Existence and Stability of Periodic Travelling-Wave Solutions of the Benjamin Equation

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Abstract

Let $\mathcal{H}$ denote the periodic Hilbert transform and $l \in \mathbb{R}$. A proof of the existence and nonlinear stability of real even periodic travelling-wave solutions for the Benjamin Equation, $u_t + uu_x + u_{xxx} + l\mathcal{H}u_{xx} = 0$, is developed.

1 Introduction

In this work we will study certain qualitative properties of the solutions of the following equation put forward by Benjamin in [9], [10],

$$u_t + uu_x + u_{xxx} + l\mathcal{H}u_{xx} = 0, \qquad (x,t) \in \mathbb{R} \times \mathbb{R}, \tag{1}$$

which is known as *the Benjamin equation*. Here $u = u(x,t) \in \mathbb{R}$, l is a real number and $\mathcal{H}$ is the Hilbert transform. This pseudo-differential equation arises as an approximate model for nonlinear dispersive waves in a two-fluid system where the interface is subject to capillarity. Benjamin was concerned with an incompressible system that, at rest, consists of a layer of depth h_1 of light fluid with density ρ_1 bounded above by a rigid plane and resting upon a layer of heavier fluid with density $\rho_2 > \rho_1$ and depth h_2, also resting on a rigid plane. Because of the density difference, waves can propagate along the interface between the two fluids. In Benjamin's theory, diffusivity is ignored, but the parameters of the system are such that capillarity can not be discarded. Here,

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$u(x,t)$ is the vertical displacement of the interface between the two fluids at the spatial point x at time t.

Various aspects of the behavior of solutions to the Benjamin equation have been analyzed in the last ten years ([3], [5], [6], [9], [10], [11], [20]). Our principal focus here is the travelling wave solutions of (1), which are solutions of the form $u(x,t) = \phi(x-ct)$, with $\phi : \mathbb{R} \to \mathbb{R}$ and $c > 0$ represents wave velocity relative to the velocity of infinitesimal long waves. The first works were concerned with the existence and nonlinear stability of travelling wave solutions called solitary waves, which are solutions $\phi(\xi)$ such that its profile and its derivatives tend to zero as $\xi \to \pm\infty$. So, ϕ must satisfy the pseudo-differential equation

$$\phi'' + l\mathcal{H}\phi' + \frac{1}{2}\phi^2 - c\phi = 0. \tag{2}$$

Benjamin, in [9], by using a Leray-Schauder degree theory approach, showed that for each value of $\gamma = l/(2\sqrt{c})$ in the range $[0,1)$, equation (2) has a solution $\phi = \phi_\gamma$ which is an even function and which has a positive Fourier transform. Benjamin also provided some formal asymptotics suggesting that, for each fixed value of γ, $|\phi(\xi)|$ should decay monotonically like $1/\xi^2$. In fact, later in Bona&Chen ([11]) it was confirmed the above predictions. Benjamin also sketched a perturbation-theoretical approach to a proof of existence of a branch of solutions of (2) defined for γ near 0, which will be orbitally stable solutions of the initial value problem associated to (1). Albert&Bona&Restrepo, in [3], complete the arguments of stability given by Benjamin, still limited to the case of small l, and generate a scheme capable of computing the solitary wave solutions numerically for all values of γ in $[0,1)$. This computer-generated approximations reveal the curious behavior that the solutions of (2) feature a finite number of oscillations. Finally, Angulo in [6] showed a theory of existence and nonlinear stability of solitary wave solutions for Benjamin's equation when γ is close to 1.

With regard to travelling wave solutions of periodic type, the only one study known by the authors is that established by Benjamin in [9]. In fact, by using again the Leray-Schauder degree theory, he showed the existence of non-trivial even solutions ϕ of (2) with period $2L$, provided L is sufficiently large. More precisely, for c, l positive numbers such that $c > l^2/4$, there is a non-constant periodic solution ϕ of the Benjamin equation provided

$$L > \frac{2\pi}{l + \sqrt{l^2 + 4c}}.$$

In this paper, our principal focus is to establish a theory of existence and nonlinear stability of periodic travelling-wave solutions of Benjamin's equation. This work has been inspired by a recent paper of Angulo&Bona ([7]) about stability of periodic travelling wave solutions to the Korteweg-de Vries equation ($l = 0$ in (1), henceforth KdV equation) called cnoidal waves (see Section 2 for a review of this theory). If $f = f(x,t)$ represents a $2L$-periodic function of x, the *periodic Hilbert transform* is defined by the Hilbert kernel as

$$\mathcal{H}f(x) = \frac{1}{2L}\text{p.v.} \int_{-L}^{L} \cot\Big[\frac{\pi(x-y)}{2L}\Big] f(y)dy,$$

where p.v. represents the Cauchy principal value of the integral. Moreover, the Fourier transform of $\mathcal{H}$ is given by

$$\widehat{\mathcal{H}f}(k) = -i\,\text{sgn}(k)\hat{f}(k), \quad k \in \mathbb{Z}, \tag{3}$$

for any $2L$-periodic, square integrable function f on $[-L, L]$, where $\text{sgn}(0) = 0$ and

$$\text{sgn}(k) = \begin{cases} -1, & k < 0, \\ \;\;\,1, & k > 0. \end{cases}$$

So, the differential operator $\mathcal{H}\partial_x$ has a Fourier transform given by the sequence $\{|k|\}_{k\in\mathbb{Z}}$. Therefore, the dispersion relation for (1) is given by $\{k^2 - l|k|\}_{k\in\mathbb{Z}}$ and so we have that the symbol of the linear part of equation (2), $c + k^2 - l|k|$, will be positive for every $k \in \mathbb{Z}$ provided $c > l^2/4$, without consideration of the period $2L$.

This paper has two basic goals. The first one is to show, via the Implicit Function Theorem, the existence of real even periodic travelling-wave solutions of the form $u(x,t) = \phi(x - ct)$, where $\phi = \phi_{l,c}$ satisfy the pseudo-differential equation

$$\phi'' + l\mathcal{H}\phi' + \frac{1}{2}\phi^2 - c\phi = A_{\phi_{l,c}}, \tag{4}$$

where $A_{\phi_{l,c}}$ is an integration constant. More precisely, by giving any fixed period $2L$ and velocity c, we establish an analytic branch $l \to \phi_{l,c}$ of $2L$-periodic solutions for l in a neighborhood of zero and such that $\phi_{l,c}$ has mean zero over $[-L, L]$. We note that in this case the integration constant satisfies

$$A_{\phi_{l,c}} = \frac{1}{4L}\int_{-L}^{L} \phi_{l,c}^2(x)dx. \tag{5}$$

Moreover, the fixed wave speed c can be taken arbitrarily from the interval $(0, \infty)$ (see the Remarks following Theorems 2.1 and 2.3, below), considering again a fixed period $2L$. Let us remark that c and L satisfy the *a priori* relation (see (14))

$$L^2 > \frac{\pi^2}{\sqrt{c^2 + 2A_{\psi_c}}}, \tag{6}$$

where ψ_c is the cnoidal wave solution of the equation

$$\psi_c'' + \frac{1}{2}\psi_c^2 - c\psi_c = A_{\psi_c},$$

given by Theorem 2.3.

Our second objective here will be to show that the orbit generated by the periodic travelling wave solution $\phi_{l,c}$, namely the set

$$\Omega_{\phi_{l,c}} = \{\phi_{l,c}(\cdot + s)\ ;\ s \in \mathbb{R}\},$$

is $H^1_{per}([-L,L])$-stable with regard to the flow of the Benjamin equation. Our approach will be based on the Grillakis&Shatah&Strauss theory in ([16]) and the theory developed in [7]. Therefore, a careful analysis of spectral type associated to the self-adjoint operator

$$T_l = -\partial_x^2 - l\mathcal{H}\partial_x - \phi_{l,c} + c$$

will be necessary. In fact, by using the Floquet theory (see [21]) and perturbation theory of operators (see [19]) we establish that for l small, T_l with domain in $H^2_{per}([-L,L])$, has exactly its three first eigenvalues simple, being the eigenvalue zero the second one with eigenfunction $\phi'_{l,c}$, and the rest of the spectrum is constituted by a discret set of eigenvalues (see Lemma 4.1). Now, let us remember that equation (1) has two basic constants of motion, namely,

$$E_B(u) = \frac{1}{2}\int_{-L}^{L} \big[(u_x)^2 - lu\mathcal{H}u_x - \frac{1}{3}u^3\big]dx \tag{7}$$

and

$$F(u) = \frac{1}{2}\int_{-L}^{L} u^2 dx, \tag{8}$$

what means that if u is a smooth solution of (1), then $E_B(u(\cdot,t)) = E_B(u(\cdot,0))$ and $F(u(\cdot,t)) = F(u(\cdot,0))$ for all t. Let us remark that, the stability criterion given by the Grillakis *et. al.* theory is usually based on the convexity property of the classical function $d(c) = E_B(\phi_{l,c}) + cF(\phi_{l,c})$. Here, we will use an alternative sufficient condition for stability (see (46)), namely, there exists $\varphi_l \in L^2_{per}([-L,L])$ such that

$$T_l\varphi_l = \phi_{l,c} + \frac{d}{dc}A_{\psi_c}, \quad \text{and} \quad (\varphi_l, T_l\varphi_l) < 0.$$

The plan of this paper is as follows. In Section 2 we review the basic points of the theory of stability of cnoidal wave solutions for the Korteweg-de Vries equation put forward by Angulo&Bona in [7]. This theory will be crucial in our study. In Section 3, we present a complete theory of existence of periodic travelling wave solutions for the Benjamin equation, which will be defined in a *a priori* fundamental interval $[-L, L]$ and with mean zero on it. Section 4 is devoted to establish the nonlinear stability of periodic solutions based on the Grillakis *et.al.* theory. For convenience of the reader, we establish here an outline of this criterion for stability. The paper concludes with a review about Jacobian elliptic integrals and some facts from perturbation theory of closed operators on Hilbert space.

