Spatial Analyticity of Solutions of a nonlocal perturbation of the KdV equation

Borys Alvarez Samaniego.

IMECC-UNICAMP, C.P. 6065, 13083-970, Campinas, SP, Brazil.

balvarez@ime.unicamp.br

14th July 2003

Abstract

Let $\mathcal{H}$ denote the Hilbert transform and $\eta \geq 0$. We show that if the initial data of the following problems $u_t + uu_x + u_{xxx} + \eta(\mathcal{H}u_x + \mathcal{H}u_{xxx}) = 0$, $u(\cdot, 0) = \phi(\cdot)$ and $v_t + \frac{1}{2}(v_x)^2 + v_{xxx} + \eta(\mathcal{H}v_x + \mathcal{H}v_{xxx}) = 0$, $v(\cdot, 0) = \psi(\cdot)$ has an analytic continuation to a strip containing the real axis, then the solution has the same property, although the width of the strip might decrease with time. We also study the case when the initial data of the first problem is complex-valued.

1 Introduction

We are interested in studying spatial analyticity of the solutions of the following problems:

$$u_t + uu_x + u_{xxx} + \eta(\mathcal{H}u_x + \mathcal{H}u_{xxx}) = 0,\ u(\cdot, 0) = \phi(\cdot), \tag{1}$$

$$v_t + \frac{1}{2}(v_x)^2 + v_{xxx} + \eta(\mathcal{H}v_x + \mathcal{H}v_{xxx}) = 0,\ v(\cdot, 0) = \psi(\cdot), \tag{2}$$

where $\mathcal{H}$ denotes the Hilbert transform given by $\mathcal{H}f(x) = \frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty} \frac{f(y)}{y-x}dy$ for $f \in \mathcal{S}(\mathbb{R})$ the Schwartz space of rapidly decreasing $C^\infty(\mathbb{R})$ functions, $\mathcal{P}$ represents the principal

[1]2000 Mathematics Subject Classification. 35Q53, 35B30.

[2]Key words and phrases. Spatial analyticity, Hilbert transform, KdV equation.

[3]Research supported by FAPESP/Brazil under grant No. 02/02522-0.

value of the integral and the parameter η is an arbitrary nonnegative number. It is known that $\widehat{(\mathcal{H}f)}(\xi) = i\text{sgn}(\xi)\hat{f}(\xi)$, for all $f \in H^s(\mathbb{R})$, where

$$\text{sgn}\,(\xi) = \begin{cases} -1, & \xi < 0, \\ 1, & \xi > 0. \end{cases}$$

The equation (1) was derived by Ostrovsky *et.al.* (see [8] for more details) to describe the radiational instability of long non-linear waves in a stratified fluid caused by internal wave radiation from a shear layer; the fourth term corresponds to the *wave amplification* and the fifth term represents *damping*. It models the motion of a homogeneous finite-thickness fluid layer with density δ_1, which moves at a constant speed U, slipping over an immobile infinitely deep stratified fluid with a density $\delta_2 > \delta_1$. The upper boundary of the layer is supposed to be rigid and the lower one is contiguous to the infinitely deep fluid. Here $u(x,t)$ is the deviation of the interface from its equilibrium position. Let us remark that some numerical results for periodic and solitary-wave solutions of equation (1) were obtained by Bao-Feng Feng and T. Kawahara [3].

The Cauchy problems associated to (1) and (2) were studied in [1], where it was proved that the problems (1) and (2) are globally well-posed in $H^s(\mathbb{R})$ for $s \geq 1$, considering real-valued solutions.

In this paper we are interested in proving that if the initial condition of the problem (1) (resp. (2)) is analytic and has an analytic continuation to a strip containing the real axis, then the solution of (1) (resp. (2)) has the same property, but the width of the strip might decrease with time.

Section 2 is devoted to studying the case when the solutions are real-valued on the real axis at any time. Hence, the results obtained in [1] about the initial value problems associated to (1) and (2) will be helpful. We use the method developed by Kato&Masuda [7] which estimates certain families of Liapunov functions for the solutions, to prove global spatial analyticity of the solutions. Moreover, proceeding as in [2], where Bona&Grujić studied some KdV-type (KdV is an abbreviation for Korteweg-de Vries) equations, we prove that if the initial state belongs to a Gevrey space, then the solution of (1) and (2) remains in this class for all time.

In Section 3 we consider the case of complex-valued initial data. Similar as in [5], where it was studied analyticity of solutions of the KdV equation, we use Banach's fixed point theorem in a suitable function space in order to find a local solution of the problem (1) that is analytic and has an analytic extension to a strip around the real axis.

Finally, it deserves remark that it was recently proved by Grujić&Kalish [4] a result on local well-posedness of the generalized KdV equation in spaces of analytic functions on a strip containing the real axis without shrinking the width of the strip in time.

Notation:

- $\hat{f} = \mathcal{F}f$: the Fourier transform of f ($\mathcal{F}^{-1}$: the inverse of the Fourier transform), where $\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int e^{-i\xi x} f(x) dx$, for $f \in L^1(\mathbb{R})$.

- $\|\cdot\|_s$, $(\cdot,\cdot)_s$: the norm and the inner product respectively in $H^s(\mathbb{R})$ (Sobolev space of order s of L^2 type), $s \in \mathbb{R}$. $\|f\|_s^2 \equiv \int (1+|\xi|^2)^s |\hat{f}(\xi)|^2 d\xi$.
  $\|\cdot\| = \|\cdot\|_0$: the $L^2(\mathbb{R})$ norm. $(\cdot,\cdot)$ denotes the inner product on $L^2(\mathbb{R})$.
  $H^\infty(\mathbb{R}) \equiv \cap H^s(\mathbb{R})$.

- $\mathcal{H}$: the Hilbert transform.

- $B(X,Y)$: set of bounded linear operators on X to Y. If $X = Y$ we write $B(X)$.

- $S(r) = \{x + iy \in \mathbb{C};\ x \in \mathbb{R},\ |y| < r\}$, for $r > 0$.
  $A(r)$: the set of all analytic functions f on $S(r)$ such that $f \in L^2(S(r'))$ for each $0 < r' < r$ and that $f(x) \in \mathbb{R}$ for $x \in \mathbb{R}$.

- $A = (I - \partial_x^2)^{1/2}$, $X^{\sigma,s} = \mathcal{D}(A^s e^{\sigma A})$ the domain of the operator $A^s e^{\sigma A}$.

- $L^p = \{f; f$ is measurable on $\mathbb{R}, \|f\|_{L^p} < \infty\}$, where $\|f\|_{L^p} = \left(\int |f(x)|^p dx\right)^{1/p}$ if $1 \le p < +\infty$, and $\|f\|_{L^\infty} = \operatorname{ess\,sup}_{x\in\mathbb{R}} |f(x)|$, f is an equivalence class.

- $L_r = \{f \in L^2; \|f\|_{L_r}^2 = (f,f)_{L_r} = (\cosh(2r\xi)\hat{f}, \hat{f}) < +\infty\}$.

- $X_r = \left\{f \in L^2; \|f\|_{X_r}^2 = (f,f)_{X_r} = \sum_{j=0}^1 (\xi^{2j}(\cosh(2r\xi) + \xi \sinh(2r\xi))\hat{f}, \hat{f}) < \infty\right\}$.

- $Y_r = \left\{f \in L^2; \partial_x f \in L^2, \|f\|_{Y_r}^2 = (f,f)_{Y_r} = \sum_{j=0}^1 (\xi^{2j+2} \cosh(2r\xi)\hat{f}, \hat{f}) < +\infty\right\}$.

- $\|f\|_{m,p}^p = \sum_{j=0}^m \|\partial_x^j f\|_{L^p}^p$, $1 \le p < +\infty$. $\|f\|_{m,\infty} = \sum_{j=0}^m \|\partial_x^j f\|_\infty$.

- $H^p(r)$: the analytic Hardy space on the strip $S(r)$.
  $H^p(r) = \{F; F$ is analytic on $S(r), \|F\|_{H^p(r)} = \sup_{|y|<r} \|F(\cdot + iy)\|_{L^p} < \infty\}$.

- $H^{m,p}(r) = \{F \in H^p(r); \|F\|_{H^{m,p}(r)}^p = \sum_{j=0}^m \|\partial_z^j F\|_{H^p(r)}^p < \infty\}$.

- $C(I;X)$: set of continuous functions on the interval I into the Banach space X.

