EXPANDING LORENZ ATTRACTORS THROUGH RESONANT DOUBLE HOMOCLINIC LOOPS

C.A.MORALES, M.J.PACIFICO & B. SAN MARTIN *

Abstract.
In this paper we study the existence of Lorenz attractors in the unfolding of resonant double homoclinic loops in dimension three. Our results generalize the ones obtained in [Rob1] in two ways. First, we obtain *attractors* instead of *weak attractors* obtained there. Second, we enlarge considerably the region in the parameter space corresponding to flows presenting expanding Lorenz attractors. The proof is based on rescaling techniques [PT] to obtain convergence to non-continuous maps.

Key words. Lorenz attractors, resonant double homoclinic loops

AMS subject classifications. 37G10, 37G15, 37G25

1. Introduction. In the serie of papers [Rob1, Rob2, Rob3] Robinson studied the existence of transitive attractors of Lorenz type in generic unfoldings of resonant double homoclinic loops in dimension three. For instance, [Rob1, Theorem 3.1 p. 130] says that under certain conditions such unfoldings produce transitive weak attractors containing the singularity. In this paper we improve this result in two ways. First, we obtain expanding Lorenz attractors instead of weak attractors as defined in [Rob1, p. 120]. Second we enlarge considerably the region in the parameter space which corresponds to flows presenting expanding Lorenz attractors. By *attractor* we mean a transitive set which is maximal invariant in a positively invariant open set. A set is *transitive* if it is the omega-limit set of one of its orbits. An attractor of a three-dimensional vector field is *expanding Lorenz* if it contains a unique singularity whose eigenvalues $\lambda_1, \lambda_2, \lambda_3$ are real and satisfy $\lambda_2 < \lambda_3 < 0 < -\lambda_3 < \lambda_1$. The classical example of an expanding Lorenz attractor is the geometric Lorenz attractor in [GW, ABS]. We shall consider parametrized families of vector fields unfolding a resonant double homoclinic loop at $\eta = \eta_0$ as in [Rob1]. We assume the same hypotheses (A1)-(A7) of [Rob1], except for (A5) that we replace for

$$B = \frac{C^+_{\eta_0} + C^-_{\eta_0}}{C^+_{\eta_0} C^-_{\eta_0}} > 1,$$

where $C^{\pm}_{\eta_0}$ are defined as in that paper: the constants $C^{\pm}_{\eta_0}$ measure the change in area within a certain bundle over Γ, the resonant double homoclinic loop. Note that (A5) in [Rob1] implies $B > 1$ but not conversely. The proof is based on rescaling techniques [PT] to obtain convergence to non-continuous maps. In a forthcoming paper we use the rescalling techniques developed here to study the existence of contracting Lorenz attractors (Rovella attractors) in the unfolding of resonant double homoclinic loops. This will answer a question posed in [Rob1, Remark 5.1 p. 138].

Before we announce in a precise way our results, let us comment on some other results concerning their Lorenz attractors and its bifurcations.

1.1. Related results and comments.

*This work is partially supported by CNPq, FAPERJ, PRONEX on Dyn. Systems, FONDECYT GRANT N 1981241, N 100047 and FUNDACION ANDES.

1.1.1. Lorenz equations and the geometrical model. The main motivation for all these results is the Lorenz attractor [Lo], given by the solutions of the polynomial vector field in $\mathbb{R}^3$:

$$X(x,y,z) = \begin{cases} \dot{x} = -\alpha x + \alpha y \\ \dot{y} = \beta x - y - xz \\ \dot{z} = -\gamma z + xy\,, \end{cases} \tag{1.1}$$

where α, β, γ are real parameters. Numerical experiments performed by Lorenz (for $\alpha = 10, \beta = 28$ and $\gamma = 8/3$) suggested the existence of a strange attractor toward which tends a full neighborhood of positive trajectories of the above system. Moreover, the strange attractor seemed to be robust: it can not be destroyed by any perturbation of the parameters. On the other hand, this attractor contains an equilibrium point $(0,0,0)$, and periodic points accumulating on it, and hence can not be hyperbolic. The book [Sp] contains an extensive presentation of analytical and numerical facts about the original Lorenz equations. We point out that was it proved [Tu1, Tu2] that the solutions of (1.1) satisfy such a property for values α, β, γ near the ones considered by Lorenz. See the feature review [Tu2, MR2003b:37055] and [V] for a survey and a discussion of Tucker's proof.