Notation: The set of natural numbers $\{0, 1, 2, 3, ...\}$ and the set of all integers are written as $\mathbb{N}$ and $\mathbb{Z}$, respectively. The set of all real numbers is denoted by $\mathbb{R}$. The norm of a function $f \in L^p(\Omega)$ (equivalence class), for Ω an open subset of $\mathbb{R}$, is written $|f|^p_{L^p(\Omega)} = \int_\Omega |f|^p dx$, $p \geqq 1$. The inner product of two functions $f, g \in L^2(\Omega)$ is written as $(f, g) = \int_\Omega f\bar{g}dx$. To define the Sobolev spaces (of L^2 type) of periodic functions, we need the following notions (for further information see Iorio&Iorio [18]). Let $\mathcal{P} = C^\infty_{per}$ denote the collection of all the functions $f : \mathbb{R} \to \mathbb{C}$ which are $C^\infty(\mathbb{R})$ and periodic with period $2L > 0$. The topological dual of $\mathcal{P}$ will be denoted by $\mathcal{P}'$, i.e., $\mathcal{P}'$ is the collection of all continuous linear functionals from $\mathcal{P}$ into $\mathbb{C}$. $\mathcal{P}'$ is called the set of all *periodic distributions*. If $\Psi \in \mathcal{P}'$ we denote the value of Ψ in φ by $\Psi(\varphi) =< \Psi, \varphi >$. Define the functions $\Theta_k(x) = \exp(ik\pi x/L)$, $k \in \mathbb{Z}$, $x \in \mathbb{R}$. The Fourier transform of $\Psi \in \mathcal{P}'$ is the function $\widehat{\Psi} : \mathbb{Z} \to \mathbb{C}$ defined by the formula

$$\widehat{\Psi}(k) = \frac{1}{2L} < \Psi, \Theta_{-k} >, \qquad k \in \mathbb{Z}.$$

So if Ψ is a periodic function with period $2L$, for example $\Psi \in L^2(-L, L)$, we have

$$\widehat{\Psi}(k) = \frac{1}{2L} \int_{-L}^{L} \Psi(x) e^{-ik\pi x/L} dx.$$

Let $s \in \mathbb{R}$. The Sobolev space $H^s_{2L}(\mathbb{R}) = H^s_{per}([-L, L])$ is the set of all $f \in \mathcal{P}'$ such that

$$\|f\|^2_{H^s_{2L}} \equiv 2L \sum_{k=-\infty}^{\infty} (1 + |k|^2)^s |\widehat{f}(k)|^2 < \infty.$$

$H^s_{2L}(\mathbb{R})$ is a Hilbert space with respect to the inner product

$$(f|g)_s = 2L \sum_{k=-\infty}^{\infty} (1 + |k|^2)^s \widehat{f}(k)\overline{\widehat{g}(k)}.$$

Let us remark that, since any function $f \in L^2([-L,L])$ can be extended periodically, with period $2L$, to all the real line, then we can identify isometrically $L^2([-L,L]) = H^0_{per}([-L,L])$. When there is no possibility of confusion, we write L^2 to denote $L^2([-L,L])$. So we have $(f|g)_0 = (f,g) = \int_{-L}^{L} f\bar{g}dx$. $H^0_{2L}(\mathbb{R})$ will be denoted by $L^2_{2L}(\mathbb{R})$ with $\|\cdot\|_{L^2_{2L}} = |\cdot|_{L^2(-L,L)}$. Since $H^s_{2L}(\mathbb{R}) \subset L^2_{2L}(\mathbb{R})$ for every $s \geqq 0$, we have via the Parseval (or Plancherel's) identity that for every $n \in \mathbb{N}$

$$\|f\|^2_{H^n_{2L}} = \sum_{j=0}^{n} \|f^{(j)}\|^2_{L^2_{2L}} = \sum_{j=0}^{n} \int_{-L}^{L} |f^{(j)}(x)|^2 \, dx$$

where $f^{(j)}$ represents the j-th derivative of f taken in the sense of $\mathcal{P}'$. Moreover, $(H^s_{2L}(\mathbb{R}))'$, the topological dual of $H^s_{2L}(\mathbb{R})$, is isometrically isomorphic to $H^{-s}_{2L}(\mathbb{R})$ for all $s \in \mathbb{R}$. The duality is implemented by the pairing

$$< f, g >_s = 2L \sum_{k=-\infty}^{\infty} \widehat{f}(k)\overline{\widehat{g}(k)}, \qquad f \in H^{-s}_{2L}(\mathbb{R}),\ g \in H^s_{2L}(\mathbb{R}).$$

So, if $f, g \in L^2_{2L}(\mathbb{R})$, it follows that $< f, g >_s = (f,g)$. Moreover, Sobolev's Lemma states that, if $s > \frac{1}{2}$ and $C_{per} = \{f : \mathbb{R} \to \mathbb{C}; f \text{ is continuous and periodic with period } 2L\}$ then $H^s_{2L}(\mathbb{R}) \hookrightarrow C_{per}$. Finally, let us write some more notations will appear around the paper. We write $H^s_{per,e}([-L,L])$ to denote the set of even, $2L$-periodic-Sobolev distributions of order $s \in \mathbb{R}$. $B(X,Y)$ is used to denote the set of bounded linear operators on X to Y, if $X = Y$, we write $B(X)$. We write $\mathcal{C}(X,Y)$ to represent the set of closed operators from X into Y, if $X = Y$, we write $\mathcal{C}(X)$. Finally we will use $\mathcal{D}(A)$ to denote the *domain* of the operator A.

2 Review about the Stability of Periodic Travelling-Wave Solutions for the KdV equation

In this section we establish the basic points of the theory put forward by Angulo&Bona in [7] (see [8] where a complete theory about the issue was not made) with regard to the nonlinear stability of periodic travelling-wave solutions for the KdV equation

$$u_t + uu_x + u_{xxx} = 0, \tag{9}$$

which will be used in our stability theory of periodic travelling-wave solutions for the Benjamin equation (1). The so called *cnoidal wave* solutions, $u(x,t) = \psi_c(x - ct)$ of equation (9), studied in [7], will satisfy the equation

$$\psi_c'' + \frac{1}{2}\psi_c^2 - c\psi_c = A_{\psi_c}, \tag{10}$$

with A_{ψ_c} an integration constant. An explicit form for ψ_c is given by

$$\psi_c(\xi;\beta_1,\beta_2,\beta_3) = \beta_2 + (\beta_3-\beta_2)\mathrm{cn}^2\Big[\sqrt{\frac{\beta_3-\beta_1}{12}}\xi;k\Big], \tag{11}$$

where cn is the *Jacobian elliptic function* (see Appendix or Byrd&Friedman [14]) and k is the *modulus* defined by

$$k^2 = \frac{\beta_3-\beta_2}{\beta_3-\beta_1}, \tag{12}$$

with $\beta_1+\beta_2+\beta_3 = 3c$, $\beta_1<\beta_2<\beta_3$ and $A_{\psi_c} = -\frac{1}{6}\sum_{i<j}\beta_i\beta_j$. Since $\mathrm{cn}(u+2K) = -\mathrm{cn}u$, where $K = K(k)$ is the *complete elliptic integral of the first kind* determined by the modulus k, we have that ψ_c has fundamental period (wavelength) T_{ψ_c}, given by

$$T_{\psi_c} \equiv \frac{4\sqrt{3}}{\sqrt{\beta_3-\beta_1}}\, K(k). \tag{13}$$

And moreover, as mentioned in [7], the fundamental period depends on the speed c and satisfies the inequality

$$T_{\psi_c}^2 > \frac{(2\pi)^2}{\sqrt{c^2+2A_{\psi_c}}}. \tag{14}$$

The first basic result established in [7] is the existence of cnoidal wave solutions with mean zero on its fundamental period (see Theorem 2.1 in [7]).

Theorem 2.1. *For every $\beta_3 > 3$ there are unique β_1, β_2 satisfying $\beta_1 < \beta_2 < 0 < \beta_3$ and $\beta_1+\beta_2+\beta_3 = 3$, such that the cnoidal wave $\psi_1(\cdot;\beta_i)$ defined in (11), for $c = 1$, has mean zero over $[-T_{\psi_1}/2, T_{\psi_1}/2]$, namely, $\beta_1 K(k) + (\beta_3-\beta_1)E(k) = 0$ where $k^2 = \frac{\beta_3-\beta_2}{\beta_3-\beta_1}$. Moreover, the map $\beta_3 \in (3,\infty) \mapsto \beta_2 \equiv \beta_2(\beta_3)$ is continuous.*

Proof. (Sketch) Let $\beta_3 > 3$. Let us denote by $\alpha \equiv \frac{\beta_3-3}{2}$, $\beta \equiv \frac{1}{\alpha}$ and I the open interval $(-\alpha, 0)$. Define for $x \in I$, the continuous function

$$J(x) = \frac{K(k(x))}{E(k(x))} + \frac{\beta_3}{3-\beta_3-x} - 1, \tag{15}$$

where $k(x) = \sqrt{\frac{\beta_3-x}{2\beta_3+x-3}}$, K and E are the complete elliptical integral of the first and second kind respectively (see Appendix). The idea of the proof is to see that for every

$\beta_3 > 3$, there is a unique $\beta_2 \equiv \beta_2(\beta_3) \in I$ such that $J(\beta_2) = 0$. For accomplishing it we define for $x \in I$

$$A(x) \equiv \int_0^1 \frac{1}{\sqrt{1-t^2}} \frac{-\beta_3 + (\beta_3 - x)t^2}{\sqrt{2\beta_3 - 3 + x - (\beta_3 - x)t^2}} dt. \tag{16}$$

It is easy to see that $J(x) = 0$ if and only if $A(x) = 0$. So, to prove that J has a unique zero in the interval I, it is enough to prove that the function A is strictly decreasing on that interval, because $\lim_{x\to-\alpha} A(x) = +\infty$ and $A(0) < 0$. We refer the reader to Angulo&Bona in [7] where a complete proof of this is done.

The continuity of the map $\beta_3 \in (3, +\infty) \mapsto \beta_2 = \beta_2(\beta_3)$ is a consequence of the Implicit Function Theorem applied to the function $A(x, \beta_3)$ given by the right hand side of (16), considering that $\partial_x A(\beta_2(\beta_3), \beta_3) < 0$. ■

Remark: Let $L > 0$ be a fixed number. It follows from Theorem 2.1 that for every $\beta_3 > 3$ will exist a unique choice $\beta_2 \in I = (\frac{3-\beta_3}{2}, 0)$ such that β_2 is the unique zero in the interval I of the function J defined by (15) and moreover the map $\beta_3 \in (3, \infty) \mapsto \beta_2 = \beta_2(\beta_3)$ is continuous. So, if we regard the fundamental period T_{ψ_1} of $\psi_1 = \psi_1(\cdot; \beta_i)$, as a function of the parameter β_3, namely,

$$T_{\psi_1}(\beta_3) = \frac{4\sqrt{3}}{\sqrt{2\beta_3 + \beta_2(\beta_3) - 3}}\, K\Big(\sqrt{\frac{\beta_3 - \beta_2(\beta_3)}{2\beta_3 + \beta_2(\beta_3) - 3}}\Big),$$

we can infer immediately that $T_{\psi_1}(\beta_3) \to +\infty$ as $\beta_3 \to 3$, since $K(k) \to +\infty$ as $k \to 1$. On the other hand, since $J(\beta_2) = 0$ and $\frac{3-\beta_3}{2} < \beta_2 < 0$ we have that

$$\begin{aligned} 0 \le T_{\psi_1}(\beta_3) &= 4\sqrt{3}\, \frac{\sqrt{2\beta_3 + \beta_2 - 3}}{\beta_3 + \beta_2 - 3}\, E(k(\beta_2)) \le 2\pi\sqrt{3}\, \frac{\sqrt{2\beta_3 + \beta_2 - 3}}{\beta_3 + \beta_2 - 3} \\ &\le 4\pi\sqrt{3}\, \frac{\sqrt{2\beta_3 + \beta_2 - 3}}{\beta_3 - 3} < 4\pi\sqrt{3}\, \frac{\sqrt{2\beta_3 - 3}}{\beta_3 - 3}, \end{aligned}$$

and so $T_{\psi_1}(\beta_3) \to 0$ as $\beta_3 \to +\infty$.