- $C^\omega(I;X)$: the set of weakly continuous functions from I to X.

- $\Re(z)$: the real part of the complex number z.

2 Real-valued initial data.

We deduce in this Section global analyticity (in space variables) of the solutions of problems (1) and (2) when the initial data and the corresponding solution take real values on the real axis and supposing moreover that the initial data has an analytic continuation that is analytic in a strip containing the real axis. We will use the fact that problems (1) and (2) are globally well possed in $H^s(\mathbb{R})$ for $s \geq 1$, when the solution of the two previously mentioned problems take real values on the real axis at any time. More precisely we have the following two theorems (for real valued solutions) which can be found in [1].

Theorem 2.1. *Let $\phi \in H^s(\mathbb{R})$, $s \geq 1$. Then, for each $\eta > 0$ there exists a unique $u = u_\eta \in C([0,\infty); H^s(\mathbb{R}))$ solution to the problem (1) such that $\partial_t u \in C([0,\infty); H^{s-3}(\mathbb{R}))$.*

Proof. See Theorem 4.2 in [1]. ■

Theorem 2.2. *Let $\psi \in H^s(\mathbb{R})$, $s \geq 1$. Then, for each $\eta > 0$ there exists a unique $v = v_\eta \in C([0,\infty); H^s(\mathbb{R}))$ solution to the problem (2) such that $\partial_t v \in C([0,\infty); H^{s-3}(\mathbb{R}))$.*

Proof. See Theorem 4.1 in [1]. ■

First (see Theorems 2.3 and 2.4 below), we prove that if the initial state has an analytic continuation that belongs to $A(r_0)$ for some $r_0 > 0$ then the solutions $u(t), v(t)$ of problems (1) and (2) respectively also have analytic continuations belonging to $A(r_1)$ for all $t \in [0,T]$, where r_1 might decrease with time. Theorem 2.3, below, is an application of the method developed by Kato&Masuda in [7] to study global analyticity (in space variables) of some partial differential equations. Similar as in the proof of Theorem 2 in [7] we consider $H^{m+5}(\mathbb{R}) \equiv Z \subset X \equiv H^{m+2}(\mathbb{R})$ and $\Phi_{\sigma;m}(v) \equiv \frac{1}{2}\|v\|^2_{\sigma,2;m}$ defined on an appropriate open set $O \subset Z$, where

$$\|f\|^2_{\sigma,s;m} \equiv \sum_{j=0}^{m} \frac{\mathrm{e}^{2j\sigma}}{(j!)^2}\|\partial_x^j f\|_s^2 \qquad \text{and} \qquad \|f\|^2_{\sigma,s} \equiv \sum_{j=0}^{\infty} \frac{\mathrm{e}^{2j\sigma}}{(j!)^2}\|\partial_x^j f\|_s^2, \quad s \in \mathbb{R}.$$

Let $F(v) \equiv -vv_x - v_{xxx} - \eta(\mathcal{H}v_x + \mathcal{H}v_{xxx})$.

Lemma 2.1. *There exist constants $c, \gamma > 0$ such that for every $v \in H^{m+5}(\mathbb{R})$,*

$$\langle F(v), D\Phi_{\sigma;m}(v)\rangle \leq c(\eta + \|v\|_2)\Phi_{\sigma;m}(v) + \gamma\sqrt{\Phi_{\sigma;m}(v)}\,\partial_\sigma \Phi_{\sigma;m}(v). \tag{3}$$

Proof. As in [7] we have that $D\Phi_{\sigma;m}(v) = \sum_{j=0}^m \frac{\mathrm{e}^{2j\sigma}}{(j!)^2}(-\partial_x)^j A^4 \partial_x^j v$, where $A = (1-\partial_x^2)^{1/2}$. Then

$$\begin{aligned}\langle F(v), D\Phi_{\sigma;m}(v)\rangle &= \sum_{j=0}^m \frac{\mathrm{e}^{2j\sigma}}{(j!)^2}\big[-\big(\partial_x^j(v\partial_x v), \partial_x^j v\big)_2 - \eta\big(\partial_x^j(\mathcal{H}\partial_x v + \mathcal{H}\partial_x^3 v), \partial_x^j v\big)_2\big]\\ &= \sum_{j=0}^m \frac{\mathrm{e}^{2j\sigma}}{(j!)^2}\big[-\big(v\partial_x^{j+1}v, \partial_x^j v\big)_2 - Q_j(v) - \eta\big(\partial_x^j(\mathcal{H}\partial_x v + \mathcal{H}\partial_x^3 v), \partial_x^j v\big)_2\big],\end{aligned}$$

where $Q_j(v) = \sum_{k=1}^j \binom{j}{k}\big(\partial_x^k v \partial_x^{j-k+1}v, \partial_x^j v\big)_2$, and $Q_0(v) \equiv 0$. By using Kato's inequality (K) in the Appendix we have that

$$|(v\partial_x^{j+1}v, \partial_x^j v)_2| \le c\|\partial_x v\|_1 \|\partial_x^j v\|_2^2 \le c\|v\|_2\|\partial_x^j v\|_2^2.$$

Moreover

$$\begin{aligned}-\big(\partial_x^j(\mathcal{H}\partial_x v + \mathcal{H}\partial_x^3 v), \partial_x^j v\big)_2 &= \int (1+\xi^2)^2 \xi^{2j}(|\xi| - |\xi|^3)|\hat{v}(\xi)|^2 d\xi\\ &\le \int (1+\xi^2)^2\xi^{2j}|\hat{v}(\xi)|^2 d\xi = \|\partial_x^j v\|_2^2.\end{aligned} \tag{4}$$

Then

$$\langle F(v), D\Phi_{\sigma;m}(v)\rangle \le c(\|v\|_2 + \eta)\Phi_{\sigma;m}(v) - \sum_{j=0}^m \frac{\mathrm{e}^{2j\sigma}}{(j!)^2} Q_j(v). \tag{5}$$

Using the Schwarz inequality and the formula $\|fg\|_2 \le \gamma\big(\|f\|_2\|g\|_1 + \|f\|_1\|g\|_2\big)$ we get

$$|Q_j(v)| \le \gamma \sum_{k=1}^j \binom{j}{k} \|\partial_x^j v\|_2 \big(\|\partial_x^k v\|_1 \|\partial_x^{j-k+1} v\|_2 + \|\partial_x^k v\|_2 \|\partial_x^{j-k+1}v\|_1\big). \tag{6}$$

We denote as in [7], $b_j \equiv \frac{\mathrm{e}^{j\sigma}}{j!}\|\partial_x^j v\|_2$, $B^2 \equiv \sum_{j=0}^m b_j^2 = 2\Phi_{\sigma;m}(v)$ and $\tilde{B}^2 \equiv \sum_{j=1}^m j b_j^2 =$

$\partial_\sigma \Phi_{\sigma;m}(v)$. By using (6) it follows, as a particular case of Lemma 3.1 in [7], that

$$\begin{aligned}
\sum_{j=0}^{m} \frac{\mathrm{e}^{2j\sigma}}{(j!)^2}|Q_j(v)| &\leq \gamma \sum_{j=1}^{m}\sum_{k=1}^{j}\big(b_j \frac{b_{k-1}}{k}(j-k+1)b_{j-k+1} + b_j b_k b_{j-k}\big) \\
&= \gamma \sum_{k=1}^{m} \frac{b_{k-1}}{k}\big(\sum_{j=k}^{m} b_j (j-k+1) b_{j-k+1}\big) + \gamma \sum_{k=1}^{m} b_k \big(\sum_{j=k}^{m} b_j b_{j-k}\big) \\
&\leq \gamma \tilde{B}^2 \big(\sum_{k=1}^{m} \frac{1}{k^2}\big)^{1/2}\big(\sum_{k=1}^{m} b_{k-1}^2\big)^{1/2} + \gamma B\tilde{B} \sum_{k=1}^{m} \frac{b_k}{\sqrt{k}} \\
&\leq 2\gamma B\tilde{B}^2 + \gamma B\tilde{B}\big(\sum_{k=1}^{m} \frac{1}{k^2}\big)^{1/2}\big(\sum_{k=1}^{m} k b_k^2\big)^{1/2} \leq 4\gamma B\tilde{B}^2. \qquad (7)
\end{aligned}$$

By replacing the inequality (7) into (5), the Lemma follows. ■

Next, we enunciate Lemma 2.4 in [7] and we give, for expository completeness, a proof of this trivial result.