However, already in the mid-seventies, the existence of robust non-hyperbolic attractors was proved for flows introduced in [ABS] and [GW], which we now call geometrical models for Lorenz attractors. At the same time the theory of uniformly hyperbolic systems was being developed, and it was increasingly clear that such systems are not dense. One of those examples was precisely the geometrical Lorenz attractor.

Although geometrical Lorenz attractors fail to be uniformly hyperbolic, they do exhibit a certain amount of hyperbolicity (called singular hyperbolicity in [MPP2, MPP3]), which has been exploited to understand their structure. The bifurcations of these attractors and their relations to homoclinic bifurcations of codimension 2 and 3 have also been intensively studied by several authors, both theoretically and a view towards applications. This included great interest in the so-called Lorenz maps, one dimensional models to which the dynamics of a geometrical Lorenz attractor can be reduced. In addition a number of bifurcation mechanisms have been found which yield attractor with similar properties. In what follows we give an overview of the literature, without trying to be exhaustive.

1.1.2. Attractors that resemble a Lorenz attractor. First, let us list some papers about the existence of chaotic attractors that resemble geometrical Lorenz models. We start with [MPP2, MPP3] where it is proved that any robust attractor of a flow in three-manifolds containing equilibria looks like a geometric Lorenz attractor. In [BPV] the authors construct a multidimensional Lorenz-like attractor that is C^r-robust (r large) and contains a singularity with at least two positive eigenvalues. They also investigatethe SBR measures of these attractors. Their construction works in dimension ≥ 5. In [ST] the authors present an example of a four-dimensional quasi-attractor and study its perturbations. The quasi-attractor is pseudo-hyperbolic, contains a singularity with a complex eigenvalue and can not be destroyed by small perturbations of the system. In [Lo84] the author reports a careful numerical study of what seems to be a strange (chaotic) attractor in four dimensions for a system of 2-degree polynomial equations. In [Rov] the author proves existence and persistence of contracting Lorenz attractors, that is, with the contracting eigenvalue condition $-\lambda_3 > \lambda_1$.

In [PRV] the authors prove that certain parametrized families of one-dimensional maps with infinitely many critical points exhibit global chaotic behavior in a persistent way. An application of the methods developed there yields a proof of existence and even persistence of global spiral attractors for smooth flows in three dimensions, to be given in [CPRV].

In [P, S] the authors propose abstract models for attractors with singularities, called generalized hyperbolic attractors, and study their properties.

1.1.3. Topological dynamics. Some aspects of the topological dynamics of the geometric model was studied in [Ko1, Ko2], where it was proved that most geometrical Lorenz attractors do not have the shadowing property, and their expansive properties are investigate. In [Kl] the author finds a topological invariant for the Lorenz attractor allowing him to exhibit an uncountable number of non-homeomorphic Lorenz attractors in the unfolding of a certain homoclinic loop. In [Ya] the author shows that the geometrical Lorenz attractor can be approximated by horseshoes with entropy close to that of the Lorenz attractor. In [BW] the knot type of the geometric model is analysed, and in [GH] the Lorenz attractor is used to investigate the existence of flows realizing all links and knots as periodic orbits in 3-manifolds and an explicit ODE with such properties is exhibited. One can also see the survey [PS].

1.1.4. Dimension theory, ergodic and statistical properties. Concerning fractal dimensions of Lorenz attractors we mention the results in [Le1, Le2] and [BL]. The first contains explicit formulae for the Liapunov dimension of the Lorenz attractor and the second a simple upper bound on the Hausdorff dimension of Lorenz attractors is given in terms of parameters σ, α, β in (1.1).

Statistical and ergodic properties of the geometrical model were investigated in [Bu]. In [Me1, Me2] it is proved the existence of SBR measures and the stochastic stability for the contracting model.