One of the main points established in [7] is the existence of a smooth curve $c \mapsto \psi_c$ of cnoidal waves with c defined in the interval $(0, \infty)$ satisfying relation (6), and ψ_c having an arbitrary fundamental period $2L$ with mean zero on $[-L, L]$. In fact, since

$$\int_0^{T_{\psi_c}} \psi_c(\xi)\, d\xi = 0 \Leftrightarrow \beta_2 + (\beta_3 - \beta_2)\frac{E - k'^2 K}{k^2 K} = 0,$$

where $E = E(k)$ is the *complete elliptic integral of the second kind* determined by the modulus k and $k' = \sqrt{1-k^2}$, then we have the following theorem of existence.

Theorem 2.2. *Let $L > 0$ be a fixed number and consider β_i, $i = 1, 2, 3$, an arbitrary but fixed trio defined in Theorem 2.1. Define $c_0 = T^2_{\psi_1}/4L^2$, where $\psi_1 = \psi_1(\cdot; \beta_i)$. Then,*

(1) *there exist an interval $I(c_0)$ around of c_0, a ball $B(\overrightarrow{\beta})$ around of $\overrightarrow{\beta} = (\beta_1, \beta_2, \beta_3)$ and a unique smooth function*

$$\begin{array}{rcl} \Pi : I(c_0) & \mapsto & B(\overrightarrow{\beta}) \\ c & \mapsto & (\alpha_1(c), \alpha_2(c), \alpha_3(c)) \end{array}$$

such that $\Pi(c_0) = (\beta_1, \beta_2, \beta_3)$ and $\alpha_i \equiv \alpha_i(c)$ satisfy the relations

$$\begin{array}{rcl} \dfrac{4\sqrt{3}}{\sqrt{\alpha_3 - \alpha_1}} K(k) & = & 2L\sqrt{c}, \\ \alpha_1 + \alpha_2 + \alpha_3 & = & 3, \\ \alpha_2 + (\alpha_3 - \alpha_2)\dfrac{E(k) - k'^2 K(k)}{k^2 K(k)} & = & 0, \end{array}$$

where $k^2 = (\alpha_3 - \alpha_2)/(\alpha_3 - \alpha_1)$.

(2) *The cnoidal wave $\psi_1 \equiv \psi_1(\cdot; \alpha_i)$, given by (11) with α_i instead of β_i, has fundamental period $2L\sqrt{c}$ and mean zero over $[-L\sqrt{c}, L\sqrt{c}]$. Moreover, for all $c \in I(c_0)$*

$$A_{\psi_1}(c) = \frac{1}{4L\sqrt{c}} \int_{-L\sqrt{c}}^{L\sqrt{c}} \psi_1^2(x)dx = -\frac{1}{6} \sum_{i<j} \alpha_i(c)\alpha_j(c). \tag{17}$$

(3) *For all $c \in I(c_0)$, the modulus-function*

$$k(c) \equiv \frac{\sqrt{\alpha_3(c) - \alpha_2(c)}}{\sqrt{\alpha_3(c) - \alpha_1(c)}}.$$

satisfies $\frac{d}{dc}k(c) > 0$.

Proof. The proof is a consequence of the Implicit Function Theorem applied to the function $\Phi : \Omega \to \mathbb{R}^3$ given by

$$\Phi(\alpha_1, \alpha_2, \alpha_3, c) = \Big(\frac{4\sqrt{3}\ K(k)}{\sqrt{\alpha_3 - \alpha_1}} - 2L\sqrt{c}, \alpha_1 + \alpha_2 + \alpha_3 - 3, \alpha_2 + (\alpha_3 - \alpha_2)\frac{E(k) - k'^2 K(k)}{k^2 K(k)}\Big)$$

where $k^2 = (\alpha_3 - \alpha_2)/(\alpha_3 - \alpha_1)$ and $\Omega = \{(\alpha_1, \alpha_2, \alpha_3, c);\ \alpha_1 < \alpha_2 < 0 < \alpha_3,\ \ \alpha_3 > 3 \text{ and } c > 0\ \} \subset \mathbb{R}^4$. We refer the reader to Lemma 2.2, Corollary 2.3 and Theorem 2.5 in [7] to see a complete proof of the Theorem. ■

Now, we establish the existence of a smooth curve of cnoidal wave solutions to the KdV equation with a fixed period $2L$ and mean zero on $[-L, L]$. The notation used in Theorem 2.3, below, will be that established in Theorem 2.2.

Theorem 2.3. *Let $L > 0$ be a fixed number and consider $c \in I(c_0)$. Then there exists $\psi_c \in H^n_{per}([-L, L])$, for each $n \in \mathbb{N}$, with fundamental period $2L$, such that $\langle \psi_c \rangle \equiv \frac{1}{2L}\int_{-L}^{L} \psi_c(x)dx = 0$ and satisfies the differential equation*

$$\psi_c''(\xi) + \frac{1}{2}\psi_c^2(\xi) - c\,\psi_c(\xi) = A_{\psi_c}, \qquad \text{for all } \xi \in \mathbb{R}, \tag{18}$$

where

$$A_{\psi_c}(c) = \frac{1}{4L}\int_{-L}^{L} \psi_c^2(\xi)\,d\xi = -\frac{c^2}{6}\sum_{i<j}\alpha_i(c)\alpha_j(c), \tag{19}$$

and $\alpha_i(c)$, $i = 1, 2, 3$ are given by Theorem 2.2. Moreover, the map $c \in I(c_0) \mapsto \psi_c \in H^n_{per}([-L, L])$ is a smooth function.

Proof. Let $\psi_1 \equiv \psi_1(\cdot; \alpha_1, \alpha_2, \alpha_3)$ be the cnoidal wave determined in Theorem 2.2 with $\alpha_i \equiv \alpha_i(c)$ for $c \in I(c_0)$, fundamental period $2L\sqrt{c}$, mean zero on $[-L\sqrt{c}, L\sqrt{c}]$, and satisfying the differential equation (10) with $c = 1$. Define

$$\psi_c(\xi) = c\,\psi_1(\sqrt{c}\,\xi), \qquad \text{for all } \xi \in \mathbb{R}.$$

Then we get that ψ_c has fundamental period $2L$, mean zero over $[-L, L]$, and satisfies the differential equation

$$\psi_c''(\xi) + \frac{1}{2}\psi_c^2(\xi) - c\psi_c(\xi) = c^2 A_{\psi_1}(c).$$

We note that ψ_c has the form in (11) with $\beta_i = c\alpha_i$ and $\sum_i \beta_i = 3c$. Next, from the properties of ψ_1 we have the relation

$$A_{\psi_1}(c) = \frac{1}{4L\sqrt{c}}\int_{-L\sqrt{c}}^{L\sqrt{c}} \psi_1^2(\xi)d\xi = \frac{1}{4Lc^2}\int_{-L}^{L} \psi_c^2(\xi)d\xi \equiv \frac{1}{c^2}A_{\psi_c}(c), \tag{20}$$

and so ψ_c satisfies (18). Moreover, from (20) and (17) we have (19). Finally, from the regularity properties of the cnoidal wave ψ_1 and α_i's we have the required regularity on the curve $c \mapsto \psi_c$. This proves the Theorem. ■

Remark: Let $L > 0$ be a fixed number. By (14), we obtain an *a priori* lower bound for $c_0 \equiv c_0(\beta_3) = T^2_{\psi_1}(\beta_3)/(4L^2) > \frac{\pi^2}{L^2\sqrt{1+2A_{\psi_1}}}$. On the other hand, it follows from the Remark following Theorem 2.1 that $T_{\psi_1}(\beta_3)$ can take arbitrarily any value from the interval $(0, \infty)$ for $\beta_3 > 3$. So, we can choose $c_0 = T^2_{\psi_1}/(4L^2)$ arbitrarily from the interval $(0, \infty)$. Now, by Theorems 2.2 and 2.3, there is an interval $I(c_0)$ around of c_0 such that there exists a well defined smooth curve $c \in I(c_0) \mapsto \psi_c \in H^n_{per}([-L, L])$ of cnoidal waves satisfying equation (18). By using the uniqueness of the map Π in Theorem 2.2, we can define the map $c \in (0, \infty) \mapsto \psi_c \in H^n_{per}([-L, L])$ as a smooth curve of cnoidal wave solutions to the KdV equation with a fixed period $2L$ and mean zero on $[-L, L]$. Note that from (14) we have the *a priori* relation (6) between L and c.

The theory of nonlinear stability of the ψ_c-orbit (generated by the cnoidal wave ψ_c, namely, $\Omega_{\psi_c} = \{\psi_c(\cdot + y) : y \in \mathbb{R}\}$), by the flow of the KdV equation (9) in $H^1_{per}([-L, L])$ was shown in [7], by using the framework established in [16] by Grillakis&Shatah&Strauss. So, an analysis of the classical function $d(c)$, defined by

$$d(c) = E_{KdV}(\psi_c) + cF(\psi_c) \equiv \frac{1}{2}\int_{-L}^{L} [(\psi_c')^2 - \frac{1}{3}\psi_c^3]dx + c\,\frac{1}{2}\int_{-L}^{L} \psi_c^2 dx,$$

and a spectral analysis of the following self-adjoint operator on $L^2_{per}([-L, L])$

$$\mathcal{L}_{KdV} = -\frac{d^2}{dx^2} + c - \psi_c, \tag{21}$$

associated to the cnoidal wave ψ_c were necessary. Now, we collect this last main information in the next Theorem.

Theorem 2.4. (1) (**Convexity of** $d(c)$) *If $c \in (0, \infty)$ then $d(c)$ is strictly convex. More exactly,*

$$d''(c) = \frac{288}{L^3}(K - E)EK\frac{dK}{dk}\frac{dk}{dc} > 0. \tag{22}$$

(2) (**Spectrum of** $\mathcal{L}_{KdV}$) *Let ψ_c be the cnoidal wave given by Theorem 2.3. Then the linear operator $\mathcal{L}_{KdV}$ defined in (21) on $H^2_{per}([-L, L])$, has exactly its three first*

eigenvalues simple, being the eigenvalue zero the second one with eigenfunction ψ_c'. Moreover, the rest of the spectrum is constituted by a discret set of eigenvalues, which are double.

Proof. We will give here a sketch of the proofs. We refer the reader to see Theorem 2.6 and Theorem 3.2 in [7] for more details.