Lemma 2.2. *Let $\phi_n \in A(r)$, $n = 1, 2, ...$ be a sequence with $\|\phi_n\|_{\sigma,2}$ bounded, where $e^\sigma < r$. If $\phi_n \to 0$ in H^∞ as $n \to \infty$, then $\|\phi_n\|_{\sigma',2} \to 0$ for each $\sigma' < \sigma$.*

Proof. Let $\sigma' < \sigma$. Choose M such that $\|\phi_n\|_{\sigma,2} \leq M$ for all n. Since $\frac{\mathrm{e}^{2j\sigma'}}{(j!)^2}\|\partial_x^j \phi_n\|_2^2 \leq \frac{M^2}{\mathrm{e}^{2j(\sigma-\sigma')}}$, it follows, by using the dominated convergence theorem, that

$$\lim_{n\to\infty} \|\phi_n\|_{\sigma',2}^2 = \sum_{j=0}^{\infty} \frac{\mathrm{e}^{2j\sigma'}}{(j!)^2}\big(\lim_{n\to\infty} \|\partial_x^j \phi_n\|_2^2\big) = 0. \ ■$$

Theorem 2.3. *Let $\eta \geq 0$ and $T > 0$. Let $u \in C([0, T]; H^\infty(\mathbb{R}))$ be a solution of (1). If $u(0) = \phi \in A(r_0)$ for some $r_0 > 0$, there exists $r_1 > 0$ such that $u \in C([0, T]; A(r_1))$.*

Proof. We proceed as in the proof of Theorem 2 in [7], to prove that $\Phi_{\sigma;m}$ is a Liapunov family for (1) on $O = Z = H^{m+5}(\mathbb{R})$, considering the functions: $\alpha(r) = \gamma\sqrt{r}$, $\beta(r) = c(\eta + M)r \equiv Kr$, for $r \geq 0$, where $M = \max_{t\in[0,T]} \|u(t)\|_2$, and $\gamma, c > 0$ are constants given by Lemma 2.1, $\rho(t) = \frac{1}{2}\|\phi\|_{b,2}^2 \mathrm{e}^{Kt}$ where $b < \sigma_0$, $\mathrm{e}^{\sigma_0} < r_0$, and

$$\sigma(t) = b - \frac{\sqrt{2}\gamma}{K}\|\phi\|_{b,2}(\mathrm{e}^{\frac{K}{2}t} - 1),\ t \in [0, T]. \qquad (8)$$

We have that $u(t) \in A(\mathrm{e}^{\sigma(t)})$, for all $t \in [0, T]$. Then $u(t) \in A(r_1)$ for all $t \in [0, T]$, where $r_1 = \mathrm{e}^{\sigma(T)}$. The continuity of u, as in the proof of Theorem 2 in [7], is a consequence of Lemma 2.2 above. ■

Theorem 2.4. *Let $\eta \geq 0$ and $T > 0$. Let $v \in C([0,T]; H^\infty(\mathbb{R}))$ be a solution of (2). If $v(0) = \psi \in A(r_0)$ for some $r_0 > 0$, there exists $r_1 > 0$ such that $v \in C([0,T]; A(r_1))$.*

Proof. We remark that $u \equiv v_x \in C([0,T]; H^\infty(\mathbb{R}))$ is a solution of (1) with $u(0) = \psi'$. Since $\psi' \in H^\infty(\mathbb{R})$ and $\|\psi'\|_{\sigma,0} \leq \|\psi\|_{\sigma,1}$ for each σ such that $\mathrm{e}^\sigma < r_0$ it follows, as a consequence of Lemma 2.2 in [7], that $\psi' \in A(r_0)$. So, by Theorem 2.3, there exists $r_1 > 0$ such that $v_x \in C([0,T]; A(r_1))$. Now, since $v(t) \in H^\infty(\mathbb{R})$ and $\|v(t)\|^2_{\sigma,0} \leq \sup_{t\in[0,T]} \|v(t)\|^2 + \mathrm{e}^{2\sigma} \sup_{t\in[0,T]} \|u(t)\|^2_{\sigma,0} < \infty$ for each σ such that $\mathrm{e}^\sigma < r_1$ and for all $t \in [0,T]$, it follows that $v(t) \in A(r_1)$ for all $t \in [0,T]$. The continuity of v follows as in the previous Theorem. ∎

Now, we make a Gevrey-class analysis of (1) and (2). Theorem 2.5, below, is proved similarly to Theorem 11 in [2], where it was also studied the rate of decrease of the uniform radius of analyticity for KdV-type equations of the form $u_t + G(u)u_x - Lu_x = 0$, where $u = u(x,t)$, for $x, t \in \mathbb{R}$, G is a function that is analytic at least in a neighborhood of zero in $\mathbb{C}$, but real-valued on the real axis, and L is a homogeneus Fourier multiplier operator defined by $\widehat{Lu}(\xi) = |\xi|^\mu \hat{u}(\xi)$, for some $\mu > 0$. In the case of problem (1) we have that $\widehat{Lu}(\xi) = [\xi^2 - \eta i(\mathrm{sgn}(\xi) - \xi|\xi|)]\hat{u}(\xi)$. Let us remark that, since each function $f \in X^{\sigma,s} = \mathcal{D}(A^s \mathrm{e}^{\sigma A})$ with $\sigma > 0$ and $s \geq 0$, satisfies $\|f\|^2_{L_\sigma} \leq \int \mathrm{e}^{2\sigma|\xi|} |\hat{f}(\xi)|^2 d\xi \leq \|f\|^2_{X^{\sigma,s}} < \infty$, it follows from Theorem 1 in [5] that f has an analytic extension $F \in H^2(\sigma)$.

Theorem 2.5. *Let $\eta \geq 0$ and $T > 0$. Suppose that $\phi \in X^{\sigma_0,s}$, for some $\sigma_0 > 0$ and $s > 5/2$. Then the solution u of the problem (1) satisfies $u \in C([0,T]; X^{\sigma(T),s})$, where $\sigma(t)$ is a positive monotone decreasing function given by (15).*

Proof. Let η, T, σ_0, and s be as in the hypothesis of the theorem. Let $r \equiv s - 1 > 3/2$. We have that $u \in C([0,T]; H^s(\mathbb{R}))$, which follows from Theorem 2.1 and from Corollary 4.7 in [6]. We give a sketch of the proof, since it is similar to the proof of Theorem 11 in [2]. Let $v = u_x$, we have that v satisfies

$$v_t + v^2 + v_{xxx} + \eta(\mathcal{H}v_x + \mathcal{H}v_{xxx}) + uv_x = 0, \;\; v(0) = \phi'. \tag{9}$$

Let $\sigma \in C^1([0,T]; \mathbb{R})$ be a positive function such that $\sigma' < 0$ and $\sigma(0) = \sigma_0$. Then

$$\frac{1}{2}\frac{d}{dt}\|v(t)\|^2_{X^{\sigma(t),r}} - \sigma'(t)\|v(t)\|^2_{X^{\sigma(t),r+1/2}} = \Re \int (1+\xi^2)^r \mathrm{e}^{2\sigma(t)(1+\xi^2)^{1/2}} \partial_t \hat{v}(t,\xi)\overline{\hat{v}(t,\xi)} d\xi.$$

It follows from the last expression and from (9) that

$$\frac{1}{2}\frac{d}{dt}\|v(t)\|^2_{X^{\sigma(t),r}} - \sigma'(t)\|v(t)\|^2_{X^{\sigma(t),r+1/2}} \leq I_1 + I_2 + \eta\|v(t)\|^2_{X^{\sigma(t),r}}, \tag{10}$$

where

$$\begin{cases} I_1 \equiv |(A^r \mathrm{e}^{\sigma(t)A} v^2(t), A^r \mathrm{e}^{\sigma(t)A} v(t))|, \\ I_2 \equiv |(A^r \mathrm{e}^{\sigma(t)A} (uv_x)(t), A^r \mathrm{e}^{\sigma(t)A} v(t))|. \end{cases}$$

I_1 and I_2 are particular cases of the corresponding ones in [2], taking $G(u) = u$. For the sake of completeness we estimate them here. Since $r > 1/2$, by using Lemmas 6 and 9 in [2], we have that

$$\begin{aligned} I_1 &\leq c_r \|A^r \mathrm{e}^{\sigma(t)A} v(t)\|^3 \leq c_r \|A^r v(t)\|^3 + \tilde{c}_r \sigma(t) \|A^{r+1/3} \mathrm{e}^{\sigma(t)A} v(t)\|^3 \\ &\leq c_r \|A^r v(t)\|^3 + \tilde{c}_r \sigma(t) \|A^r \mathrm{e}^{\sigma(t)A} v(t)\| \|A^{r+1/2} \mathrm{e}^{\sigma(t)A} v(t)\|^2. \end{aligned} \tag{11}$$