1.1.5. Lorenz maps of the interval. Lorenz-like maps of the interval and their bifurcations have being studied in [LM1, LM2]. See also references therein. In [AL] the authors describe the use of kneading theory to study the dynamics of Lorenz maps. In [HS] the Lorenz maps are classified, up to topological conjugacy, by their kneading invariants. In [MdeM] it is introduced the notion of monotone Lorenz families and the authors prove that all possible topological dynamics behavior of Lorenz maps is realized within every monotone Lorenz family. In [KS] the authors study the case where the Lorenz map has negative Schwarzian derivative and the derivative vanishes at both sides of the discontinuity point. The paper has both a survey and a research flavor. Among other results they characterize the global attractor associated in terms of renormalization properties of the Lorenz map. A modified Lorenz interval map is studied in [LV], combining a finite number of critical points and a finite number of discontinuities with infinite derivative. The authors prove that the associated attractor is nonuniformly hyperbolic. In [DY] the authors study the asymptotic periodicity of a Lorenz interval map. They prove that a Lorenz-like map is asymptotically periodic if the derivative set of the pre-image set of the discontinuity point is countable. See also the book [AH] and references therein for more about this topic.

1.1.6. Bifurcations. An extensive serie of works have been dedicated to the bifurcations of systems leading to the appearence of Lorenz attractors, contracting or not, besides the papers by Robinson quoted before. We start with the Shimizu-Morioka model, a 3-dimensional system of first-order ordinary differential equations which depend on two parameters. It has been derived to study the Lorenz system for large Rayleigh number. In [SM] the authors proved that there are two types of Lorenz-like attractors in this model. In [Sh], bifurcations near two codimension-two points in the parameter plane are analyzed in great detail. These points lie on the boundary of the region of existence of the Lorenz-like attractors. In [MPP1] it is studied a bifurcation of hyperbolic vector fields on 3-dimensional manifolds leading to robust strange attractors with singularities. In [DKO] the authors prove that bifurcations of a certain double homoclinic loop associated to a degenerated singularity can produce geometrical Lorenz attractors, and they also exhibit an explicit ODE presenting such attractors. [Ry] gives sufficient conditions for a butterfly inclination-flip loop to generate geometrical Lorenz attractors. In [ACL] the authors investigate the existence of Lorenz attractors in the unfolding of a certain singular cycle envolving a saddle-node periodic orbit and a Lorenz-like singularity. In [MPu] the authors show the appearence of Lorenz arttractors in the unfolding of a cycle formed by a Lorenz-like singularity and a saddle-node periodic orbit, and in [Mo] it is shown the appearence of Lorenz attractors through a saddle-node bifurcation. See also [KKO] and the references therein for more about bifurcations generating geometrical Lorenz attractors.

1.2. Our hypotheses. Let us state our hypotheses in a precise way. In what follows X_η is a family of C^r, $r \geq 1$, vector fields on $\mathbb{R}^3$ satisfying the following conditions:

(A1) *For every η, X_η has a hyperbolic singularity Q_η such that the eigenvalues of $DX_\eta(Q_\eta)$ are real with $\lambda_{ss}(\eta) < \lambda_s(\eta) < 0 < \lambda_u(\eta)$, and with eigenvectors v^{ss}, v^s, and v^u, respectively.*

With this assumption, there are several invariant manifolds for the singularity Q_η. We denote the one-dimensional unstable manifold tangent to v^u by $W^u(Q_\eta, \eta)$, and the two-dimensional stable manifold tangent to v^{ss} and v^s by $W^s(Q_\eta, \eta)$. Next, there is a one-dimensional strong stable manifold $W^{ss}(Q_\eta, \eta)$. This latter manifold is made of points which converge to Q_η at an asymptotic rate determined by the eigenvalue λ_{ss}. All these manifolds are C^r if the vector field is C^r. Finally, there is a two-dimensional extended stable manifold tangent to v^s and v^u, which we denote by $W^{cu}(Q_\eta, \eta)$. The later manifold is at least C^1. With this notation we can make the second assumption about the existence of a double homoclinic connection.