(1) Since the map $c \in (0,\infty) \mapsto \psi_c \in H^n_{per}([-L,L])$ is C^1, ψ_c satisfies (18), and $\langle \frac{d}{dc}\psi_c \rangle = 0$, it follows from (19) that

$$\begin{aligned} d'(c) &= < E'_{_{KdV}}(\psi_c) + cF'(\psi_c), \frac{d}{dc}\psi_c >_1 + F(\psi_c) = (-A_{\psi_c}, \frac{d}{dc}\psi_c) + F(\psi_c) \\ &= F(\psi_c) = \frac{1}{2}\int_{-L}^{L} \psi_c^2(x)\, dx = -\frac{Lc^2}{3}\sum_{i<j} \alpha_i(c)\alpha_j(c). \end{aligned} \tag{23}$$

So, we only need to find expressions for the α_i's. In fact, from Theorem 2.2 we have that

$$\alpha_1 = -\frac{12K}{cL^2}E, \quad \alpha_2 = -\frac{12K}{cL^2}[E - k'^2K], \quad \alpha_3 = \frac{12K}{cL^2}[K - E]. \tag{24}$$

Now, since K and E satisfy the following *hypergeometric differential equation*

$$\begin{cases} kk'^2 \frac{d^2K}{dk^2} + (1-3k^2)\frac{dK}{dk} - kK = 0, \\ kk'^2 \frac{d^2E}{dk^2} + k'^2\frac{dE}{dk} + kE = 0, \end{cases}$$

it follows from (23) that

$$d''(c) = \frac{288}{L^3}(K-E)EK\frac{dK}{dk}\frac{dk}{dc},$$

which is strictly positive since $K(k) > E(k)$ for $k \in (0,1)$, $K(\cdot)$ is strictly increasing and $\frac{dk}{dc} > 0$, where the last expression follows from (3) in Theorem 2.2.

(2) Consider the periodic eigenvalue problem on $[-L,L]$ associated to $\mathcal{L}_{_{KdV}}$,

$$\begin{cases} \mathcal{L}_{_{KdV}}\varphi = \lambda\varphi \\ \varphi(-L) = \varphi(L), \quad \varphi'(-L) = \varphi'(L). \end{cases} \tag{25}$$

We shall show that the problem (25) determines exactly the existence of a single negative eigenvalue $\lambda = \lambda_0$ which is simple and that $\lambda = 0$ is a simple eigenvalue with eigenfunction ψ_c'. At this point we will use the *Floquet theory* (Ince [17], Erdélyi [15], Magnus&Winkler [21]). First, it follows from the classical theory of compact symmetric linear operators applied to the problem (25) that the spectrum of $\mathcal{L}_{_{KdV}}$ is a countable infinite set of eigenvalues $\{\lambda_n; n = 0, 1, 2, ...\}$ such that

$$\lambda_0 \leqq \lambda_1 \leqq \lambda_2 \leqq \lambda_3 \leqq \lambda_4 \leqq \cdots, \tag{26}$$

where each double eigenvalue is counted twice and $\lambda_n \to \infty$ as $n \to \infty$. We shall denote by φ_n the eigenfunction associated to the eigenvalue λ_n. By the conditions $\varphi(-L) = \varphi(L)$, $\varphi'(-L) = \varphi'(L)$, φ_n can be extended to the whole of $(-\infty, \infty)$ as a continuously differentiable function with period $2L$. Moreover, from the Floquet theory (see Magnus&Winkler [21]) we have that

$$\begin{cases} \varphi_0 \ \text{ has no zeros in } \ [-L, L], \\ \varphi_{2n+1} \text{ and } \varphi_{2n+2} \ \text{ have exactly } \ 2n+2 \ \text{ zeros in } \ [-L, L). \end{cases} \tag{27}$$

Then, since $\mathcal{L}_{_{KdV}}\psi_c' = 0$ and ψ_c' has 2 zeros in $[-L, L)$, it follows from (27) that the eigenvalue 0 is either λ_1 or λ_2. We claim that $0 = \lambda_1 < \lambda_2$. In fact, via the transformation $\Lambda(x) \equiv \varphi(\eta x)$ with $\eta^2 = 12/(\beta_3 - \beta_1)$, we have that (25) is equivalent to the eigenvalue problem

$$\begin{cases} \frac{d^2}{dx^2}\Lambda + [\rho - 12k^2\mathrm{sn}^2(x)]\Lambda = 0 \\ \Lambda(-K) = \Lambda(K), \ \ \Lambda'(-K) = \Lambda'(K), \end{cases} \tag{28}$$

where the relation between ρ and λ is given by

$$\rho = -12[c - \beta_3 - \lambda]/(\beta_3 - \beta_1). \tag{29}$$

We recall that the second order differential equation in (28) is called the *Jacobian form of Lamé's equation*. Now, from Floquet theory (Magnus&Winkler [21], Theorem 7.8) it follows that Lamé's equation has exactly 4 intervals of instability. Therefore, we have that the three first eigenvalues ρ_0, ρ_1, ρ_2, associated to (28) are simple. Hence, since $\rho_1 = 4 + 4k^2$ is an eigenvalue to (28) with eigenfunction $\Lambda_1(x) = \mathrm{cn}(x)\mathrm{sn}(x)\mathrm{dn}(x) = \mathrm{const} \cdot \psi_c'(\eta x)$, it follows from (29) that $\lambda = 0$ is a simple eigenvalue to (25) with eigenfunction ψ_c'. Now, from Ince [17] we have that the following functions with period $2K$ (called *Lamé polynomials*)

$$\begin{aligned} \Lambda_0(x) &= \mathrm{dn}(x)[1 - (1 + 2k^2 - \sqrt{1 - k^2 + 4k^4}\ )\mathrm{sn}^2(x)], \\ \Lambda_2(x) &= \mathrm{dn}(x)[1 - (1 + 2k^2 + \sqrt{1 - k^2 + 4k^4}\ )\mathrm{sn}^2(x)], \end{aligned} \tag{30}$$

are the eigenfunctions associated to the others two eigenvalues ρ_1, ρ_2 which must satisfy the equation

$$\rho = k^2 + \frac{5k^2}{1 + \frac{9}{4}k^2 - \frac{1}{4}\rho}. \tag{31}$$

Note that Λ_0 has no zeros in $[-K, K]$ and Λ_2 has exactly 2 zeros in $[-K, K)$. Since the equation in (31) has two different roots, namely,

$$\rho_0 = 2 + 5k^2 - 2\sqrt{1 - k^2 + 4k^4}, \quad \rho_2 = 2 + 5k^2 + 2\sqrt{1 - k^2 + 4k^4},$$

it follows that Λ_0 is the eigenfunction associated to ρ_0 which will be the first one eigenvalue to (28). Moreover, since $\rho_0 < \rho_1$ for every $k^2 \in (0, 1)$, we obtain from (31) and from $(\beta_3 - \beta_1)(1 + k^2) = 3(\beta_3 - c)$ that

$$12\lambda_0 = 3\frac{\alpha_3 - c}{k^2 + 1}\rho_0 + 12(c - \alpha_3) < 0 \Leftrightarrow \rho_0 < \rho_1$$

and so λ_0 is the first negative eigenvalue to $\mathcal{L}_{_{KdV}}$ with eigenfunction $\varphi_0(x) = \Lambda_0(\frac{1}{\eta}x)$. Now, since $\rho_1 < \rho_2$ for every $k^2 \in (0, 1)$, we obtain from (31) that

$$12\lambda_2 = 3\frac{\alpha_3 - c}{k^2 + 1}\rho_2 + 12(c - \alpha_3) > 0 \Leftrightarrow \rho_2 > \rho_1$$

and so λ_2 is the third eigenvalue to $\mathcal{L}_{_{KdV}}$ with eigenfunction $\varphi_2(x) = \Lambda_2(\frac{1}{\eta}x)$. Finally, by using the Floquet theory ([21]) we can see that the rest of the spectrum of $\mathcal{L}_{_{KdV}}$ is constituted by a discret set of eigenvalues, which are double. This completes the proof of the Theorem. ■

Finally, by using Theorem 2.4 and the Grillakis *et.al.* theory in [16] we obtain the following result of nonlinear stability of periodic travelling-wave solutions of the KdV equation (see Angulo&Bona in [7] for a complete proof of it).

Theorem 2.5. *Let $c \in (0, \infty)$ and consider the cnoidal wave ψ_c given by Theorem 2.3. Then, the orbit $\Omega_{\psi_c} = \{\psi_c(\cdot + s)\ ;\ s \in \mathbb{R}\}$ is $H^1_{per}([-L, L])$-stable with regard to the flow of the KdV equation .*

3 Existence of Periodic Travelling-Wave Solutions for the Benjamin Equation

In this section we establish a theory of existence of even periodic solutions $\phi = \phi_{l,c}$ to the equation

$$\phi'' + l\mathcal{H}\phi' + \frac{1}{2}\phi^2 - c\phi = A_{\phi_{l,c}}, \tag{32}$$

where $A_{\phi_{l,c}}$ is an integration constant, such that the solution ϕ is defined in an *a priori* interval $[-L, L]$ and has mean zero on it. This last one property, as we will see, is crucial in our nonlinear stability theory.

We start by establishing some notations. For every $L > 0$ fixed, we denote by X_0 the space of functions in $H^2_{per,e}([-L, L])$ with mean equals zero, namely,

$$X_0 = \{f \in H^2_{per,e}([-L, L]); \langle f\rangle \equiv \frac{1}{2L}\int_{-L}^{L} f(x)dx = 0\},$$

and we denote by Y_0 the space $\{f \in L^2_e([-L, L]); \langle f\rangle = 0\}$. Let ψ_c be the *cnoidal wave* solution associated to the KdV equation obtained in Theorem 2.3 for every $c \in (0, \infty)$, and given by

$$\psi_c(\xi) = \beta_2 + (\beta_3 - \beta_2)\mathrm{cn}^2\Big[\sqrt{\frac{\beta_3 - \beta_1}{12}}\xi; k\Big], \tag{33}$$

where $k^2 = \frac{\beta_3-\beta_2}{\beta_3-\beta_1}$, $\beta_1 + \beta_2 + \beta_3 = 3c$ and $\beta_1 < \beta_2 < 0 < \beta_3$. We take $\phi_{0,c} \equiv \psi_c$ that is a solution of (32) with $l = 0$. We will write A_{ψ_c} to denote $A_{\phi_{0,c}}$. Therefore, $A_{\psi_c} = \frac{1}{4L}\int_{-L}^{L}\psi_c^2(x)dx$. We also remark, as a consequence of Theorem 2.3, that the solution ψ_c satisfies the following relations

$$0 = \frac{d}{dc}\int_{-L}^{L}\psi_c(x)dx = \int_{-L}^{L}\frac{d}{dc}\psi_c(x)dx, \tag{34}$$

and

$$\mathcal{L}_{KdV}\big(-\frac{d}{dc}\psi_c\big) = \psi_c + \frac{d}{dc}A_{\psi_c}. \tag{35}$$

Next Theorem proves the existence of a curve of real even periodic travelling-wave solutions of the Benjamin equation (1) for l in a neighborhood of zero. Our approach use the Implicit Function Theorem.