Since $r > 3/2$, by using Lemma 10 in [2], we obtain

$$I_2 \leq c_r \|A^{r+1} u(t)\| \|A^r v(t)\|^2 + \tilde{c}_r \sigma(t) \|A^{r+1} \mathrm{e}^{\sigma(t)A} u(t)\| \|A^{r+1/2} \mathrm{e}^{\sigma(t)A} v(t)\|^2. \tag{12}$$

Since $u \in C([0,T]; H^s(\mathbb{R}))$, it follows that $\|A^{r+1} u(t)\| \leq c_{(r,T)}$. Moreover

$$\begin{aligned} &\|A^{r+1} \mathrm{e}^{\sigma(t)A} u(t)\|^2 = \int (1+\xi^2)^r \mathrm{e}^{2\sigma(t)(1+\xi^2)^{1/2}} |\hat{u}(\xi,t)|^2 d\xi + \|A^r \mathrm{e}^{\sigma(t)A} v(t)\|^2 \\ &\leq \mathrm{e}^{2\sqrt{2}\sigma_0} \|u(t)\|_r^2 + 2\|A^r \mathrm{e}^{\sigma(t)A} v(t)\|^2 \leq c_{(r,T,\sigma_0)} + 2\|A^r \mathrm{e}^{\sigma(t)A} v(t)\|^2, \end{aligned} \tag{13}$$

where $c_{(r,T,\sigma_0)}$ is a positive constant depending only on r, T and σ_0. Replacing the last inequalities into (12) and since $v \in C([0,T]; H^r(\mathbb{R}))$, we get

$$I_2 \leq c_{(r,T)} + \tilde{c}_r \sigma(t) \big(c_{(r,T,\sigma_0)} + \sqrt{2} \|A^r \mathrm{e}^{\sigma(t)A} v(t)\| \big) \|A^{r+1/2} \mathrm{e}^{\sigma(t)A} v(t)\|^2. \tag{14}$$

Now, by using (11) and (14) into (10), we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt} \|A^r \mathrm{e}^{\sigma(t)A} v(t)\|^2 - \sigma'(t) \|A^{r+1/2} \mathrm{e}^{\sigma(t)A} v(t)\|^2 \\ \leq\ & c_{(r,T)} + c_{(r,T,\sigma_0)} \sigma(t) \big(1 + \|A^r \mathrm{e}^{\sigma(t)A} v(t)\|\big) \|A^{r+1/2} \mathrm{e}^{\sigma(t)A} v(t)\|^2 + \eta \|A^r \mathrm{e}^{\sigma(t)A} v(t)\|^2. \end{aligned}$$

Then

$$\begin{aligned} &\frac{d}{dt} \|A^r \mathrm{e}^{\sigma(t)A} v(t)\|^2 + 2\big(-\sigma'(t) - c_{(r,T,\sigma_0)} \sigma(t) (1 + \|A^r \mathrm{e}^{\sigma(t)A} v(t)\|) \big) \|A^{r+1/2} \mathrm{e}^{\sigma(t)A} v(t)\|^2 \\ &\leq c_{(r,T)} + 2\eta \|A^r \mathrm{e}^{\sigma(t)A} v(t)\|^2. \end{aligned}$$

The last inequality implies that $v(t) \in \mathcal{D}(A^r \mathrm{e}^{\sigma(t)A}) = X^{\sigma(t),r}$ for all $t \in [0,T]$, where

$$\sigma(t) = \sigma_0 e^{-Kt}, \tag{15}$$

$K = K(r, T, \sigma_0, \phi, \eta) = c_{(r,T,\sigma_0)}(1 + c_{(r,T,\sigma_0,\phi)} \mathrm{e}^{\eta T})$, and $c_{(r,T,\sigma_0,\phi)} = \big(\|A^r \mathrm{e}^{\sigma_0 A} \phi'\|^2 + c_{(r,T)} T \big)^{1/2}$. More precisely we have that $\|A^r \mathrm{e}^{\sigma(t)A} v(t)\| \leq c_{(r,T,\sigma_0,\phi)} \mathrm{e}^{\eta T}$. Finally, it follows from (13) that $u(t) \in X^{\sigma(t),s} \subset X^{\sigma(T),s}$ for all $t \in [0,T]$. ■

Theorem 2.6. *Let $\eta \geq 0$ and $T > 0$. Suppose that $\psi \in X^{\sigma_0,s}$, for some $\sigma_0 > 0$ and $s > 7/2$. Then the solution v of the problem (2) satisfies $v \in C([0,T]; X^{\sigma(T),s})$, where $\sigma(t)$ is a positive monotone decreasing function given by (15).*

Proof. Since $\psi' \in X^{\sigma_0,s-1}$, and $s-1 > 5/2$, it follows from Theorem 2.5 that $u \equiv v_x \in C([0,T]; X^{\sigma(T),s-1})$, where σ is given by (15). Making similar calculations to (13), we see that $v(t) \in X^{\sigma(t),s}$ for all $t \in [0,T]$. Finally, since

$$\|v(t)-v(\tau)\|^2_{X^{\sigma(T),s}} \leq \mathrm{e}^{2\sqrt{2}\sigma(T)}\|v(t)-v(\tau)\|^2_{s-1} + \|u(t)-u(\tau)\|^2_{X^{\sigma(T),s-1}},$$

it follows that $v \in C([0,T]; X^{\sigma(T),s})$. ■

Remark: For $r > 0$, we have

$$\begin{cases} X^{r,s} \subset X_r, \ \text{ if } \ s \geq 3/2. \\ X_r \subset L_r. \\ \text{If } \ f \in L_r, \ \text{ and } \ f(x) \in \mathbb{R} \ \text{ for } \ x \in \mathbb{R}, \ \text{ then } \ f \in A(r). \end{cases} \tag{16}$$

The first part in (16) follows from the inequality $\|f\|^2_{X_r} \leq 4\|f\|^2_{X^{r,s}}$ for $s \geq 3/2$. The fact that $X_r \subset L_r$ is obvious. Finally, suppose that $f \in L_r$ and $f(x) \in \mathbb{R}$ for $x \in \mathbb{R}$; then we already know (see the paragraph before Theorem 2.5) that f has an analytic extension $F \in H^2(r)$; moreover, for every $0 < r' < r$ we have

$$\int_{-r'}^{r'}\int_{-\infty}^{+\infty} |F(x+iy)|^2 dxdy \leq \int_{-r'}^{r'} \big(\sup_{|y|<r'} \int_{-\infty}^{+\infty} |F(x+iy)|^2 dx \big) dy \leq 2r\|F\|^2_{H^2(r)}.$$

3 Complex-valued initial data.

In this section we show that if the initial data ϕ of (1) is analytic and has an analytic continuation to a strip containing the real axis, then there exists a $T > 0$ such that the solution $u(t)$ of (1) has the same property for all $t \in [0,T]$, but the width of the strip may decrease as a function of time. Similar to the proof of Theorem 2 in [5], we establish in Theorem 3.1, below, existence and analyticity of the solution of problem (1) simultaneously including the case when the initial condition is complex-valued. Let us prove before some previous Lemmas.

