetilde{\mathcal{F}}$ are given by $\widetilde{N_j(\mathcal{F})}$, i.e. $N_j(\widetilde{\mathcal{F}}) = \widetilde{N_j(\mathcal{F})}$.*

Proof. As $\pi : (\mathcal{H}, D)\backslash D \longrightarrow (\mathbb{C}^2, 0)\backslash\{0\}$ is a biholomorphism, then it only rests to prove that $\widetilde{N_j(\mathcal{F})} \underset{1/p}{\longrightarrow} \widetilde{\widehat{N}(\mathcal{F})}$. For, recall that $\widehat{N}(\mathcal{F})(x, y) = (x + \sum_{j\geq 1} \underline{a_j}(x)y^j, y)$, where the series diverges in y, and all functions $\underline{a_j}$ are holomorphic with the same radius of convergence in some neighborhood of the origin, and that $N_j(\mathcal{F}) = (x + \sum_{j\geq 1} a_j(x)y^j, y)$ where $a_j \in \mathcal{O}(U_j)$. Hence

$$\widetilde{N_j(\mathcal{F})}(u, y) = \pi^* N_j(\mathcal{F})(u, y) = N_j(\mathcal{F})(uy, y) = (uy + \sum_{j\geq 1} a_j(uy)y^j, y)$$

and then as, $N_j(\mathcal{F}) \underset{1/p}{\longrightarrow} \widehat{N}(\mathcal{F})$, we have that

$$\begin{aligned} ||\widetilde{N_j(\mathcal{F})}(u, y) - (\sum_{j=1}^{k-1} \underline{a_j}(uy)y^j, y)|| &= ||\sum_{j\geq 1} a_j(uy)y^j - \sum_{j=1}^{k-1} \underline{a_j}(uy)y^j|| \\ &\leq C_j(k!)^{p-1}A_j^k|y|^k. \end{aligned}$$

□

Therefore we have that:

Proposition 4. *Let $\mathbb{H} : (p : \mathcal{H} \to \mathbb{CP}(1))$ be a Hopf bundle, $\widetilde{\mathcal{F}}^i$, $i = 1 \ldots 2$, be two foliations of type (*) defined in $\mathcal{H}$, whose parallel saddle-node has the same formal type. Further let $\varphi \in PSL(2, \mathbb{C})$ a conjugation between the trees of singularities of $\mathcal{F}_1$ and $\mathcal{F}_2$. Then $\mathcal{F}_1$ and $\mathcal{F}_2$ are analytic conjugated if and only if their projective holonomies have a conjugation subordinated to φ.*

Proof. First notice that by lemma 3 we may suppose that both the parallel saddle node of $\widetilde{\mathcal{F}}^i$ as the saddle-nodes $\mathcal{F}^i := \pi_*(\widetilde{\mathcal{F}}^i)$, are in the Dulac analytic normal form. Thence by [19] we have that the Stokes Phenomenon of $\widetilde{\mathcal{F}}^1$ and $\widetilde{\mathcal{F}}^2$ are equivalent, that is, locally we have that $N_{j+1,j}(\widetilde{\mathcal{F}}^1) = N_{j+1,j}(\widetilde{\mathcal{F}}^2)$, but by the identity theorem ([12]), we have that the equality holds in $(\mathcal{H}, D)\backslash(t = 0)$. But, by the uniqueness of sectorial charts (c.f. [15]), we have that $\widetilde{N_j(\mathcal{F}^i)} = N_j(\widetilde{\mathcal{F}}^i)$. So after a blow-down, the result follows as

$$N_{j+1,j}(\mathcal{F}^i) = \widetilde{N_{j+1}(\mathcal{F}^i)} \circ \pi^{-1} \circ \pi \circ \left(\widetilde{N_j(\mathcal{F}^i)}\right)^{-1} = N_{j+1,j}(\widetilde{\mathcal{F}}^i).$$

□

5.3. **Transversal saddle-nodes.** We describe necessary and sufficient conditions for two solved non-dicritical cells, whose possible saddle-nodes are transversal, to be analytically equivalent. In fact we relate their trees of singularities, reduced singularities first jets (for non-degenerated singularities) and formal type (for saddle-nodes), and projective holonomies.

Proposition 5. *Let $\widetilde{\mathcal{F}}^i$, $i = 1, 2$, be two solved and rectified non-dicritical cells which are cell-wise equivalent up to first order (for non-degenerated singularities) and formal type (for saddle-nodes) by a conjugation, say $\varphi : D^1 \longrightarrow D^2$, and whose possible saddle-nodes are all transversal. Then they are analytically equivalent if and only if they satisfy*

i) *The Stokes phenomenon of two equivalent (up to formal type) saddle-nodes have the same order of symmetry;*
ii) *The projective holonomies of $\widetilde{\mathcal{F}}^1$ and $\widetilde{\mathcal{F}}^2$ have a conjugation, namely ϕ, subordinated to φ;*
iii) *ϕ commutes with the stokes phenomenon of $\widetilde{\mathcal{F}}^1$ and $\widetilde{\mathcal{F}}^2$.*

Proof. Clearly if $\widetilde{\mathcal{F}}^1$ and $\widetilde{\mathcal{F}}^2$ have a global conjugation then conditions i) and ii) hold. On the other hand, condition iii) comes from proposition 2 in view of the compatibility equations (4.1). In fact we just have to restrict the global conjugation to fibers of $\mathbb{H}^1$, and $\mathbb{H}^2$, equivalent under φ. Now let us verify the converse. In fact, as the foliations are rectified, then by i) and ii) one can successively apply path lifting arguments as in [21], and Riemann extension theorem ([12]) to extend ϕ to a fibered conjugation, namely Φ, between $\widetilde{\mathcal{F}}^1$ and $\widetilde{\mathcal{F}}^2$, except for a fiber neighborhood of their (equivalent transversal) saddle-nodes. Now, by iii) one can apply lemma 1 and Riemann extension theorem to extend Φ to a whole neighborhoods of the zero section of the bundles. □

6. Dicritical Cells and Möbius conjugation

We describe the necessary and sufficient conditions for two "solved and rectified" dicritical cells to be analytic equivalent, relating their separatrices, (possible) transversal saddle-nodes, and projective holonomies.