Theorem 3.1. *Let $L > 0$ and $c \in (0, \infty)$ be fixed numbers satisfying the a priori relation (6). Then, there exists $\delta > 0$ such that for every $l \in (-\delta, \delta)$ equation (32) has a solution $\Phi_l = \phi_{l,c} \in X_0$. Moreover, $\Phi_0 = \psi_c$ and the map $l \mapsto \Phi_l$ is analytic.*

Proof. Let us consider, for some $l_1 > 0$, the following functional

$$\begin{aligned} F : (-l_1, l_1) \times X_0 &\mapsto Y_0 \\ (l, \phi) &\mapsto F(l, \phi) = -\phi'' - l\mathcal{H}\phi' - \frac{1}{2}\phi^2 + c\phi + \frac{1}{4L}\int_{-L}^{L}\phi^2(x)dx. \end{aligned} \tag{36}$$

Then, we have from Theorem 2.3 that $F(0, \psi_c) = 0$. Moreover, it is not difficult to see that the Fréchet derivative $\partial_2 F$ exists on $(-l_1, l_1) \times X_0$ and is defined as a map from $(-l_1, l_1) \times X_0$ to $B(X_0, Y_0)$ by

$$\partial_2 F(l, \phi)\varphi = (-\partial_x^2 - l\mathcal{H}\partial_x + c - \phi)\varphi + \frac{1}{2L}(\phi, \varphi), \tag{37}$$

So, it follows in particular from (37) and (21) that $\partial_2 F(0, \psi_c)$ has the form

$$\partial_2 F(0, \psi_c) = \mathcal{L}_{_{KdV}} + \frac{1}{2L}(\psi_c, \cdot). \tag{38}$$

Next, we will prove that $\partial_2 F(0, \psi_c)$ is a bijection from X_0 onto Y_0. So, by applying the Implicit Function Theorem we will have the result. We start with the injectivity. Let $\varphi \in X_0$ and suppose that $\partial_2 F(0, \psi_c)\varphi = 0$. Then,

$$\mathcal{L}_{_{KdV}}\varphi = -\frac{1}{2L}(\psi_c, \varphi). \tag{39}$$

We assert that there is no $f \in X_0$ such that $\mathcal{L}_{_{KdV}} f = 1$. If the last assertion is true, it follows from (39) that $(\psi_c, \varphi) = 0$, which implies that $\mathcal{L}_{_{KdV}}\varphi = 0$. Now, from Theorem 2.4 we have that $\mathcal{L}_{_{KdV}} : H^2_{per,e}([-L, L]) \mapsto L^2_e([-L, L])$ is one to one, in fact, since zero is a simple eigenvalue of $\mathcal{L}_{_{KdV}}$ defined on $H^2_{per}([-L, L]) \subset L^2([-L, L])$ with eigenfunction ψ_c', which is an odd function, we have that if $\mathcal{L}_{_{KdV}}\varphi = 0$ for $\varphi \in H^2_{per,e}([-L, L])$ then $\varphi = \lambda\psi_c'$, for some $\lambda \in \mathbb{R}$, which obviously implies that $\lambda = 0$. Now let us prove, by contradiction, the assertion we did before. Suppose that there is $f \in X_0$ such that $\mathcal{L}_{_{KdV}} f = 1$. Then, $-f'' - \psi_c f + cf = 1$, and integrating the last equation on $[-L, L]$, it follows that

$$-(\psi_c, f) = 2L. \tag{40}$$

Now, since $\partial_2 F(0, \psi_c)$ is a self-adjoint operator with domain $H^2_{per}([-L, L]) \subset L^2([-L, L])$ and using (34), we get

$$(\partial_2 F(0, \psi_c)\big(\frac{d}{dc}\psi_c\big), f) = (\frac{d}{dc}\psi_c, 1 + \frac{1}{2L}(\psi_c, f)) = 0. \tag{41}$$

On the other hand, using (35) and the value of A_{ψ_c}, we have that

$$\partial_2 F(0, \psi_c)\big(\frac{d}{dc}\psi_c\big) = \mathcal{L}_{_{KdV}}\big(\frac{d}{dc}\psi_c\big) + \frac{1}{2L}(\psi_c, \frac{d}{dc}\psi_c) = -\psi_c.$$

Now, using the last expression and (41), it follows that

$$(\partial_2 F(0, \psi_c)\big(\frac{d}{dc}\psi_c\big), f) = -(\psi_c, f) = 0,$$

which is a contradiction with (40). So, $\partial_2 F(0,\psi_c)$ is one to one.

Now, we prove that $\partial_2 F(0,\psi_c)$ is a surjective map onto Y_0. Let us remark that $\partial_2 F(0,\psi_c) - \mathcal{L}_{_{KdV}}$ is a compact operator (dimension of the range is one), then by Theorem 5.35 in [19] (also known as Kato-Rellich's Theorem) and by Theorem 2.4 we have that

$$\sigma_{ess}(\partial_2 F(0,\psi_c)) = \sigma_{ess}(\mathcal{L}_{_{KdV}}) = \emptyset,$$

where σ_{ess} is used to denote the *essential spectrum*. On the other hand, for any self-adjoint operator A, we have that (see [19]), $\sigma(A) = \sigma_{ess}(A) \cup \sigma_{disc}(A)$, where $\sigma_{disc}(A)$ denotes the *discrete spectrum* of the operator A, and $\sigma_{ess}(A)$, $\sigma_{disc}(A)$ are disjoint sets. Since $\partial_2 F(0,\psi_c)$ is one to one, it follows that zero is not an eigenvalue of $\partial_2 F(0,\psi_c)$, then $0 \in \rho(\partial_2 F(0,\psi_c))$, where ρ is used to denote the *resolvent* set of an operator. Hence $\partial_2 F(0,\psi_c)$ is surjective.

Finally, since F is a C^1-map on an open neighborhood of the point $(0,\psi_c)$, it follows from the Implicit Function Theorem that there exist $\delta > 0$, $\epsilon > 0$ and a unique C^1-map

$$\begin{array}{rcl} \Phi : (-\delta,\delta) & \mapsto & B(\psi_c,\epsilon) \subset X_0 \\ l & \mapsto & \Phi_l \end{array}$$

such that $F(l,\Phi_l) = 0$, where $\Phi_0 = \psi_c$. Moreover, since $F(l,\phi)$ depends analytically on l, the map $l \to \Phi_l$ is analytic as well. This concludes the proof. ■

4 Stability of Periodic Travelling-Wave Solutions for the Benjamin Equation

In this section we establish a theory of nonlinear stability of the periodic travelling-wave for the Benjamin equation determined in Theorem 3.1 above. Our approach will be based on the Grillakis *et.al.* theory ([16]). Since many basic changes in this theory are necessary when it is applied in the case of periodic solutions, we will establish here an outline of this criterion for stability. We start by establishing the spectral information necessary to the self-adjoint operator

$$T_l = -\partial_x^2 - l\mathcal{H}\partial_x - \phi_{l,c} + c, \tag{42}$$

where $\phi_{l,c}$ is the periodic travelling-wave solution for the Benjamin equation obtained in Theorem 3.1 for $c \in (0,\infty)$ and $l \in (-\delta,\delta)$. Let us define, for $l \in (-\delta,\delta)$, the following self-adjoint operator

$$\mathcal{L}_l = -\partial_x^2 - l\mathcal{H}\partial_x - \phi_{l,c} + c + \frac{1}{2L}(\phi_{l,c},\cdot).$$

So, we have that

$$T_l = \mathcal{L}_l - \frac{1}{2L}(\phi_{l,c},\cdot), \quad \mathcal{L}_0 = \mathcal{L}_{KdV} + \frac{1}{2L}(\psi_c,\cdot).$$

The next Lemma shows us that T_l varies continuously with l, in the topology of generalized convergence on the space $\mathcal{C}(L^2)$ of closed operators on $L^2([-L,L])$ (see Appendix or section IV-2 of Kato's book [19]). Hence the eigenvalues of $\mathcal{L}_l$ depend continuously on l.

Lemma 4.1. *Let $c \in (0,\infty)$ and $L > 0$. Let T_l be the operator defined above, for $l \in (-\delta,\delta)$, where $\delta > 0$ is given by Theorem 3.1. Then*

$$\lim_{l\to 0} \hat{\delta}(T_l, \mathcal{L}_{KdV}) = 0$$

where $\hat{\delta}$ is a metric defined on $\mathcal{C}(L^2)$. Therefore, for l small, we have that the linear operator T_l with domain $H^2_{per}([-L,L])$, has exactly its three first eigenvalues simple, being the eigenvalue zero the second one with eigenfunction $\phi'_{l,c}$. Moreover, the rest of the spectrum is constituted by a discret set of eigenvalues.

Proof. Using Theorem A.1 in the Appendix, it follows that

$$\begin{aligned} \hat{\delta}(T_l, \mathcal{L}_{KdV}) &= \hat{\delta}(-\partial_x^2 - \psi_c + c - l\mathcal{H}\partial_x + (\psi_c - \phi_{l,c}), -\partial_x^2 - \psi_c + c) \\ &\leq 2(1 + \|\psi_c\|^2_{L^\infty})\, \hat{\delta}(T + A_l, T), \end{aligned} \tag{43}$$

where $T = -\partial_x^2 + c$ and $A_l = -l\mathcal{H}\partial_x + (\psi_c - \phi_{l,c})$. On the other hand, for every $f \in \mathcal{D}(T) = H^2_{per} \subset H^1_{per} = \mathcal{D}(A_l)$ we have that

$$\begin{aligned} \|A_l f\| &\leq \|\psi_c - \phi_{l,c}\|_{L^\infty}\|f\| + |l|\Big(\sum_{|k|\geq 1} k^2|\hat{f}(k)|^2\Big)^{1/2} \\ &\leq \|\psi_c - \phi_{l,c}\|_{L^\infty}\|f\| + |l|\|Tf\|, \end{aligned}$$

where in the last inequality it was used that $|k| \leq c + k^2$ for all integer k. Taking $|l| < 1$ in the last inequality we have that A_l is T-bounded with relative bound less than one, so it follows from (43) and Theorem A.2 that

$$\hat{\delta}(T_l, \mathcal{L}_{KdV}) \leq 2(1 + \|\psi_c\|^2_{L^\infty})\frac{\sqrt{\|\psi_c - \phi_{l,c}\|^2_{L^\infty} + l^2}}{1 - |l|}. \tag{44}$$

Since, by Theorem 3.1, $\phi_{l,c} \to \psi_c$ as $l \to 0$ in $H^2_{per}([-L,L])$, it follows from the Sobolev embedding theorem that $\|\psi_c - \phi_{l,c}\|_{L^\infty} \to 0$ as $l \to 0$. Hence, the inequality (44) implies that $\hat{\delta}(T_l, \mathcal{L}_{KdV})$ tends to zero as $l \to 0$.

The last part of the Theorem follows immediately from Theorem A.6 in the Appendix (see also Albert&Bona&Henry [2], Proposition 4.4). ■

Next, we have that the map $l \mapsto T_l^{-1}(\phi_{l,c})$ is a continuous $B(L_e^2)$-valued map for l in some neighborhood of zero.

Lemma 4.2. *Let $c \in (0,\infty)$ and $L>0$. Let $l \in (-\delta,\delta)$ and T_l be the operator defined in (42). Then,*

$$T_l^{-1} \to \mathcal{L}_{_{KdV}}^{-1} \quad as \quad l \to 0,$$

in the topology of the operator norm on $B(L_e^2)$. In particular, $T_l^{-1}(\phi_{l,c}) \to \mathcal{L}_{_{KdV}}^{-1}(\psi_c)$ as $l \to 0$ in L^2-norm.