Lemma 3.1. *Y_r is a dense subset of X_r, for $r > 0$.*

Proof. Let $f \in Y_r$. We have that

$$\int \cosh(2r\xi)|\hat{f}(\xi)|^2 d\xi \leq \cosh(2r)\|f\|^2 + \int_{|\xi|>1} \cosh(2r\xi)|\hat{f}(\xi)|^2 d\xi \leq \cosh(2r)(\|f\|^2 + \|f\|^2_{Y_r}),$$

$$\int \xi \sinh(2r\xi)|\hat{f}(\xi)|^2 d\xi \leq \sinh(2r)\|f\|^2 + \int_{|\xi|>1} \xi^2 \cosh(2r\xi)|\hat{f}(\xi)|^2 d\xi \leq \cosh(2r)(\|f\|^2 + \|f\|^2_{Y_r}),$$

and similarly $\int \xi^3 \sinh(2r\xi)|\hat{f}(\xi)|^2 d\xi \leq \cosh(2r)(\|f\|^2 + \|f\|^2_{Y_r})$. So, using the last three inequalities, it is not difficult to prove that

$$\|f\|^2_{X_r} \leq 4\cosh(2r)(\|f\|^2 + \|f\|^2_{Y_r}). \tag{17}$$

It follows from the last inequality that $Y_r \subset X_r$. Now, let us consider the set

$$L_{2,0}(\mathbb{R}) = \big\{g \in L^2(\mathbb{R});\ \exists K > 0,\ |\hat{g}(\xi)| \leq K \text{ a.e. in } \mathbb{R},\ \hat{g}(\xi) = 0 \text{ a.e. in } \{\xi \in \mathbb{R}; |\xi| > K\}\big\}.$$

Let $g \in L_{2,0}(\mathbb{R})$, then $\sum_{j=0}^{1} \int \xi^{2j+2} \cosh(2r\xi)|\hat{g}(\xi)|^2 d\xi \leq (K^2 + K^4)\cosh(2rK)\|g\|^2$, so $g \in Y_r$. Now we will prove that $L_{2,0}(\mathbb{R})$ is a dense set in X_r. Let f be an element of X_r. For each $n \in \mathbb{N}$, take $f_n \in L_{2,0}(\mathbb{R})$ given by

$$\widehat{f_n}(\xi) = \begin{cases} \hat{f}(\xi), \text{ if } |\xi| \leq n \text{ and } |\hat{f}(\xi)| \leq n \\ 0, \text{ otherwise.} \end{cases}$$

It follows easily from the definition of f_n that $|\widehat{f_n}(\xi)| \leq |\hat{f}(\xi)|$, for all $n \in \mathbb{N}$ and $\xi \in \mathbb{R}$. Moreover, $\widehat{f_n}(\xi) \to \hat{f}(\xi)$ as $n \to \infty$, for all $\xi \in \mathbb{R}$. So, by the dominated convergence theorem, we have that $\|f_n - f\|_{X_r} \to 0$ as $n \to \infty$. This concludes the proof. ∎

Lemma 3.2. *(i.) Suppose that $F \in H^{2,2}(r)$. Let f be the trace of F on the real line. Then $f \in Y_r$ and*

$$\|f\|_{Y_r} \leq \sqrt{2}\|F\|_{H^{2,2}(r)}. \tag{18}$$

(ii.) Conversely, suppose that $f \in Y_r$. Then f has an analytic extension $F \in H^{2,2}(r)$ and

$$\|F\|_{H^{2,2}(r)} \leq \sqrt{10\cosh(2r)}(\|f\| + \|f\|_{Y_r}). \tag{19}$$

Proof. (i.) Since $F \in H^{2,2}(r)$, we see that $\partial_z F, \partial_z^2 F \in H^2(r)$. Then, by using Theorem 1 in [5], we have that $f', f'' \in L_r$ and moreover

$$\|f\|^2_{Y_r} = \|f'\|^2_{L_r} + \|f''\|^2_{L_r} \leq 2(\|\partial_z F\|^2_{H^2(r)} + \|\partial_z^2 F\|^2_{H^2(r)}) \leq 2\|F\|^2_{H^{2,2}(r)}.$$

(ii.) By Lemma 3.1, $Y_r \subset X_r$. So, it follows from Lemma 2.1 in [5] that f has an analytic extension F on $S(r)$ such that $\|\partial_z F\|_{H^{1,2}(r)} \leq \sqrt{2}\|f\|_{Y_r}$ and $\|F\|_{H^{1,2}(r)} \leq \sqrt{2}\|f\|_{X_r}$. Now, using the last inequalities, we have that

$$\begin{aligned}\|F\|^2_{H^{2,2}(r)} &= \|F\|^2_{H^{1,2}(r)} + \|\partial_z^2 F\|^2_{H^2(r)} \leq \|F\|^2_{H^{1,2}(r)} + \|\partial_z F\|^2_{H^{1,2}(r)} \\ &\leq 2(\|f\|^2_{X_r} + \|f\|^2_{Y_r}) \leq 10\cosh(2r)(\|f\|^2 + \|f\|^2_{Y_r}),\end{aligned}$$

where in the last inequality it was used (17). ■

Lemma 3.3. *Let $r > 0$. Suppose that $F, G \in H^{2,2}(r)$. Then $FG \in H^{2,2}(r)$ and*

$$\|FG\|_{H^{2,2}(r)} \leq c\|F\|_{H^{2,2}(r)}\|G\|_{H^{2,2}(r)}. \tag{20}$$

Proof. Using Leibniz's rule and Sobolev's inequality ($\|G\|_{H^\infty(r)} \leq c\|G\|_{H^{1,2}(r)}$) we have that

$$\begin{aligned}\|\partial_z^2(FG)\|_{H^2(r)} &\leq \|\partial_z^2 F\|_{H^2(r)}\|G\|_{H^\infty(r)} + 2\|\partial_z F\|_{H^\infty(r)}\|\partial_z G\|_{H^2(r)} + \|F\|_{H^\infty(r)}\|\partial_z^2 G\|_{H^2(r)}. \\ &\leq c\|F\|_{H^{2,2}(r)}\|G\|_{H^{2,2}(r)}.\end{aligned}$$

Moreover, by Lemma 2.3 in [5], we have that $\|FG\|_{H^{1,2}(r)} \leq c\|F\|_{H^{1,2}(r)}\|G\|_{H^{1,2}(r)}$. Since $\|FG\|^2_{H^{2,2}(r)} = \|FG\|^2_{H^{1,2}(r)} + \|\partial_z^2(FG)\|^2_{H^2(r)}$, the proof of the Lemma follows from the last two inequalities. ■

Lemma 3.4 and Corollary 3.1, below, are particular cases of Lemma 2.4 and its Corollary in Hayashi [5] respectively.

Lemma 3.4. *There exists a polynomial $\tilde{a}$ of which coefficients are all nonnegative with the following property: If $r > 0$ and $f, g, v \in X_r \cap Y_r$, then*

$$|(f\partial_x f - g\partial_x g, v)_{X_r}| \leq \tilde{a}(\|f\|_{X_r}, \|g\|_{X_r})\big(\|f\|_{Y_r}\|f-g\|_{X_r} + \|f-g\|_{Y_r}\big)\big(\|v\|_{X_r} + \|v\|_{Y_r}\big).$$

Corollary 3.1. *There exists a polynomial $\tilde{a_1}$ with nonnegative coefficients such that if $f, v \in X_r \cap Y_r$, then*

$$|(f\partial_x f, v)_{X_r}| \leq \tilde{a_1}(\|f\|_{X_r})\|f\|_{Y_r}\big(\|v\|_{X_r} + \|v\|_{Y_r}\big). \tag{21}$$

Now, we state the main theorem of this Section.

Theorem 3.1. *Let $\eta \geq 0$. Suppose that $\phi \in X_{\sigma_0}$, for some $\sigma_0 > 0$. Then there exist a $T = T(\|\phi\|_{X_{\sigma_0}}, \eta, \sigma_0) > 0$ and a positive monotone decreasing function $\sigma(t)$ satisfying $\sigma(0) = \sigma_0$ and such that (1) has a unique solution $u \in C^\omega([0,T]; X_{\sigma(T)})$.*

Proof. Let $\eta \geq 0$, $\sigma_0 > 0$, and $\phi \in X_{\sigma_0}$. Let $\sigma(t) = \sigma_0 e^{-At/\sigma_0}$, where the positive constant A will be conveniently chosen later. For $T > 0$, consider the space

$$B(T) = \left\{ f : [0,T] \times \mathbb{R} \mapsto \mathbb{C}; \|f\|_{B(T)}^2 = \sup_{0 \leq t \leq T} \|f(t)\|_{X_{\sigma(t)}}^2 + A \int_0^T \|f(t)\|_{Y_{\sigma(t)}}^2 dt < +\infty \right\}$$

and

$$B_\rho(T) = \{ f \in B(T); \|f\|_{B(T)} \leq \rho \}.$$

Let $\rho = 4\|\phi\|_{X_{\sigma_0}}$. For $v \in B_\rho(T)$, we define the mapping M by $u = Mv$, where u is the solution of the linearized problem

$$\begin{cases} \partial_t u + \partial_x^3 u + \eta(\mathcal{H}\partial_x u + \mathcal{H}\partial_x^3 u) = -v\partial_x v \\ u(0) = \phi. \end{cases} \tag{22}$$