First let us introduce some notation in order to clarify the ideas. Let $\mathbb{H} : (p : \mathcal{H} \longrightarrow D)$ be a Hopf bundle, we shall say that a fibration $f : \mathcal{H} \longrightarrow D$, has a **normal fiber** if there is a bundle atlas (t, x), (u, y) such that $f^{-1}(0) = (u = 0)$.

Proposition 6. *Let $\widetilde{\mathcal{C}}^i = \widetilde{\mathcal{F}}^i_0 \cup \widetilde{\mathcal{F}}^i_l$, $i = 1, 2$, $l = 1 \cdots k$, be two solved and rectified dicritical cells whose singularities trees are equivalent up to first order (for non-degenerated singularities) and formal type (for saddle-nodes) by a collection of homographies, say $\varphi := \varphi^i_l : D^1_l \longrightarrow D^2_l$. Further, suppose that each $\widetilde{\mathcal{F}}^i_0$ has a normal fiber, in two equivalent points. Then they are analytically equivalent if and only if they satisfy:*

i) *The Stokes phenomenon of two equivalent (up to formal type) saddle-nodes have the same order of symmetry;*
ii) *$\widetilde{\mathcal{C}}^1$ and $\widetilde{\mathcal{C}}^2$ have a Möbius conjugation, namely ϕ, subordinated to φ;*
iii) *ϕ commutes with the stokes phenomenon of $\widetilde{\mathcal{F}}^1_l$ and $\widetilde{\mathcal{F}}^2_l$.*

Proof. Clearly if $\widetilde{\mathcal{C}}^1$ and $\widetilde{\mathcal{C}}^2$ have a global conjugation then conditions i) and ii) hold. On the other hand, condition iii) comes from proposition 2 and the compatibility equations (4.1). Now let us verify the converse. First choose bundle charts for $\mathcal{H}^i_0$, say (t_i, x_i), (u_i, y_i) such that $(u_i = 0)$ is a normal fiber of $\widetilde{\mathcal{F}}^i_0$. Now consider in the chart (t_i, x_i), the fibration given by $(x_i = const.)$, which is transverse to $\widetilde{\mathcal{F}}^i_0$. Then by iii) one can extend the conjugation $\phi : D^1_0 \longrightarrow D^2_0$ by path lifting a curve in a given leaf of $\widetilde{\mathcal{F}}^i_0$ along this fibration, thus obtaining the conjugation $\Phi_\varepsilon : A^1_\varepsilon \longrightarrow A^2_\varepsilon$, where $A^i_\varepsilon = \{(t_i, x_i) \in \mathbb{C}^2 : |t_i| < \varepsilon,\ |x_i| < \frac{1}{\varepsilon}\}$, between $\widetilde{\mathcal{F}}^1_0|_{A^1_\varepsilon}$ and $\widetilde{\mathcal{F}}^2_0|_{A^2_\varepsilon}$.

On the other hand, note that in the bundle chart (u_i, t_i) the "fibration" $(x_i = const)$ turns out to be the singular foliation $\mathcal{G}_i$:$(u_i^k y_i = const)$, where $-k = -k^i_0$ is the Chern class of $\mathbb{H}^i_0$ (notice that they coincide as their trees of singularities are equivalent). Indeed $\widetilde{\mathcal{F}}^i_0$ is transversal to $\mathcal{G}^i$ except for the common leaf $(u_i = 0)$. Therefore, if we revert our standpoint, that is look $\mathcal{G}_i$ as the singular foliation and $\widetilde{\mathcal{F}}^i_0$ as a local transversal fibration (see figure 5), then we have Φ_ε preserving both two (by construction). So, applying the path lifting argument in [21] (now with $\widetilde{\mathcal{F}}_{j,0}$ as the transversal fibration and $\mathcal{G}^i$ as the singular foliation), we can extend Φ_ε to a neighborhood of the normal fibers in $\mathcal{H}^i_0$. Finally, applying ii) and iii) we can extend Φ_ε to a neighborhood of D^i_l, $l = 1 \cdots n$, in $\mathcal{H}^i_l$ and thus to the whole cells. □

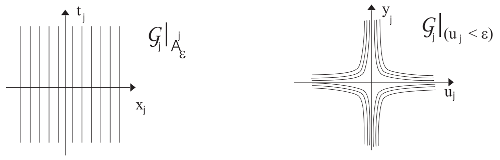


Figure 5

7. Rectifier resolution and cells

7.1. **Non-dicritical cells and blowing-ups.** We verify the behavior of a solved non-dicritical cell after blowing-up some of their (already reduced) singularities.

7.1.1. *Reduced non-degenerated singularities.* Let us recall from [3], what happens when we blow-up a reduced non-degenerated singularity.

Lemma 5 ([3])**.** *Let $(\mathcal{F} : \omega = 0)$ be the reduced foliation ($\lambda \notin Q^-$) leaving the x-axis invariant given by $\omega(x, y) = (x + A(x, y))dy + \lambda y(1 + B(x, y))dx)$, where $A = 0 \bmod O(|(x, y)|^2)$ and $B = 0 \bmod O(|(x, y)|)$. Then after one blow-up we obtain a foliation with just two singularities at $\mathbb{CP}(1)$ in $\mathcal{H}(-1)$, given by:*

$$\begin{aligned}
\widetilde{\omega}(t, x) &= t\left(1 + \lambda + \tfrac{A(x,tx)}{x} + \lambda B(x, tx)\right) dx + (x + A(x, tx))\, dt,\\
\widetilde{\omega}(u, y) &= \left((1 + \lambda)u + \tfrac{A(uy,y)}{y} + \lambda u B(uy, y)\right) dy + \lambda y\,(1 + B(uy, y))\, du.
\end{aligned}$$

7.1.2. *Parallel saddle-nodes.* Now let us verify the behavior of parallel saddle-node after a blow-up