Proof. It is not difficult to see that $\mathcal{L}_{_{KdV}}, T_l \in \mathcal{C}(L_e^2)$. Moreover, since $\sigma_{ess}(\mathcal{L}_{_{KdV}}) = \emptyset$ and $\mathcal{L}_{_{KdV}}$ is injective on $L_e^2([-L,L])$ (see proof of Theorem 3.1 above), $\mathcal{L}_{_{KdV}}^{-1} \in B(L_e^2, H_e^2) \subset B(L_e^2)$. On the other hand, by (44), there exists $l_0>0$ such that

$$\hat{\delta}(T_l, \mathcal{L}_{_{KdV}}) < (1 + \|\mathcal{L}_{_{KdV}}^{-1}\|_{B(L_e^2)}^2)^{-1/2},$$

for $l \in (-l_0, l_0)$. So, Theorem A.3 implies that $T_l^{-1} \in B(L_e^2)$, for $l \in (-l_0,l_0)$. Now, by Theorem A.5, we have that

$$\hat{\delta}(T_l^{-1}, \mathcal{L}_{_{KdV}}^{-1}) = \hat{\delta}(T_l, \mathcal{L}_{_{KdV}}) < (1 + \|\mathcal{L}_{_{KdV}}^{-1}\|_{B(L^2)}^2)^{-1/2},$$

for all $l \in (-l_0,l_0)$. The last inequality and Theorem A.4, imply that

$$\|T_l^{-1} - \mathcal{L}_{_{KdV}}^{-1}\|_{B(L_e^2)} \leq \frac{(1+\|\mathcal{L}_{_{KdV}}^{-1}\|^2)\ \hat{\delta}(T_l, \mathcal{L}_{_{KdV}})}{1-(1+\|\mathcal{L}_{_{KdV}}^{-1}\|^2)^{1/2}\ \hat{\delta}(T_l, \mathcal{L}_{_{KdV}})}, \tag{45}$$

for all $l \in (-l_0,l_0)$. Using Lemma 4.1 into (45), finishes the proof. ■

The main Theorem of this paper is the following, where we will use the notations established by Theorem 3.1.

Theorem 4.1. *Let $L>0$ and $c \in (0,\infty)$ be fixed numbers satisfying (6). Consider the periodic travelling-wave solution $\phi_{l,c}$ given by Theorem 3.1 for l in a neighborhood of zero . Then, the orbit $\Omega_{\phi_{l,c}} = \{\phi_{l,c}(\cdot + s)\ ;\ s \in \mathbb{R}\}$ is $H^1_{per}([-L,L])$-stable with regard to the flow of the Benjamin equation .*

For the proof of Theorem 4.1, we will need of the following Lemmas. Initially, for any real number s let τ_s denote the translation by s, that is, $\tau_s f(x) = f(x+s)$ for all $x \in \mathbb{R}$. For $\epsilon > 0$, consider the tubular neighbourhood around the $\phi_{l,c}$-orbit, $\Omega_{\phi_{l,c}} = \{\phi_{l,c}(\cdot + s)\ ;\ s \in \mathbb{R}\}$,

$$U_\epsilon = \{u \in H^1_{per}([-L,L])\ ;\ \inf_{s\in\mathbb{R}} \|u - \tau_s\phi_{l,c}\|_1 < \epsilon\}.$$

So, following Grillakis *et.al.* we have the following.

Lemma 4.3. *There exist $\epsilon > 0$ and a C^1 map $\alpha : U_\epsilon \mapsto \mathbb{R}/2L$ such that, for all $u \in U_\epsilon$ and $r \in \mathbb{R}$,*

$$\begin{array}{ll} (i) & (u(\cdot + \alpha(u)), \phi'_{l,c}) = 0. \\ (ii) & \alpha(u(\cdot + r)) = \alpha(u) - r, \textit{ modulo the period.} \end{array}$$

Proof. The proof is standard. Consider the functional

$$G(u,\alpha) = \int_{-L}^{L} u(x+\alpha)\phi'_{l,c}(x)\, dx.$$

for $(u,\alpha) \in L^2_{per}([-L,L])\times\mathbb{R}$. Since $G(\phi_{l,c},0) = 0$ and $\partial_\alpha G(\phi_{l,c},0) = \int_{-L}^{L} [\phi'_{l,c}(x)]^2\, dx \neq 0$, by the Implicit Function Theorem there is a unique C^1 functional $\alpha(u)$ satisfying $G(u,\alpha(u)) = 0$ for u in a ball $B_\epsilon(\phi_{l,c})$ of $\phi_{l,c}$. Now, we show that (ii) is true within $B_\epsilon(\phi_{l,c})$. Let $u \in B_\epsilon(\phi_{l,c})$ and $\eta \in [-L,L]$ such that $u(\cdot + \eta) \in B_\epsilon(\phi_{l,c})$, then since $0 = G(u,\alpha(u)) = G(\tau_\eta u, \alpha(u) - \eta)$ we have by the uniqueness of $\alpha(u)$ that, $\alpha(\tau_\eta u) = \alpha(u) - \eta$. So, since every real number r can be written in the form $r = \eta + m2L$, $m \in \mathbb{Z}$, it follows immediately (ii). Finally, we extend the definition of $\alpha(u)$ to $u \in U_\epsilon$. If $\|u - \tau_{s_0}\phi_{l,c}\|_1 < \epsilon$ for some $s_0 \in \mathbb{R}$, we define $\alpha(u) \equiv \alpha(\tau_{-s_0}u) - s_0$. This definition does not depend of the choice of s_0. In fact, let s_1 be such that $\|u - \tau_{s_1}\phi_{l,c}\|_1 < \epsilon$, then $\tau_{-s_0}u$ and $\tau_{-s_1}u$ belong to $B_\epsilon(\phi_{l,c})$. Since (ii) is true within $B_\epsilon(\phi_{l,c})$, we have $\alpha(\tau_{s_0-s_1}\tau_{-s_0}u) = \alpha(\tau_{-s_0}u) - (s_0 - s_1)$, modulo the period. Therefore, $\alpha(\tau_{-s_1}u) - s_1 = \alpha(\tau_{-s_0}u) - s_0$, modulo the period. Hence, $\alpha(u)$ is defined for all $u \in U_\epsilon$ and satisfies properties $(i) - (ii)$. ■

In the next Lemma we will use the special structure of the spectrum of T_l given by Lemma 4.1.

Lemma 4.4. *Let $c \in (0,\infty)$, $L>0$ and $l \in (-\delta,\delta)$ where $\delta>0$ is given by Theorem 3.1. Suppose that there exists a $\varphi_l \in L^2_{per}([-L,L])$ such that*

$$T_l\varphi_l = \phi_{l,c} + \frac{d}{dc}A_{\psi_c}, \quad \textit{and} \quad (\varphi_l, T_l\varphi_l) = \int_{-L}^{L} \varphi_l[\ phi_{l,c} + \frac{d}{dc}A_{\psi_c}]dx < 0. \tag{46}$$

Define $\mathcal{A} = \{\varphi \in H^1_{per}; \langle\varphi\rangle = 0,\ (\varphi,\phi_{l,c}) = (\varphi,\phi'_{l,c}) = 0,\ \text{and}\ (\varphi,\varphi)=1\}$. Then

$$\zeta = \inf\{< T_l\varphi, \varphi >_1 :\ \varphi \in \mathcal{A}\} > 0.$$

Therefore, there is a positive constant D such that $< T_l\varphi,\varphi >_1 \geqq D\|\varphi\|_1^2$ for all $\varphi \in \mathcal{A}$.

Proof. Initially we show that $\zeta \geqq 0$. In fact, from Lemma 4.1 we can write $L^2_{per}([-L,L])$ as $L^2_{per} = [\chi_0] \oplus [\phi'_{l,c}] \oplus P$ (see Theorem III.6.17 in Kato [19]), with $\chi_0 \in \mathcal{D}(T_l)$ satisfying $\|\chi_0\| = 1$ and $T_l\chi_0 = -\lambda_0\chi_0$ with $\lambda_0 > 0$. We remember that $P_l = \mathcal{D}(T_l) \cap P$ is called the positive subspace of T_l and there is a $\eta > 0$ such that $(T_lp,p) \geqq \eta\|p\|^2$ for $p \in P_l$. So, we have the following decomposition $\varphi_l = a_0\chi_0 + b_0\phi'_{l,c} + p_0$, where $< T_lp_0, p_0 >_1 > 0$ and $T_l\phi'_{l,c} = 0$. It follows then from (46) that

$$\begin{aligned} < T_lp_0, p_0 >_1 &= < T_l\varphi_l, p_0 >_1 = < T_l\varphi_l, \varphi_l - a_0\chi_0 - b_0\phi'_{l,c} >_1 \\ &= < \varphi_l, T_l\varphi_l + a_0\lambda_0\chi_0 >_1 = (\varphi_l, T_l\varphi_l) + a_0\lambda_0(\varphi_l,\chi_0) \\ &= (\varphi_l, T_l\varphi_l) + a_0^2\lambda_0 < a_0^2\lambda_0. \end{aligned}$$

Next let $\varphi \in \mathcal{A}$ and write $\varphi = a\chi_0 + p$ with $p \in P$. So, $p \in H^1_{per}([-L,L])$. Then we have

$$0 = (\phi_{l,c},\varphi) = (\phi_{l,c} + \frac{d}{dc}A_{\psi_c}, \varphi) = < T_l\varphi_l, \varphi >_1 = -a_0a\lambda_0 + < T_lp_0, p >_1,$$

which implies

$$\begin{aligned} < T_l\varphi, \varphi >_1 &= -a^2\lambda_0 + < T_lp, p >_1 \geqq -a^2\lambda_0 + \frac{< T_lp, p_0 >_1^2}{< T_lp_0, p_0 >_1} \\ &> -a^2\lambda_0 + \frac{(a_0a\lambda_0)^2}{a_0^2\lambda_0} = 0. \end{aligned}$$

Therefore $\zeta \geqq 0$. Now suppose $\zeta = 0$, then following a similar analysis as that done in Albert&Bona ([1]) we obtain that there is a $\varphi^* \in \mathcal{A}$ such that $< T_l\varphi^*, \varphi^* >_1 = 0$, which is a contradiction. So $\zeta > 0$, and for $\varphi \in H^1_{per}([-L,L])$ such that $(\varphi,\phi_{l,c}) = (\varphi,\phi'_{l,c}) = 0 = \langle\varphi\rangle$ we have that

$$< T_l\varphi, \varphi >_1 \geqq \zeta\|\varphi\|^2.$$

Finally, it follows easily from the specific form of the operator T_l that we can change the L^2_{per}-norm on the right hand side of the last inequality by the H^1_{per}-norm. This finishes the proof of the Lemma. ■

Now we obtain the key inequality for obtaining our stability result.