More precisely u can be obtained as follows. Take the Fourier transform to (22), then

$$\partial_t \hat{u}(t,\xi) - i\xi^3 \hat{u}(t,\xi) + \eta(-|\xi| + |\xi|^3)\hat{u}(t,\xi) = -\widehat{vv_x}(t,\xi).$$

Now, integrating the last expression between 0 and t, it follows that

$$\hat{u}(t,\xi) = e^{(i\xi^3 + \eta(|\xi| - |\xi|^3))t} \hat{\phi}(\xi) - \int_0^t e^{(i\xi^3 + \eta(|\xi| - |\xi|^3))(t-\tau)} \widehat{vv_x}(\tau,\xi) d\tau, \tag{23}$$

where the last integral is well defined, since

$$\begin{aligned} \int_0^t |\widehat{vv_x}(\tau,\xi)| d\tau &= \int_0^t \left| \left(\widehat{v(\tau)} * \widehat{v_x(\tau)} \right)(\xi) \right| d\tau = \int_0^t \left| \int \hat{v}(\tau, \xi - \omega) \omega \hat{v}(\tau,\omega) d\omega \right| d\tau \\ &\leq \int_0^t \|v(\tau)\| \|v(\tau)\|_1 d\tau \leq \rho^2 t, \end{aligned}$$

and in the last inequality it was used that

$$\|v(\tau)\|_1 \leq \|v(\tau)\|_{X_{\sigma(\tau)}} \leq \|v\|_{B(T)} \leq \rho.$$

Let us choose $A, T > 0$ such that

$$e^{-AT/\sigma_0} > \frac{1}{2}. \tag{24}$$

Let $v \in B_\rho(T)$. It will be proved that $u = Mv \in B_\rho(T)$, for suitable chosen $T, A > 0$. Now, it is not difficult to see that

$$\begin{aligned}\frac{d}{dt}\|u(t)\|^2_{X_{\sigma(t)}} &= 2\sigma'(t)\sum_{j=0}^{1}\int \xi^{2j}\big(\xi\sinh(2\sigma(t)\xi)+\xi^2\cosh(2\sigma(t)\xi)\big)|\hat{u}(t,\xi)|^2 d\xi\\ &\quad +2\Re\sum_{j=0}^{1}\int \xi^{2j}\big(\cosh(2\sigma(t)\xi)+\xi\sinh(2\sigma(t)\xi)\big)\partial_t\hat{u}(t,\xi)\overline{\hat{u}(t,\xi)}d\xi.\end{aligned}$$

Taking the Fourier transform to (22), multiplying both sides of the obtained equation by $\xi^{2j}\big(\cosh(2\sigma(t)\xi)+\xi\sinh(2\sigma(t)\xi)\big)\overline{\hat{u}(t,\xi)}$, integrating with respect to ξ, summing the terms for $j = 0, 1$ and finally taking the real part we get

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{X_{\sigma(t)}} - \sigma'(t)\sum_{j=0}^{1}\int \xi^{2j}\big(\xi\sinh(2\sigma(t)\xi)+\xi^2\cosh(2\sigma(t)\xi)\big)|\hat{u}(t,\xi)|^2 d\xi$$

$$+\eta\sum_{j=0}^{1}\int \xi^{2j}\big(\cosh(2\sigma(t)\xi)+\xi\sinh(2\sigma(t)\xi)\big)(-|\xi|+|\xi|^3)|\hat{u}(t,\xi)|^2 d\xi = -\Re(vv_x, u)_{X_{\sigma(t)}}.$$

Since $\sigma'(t) = -Ae^{-\frac{At}{\sigma_0}}$ and $|\xi| - |\xi|^3 \le 1$, for all $\xi \in \mathbb{R}$, it follows that

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{X_{\sigma(t)}} + Ae^{-\frac{At}{\sigma_0}}\|u(t)\|^2_{Y_{\sigma(t)}} \le -\Re(vv_x, u)_{X_{\sigma(t)}} + \eta\|u(t)\|^2_{X_{\sigma(t)}}.$$

Using (24) and Corollary 3.1, it follows from the last inequality that

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{X_{\sigma(t)}} + \frac{A}{2}\|u(t)\|^2_{Y_{\sigma(t)}} \le a(\rho)\|v(t)\|_{Y_{\sigma(t)}}(\|u(t)\|_{X_{\sigma(t)}} + \|u(t)\|_{Y_{\sigma(t)}}) + \eta\|u(t)\|^2_{X_{\sigma(t)}},$$

where $a(\cdot)$ is a polynomial with nonnegative coefficients. Now integrating the last inequality from 0 to t we get

$$\sup_{0\le t\le T}\|u(t)\|^2_{X_{\sigma(t)}} + A\int_0^T \|u(t)\|^2_{Y_{\sigma(t)}}dt \le 2\big[\|\phi\|^2_{X_{\sigma_0}} + 2a(\rho)\big(\int_0^T \|v(t)\|^2_{Y_{\sigma(t)}}dt\big)^{1/2}$$

$$\cdot\big[\big(\int_0^T \|u(t)\|^2_{X_{\sigma(t)}}dt\big)^{1/2} + \big(\int_0^T \|u(t)\|^2_{Y_{\sigma(t)}}dt\big)^{1/2}\big] + 2\eta\int_0^T \|u(t)\|^2_{X_{\sigma(t)}}dt\big].$$

Then,

$$\begin{aligned}\|u\|^2_{B(T)} &\le \frac{\rho^2}{8} + \frac{4\rho a(\rho)}{\sqrt{A}}\big(\sqrt{T}+\frac{1}{\sqrt{A}}\big)\|u\|_{B(T)} + 4\eta T\|u\|^2_{B(T)}\\ &\le \frac{\rho^2}{8} + \frac{1}{2}\big[\frac{4\rho a(\rho)}{\sqrt{A}}\big(\sqrt{T}+\frac{1}{\sqrt{A}}\big)\big]^2 + \big(\frac{1}{2}+4\eta T\big)\|u\|^2_{B(T)}.\end{aligned}$$

By choosing $T > 0$ small enough such that

$$\eta T < \frac{1}{12}, \tag{25}$$

we have that

$$\|u\|_{B(T)}^2 \leq \Big(\frac{3}{4} + \frac{48a^2(\rho)}{A}\big(\sqrt{T} + \frac{1}{\sqrt{A}}\big)^2\Big)\rho^2.$$

Now we take $A, T > 0$ such that

$$\frac{a(\rho)}{\sqrt{A}}\big(\sqrt{T} + \frac{1}{\sqrt{A}}\big) < \frac{1}{8\sqrt{3}}. \tag{26}$$

So, choosing $A, T > 0$ such that (24)-(26) are satisfied, it follows that $u = Mv \in B_\rho(T)$. Now let us prove that the mapping M is a contraction, defined from $B_\rho(T)$ into itself. Let $v_1, v_2 \in B_\rho(T)$, $w = Mv_1 - Mv_2 = u_1 - u_2$, where $u_1(0) = u_2(0) = \phi$. Then,

$$\partial_t \hat{w}(t,\xi) - i\xi^3 \hat{w}(t,\xi) + \eta(-|\xi| + |\xi|^3)\hat{w}(t,\xi) = \mathcal{F}(-v_1\partial_x v_1 + v_2\partial_x v_2)(t,\xi).$$