Lemma 6. *Let $(\mathcal{F} : \omega = 0)$ be a saddle node with the x-axis as the strong variety, given by $\omega(x, y) = y^{p+1}dx + (x + yA(x, y))dy$, where $A(0, 0) = 0$ and $p \geq 1$. Then after one blow-up we obtain a foliation with just two singularities at $\mathbb{CP}(1) \subset \mathcal{H}(-1)$, given by:*

$$\begin{aligned}
\widetilde{\omega}(t, x) &= t(1 + t + A(x, tx) + t^p x^p)dx + x(1 + tA(x, tx))dt,\\
\widetilde{\omega}(u, y) &= (u + A(uy, y) + uy^p)dy + y^{p+1}du.
\end{aligned}$$

Proof. In fact after one blow-up we obtain

$$\begin{aligned}
\omega(x, tx) &= t^{p+1}x^{p+1}dx + (x + txA(x, tx))(tdx + xdt)\\
&= tx\,(1 + tA(x, tx) + t^p x^p)\, dx + x^2\,(1 + tA(x, tx))\, dt
\end{aligned}$$

and

$$\begin{aligned}
\omega(uy, y) &= y^{p+1}(udy + ydu) + (uy + yA(uy, y)))dy\\
&= y\,(u + A(uy, y) + uy^p)\, dy + y^{p+2}du.
\end{aligned}$$

And the result follows by dividing the equations by x any y respectively. □

7.1.3. *Transversal saddle-nodes.* Now let us verify what happens when we blow-up a transversal saddle-node.

Lemma 7. *Let $(\mathcal{F} : \omega = 0)$ be a saddle node with the y-axis as the strong variety, given by $\omega(x, y) = x^{p+1}dy + (y + xA(x, y))dx$, where $A(0, 0) = 0$ and $p \geq 1$. Then after one blow-up we obtain a foliation with just two singularities at $\mathbb{CP}(1) \subset \mathcal{H}(-1)$, given by:*

$$\begin{aligned}
\widetilde{\omega}(t, x) &= x^{p+1}dt + (t + tx^p + A(x, tx))dx,\\
\widetilde{\omega}(u, y) &= (u + u^2A(uy, y) + u^{p+1}y^p)dy + (y + uyA(uy, u))du.
\end{aligned}$$

Proof. In fact after one blow-up we obtain

$$\begin{aligned}\omega(x, tx) &= x^{p+1}(tdx + xdt) + (tx + xA(x, tx))dx \\ &= x\left(t + x^p + A(x, tx) + t^p x^p\right)dx + x^{P+2}dt\end{aligned}$$

and

$$\begin{aligned}\omega(uy, y) &= u^{p+1}y^{p+1}dy + (y + uyA(uy, y))(udy + ydu) \\ &= uy(1 + uA(uy, y) + u^p y^p)dy + y(y + uyA(uy, y))du.\end{aligned}$$

And the result follows by dividing the equations by x any y respectively. □

7.1.4. *Former Hopf component.* Now let us recall from [3] the behavior of a Hopf component of the minimal resolution after we blow up some of their non-corner singularities.

Lemma 8 ([3]). *Let $\mathbb{H}(-k) : \left(p_{(k)} : \mathcal{H}(-k) \to \mathbb{CP}(1)\right)$ be a Hopf bundle of class $-k$, and $(\widetilde{\mathcal{F}} : \widetilde{\omega} = 0)$ be a germ of solved non-dicritical singular foliation in $\mathbb{CP}(1) \subset \mathcal{H}(-k)$. Then, after a blow-up in each of its m reduced singularities, $\mathcal{H}(-k)$ turns out to be $\mathcal{H}(-k-m)$ and the Hopf component of the transform of $\widetilde{\mathcal{F}}$, namely $\widetilde{\widetilde{\mathcal{F}}} = \pi^*(\widetilde{\mathcal{F}})$, has its separatrix' Hopf component contained in fibers of $\mathbb{H}(-k-m) : \left(p_{(k+m)} : \mathcal{H}(-k-m) \to \mathbb{CP}(1)\right)$, where $p_{(k+m)} = \pi^{-1} \circ p_{(k)} \circ \pi$, with π being the proper map obtained by the composition of the m blow-ups.*

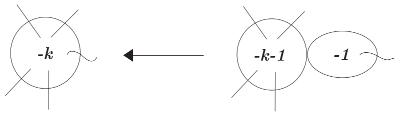


Figure 6: A rectifier blow-up

7.2. **The rectifier resolutions.** Now we show how the above information fit together in order to obtain the rectifier resolutions.

Proof of proposition 1. First let us consider the minimal resolution of a germ of non-dicritical foliation $\mathcal{F}$, say $\widetilde{\mathcal{F}}$. Then each non-dicritical Hopf component of $\mathcal{F}$, namely $\widetilde{\mathcal{F}}_j$, has $n_j := \#Sing(\widetilde{\mathcal{F}}_j)$ reduced singularities. Then, after we blow-up each of these n singularities we obtain a Hopf bundle with Chern class $c(\mathcal{H}_j) - n$ with only corner singularities, say $\mathcal{H}_{j,0}$, and n Hopf bundle of Chern class -1, say $\mathcal{H}_{j,l}$, $l = 1 \cdots n$ intercepting the previous one by normal crossing at corner singularities (see figure 6). In particular, the new resolution will satisfy conditions i) and ii) of the definition (see introduction) by lemmas 5, 6, 7 and 8. Finally, in order to obtain the extended rectifier resolution we have first to recall that in [2] it is proved that any dicritical foliation $\mathcal{F}$ has an extended resolution $\widetilde{\mathcal{F}}$. That is, after a finite number of blow-up we obtain a germ of foliation in a neighborhood of the exceptional divisor of the transform, such that:

a) The non-dicritical components of $\widetilde{\mathcal{F}}$ are solved, that is, they have just reduced singularities;
b) The dicritical components of $\widetilde{\mathcal{F}}$ are pair-wise non-intercepting, and have no singular or tangential points in the zero section of its Hopf bundle.