Lemma 4.5. *Let $\delta > 0$ be a positive number given by Theorem 3.1. Let $c \in (0, \infty)$ and $L > 0$. Consider $l \in (-\delta, \delta)$ and $\varphi_l \in L^2_{per}([-L, L])$ satisfying (46). Then, there exist constants $\beta > 0$ and $\epsilon > 0$ such that*

$$E_B(u) - E_B(\phi_{l,c}) \geqq \beta \|u(\cdot + \alpha(u)) - \phi_{l,c}\|_1^2 \tag{47}$$

for all $u \in U_\epsilon$ which satisfy $F(u) = F(\phi_{l,c})$ and $\langle u \rangle = 0$.

Proof. Write u in the form $u(\cdot + \alpha(u)) = (1 + a)\phi_{l,c} + \varphi$ where $(\varphi, \phi_{l,c}) = 0$ and a is a scalar. Note that $\langle \varphi \rangle = 0$. Now, by the translation invariance of F and Taylor's Theorem

$$F(\phi_{l,c}) = F(u(\cdot + \alpha(u))) = F(\phi_{l,c}) + (\phi_{l,c}, u(\cdot + \alpha(u)) - \phi_{l,c}) + O(\|u(\cdot + \alpha(u)) - \phi_{l,c}\|_1^2).$$

Then we have $a = O(\|u(\cdot + \alpha(u)) - \phi_{l,c}\|_1^2)$. Consider now $L(u) = E_B(u) + cF(u)$. Another Taylor expansion gives

$$L(u) = L(u(\cdot + \alpha(u))) = L(\phi_{l,c}) + < L'(\phi_{l,c}), v >_1 + \frac{1}{2} < L''(\phi_{l,c})v, v >_1 + o(\|v\|_1^2),$$

where $v \equiv u(\cdot + \alpha(u)) - \phi_{l,c} = a\phi_{l,c} + \varphi$. Since $F(u) = F(\phi_{l,c})$, $\langle v \rangle = 0$, $< L'(\phi_{l,c}), v >_1 = (-A_{\phi_{l,c}}, v) = 0$ and $L''(\phi_{l,c}) = T_l$, we obtain that

$$E_B(u) - E_B(\phi_{l,c}) = \frac{1}{2} < T_l v, v >_1 + o(\|v\|_1^2) = \frac{1}{2} < T_l \varphi, \varphi >_1 + o(\|v\|_1^2).$$

By Lemma 4.3 we have that $(\varphi, \phi'_{l,c}) = 0$. So, from Lemma 4.4 and taking $\|v\|_1$ small we have finally

$$E_B(u) - E_B(\phi_{l,c}) \geqq \frac{1}{2} D \|\varphi\|_1^2 + o(\|v\|_1^2) \geqq \frac{1}{4} D \|v\|_1^2.$$

Therefore, Lemma 4.5 is established. ■

The proof of Theorem 4.1 will be based on Lemma 4.5, so initially we verify the validity of assumption (46). In fact, since $\phi_{l,c} + \frac{d}{dc} A_{\psi_c} \in L^2_e([-L, L])$ we define, for c fixed,

$$\varphi_{l,c} \equiv T_l^{-1}(\phi_{l,c} + \frac{d}{dc} A_{\psi_c}).$$

Then, by Lemma 4.2 we have that $\varphi_{l,c} \to \mathcal{L}^{-1}_{KdV}(\psi_c + \frac{d}{dc}A_{\psi_c})$ as $l \to 0$. Therefore, by using Theorem 3.1, (35), (34) and the convexity of function $d(c)$ (Theorem 2.4) we have that for $l \to 0$

$$\begin{aligned}(\varphi_{l,c}, T_l\varphi_{l,c}) &\to (\mathcal{L}^{-1}_{KdV}[\psi_c + \frac{d}{dc}A_{\psi_c}], \psi_c + \frac{d}{dc}A_{\psi_c}) \\ &= (-\frac{d}{dc}\psi_c, \psi_c + \frac{d}{dc}A_{\psi_c}) = -\frac{1}{2}\frac{d}{dc}\|\psi_c\|^2 < 0. \end{aligned} \tag{48}$$

Therefore, we have that (46) is hold for l in a neighborhood of zero. We note that the value of the product $(\mathcal{L}^{-1}_{KdV}f, f)$, for $f = \psi_c + \frac{d}{dc}A_{\psi_c}$ is independent of the value of $\mathcal{L}^{-1}_{KdV}f$. Indeed, if $g = \mathcal{L}^{-1}_{KdV}f$, $h = \mathcal{L}^{-1}_{KdV}f$ then $\mathcal{L}_{KdV}(g-h) = 0$ and so we have $g - h = \lambda\psi'_c$ and $(g-h, f) = 0$.

Next we establish a Theorem of global well-posedness of the Benjamin equation (1), in the periodic case, which will be sufficient to our interest. This result has been established by Linares in [20], which is based on the new function spaces introduced by Bourgain ([12], [13]). To state the results we recall the following notations. For $s, b \in \mathbb{R}$, $Y_{s,b}$ denotes the completion of the space Y with respect to the norm

$$\|f\|_{Y_{s,b}} = \Big(\sum_{n\neq 0}\int_{-\infty}^{\infty}(1 + |\tau - n^3 + l\,n|n|\,|)^{2b}|n|^{2s}|\widehat{f}(n,\tau)|^2\,d\tau\Big)^{1/2},$$

where Y is the space of all f such that

1. $f : [-L, L] \times \mathbb{R} \to \mathbb{C}$;
2. $f(x, \cdot) \in \mathcal{S}(\mathbb{R})$, for each $x \in [-L, L]$;
3. $x \to f(x, \cdot)$ is C^∞;
4. $\widehat{f}(0, \tau) = 0$, for all $\tau \in \mathbb{R}$.

So, we gather in a same theorem the results of well-posedness to the periodic Benjamin equation given in [20].

Theorem 4.2. *(i) Let $u_0 \in L^2([-L, L])$ with $\langle u_0\rangle = 0$. Then there exists $T = T(\|u_0\|_2, l) > 0$ and a unique solution u of the periodic initial value problem*

$$\begin{cases} u_t + uu_x + u_{xxx} + l\mathcal{H}u_{xx} = 0 \\ u(0) = u_0, \end{cases}$$

such that $u \in C([0,T]; L^2([-L,L])) \cap Y_{0,b}$ *with* $\partial_x(u^2) \in Y_{0,b-1}$.

Moreover, given $T' \in (0,T)$ *the map* $u_0 \mapsto u(t)$ *is Lipschitz continuous from* $L^2([-L,L]) \cap \{f; \langle f\rangle = 0\}$ *to* $C([0,T']; L^2([-L,L])) \cap Y_{0,b}$.

(ii) For $s \geq 0$ *and initial data* $u_0 \in H^s_{per}([-L,L])$ *with arbitrary mean value, we obtain the same results as in (i).*

(iii) From the conservation laws (7) and (8) we can choose $T = +\infty$ *at least for* $s = 0$ *and* $s = 1$.

Now, we will prove the Theorem 4.1.

Proof. Initially we will show stability of the $\Omega_{\phi_{l,c}}$-orbit in the closed space $\mathcal{X}_0 \equiv H^1_{per}([-L,L]) \cap \mathcal{V}$ where $\mathcal{V} \equiv \{f \in H^1_{per}([-L,L])\ ;\ \langle f\rangle = 0\}$. In fact, assume that $\Omega_{\phi_{l,c}}$ is $\mathcal{X}_0$-unstable and choose initial data $u_k(0) \in U_{1/k} \cap \mathcal{V}$ such that $u_k(t) \notin U_\epsilon$ for some $\epsilon > 0$ and some $t \in (0,\infty)$, where $u_k(t)$ is the solution of the Benjamin equation (1) with initial data $u_k(0)$ (see Theorem 4.2). Since $H(u) = \int_{-L}^{L} u\, dx$ is a conservation law to equation (1), we get that $H(u_k(t)) = H(u_k(0)) = 0$ for all t. Now, by continuity in t we can find t_k, k sufficiently large, such that

$$\inf_{y\in\mathbb{R}} \|u_k(t_k) - \phi_{l,c}(\cdot + y)\|_1 = \epsilon/2. \tag{49}$$

Since E_B and F are continuous functionals over $H^1_{per}([-L,L])$, conservation laws of (1) and translation invariants, we get $E_B(u_k(t_k)) = E_B(u_k(0)) \to E_B(\phi_{l,c})$ and $F(u_k(t_k)) = F(u_k(0)) \to F(\phi_{l,c})$ as $k \to \infty$. Next choose $w_k \in U_\epsilon \cap \mathcal{V}$ so that $F(w_k) = F(\phi_{l,c})$ and $\|w_k - u_k(t_k)\|_1 \to 0$. Then, by Lemma 4.5

$$0 \leftarrow E_B(w_k) - E_B(\phi_{l,c}) \geqq D\|w_k(\cdot + \alpha(w_k)) - \phi_{l,c}\|_1^2 = D\|w_k - \phi_{l,c}(\cdot - \alpha(w_k))\|_1^2,$$

and therefore $\|u_k(t_k) - \phi_{l,c}(\cdot - \alpha(w_k))\|_1 \to 0$. Since this contradicts (49) we must have that $\Omega_{\phi_{l,c}}$ is $\mathcal{X}_0$-stable.

Now, we consider the general case. Let $u_0 \in U_\delta$ (where δ is given by the $\mathcal{X}_0$-stability) with $\theta = \int_{-L}^{L} u_0 dx$ and consider $u \in C([0,\infty); H^1_{per}([-L,L]))$ the solution of the Benjamin equation (1) with initial data u_0 given by Theorem 4.2. Then, by defining

$$v(x,t) = u\Big(x + \frac{\theta}{2L}t, t\Big) - \frac{\theta}{2L},$$

we have that v satisfies the initial value problem

$$\begin{cases} v_t + vv_x + v_{xxx} + l\mathcal{H}v_{xx} = 0 \\ v(0) = u_0 - \frac{\theta}{2L}, \end{cases}$$

with $v_0 \equiv v(0)$ having media zero over $[-L, L]$. Therefore, using the following Poincaré's inequality for $H^1([-L, L])$

$$\int_{-L}^{L} |h(x)|^2 \, dx - \frac{1}{2L}\Big| \int_{-L}^{L} h(x) \, dx \Big|^2 \leqq C(L) \int_{-L}^{L} |h'(x)|^2 \, dx, \tag{50}$$

we have for every $y \in \mathbb{R}$ that

$$\int_{-L}^{L} |v_0 - \tau_y \phi_{l,c}|^2 \, dx \leqq C(L) \int_{-L}^{L} |u_0' - \tau_y \phi_{l,c}'|^2 \, dx,$$

and so

$$\inf_{y\in\mathbb{R}} \|v_0 - \tau_y \phi_{l,c}\|_1^2 \leqq [C(L)+1] \inf_{y\in\mathbb{R}} \|u_0' - \tau_y \phi_{l,c}'\|^2 < [C(L)+1]\delta^2.$$