Multiplying both sides of the last equation by $\xi^{2j}\big(\cosh(2\sigma(t)\xi) + \xi\sinh(2\sigma(t)\xi)\big)\overline{\hat{w}(t,\xi)}$, integrating in ξ, summing the terms for $j = 0, 1$ and taking the real part we obtain

$$\begin{aligned}
\frac{d}{dt}\|w(t)\|_{X_{\sigma(t)}}^2 + A\|w(t)\|_{Y_{\sigma(t)}}^2 &\leq -2\Re(v_1\partial_x v_1 - v_2\partial_x v_2, w)_{X_{\sigma(t)}} + 2\eta\|w(t)\|_{X_{\sigma(t)}}^2 \\
&\leq 2\tilde{a}(\rho,\rho)\big(\|v_1(t)\|_{Y_{\sigma(t)}}\|v_1 - v_2\|_{B(T)} + \|v_1(t) - v_2(t)\|_{Y_{\sigma(t)}}\big) \\
&\quad \cdot\big(\|w(t)\|_{X_{\sigma(t)}} + \|w(t)\|_{Y_{\sigma(t)}}\big) + 2\eta\|w(t)\|_{X_{\sigma(t)}}^2,
\end{aligned}$$

where the last inequality is a consequence of Lemma 3.4 and $\tilde{a}(\cdot,\cdot)$ is a polynomial with nonnegative coefficients. Integrating the last expression on $[0,t]$ and applying the Cauchy Schwarz inequality, we get

$$\begin{aligned}
\|w\|_{B(T)}^2 &\leq 4\tilde{a}(\rho,\rho)\big[\|v_1 - v_2\|_{B(T)}\big(\int_0^T \|v_1(t)\|_{Y_{\sigma(t)}}^2 dt\big)^{1/2} + \big(\int_0^T \|v_1(t) - v_2(t)\|_{Y_{\sigma(t)}}^2 dt\big)^{1/2}\big] \\
&\quad \cdot\big[\big(\int_0^T \|w(t)\|_{X_{\sigma(t)}}^2 dt\big)^{1/2} + \big(\int_0^T \|w(t)\|_{Y_{\sigma(t)}}^2 dt\big)^{1/2}\big] + 4\eta T\|w\|_{B(T)}^2 \\
&\leq \frac{4\tilde{a}(\rho,\rho)}{\sqrt{A}}(\|v_1\|_{B(T)} + 1)\|v_1 - v_2\|_{B(T)}\big(\sqrt{T} + \frac{1}{\sqrt{A}}\big)\|w\|_{B(T)} + 4\eta T\|w\|_{B(T)}^2.
\end{aligned}$$

By using (25) in the last inequality we obtain

$$\|w\|_{B(T)} \leq \frac{6\tilde{a}(\rho,\rho)}{\sqrt{A}}(1+\rho)\big(\sqrt{T} + \frac{1}{\sqrt{A}}\big)\|v_1 - v_2\|_{B(T)}.$$

If we take $A, T > 0$ such that

$$\frac{6\tilde{a}(\rho,\rho)}{\sqrt{A}}(1+\rho)\big(\sqrt{T}+\frac{1}{\sqrt{A}}\big) < 1, \tag{27}$$

then M is a contraction. So, by choosing $A, T > 0$ such that (24)-(27) are satisfied, the mapping M has a unique fixed point $u \in B_\rho(T)$ that is the solution of problem (1). Since $\|u(t)\|_{X_{\sigma(T)}} \leq \|u(t)\|_{X_{\sigma(t)}}$, for all $t \in [0,T]$, then $u(t) \in X_{\sigma(T)}$, for all $t \in [0,T]$.

Now we will prove that $u \in C^\omega([0,T]; X_{\sigma(T)})$. Let $t \in [0,T]$, without loss of generality we may assume $0' \leq t' < t \leq T$. Since by Lemma 3.1, $Y_{\sigma(T)}$ is a dense subset of $X_{\sigma(T)}$, it is enough to prove that $(u(t)-u(t'), f)_{X_{\sigma(T)}} \to 0$ as $t' \uparrow t$, for all $f \in Y_{\sigma(T)}$. So, let f be an arbitrary but fixed element of $Y_{\sigma(T)}$. Let us denote by $F_\eta(s,\xi) \equiv e^{(i\xi^3+\eta(|\xi|-|\xi|^3))s}$ and $h_j(\xi,T) \equiv \xi^{2j}\big(\cosh(2\sigma(T)\xi) + \xi\sinh(2\sigma(T)\xi)\big)$. First, let us remark that $|\hat{u}(t,\xi) - \hat{u}(t',\xi)| \to 0$ as $t' \uparrow t$, for all $\xi \in \mathbb{R}$. In fact, by using (23), we have that $\hat{u}(t,\xi) - \hat{u}(t',\xi) = \sum_{j=1}^{3} I_j(t,t',\xi)$, where

$$I_1(t,t',\xi) \equiv \big(F_\eta(t,\xi) - F_\eta(t',\xi)\big)\hat{\phi}(\xi) \longrightarrow 0 \text{ as } t' \uparrow t,$$

$$\begin{aligned}
|I_2(t,t',\xi)| &\equiv \Big|\int_{t'}^{t} F_\eta(t-\tau,\xi)\widehat{uu_x}(\tau,\xi)d\tau\Big| \leq \frac{e^{\eta T}|\xi|}{2}\int_{t'}^{t}|\widehat{u(\tau)^2}(\xi)|d\tau \\
&\leq \frac{e^{\eta T}|\xi|}{2\sqrt{2\pi}}\int_{t'}^{t}\|u(\tau)\|^2 d\tau \leq ce^{\eta T}|\xi|\|u\|^2_{B(T)}(t-t') \longrightarrow 0 \text{ as } t' \uparrow t,
\end{aligned}$$

$$|I_3(t,t',\xi)| \equiv \Big|\int_0^{t'}\big(F_\eta(t-\tau,\xi) - F_\eta(t'-\tau,\xi)\big)\widehat{uu_x}(\tau,\xi)d\tau\Big| \longrightarrow 0 \text{ as } t' \uparrow t,$$

where the last convergence is a consequence of $\big|F_\eta(t-\tau,\xi) - F_\eta(t'-\tau,\xi)\big||\widehat{uu_x}(\tau,\xi)| \leq 2e^{\eta T}|\widehat{uu_x}(\tau,\xi)| \in L^1([0,t], d\tau)$, and the dominated convergence theorem.
On the other hand we have that

$$|(u(t)-u(t'), f)_{X_{\sigma(T)}}| \leq \sum_{j=0}^{1} I_1^j(t,t') + I_2(t,t') + I_3(t,t'),$$

where

$$I_1^j(t,t') \equiv \Big|\int h_j(\xi,T)(F_\eta(t,\xi) - F_\eta(t',\xi))\hat{\phi}(\xi)\overline{\hat{f}(\xi)}d\xi\Big|,$$

which, by the dominated convergence theorem, tends to zero as $t' \uparrow t$.
Now we estimate $I_2(t,t')$ defined below. First we see that

$$\begin{aligned}\int |\xi||h_j(\xi,T)||\hat{f}(\xi)|^2 d\xi &\le \cosh(2\sigma(T))\|f\|^2 + \int_{|\xi|>1} |\xi|^{2j+1}\cosh(2\sigma(T)\xi)|\hat{f}(\xi)|^2 d\xi \\ &\quad + \sinh(2\sigma(T))\|f\|^2 + \int_{|\xi|>1} |\xi|^{2j+2}\cosh(2\sigma(T)\xi)|\hat{f}(\xi)|^2 d\xi \\ &\le e^{2\sigma(T)}\|f\|^2 + 2\|f\|^2_{Y_{\sigma(T)}}. \end{aligned} \tag{28}$$

Moreover, for every $\tau \in [0,T]$, we have that

$$\|u(\tau)^2\| \le \sup_{x\in\mathbb{R}} |u(\tau,x)|\|u(\tau)\| \le c\|u(\tau)\|_1\|u(\tau)\| \le c\|u(\tau)\|^2_{X_{\sigma(T)}} \tag{29}$$

and

$$\begin{aligned}\|u(\tau)^2\|_{Y_{\sigma(T)}} &\le \sqrt{2}\|U(\tau)^2\|_{H^{2,2}(\sigma(T))} \le c\|U(\tau)\|^2_{H^{2,2}(\sigma(T))} \\ &\le c(T)\big(\|u(\tau)\|^2 + \|u(\tau)\|^2_{Y_{\sigma(T)}}\big),\end{aligned} \tag{30}$$

where in the last inequalities we have used Lemmas 3.2 and 3.3, and $U(\tau)$ is the analytic extension of $u(\tau)$ on $S(\sigma(T))$. $I_2(t,t')$ is given by

$$\begin{aligned} I_2(t,t') &\equiv \Big|\sum_{j=0}^{1}\int h_j(\xi,T)\int_{t'}^{t} F_\eta(t-\tau,\xi)\widehat{uu_x}(\tau,\xi)d\tau\overline{\hat{f}(\xi)}d\xi\Big| \\ &= \Big|\sum_{j=0}^{1}\int_{t'}^{t}\int h_j(\xi,T)F_\eta(t-\tau,\xi)\widehat{uu_x}(\tau,\xi)\overline{\hat{f}(\xi)}d\xi d\tau\Big|. \end{aligned} \tag{31}$$