Hence, in order to obtain a rectifier resolution for a dicritical singularity, we just have to (possibly) repeat the process of blowing-up some singularities of the non-dicritical components of $\widetilde{\mathcal{F}}$, as in the non-dicritical case, in order to obtain the desired result. □

8. Analytic classification

We analytically classify the planar singularities by the trees of singularities of one of their minimal rectifier resolutions up to order one (for non-degenerated singularities) and formal type (for saddle-nodes), the projective holonomies of the Hopf components of these resolutions and some analytical cocycles that appear as obstructions to extend component-wise conjugacies.

8.1. **Component-wise isomorphisms.** We find conditions to determine if two component-wise isomorphic germs of foliations up to first order (for non-degenerated singularities) and formal type (for saddle-nodes) are indeed component-wise isomorphic, that is determine the moduli space $\sum_{\omega}^{\mathrm{c},1,\mathrm{F}} / \sum_{\omega}^{\mathrm{c}}$ (recall notation 1) for a minimal rectifier resolution of $(\mathcal{F} : \omega = 0)$. Finally we verify the uniqueness of the ambient surface for the minimal rectifier resolution of any element of $\sum_{\omega}^{\mathrm{c},1,\mathrm{F}}$.

First, recall that $\sum_{\omega}^{\mathrm{c}}$ (respect. $\sum_{\omega}^{\mathrm{c},1,\mathrm{F}}$) is the set of germs of foliations whose minimal rectifier resolution is component-wise analytically equivalent (respect. up to order one for non-degenerated singularities and formal type for saddle-nodes) to the minimal rectifier resolution of $(\mathcal{F} : \omega = 0)$. Further we denote as $\sum_{\omega}^{\mathrm{c},1,\mathrm{F}}$ the subset of $\sum_{\omega}^{\mathrm{c},1}$ whose separatrix set has the same analytic type of the curve $f^{-1}(0)$.

Now as a straightforward consequence of propositions 4, 5, 6, and Poincaré's linearization theorem (in case of a radial foliation) we have that:

Proposition 7. *Let $\mathcal{F}, \mathcal{F}'$ belong to the same class in $\in \sum_{\omega}^{c,1,F}$, with tree isomorphism given by $\varphi : \Gamma \longrightarrow \Gamma'$. Then they belong to the same class in $\sum_{\omega}^{c}$ if and only if they satisfy:*

i) *The Stokes phenomenon of two equivalent (up to formal type) saddle-nodes have the same order of symmetry;*
ii) *Any two non-dicritical cells, say $\widetilde{\mathcal{F}}_1$ and $\widetilde{\mathcal{F}}_2$, have a conjugation, namely ϕ, subordinated to φ;*
iii) *Any two equivalent (up to order one for non-reduced singularities and formal type for saddle-nodes) dicritical cells, say $\widetilde{\mathcal{C}}^1$ and $\widetilde{\mathcal{C}}^2$ have a Möbius conjugation, namely ϕ, subordinated to φ;*
iv) *ϕ commutes with the stokes phenomenon of $\widetilde{\mathcal{F}}_l^1$ and $\widetilde{\mathcal{F}}_l^2$, $l = 1 \cdots k$.*

So, once we have two germs of foliations in $\sum_{\omega}^{\mathrm{c}}$ we want to determine under what conditions they are in fact globally holomorphically conjugated. For this sake, it is necessary first to verify that indeed the minimal rectifier resolution of any element of $\sum_{\omega}^{\mathrm{c}}$ must be contained in the same surface (up to biholomorphism). But, this is assured in [3]. In fact we have:

Definition 2 ([3])**.** *We say that a complex surface is resolution-like, whether it is obtained by a holomorphic pasting of Hopf bundles with negative Chern class, in such a way that the union of their zero sections become a tree of projective lines isomorphic to the exceptional divisor of the composition of a finite numbers of blowing-up process applied to $(\mathbb{C}^2, 0)$.*

Of course, this definition is given in such a way that every resolution surface of some singularity became resolution-like. In fact, any resolution-like surface is biholomorphic to the resolution surface of some singularity.

Proposition 8 ([3])**.** *Let M_j, $j = 1, 2$ be two resolution-like surfaces with isomorphic trees of projective lines D_j, $j = 1, 2$. Then (M_1, D_1) is biholomorphic to (M_2, D_2).*

8.2. **Analytic cocycles.** We define a characteristic class for germs of singular foliations in $(\mathbb{C}^2,0)$ which are the remaining elements to determine their analytic classification. For this sake we use the concept of non-abelian cohomology as in [11], and [19].

First let $(\mathcal{F} : \omega = 0)$, then fix a foliation $\mathcal{F}^{\mathrm{o}}$ cell-wise isomorphic to $\mathcal{F}$, such that in the coordinates given by the minimal (extended) rectifier resolution, $Sep(\widetilde{\mathcal{C}}_j^{\mathrm{o}})$ is contained in the fibers of $\mathbb{H}_j$ for each non-dicritical Hopf component $\widetilde{\mathcal{C}}_j^{\mathrm{o}}$ (this foliation does exist by [17]), called a **fixed**

model for $\mathcal{F}$, and consider the elements $\Phi_j \in Diff(\widetilde{\mathcal{C}}_j, \widetilde{\mathcal{C}}_j^{\mathrm{o}})$, called a **cell chart** for the j^{th} cell of the fixed model. Then it is straightforward that:

Lemma 9. *For each cell $\widetilde{\mathcal{C}}_j = \widetilde{\mathcal{C}}|_{\mathcal{H}_j, \mathbb{CP}(1)}$ and each fixed model $\widetilde{\mathcal{C}}_j^o$, there exists only one cell chart, up to left composition by an element of $Iso(\widetilde{\mathcal{C}}_j^o)$.*

So, consider the sheaf of non-abelian groups $\Lambda^{\mathrm{o}} := Iso(\widetilde{\mathcal{C}}^{\mathrm{o}})$, then we say that $\mathcal{U} := \cup U_j$ is a good covering for Λ^{o} if U_j are neighborhoods of $D_j \subset \mathcal{H}_j$. Therefore, consider the first cohomology set $H^1(\mathcal{U}, \Lambda^{\mathrm{o}})$ associated to the good covering $\mathcal{U}$, and set $H^1(D, \Lambda^{\mathrm{o}})$ as the direct limit of $H^1(\mathcal{U}, \Lambda^{\mathrm{o}})$ for the good coverings of $\widetilde{\mathcal{F}}$, associated to $D = \cup D_j$, the exceptional divisor of the minimal rectifier resolution of $\mathcal{F}$. Hence by proposition 8 the map

$$\begin{array}{ccc} \sum_\omega^{\mathrm{c}} & \xrightarrow{\Theta} & H^1(D, \Lambda^{\mathrm{o}}) \\ \mathcal{F} & \mapsto & (\Phi_{i,j}) \end{array}$$

where $\Phi_{i,j} := \Phi_i \circ \Phi_j^{-1}$, is well defined and onto. Note that Θ does not depends on the fixed models up to cell-wise conjugacy class.