Therefore, using that $\Omega_{\phi_{l,c}}$ is $H^1_{per}([-L, L]) \cap \mathcal{V}$-stable we have that $v(t) \in U_\epsilon \cap \mathcal{V}$ for all $t \geqq 0$. Finally, we will see that $u(t) \in U_\epsilon$ for all $t \geqq 0$. In fact, using inequality (50) we have for all $y \in \mathbb{R}$ and $t \geqq 0$

$$\int_{-L}^{L} |u(t) - \tau_y \phi_{l,c}|^2 \, dx \leqq C(L) \int_{-L}^{L} |u'(t) - \tau_y \phi_{l,c}'|^2 \, dx + \frac{1}{2L}\theta^2.$$

Now using again (50) and $u_0 \in U_\delta$, we obtain the following estimative for $|\theta|$. Let $y \in \mathbb{R}$, then by Hölder's inequality

$$\begin{aligned} \frac{1}{2L}\theta^2 &\leqq 2\int_{-L}^{L} |v_0 - \tau_y \phi_{l,c}|^2 + |u_0 - \tau_y \phi_{l,c}|^2 \, dx \\ &\leqq 2[(1 + C(L)] \int_{-L}^{L} |u_0' - \tau_y \phi_{l,c}'|^2 + |u_0 - \tau_y \phi_{l,c}|^2 \, dx \leqq 2[1 + C(L)]\delta^2. \end{aligned}$$

So, using that $v(t) \in U_\epsilon$ (which implies that $\inf_{y\in\mathbb{R}} \ \|u'(t) - \tau_y \phi_{l,c}'\|^2 < \epsilon^2$) we have finally that

$$\inf_{y\in\mathbb{R}} \ \|u(t) - \tau_y \phi_{l,c}\|_1^2 \leqq [1 + C(L)][\epsilon^2 + 2\delta^2],$$

for all $t \geqq 0$. This completes the proof of the Theorem. ■

5 Appendix

In this Appendix, we state initially some properties about Jacobian Elliptic Integrals that have been used in this work. We define *the normal elliptic integral of the first kind*:

$$\int_0^y \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} = \int_0^\varphi \frac{d\theta}{\sqrt{1-k^2 \ \sin^2\theta}} \equiv F(\varphi, k)$$

where $y = \sin\varphi$, and *the normal elliptic integral of the second kind*,

$$\int_0^y \sqrt{\frac{1-k^2t^2}{1-t^2}}\,dt = \int_0^{\varphi} \sqrt{1-k^2\ \sin^2\theta}\,d\theta \equiv E(\varphi, k).$$

In their algebraic forms, these two integrals have the following properties: The first one is finite for all real (or complex) values of y, including infinity; the second one has a simple pole of order 1 for $y = \infty$. The number k is called the *modulus*. This number may take any real or imaginary value. Here we will take $0 < k < 1$. The number k' is called the *complementary modulus* and is related to k by $k' = \sqrt{1-k^2}$. The variable limit y or φ is the *argument* of the normal elliptic integrals, and may be of course either real or complex, but it is usually understood that $0 < y \leqq 1$ or $0 < \varphi \leqq \pi/2$. When $y = 1$, the integrals above are said to be *complete*. In this case, one writes:

$$\int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\ \sin^2\theta}} = F(\pi/2, k) \equiv K(k) \equiv K,$$

and

$$\int_0^1 \sqrt{\frac{1-k^2t^2}{1-t^2}}\,dt = \int_0^{\pi/2} \sqrt{1-k^2\ \sin^2\theta}\,d\theta = E(\pi/2, k) \equiv E(k) \equiv E.$$

So, we have some special values of K and E: $K(0) = E(0) = \pi/2$, $E(1) = 1$ and $K(1) = +\infty$. For $k \in (0,1)$, $K'(k) > 0$, $K''(k) > 0$, and $E'(k) < 0$, $E''(k) < 0$. Moreover, $E(k) < K(k)$ for all $k \in (0,1)$.

Next, we define the *Jacobian Elliptic Functions*. Initially, we consider the elliptic integral

$$u(y_1; k) \equiv u = \int_0^{y_1} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} = \int_0^{\varphi} \frac{d\theta}{\sqrt{1-k^2\ \sin^2\theta}} = F(\varphi, k)$$

which is a strictly increasing function in the variable y_1 (real), hence we can define its inverse function by $y_1 = \sin\varphi \equiv \mathrm{sn}(u; k)$, or briefly $y_1 = \mathrm{sn}\ u$ when it is not necessary to emphasize the modulus. The function sn u is an odd elliptic function. Other two basic functions can also be defined by

$$\begin{cases} \mathrm{cn}\ (u; k) = \sqrt{1-y_1^2} = \sqrt{1-\mathrm{sn}^2(u; k)} \\ \mathrm{dn}\ (u; k) = \sqrt{1-k^2y_1^2} = \sqrt{1-k^2\mathrm{sn}^2(u; k)}, \end{cases}$$

requiring that $\mathrm{sn}(0, k) = 0$, $\mathrm{cn}(0, k) = 1$ and $\mathrm{dn}(0, k) = 1$. Therefore cn u and dn u are even functions. The functions sn u, cn u, and dn u are called *Jacobian elliptic functions* and are one-valued functions of the argument u. These functions have a real period, namely, $4K, 4K$ and $2K$ respectively. Some fundamental relations of the Jacobian elliptic functions are the following: $\text{sn}^2 u + \text{cn}^2 u = 1$, $-1 \leqq \text{sn}\ u \leqq 1$, $-1 \leqq \text{cn}\ u \leqq 1$, $k'^2 \leqq \text{dn}\ u \leqq 1$, $\text{sn}\ K = 1$, $\text{cn}\ K = 0$, $\text{sn}(u+4K) = \text{sn}\ u$, $\text{cn}(u+4K) = \text{cn}\ u$, $\text{dn}(u+2K) = \text{dn}\ u$, $\text{sn}(u+2K) = -\text{sn}\ u$, $\text{cn}(u+2K) = -\text{cn}\ u$, $\text{sn}(u,0) = \sin\ u$, $\text{cn}(u,0) = \cos\ u$, $\text{sn}(u,1) = \tanh\ u$, $\text{cn}(u,1) = \text{sech}\ u$. Finally we have the following differentiation relations, $\frac{\partial}{\partial u}\text{sn}(u) = \text{cn}\ u\ \text{dn}\ u$, $\frac{\partial}{\partial u}\text{cn}(u) = -\text{sn}\ u\ \text{dn}\ u$, $\frac{\partial}{\partial u}\text{dn}(u) = -k^2 \text{sn}\ u\ \text{cn}\ u$.

Now we have some facts from Perturbation Theory of closed linear operators on Hilbert spaces that we have used along this work (see Kato [19] for details). Let X and Y be Banach spaces. Along this paper we considered the particular case $X = Y = L^2([-L, L])$. Moreover, $L^2([-L, L]) \times L^2([-L, L])$ is the Hilbert space with norm defined by $|||(f, g)||| = (\|f\|^2 + \|g\|^2)^{1/2}$, and for any closed operator T on $L^2([-L, L])$ with domain $\mathcal{D}(T)$, its graph is defined by, $G(T) = \{(f, g) \in L^2([-L, L]) \times L^2([-L, L]); f \in \mathcal{D}(T),\ T(f) = g\}$. Then a metric $\widehat{\delta}$ on $\mathcal{C}(L^2([-L, L]))$, the space of closed operators on $L^2([-L, L])$, may be defined as follows: for any $S, T \in \mathcal{C}(L^2([-L, L]))$,

$$\widehat{\delta}(S, T) = \|P_S - P_T\|_{B(L^2 \times L^2)},$$

where P_S and P_T are the orthogonal projections on $G(S)$ and $G(T)$ respectively, and $\|\cdot\|_{B(L^2 \times L^2)}$ denotes the operator norm on the space of bounded operators on $L^2([-L, L]) \times L^2([-L, L])$. The following theorems and their proofs may be found in Chapter IV of [19].

Theorem A.1. *Let S, $T \in \mathcal{C}(X, Y)$ and $A \in B(X, Y)$ with operator norm $\|A\|$. Then,*

$$\hat{\delta}(S + A, T + A) \leq 2(1 + \|A\|^2)\hat{\delta}(S, T).$$

Theorem A.2. *Let $T \in \mathcal{C}(X, Y)$ and let A be $T-$bounded with relative bound less than 1, so that we have the inequality $\|Au\| \leq a\|u\| + b\|Tu\|$, $u \in \mathcal{D}(T) \subset \mathcal{D}(A)$, with $b < 1$. Then $S = T + A \in \mathcal{C}(X, Y)$ and*

$$\hat{\delta}(S, T) \leq \frac{\sqrt{a^2 + b^2}}{1 - b}.$$

In particular if $A \in B(X, Y)$, then

$$\hat{\delta}(T + A, T) \leq \|A\|.$$

Next, we have *the principle of stability of bounded invertibility.*

Theorem A.3. *Let $T \in \mathcal{C}(X,Y)$ be invertible (means one to one) with $T^{-1} \in B(Y,X)$. If $S \in \mathcal{C}(X,Y)$ with $\hat{\delta}(S,T) < (1+\|T^{-1}\|^2)^{-1/2}$, then S is invertible and $S^{-1} \in B(Y,X)$.*

The following result is a consequence of Theorem IV.2.13 in [19].

Theorem A.4. *Let $T \in B(X,Y)$. If $S \in \mathcal{C}(X,Y)$ is such that $\hat{\delta}(S,T) < (1+\|T\|^2)^{-1/2}$, then $S \in B(X,Y)$ and*

$$\|S - T\| \leq \frac{(1 + \|T\|^2)\hat{\delta}(S,T)}{1 - (1 + \|T\|^2)^{1/2}\hat{\delta}(S,T)}.$$

Theorem A.5. *If $T, S \in \mathcal{C}(X,Y)$ are invertible, then*

$$\hat{\delta}(S^{-1}, T^{-1}) = \hat{\delta}(S,T).$$

As a consequence of Theorem IV.3.16 in [19] we obtain the following Theorem.

Theorem A.6. *Let $T \in \mathcal{C}(X)$ and let $\mathcal{U}$ denote an open subset of the complex plane whose boundary is a smooth contour Γ. Suppose that $\sigma(T) \cap \Gamma = \emptyset$ and $\sigma(T) \cap \mathcal{U}$ consists of a finite number of eigenvalues of T, each with finite (algebraic) multiplicity. Then there exists a $\delta > 0$, depending on T and Γ, such that if $S \in \mathcal{C}(X)$ and $\hat{\delta}(S,T) < \delta$, then $\sigma(S) \cap \mathcal{U}$ consists of a finite number of eigenvalues of finite multiplicity, the sum of their multiplicities being equal to the sum of the multiplicities of the eigenvalues of T in $\mathcal{U}$.*

In particular, suppose that $\sigma(T) \cap \mathcal{U}$ consists of a single, simple eigenvalue λ_0 with eigenfunction f_0. If $\{S_n\}$ is a sequence in $\mathcal{C}(X)$ such that $\hat{\delta}(S_n, T) \to 0$ as $n \to \infty$, then for n large, $\sigma(S_n) \cap \mathcal{U}$ consists of a single simple eigenvalue λ_n, and $\lambda_n \to \lambda_0$ as $n \to \infty$. Moreover, there is an eigenfunction f_n associated to λ_n such that $f_n \to f_0$ as $n \to \infty$.

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