The interchange of the integrals in (31) is a consequence of Fubini's Theorem since

$$\begin{aligned} &\int_{t'}^{t}\int h_j(\xi,T)|F_\eta(t-\tau,\xi)||\widehat{uu_x}(\tau,\xi)||\overline{\hat{f}(\xi)}|d\xi d\tau \\ \le\ & \frac{e^{\eta T}}{2}\int_{t'}^{t}\Big(\int |\xi||h_j(\xi,T)||\hat{f}(\xi)|^2 d\xi\Big)^{1/2}\Big(\int |\xi||h_j(\xi,T)||\widehat{u(\tau)^2}(\xi)|^2 d\xi\Big)^{1/2} d\tau \\ \le\ & c(T)(\|f\| + \|f\|_{Y_{\sigma(T)}})\int_0^T \big(\|u(\tau)^2\| + \|u(\tau)^2\|_{Y_{\sigma(T)}}\big)d\tau \\ \le\ & c(T)(\|f\| + \|f\|_{Y_{\sigma(T)}})\int_0^T \big(\|u(\tau)\|^2_{X_{\sigma(T)}} + \|u(\tau)\|^2 + \|u(\tau)\|^2_{Y_{\sigma(T)}}\big)d\tau \\ \le\ & c(T)(\|f\| + \|f\|_{Y_{\sigma(T)}})\big(\|u\|^2_{B(T)}T + \frac{\|u\|^2_{B(T)}}{A}\big) < +\infty, \end{aligned} \tag{32}$$

where the second and third last inequalities were consequence of (28)-(30). So, it follows from (31) that

$$I_2(t,t') = \Big| \int_{t'}^{t} \big(u(\tau)\partial_x u(\tau), g(t,\tau)\big)_{X_{\sigma(T)}} d\tau \Big|,$$

where $\overline{\widehat{g(t,\tau)}(\xi)} \equiv F_\eta(t-\tau,\xi)\overline{\hat{f}(\xi)}$. Then $\|g(t,\tau)\|_{X_{\sigma(T)}} \leq e^{\eta T}\|f\|_{X_{\sigma(T)}}$ and the same thing for the $Y_{\sigma(T)}$-norm. By Corollary 3.1 we have that

$$\begin{aligned} I_2(t,t') &\leq \int_{t'}^{t} \tilde{a_1}(\|u(\tau)\|_{X_{\sigma(T)}})\|u(\tau)\|_{Y_{\sigma(T)}}\big(\|g(t,\tau)\|_{X_{\sigma(T)}} + \|g(t,\tau)\|_{Y_{\sigma(T)}}\big)d\tau \\ &\leq \tilde{a_1}(\|u\|_{B(T)})e^{\eta T}\big(\|f\|_{X_{\sigma(T)}} + \|f\|_{Y_{\sigma(T)}}\big)\Big(\int_{t'}^{t} \|u(\tau)\|^2_{Y_{\sigma(T)}} d\tau\Big)^{1/2}\sqrt{t-t'}, \end{aligned}$$

where $\tilde{a_1}$ is a polynomial with nonnegative coefficients. Last inequality implies that $I_2(t,t')$ tends to zero as $t' \uparrow t$.
Now we estimate $I_3(t,t')$, given by

$$\begin{aligned} I_3(t,t') &\equiv \Big|\sum_{j=0}^{1} \int h_j(\xi,T)\int_0^{t'} \big(F_\eta(t-\tau,\xi) - F_\eta(t'-\tau.\xi)\big)\widehat{uu_x}(\tau,\xi)d\tau\overline{\hat{f}(\xi)}d\xi\Big| \\ &= \Big|\sum_{j=0}^{1} \int_0^{t'}\int h_j(\xi,T)\big(F_\eta(t-\tau,\xi) - F_\eta(t'-\tau.\xi)\big)\widehat{uu_x}(\tau,\xi)\overline{\hat{f}(\xi)}d\xi d\tau\Big|. \end{aligned}$$

The interchange of the integrals in the last expression was a consequence of Fubini's Theorem (similar to (31)). So, by the dominated convergence theorem, we have that $I_3(t,t')$ tends to zero as $t' \uparrow t$ since

$$\begin{aligned} &\big|h_j(\xi,T)\big(F_\eta(t-\tau,\xi) - F_\eta(t'-\tau,\xi)\big)\widehat{uu_x}(\tau,\xi)\overline{\hat{f}(\xi)}\big| \\ \leq\ & e^{\eta T}h_j(\xi,T)|\xi|\big|\widehat{u(\tau)^2}(\xi)\big|\big|\hat{f}(\xi)\big| \in L^1(\mathbb{R}\times[0,t], d\xi d\tau), \end{aligned}$$

the last expression is obtained with similar calculations to (32).
Hence $|(u(t)-u(t'),f)_{X_{\sigma(T)}}| \to 0$ as $t' \uparrow t$ (similarly as $t' \downarrow t$). Then $u \in C^\omega([0,T];X_{\sigma(T)})$.

Finally let us remark that when $\eta = 0$, $u \in C([0,T];X_{\sigma(T)})$, as it was already mentioned by Hayashi in Theorem 2 of [5]. In fact in this case, we have that $\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{X_{\sigma(T)}} = -\Re(u(t)\partial_x u(t), u(t))_{X_{\sigma(T)}}$. Integrating the last expression between t' ant t and using again Corollary 3.1, we obtain

$$
\begin{aligned}
\big|\|u(t)\|^2_{X_{\sigma(T)}} - \|u(t')\|^2_{X_{\sigma(T)}}\big| &= 2\Big|\int_{t'}^{t} \Re(u(\tau)\partial_x u(\tau), u(\tau))_{X_{\sigma(T)}} d\tau\Big| \\
&\leq 2\tilde{a}_1(\|u\|_{B(T)})\big[\|u\|_{B(T)}\Big(\int_{t'}^{t} \|u(\tau)\|^2_{Y_{\sigma(T)}} d\tau\Big)^{1/2}\sqrt{t-t'} \\
&\quad + \int_{t'}^{t} \|u(\tau)\|^2_{Y_{\sigma(T)}} d\tau\big] \longrightarrow 0 \text{ as } t' \uparrow t,
\end{aligned}
$$

and similarly as $t' \downarrow t$. Since $X_{\sigma(T)}$ is a Hilbert space, it follows that $u \in C([0,T]; X_{\sigma(T)})$ when $\eta = 0$.■

Appendix

Kato's Inequality: (See [6]).
Let $s > 3/2$, $t \geq 1$. If f and u are real-valued, then

$$
|(f\partial_x u, u)_t| \leq C\big(\|\partial_x f\|_{s-1}\|u\|_t^2 + \|\partial_x f\|_{t-1}\|u\|_s\|u\|_t\big). \tag{K}
$$

Acknowledgements. This paper was carried out while the author had a Post-Doctoral position at IMECC-UNICAMP. The author wants to express his gratitude to Prof. Marcia Scialom for her interest on this work and valuable comments.

References

[1] B. Alvarez Samaniego, *The Cauchy Problem for a nonlocal perturbation of the KdV Equation* (to appear, Differential and Integral Equations).

[2] J. Bona and Z. Grujić, *Spatial Analyticity Properties of Nonlinear Waves* (to appear, Mathematical Models and Methods in Applied Sciences).

[3] Bao-Feng Feng and T. Kawahara, *Multi-hump stationary waves for a Korteweg-de Vries equation with nonlocal perturbations*, Physica D 137 (2000), 237-246.

[4] Z. Grujić and H. Kalish, *Local Well-Posedness of the Generalized KdV Equation in Spaces of Analytic Functions*, Diff. Integral Eq. 15 (2002), 1325-1334.

[5] N. Hayashi, *Analyticity of Solutions of the Korteweg-de Vries Equation*, SIAM J. Math. Anal. 22 (1991), 1738-1743.

[6] T. Kato, *On the Cauchy Problem for the (Generalized) Korteweg-de Vries Equation*, Studies in Applied Mathematics, Advances in Mathematics Supplementary Studies, Academic Press Vol. 8 (1983), 93-128.

[7] T. Kato and K. Masuda, *Nonlinear Evolution Equations and Analyticity I*, Ann. Inst. Henri Poincaré, Anal. Non Linéaire 3 (1986), 455-467.

[8] L.A. Ostrovsky, Yu. A. Stepanyants and L. Sh. Tsimring, *Radiation instability in a stratified shear flow*, Int. J. Non-Linear Mech. 19 (1983), No. 2, 151-161.

Borys Alvarez Samaniego: Post-doctoral Researcher, Departamento de Matemática, IMECC-UNICAMP, C.P. 6065, CEP 13083-970, Campinas, SP, Brazil, balvarez@ime.unicamp.br