Note that by the definition of the fixed model we have that $\widetilde{\omega}_j^{\mathrm{o}}(u_j, y_j) = \widetilde{\omega}_{j+1}^{\mathrm{o}}(t_{j+1}, x_{j+1})$, where $(\widetilde{\mathcal{F}}_l^{\mathrm{o}} : \widetilde{\omega}_l^{\mathrm{o}} = 0)$, $l = \{1 \cdots k\}$ (see figure 2).

Proposition 9. *Two singular foliations $\mathcal{F}, \mathcal{G} \in \sum_\omega^c$ are analytically equivalent if and only if $\Theta(\mathcal{F}) = \Theta(\mathcal{G})$.*

Proof. Let $\Theta(\mathcal{F}) = (\Phi_1 \circ \Phi_2^{-1}, \cdots, \Phi_{k-1} \circ \Phi_k^{-1})$, and $\Theta(\mathcal{G}) = (\Psi_1 \circ \Psi_2^{-1}, \cdots, \Psi_{k-1} \circ \Psi_k^{-1})$. First let us verify the necessary part. Suppose that H is a global conjugation between $\mathcal{F}$ and $\mathcal{G}$, that is $H^*(\mathcal{G}) = \mathcal{F}$. Then, by lemma 9 $\Psi_j = \alpha_j \circ \Phi_j \circ H$ for some $\alpha_j \in Iso(\widetilde{\mathcal{C}}_j^{\mathrm{o}})$, and thus $\Psi_{j-1} \circ \Psi_j^{-1} = \alpha_{j-1} \circ \Phi_{j-1} \circ H \circ H^{-1} \circ \Phi_j^{-1} \circ \alpha_j^{-1} = \alpha_{j-1} \circ \Phi_{j-1} \circ \Phi_j^{-1} \circ \alpha_j^{-1}$.

Now let us see the sufficient part. By hypothesis $\mathcal{F}$ and $\mathcal{G}$ have the same fixed models, so, if $(\Phi_1 \circ \Phi_2^{-1}, \cdots, \Phi_{k-1} \circ \Phi_k^{-1}) = \Theta(\mathcal{F}) = \Theta(\mathcal{G}) = (\Psi_1 \circ \Psi_2^{-1}, \cdots, \Psi_{k-1} \circ \Psi_k^{-1})$, then there is a collection (α_j), with $\alpha_j \in Iso(\widetilde{\mathcal{C}}_j^{\mathrm{o}})$, such that $\Psi_{j-1} \circ \Psi_j^{-1} = \alpha_{j-1} \circ \Phi_{j-1} \circ \Phi_j^{-1} \circ \alpha_j^{-1}$. Hence, $(\alpha_{j-1} \circ \Phi_{j-1})^{-1} \circ \Psi_{j-1} = (\alpha_j \circ \Phi_j)^{-1} \circ \Psi_j$, so we can define a global conjugation between $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{G}}$ by taking $H := (\alpha_j \circ \Phi_j)^{-1} \circ \Psi_j$, for $j = 1 \cdots k$. ☐

Hence, by the construction made in this section, theorem 1 follows from proposition 1, 7, and 9.

9. Applications:

9.1. **Simple resolution foliations with companion fibration.** We state a result which is the main tool in order to establish the triviality of the characteristic classes for non-dicritical foliations with simple rectifier resolution.

Let $\widetilde{\mathcal{F}}$ be a simple non-dicritical branch of a solved and rectified foliation. Then, recall from definition that a dicritical singular foliation, say $\widetilde{\mathcal{G}}$, is a companion fibration for $\widetilde{\mathcal{F}}$, if $\widetilde{\mathcal{G}}$ satisfies the following properties:

a) $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{G}}$ have the same chain of projective lines (see [3]);
b) $\widetilde{\mathcal{G}}$ is simple and its (only) dicritical component is a transversal fibration (see §5.1) to the principal Hopf component of $\widetilde{\mathcal{F}}$;

Notice that item b) in the referred definition determines that $Sep(\widetilde{\mathcal{F}}_{j_0}) \subset \widetilde{\mathcal{G}}_{j_0}$ where $\widetilde{\mathcal{F}}_{j_0}$ is the principal Hopf component of $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{G}}_{j_0}$ the dicritical component of $\widetilde{\mathcal{G}}$. Further, the non-dicritical Hopf components of $\widetilde{\mathcal{G}}$ have at most two singularities. By abuse we shall consider a fixed model $\widetilde{\mathcal{F}}^{\mathrm{o}}$ for a non-dicritical branch $\widetilde{\mathcal{F}}$ and define $\Theta(\widetilde{\mathcal{F}})$ in a similar fashion we made for the whole resolution.

Proposition 10. *Let $\mathcal{F}$ a germ of non-dicritical foliations in $(\mathbb{C}^2, 0)$ whose minimal rectifier resolutions, say $\widetilde{\mathcal{F}}$, is simple and such that $Sep(\widetilde{\mathcal{F}})$ is contained in the principal projective line of $\widetilde{\mathcal{F}}$. Then $\Theta(\mathcal{F})$ is trivial whenever $\mathcal{F}$ has a companion fibration.*

The proof depends on the following two lemmas. But first we need a geometric construction to support them.

Let us order the first projective line to arise along resolution with 1, and increasingly with 2 the next one to intercept it — in the minimal rectifier resolution — and so on (see the figure 7). In particular we denote the principal projective line by D_l. As $D_1 \backslash Sing(\widetilde{\mathcal{F}}_1)$, is simply connected, then the holonomy of $\widetilde{\mathcal{F}}_1$ is trivial, hence, the germ of foliation $\widetilde{\mathcal{F}}_{1,2}$, determined by the restriction of $\widetilde{\mathcal{F}}$ to a neighborhood of $t_{1,2} := D_1 \cap D_2$, has a (local) holomorphic first integral. So the holonomy of $\widetilde{\mathcal{F}}_{1,2}$ with respect to D_2 is linearizable. Note that for $j \neq l$, $\widetilde{\mathcal{F}}_j$ has at most two singularities, so if $l > 2$ then the projective holonomy of $\widetilde{\mathcal{F}}_2$ coincides with the local holonomy of $\widetilde{\mathcal{F}}_{1,2}$ with respect to D_2 and hence it is also linearizable.

Now let $(\widetilde{\mathcal{F}}_j^{\mathrm{o}} : \widetilde{\omega}_j^{\mathrm{o}} = 0)$, $j \neq l$, be the foliation with the (global) holomorphic first integral:

$$\begin{cases} \widetilde{\omega}_j^{\mathrm{o}}(\tau, \xi) \wedge d(\tau^{p_j} \xi^{q_j}) = 0 \\ \widetilde{\omega}_j^{\mathrm{o}}(\upsilon, \zeta) \wedge d(\upsilon^{k_j q_j - p_j} \zeta^{q_j}) = 0 \end{cases} \tag{9.1}$$

where (τ, ξ), (υ, ζ) are affine charts for the Hopf bundle $\mathbb{H}_j(-k_j)$, and p_j, $q_j \in \mathbb{N}$ are relatively prime. Hence, by Camacho-Sad index theorem one has that $\widetilde{\mathcal{F}}_2$ and $\widetilde{\mathcal{F}}_2^{\mathrm{o}}$ are isomorphic up to order one. In fact, as the projective holonomy of $\widetilde{\mathcal{F}}_2$ is linearizable, then proposition 3 guarantees that they are isomorphic. Applying the same arguments inductively we have that $\widetilde{\mathcal{F}}_j$ is isomorphic to $\widetilde{\mathcal{F}}_j^{\mathrm{o}}$ for all $j \neq l$. In particular each Hopf component $\widetilde{\mathcal{F}}_j$ for all $j \neq l$ has a (global) holomorphic first integral.

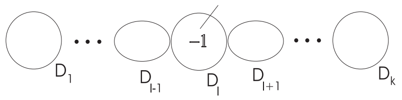


Figure 7

On the other hand, as $\widetilde{\mathcal{G}}$ is a companion fibration, thus $\widetilde{\mathcal{G}}_{l-1}$ has just one singularity (see figure 3) and by similar arguments as above, each $\widetilde{\mathcal{G}}_j$, $j \neq l$, has a (global) holomorphic first integral of the form

$$\widetilde{\eta}(t, x) = d(t^{r_j} x^{s_j}) + \cdots$$
$$\widetilde{\eta}(u, y) = d(u^{k_j s_j - r_j} y^{s_j}) + \cdots$$

(here the dots denote higher order terms) where (t, x), (u, y) are affine charts for the Hopf bundle $\mathbb{H}_j(-k_j)$, and p_j, $q_j \in \mathbb{N}$ are relatively prime. Applying the same reasoning to $j > l$, in case $p \neq l$, one obtains the same result for all $j \neq l$. Now, comparing $\widetilde{\mathcal{F}}_j$ and $\widetilde{\mathcal{G}}_j$ (starting from $l - 1$ to 1) in view of Camacho-Sad's index theorem one obtain that $p_j \neq r_j$ and $q_j \neq s_j$. In particular we have that $p_j s_j - q_j r_j \neq 0$, for all $j \neq l$, since $g.c.d(p_j, q_j) = g.c.d.(r_j, s_j) = 1$.

Lemma 10. *For all $j = 1 \cdots k$, there exists a cell chart $\Phi_j \in Diff_{\widetilde{\mathcal{F}}_j, \widetilde{\mathcal{F}}_j^{\mathrm{o}}}(\mathcal{H}_j, \mathbb{CP}(1))$ such that $\Phi_j \in Iso(\widetilde{\mathcal{G}}_j)$.*

Proof. First let us denote by $\widetilde{\mathcal{G}}_j^{\mathrm{o}}$ the linearization of $\widetilde{\mathcal{G}}_j$ (in bundle charts (t, x), (u, y)). Then we may apply a linearization theorem of [18] and path lifting arguments (as the foliations are rectified) in order to obtain $\Psi_j \in Diff_{\widetilde{\mathcal{F}}_j, \widetilde{\mathcal{F}}_j^{\mathrm{o}}}(\mathcal{H}_j, D_j)$. So let $\widetilde{\mathcal{F}}_j' := (\Psi_j)_* \widetilde{\mathcal{F}}_j$. Now, recall that, in [3] the desired result is proved for the particular case of $\widetilde{\mathcal{G}}_j = \widetilde{\mathcal{G}}_j^o$. Hence we can obtain $\Phi_{\widetilde{\mathcal{F}}'_j} \in Diff_{\widetilde{\mathcal{F}}'_j, \widetilde{\mathcal{F}}_j^o}(\mathcal{H}_j, D_j) \cap Iso(\widetilde{\mathcal{G}}_j^o)$ and $\Phi_{\widetilde{\mathcal{G}}_j} \in Diff_{\widetilde{\mathcal{G}}_j, \widetilde{\mathcal{G}}_j^o}(\mathcal{H}_j, D_j) \cap Iso(\widetilde{\mathcal{F}}_j^o)$. Thus $\Phi_j := \Psi_j \circ \Phi_{\widetilde{\mathcal{F}}'_j} \circ \Phi^{-1}_{\widetilde{\mathcal{G}}_j}$ turns commutative the following diagram:

$$\begin{array}{ccc} \left(\widetilde{\mathcal{F}}_j, \widetilde{\mathcal{G}}_j\right) & \xrightarrow{\Psi_j} & \left(\widetilde{\mathcal{F}}'_j, \widetilde{\mathcal{G}}_j^o\right) \\ \Phi_j \downarrow & & \Phi_{\widetilde{\mathcal{F}}'_j} \downarrow \\ \left(\widetilde{\mathcal{F}}_j^o, \widetilde{\mathcal{G}}_j\right) & \xrightarrow{\Phi_{\widetilde{\mathcal{G}}_j}} & \left(\widetilde{\mathcal{F}}_j^o, \widetilde{\mathcal{G}}_j^o\right) \end{array}$$

□

Now let us denote by $\widetilde{\mathcal{G}}_{j,j+1}$ the germ of foliation defined by the restriction of $\widetilde{\mathcal{G}}$ to a neighborhood of the corner $t_{j,j+1} := D_j \cap D_{j+1}$.

Lemma 11. *Let $\Phi_{j-1,j} \in Iso(\widetilde{\mathcal{F}}^o_{j-1,j}) \cap Iso(\widetilde{\mathcal{G}}_{j-1,j})$ for $j = 2 \cdots k$. Then $\Phi_{j-1,j}$ has a unique extension to $\Phi_j \in Iso(\widetilde{\mathcal{F}}^o_j)$.*

Proof. First remark that the statement is already proved in [3] for the particular case $\widetilde{\mathcal{G}}_{j-1,j} = \widetilde{\mathcal{G}}^o_{j-1,j}$. But $Ad(\Phi^{-1}_{\widetilde{\mathcal{G}}_{j-1}})\Phi_{j-1,j} \in Iso(\widetilde{\mathcal{F}}^o_{j-1,j}) \cap Iso(\widetilde{\mathcal{G}}^o_{j-1,j})$ and thus it has a extension $\Phi^o_j \in Iso(\widetilde{\mathcal{F}}^o_j) \cap Iso(\widetilde{\mathcal{G}}^o_j)$. So we take $\Phi_j := Ad(\Phi_{\widetilde{\mathcal{G}}_j})\Phi^o_j \in Iso(\widetilde{\mathcal{F}}^o_j) \cap Iso(\widetilde{\mathcal{G}}_j)$. In fact, by construction (see [3]) $\Phi^{-1}_{\widetilde{\mathcal{G}}_{j-1}}$ and $\Phi^{-1}_{\widetilde{\mathcal{G}}_j}$ agree in a neighborhood of $t_{j,j+1}$, so in this neighborhood we have that $\Phi_j = Ad(\Phi_{\widetilde{\mathcal{G}}_j} \circ \Phi^{-1}_{\widetilde{\mathcal{G}}_{j-1}})\Phi_{j-1,j} = \Phi_{j-1,j}$, as desired. □

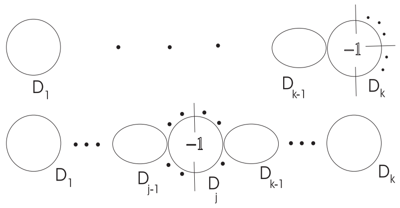
Now we have all the necessary ingredients to the proof of the proposition.

Proof of proposition 10. For simplicity in the induction argument, we prove the statement in case the principal projective line is in the "edge" of the resolution (see figure 8, for $p = 1$), that is, it intercepts just one projective line. Let $\mathcal{F}, \mathcal{F}' \in \sum^{c,1}_{\omega,f}$, with cell charts $\Phi_j \in Diff_{\widetilde{\mathcal{F}}_j, \widetilde{\mathcal{F}}^o_j}(\mathcal{H}_j, \mathbb{CP}(1))$ and $\Phi'_j \in Diff_{\widetilde{\mathcal{F}}'_j, \widetilde{\mathcal{F}}^o_j}(\mathcal{H}_j, \mathbb{CP}(1))$ respectively, for $j = 1 \cdots l-1$. Then by lemma 10 we may suppose that $\Phi_j, \Phi'_j \in Iso(\widetilde{\mathcal{G}}_j)$. Now we construct by a decreasing induction, a collection (α_j), with $\alpha_j \in Iso(\widetilde{\mathcal{F}}^o_j) \cap Iso(\widetilde{\mathcal{G}}_{j,j+1})$, such that $\Phi'_{j-1} \circ \Phi'^{-1}_j = \alpha_{j-1} \circ \Phi_{j-1} \circ \Phi^{-1}_j \circ {\alpha_j}^{-1}$. First let $\alpha_l = \mathbf{I}|_{Iso(\widetilde{\mathcal{F}}^o_l)}$ (here l stands for the principal projective line) and suppose that α_j $(j < l)$ is already defined, then let $\alpha_{j-1,j} := \Phi'_{j-1} \circ \Phi'^{-1}_j \circ \alpha_j \circ (\Phi_{j-1} \circ \Phi^{-1}_j)^{-1}$. By construction, $\alpha_{j-1,j} \in Iso(\widetilde{\mathcal{F}}^o_{j-1,j}) \cap Iso(\widetilde{\mathcal{G}}_{j-1,j})$ and therefore, by lemma 11, it has an unique extension, namely $\alpha_{j-1} \in Iso(\widetilde{\mathcal{F}}^o_{j-1})$. Then we have that $\Phi'_{j-1} \circ \Phi'^{-1}_j = \alpha_{j-1} \circ \Phi_{j-1} \circ \Phi^{-1}_j \circ \alpha^{-1}_j$ for all $j = 1 \cdots l-1$. The other case can be treated similarly, just differing by the induction arguments. □

9.2. **Projective holonomy and some non-dicritical foliations with fixed separatrices.** We classify the set $\sum^{c,1,F}_{\omega,f}$ for (reduced) $f(x,y) = \prod^d_{j=1}(y^p - \lambda_j x^q)$, where $1 < p < q \in N^*$, $g.c.d.(p,q) = 1$, and $\lambda_j \in \mathbb{C}^*$, generalizing some classification results of [6], [21] and [3]. In fact, we use the geometry of the exceptional divisor of the minimal resolution (which coincides with the minimal rectifier one) to show that their characteristic classes vanish and hence that the projective holonomy of the principal projective line classifies such germs of singular foliations.

First let us recall the geometry of the exceptional divisor of the minimal rectifier resolution of the Milnor fibrations $f(x,y) = \prod^d_{j=1}(y^p - \lambda_j x^q)$, where $p < q \in N^*$, $\lambda_j \in \mathbb{C}^*$, and $g.c.d.(p,q) = 1$.

Lemma 12 ([3]). *Every $\mathcal{F} \in \sum^{c,1,F}_{\omega,f}$ has a companion fibration given by $\frac{y^p}{x^q} \equiv const$. In particular, $\mathcal{F}$ has a simple resolution with one of the following "schemes" of resolutions:*



Respectively if $p = 1$ *or not.*

Now, in view of the above lema and proposition 10, one has that $\Theta(\mathcal{F})$ vanishes. Then theorem 2 follows from theorem 1.

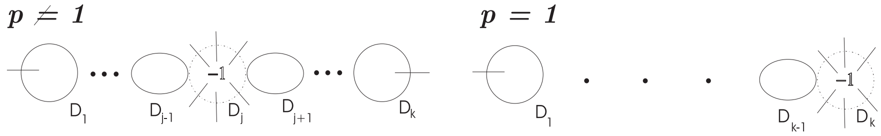


Figure 8 : Companion fibration $G$to the foliation $F \in \sum_{\omega,f}^{\mathrm{c},1,\mathrm{F}}$

9.3. **Möbius conjugation and some dicritical foliations.** Here we prove theorem 3 and a corollary which generalizing some results of ([16]).

Proof of theorem 3. First notice that the necessary part is straightforward, just by considering the transform after several blow-ups of an analytic equivalence between two germs of singular foliations as above.

Now let us verify the converse. Let $\mathcal{F}, \mathcal{F}' \in \sum_{\omega}^{\mathrm{c},1,\mathrm{F}}$ be two dicritical foliation whose extended rectifier resolution is simple. Further let φ be the isomorphism between the trees of singularities of $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{F}}'$, with $t'_{ij} = \varphi(t_{ij})$, and $\widetilde{\mathcal{F}}|_{(D_i,t_{ij})}$ and $\widetilde{\mathcal{F}}'|_{(D_i,t'_{ij})}$ having the same order of symmetry, and ϕ a Möbius conjugacy between $\bigoplus_{k=1}^{n} Hol_{D_0}(\widetilde{\mathcal{F}}_k, D_k)$ and $\bigoplus_{k=1}^{n} Hol_{D_0}(\widetilde{\mathcal{F}}'_k, D_k)$, subordinated to φ. Then by proposition 6 we have that $\widetilde{\mathcal{C}}$ and $\widetilde{\mathcal{C}}'$ are analytically equivalent. Finally, as the resolutions are simple, then the principal Hopf component of each non-dicritical branch of $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{F}}'$ intercept the dicritical components $\widetilde{\mathcal{F}}_0$ and $\widetilde{\mathcal{F}}'_0$. Hence, the result follows from proposition 10 as each non-dicritical branch has a companion fibration. □

The following proposition gives an immediate application for theorem 3, reinterpreting — in terms of projective holonomy — and generalizing a result obtained in [16] about dicritical foliations.

Proposition 11. *Let* $\mathcal{F}$ *be a germ of dicritical foliation in* $(\mathbb{C}^2, 0)$ *such that after one blow-up we obtain a foliation* $\widetilde{\mathcal{F}}^1 := \pi^*\mathcal{F}$ *whose (possible vanishing) first tangent cone is composed just by singularities of type* $\Sigma_{\omega,f}^{1,F}$, *with* f *being the Milnor fibration*

$$f(x, y) = \Pi_{j=1}^{d}(y - \lambda_j x^q)$$

where, $0 \neq q \in \mathbb{N}$, $d > 1$, $\lambda_j \in \mathbb{C}^*$ *and* $(x = 0)$ *represents the dicritical divisor. Then* $\mathcal{F}$ *has a simple minimal rectifier resolution whose non-dicritical branches have companion fibrations.*

Proof. First remark that by [3], the singularities of $\widetilde{\mathcal{F}}^1$ will generate just simple non-dicritical branches with companion fibrations. So, in view of [2] it only rests to prove that the blow-up of tangential points will generate just non-dicritical branches with companion fibration. But this is assured by the following: □

Lemma 13. *Let* $(\mathcal{F} : \omega = 0)$, *with* $\omega(x, y) = dy + x^p h(x, y)dx$, *and* $h(0) \neq 0$. *Then* $\mathcal{F}$ *has a companion fibration.*

Proof. In fact, as $\mathcal{F}$ is a regular foliation tangent to $(x = 0)$, then one has that $\omega = df$, where $f \in \mathcal{O}^2$ and $\frac{\partial f}{\partial x}(0) \neq 0$. Hence, by the implicit function theorem, there exists a local coordinate system such that $y = ax^n + \cdots$, with $a \in \mathbb{C}^*$ and $n = ord(g)$. Thus, after a change of coordinates (and some induction steps) we may suppose that $y = ax^n + \cdots + bx^m + \cdots$, where $b \in \mathbb{C}$ and m is so big as we want. But, by a theorem due to J. Mather (c.f. [25]), $f^{-1}(0)$ is determined just by a finite jet of f. Hence, we have a system of coordinates such that $y = \lambda x^n$, with $\lambda \in \mathbb{C}^*$. Now let $\mathcal{G}$ be the fibration given by $\frac{y}{x^n} \equiv const$. Then, it is not difficult to verify that $\mathcal{G}$ solves together with the fiber $\frac{y}{x^n} \equiv \lambda$, and thus together with $\mathcal{F}$. A straightforward verification shows, $\mathcal{G}$ is a companion fibration to $\mathcal{F}$. □

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