(A2) *For the bifurcation value η_0, there is a double homoclinic connection with the unstable manifold of Q_{η_0} contained in the stable manifold but outside the strong stable manifold, that is,*

$$\Gamma = W^u(Q_{\eta_0}, \eta_0) \subset W^s(Q_{\eta_0}, \eta_0) \setminus W^{ss}(Q_{\eta_0}, \eta_0).$$

In fact, we assume that the two branches $\Gamma^{\pm}$ of $\Gamma \setminus \{Q_{\eta_0}\}$ are contained in the same component of $W^s(Q_{\eta_0}, \eta_0) \setminus W^{ss}(Q_{\eta_0}, \eta_0)$. Note that $\Gamma = \{Q_{\eta_0}\} \cup \Gamma^+ \cup \Gamma^-$.

(A3) *For η_0, the central manifold $W^{cu}(Q_{\eta_0}, \eta_0)$ is transverse to the stable manifold $W^s(Q_{\eta_0}, \eta_0)$ along Γ.*

Let

$$P(q) = T_q W^{cu}(Q_{\eta_0}, \eta_0) \quad \text{for} \quad q \in \Gamma.$$

The transversality condition in (A3) with condition

$$W^u(Q_{\eta_0}, \eta_0) \cap W^{ss}(Q_{\eta_0}, \eta_0) = Q_{\eta_0}$$

in Assumption (A2) implies that $P(q)$ converges to $P(Q_{\eta_0})$ as q converges to Q_{η_0} along Γ (by the Inclination Lemma [dMP]). Therefore, $\{P(q) : q \in \Gamma\}$ is a continuous bundle over Γ. Considering one half of the homoclinic connection $\Gamma^+ \cup Q_{\eta_0}$, let $\nu^+ = 1$ if the bundle $\{P(q) : q \in \Gamma^+ \cup Q_{\eta_0}\}$ is orientable and $\nu^+ = -1$ if the bundle is non-orientable. In the same way, considering the other half of the homoclinic connection $\Gamma^- \cup Q_{\eta_0}$, let $\nu^- = \pm 1$ whenever the bundle $\{P(q) : q \in \Gamma^- \cup Q_{\eta_0}\}$ is orientable or non-orientable respectively.

(A4) *We assume that*

$$\lambda_{ss}(\eta_0) - \lambda_s(\eta_0) + \lambda_u(\eta_0) < 0, \quad \textit{and} \quad \lambda_{ss}(\eta_0) < 2\lambda_s(\eta_0).$$

We shall use the notation $\alpha(\eta) = -\frac{\lambda_s(\eta)}{\lambda_u(\eta)}$ and $\beta(\eta) = -\frac{\lambda_{ss}(\eta)}{\lambda_u(\eta)}$.

These are open conditions and so do not add a codimension to the bifurcation. The second inequality in (A4) assures that $W^{cu}(Q_{\eta_0}, \eta_0)$ is C^2.

Let $q^\pm(t)$ be a parametrization of the solution along $\Gamma^\pm$ and $\mathrm{div}_2(q^\pm(t))$ the Jacobian of X_{η_0} at t restricted to $T_{\Gamma^\pm} W^{cu}$. Define $C^\pm_{\eta_0}$ by

$$C^\pm_{\eta_0} = \exp\left(\int_{-\infty}^{\infty} \mathrm{div}_2(q^\pm(t))dt\right).$$

The quantity $C^\pm_{\eta_0}$ is the change in area within the -planes $P(q)$ along the whole length of $\Gamma^\pm$.

(A5) $B > 1$, *where*

$$B = \frac{C^+_{\eta_0} + C^-_{\eta_0}}{C^+_{\eta_0} C^-_{\eta_0}}.$$

We observe that the condition (A5) in [Rob1] requires either $0 < C^\pm_{\eta_0} < 1$ or $0 < C^\pm_{\eta_0} < 2$ and $\frac{C^+_{\eta_0}}{C^-_{\eta_0}} \in [(1+\sqrt{2})^{-1}, 1+\sqrt{2}]$. Robinson also has a polynomial vector field realizing such conditions.

We, on the contrary, do not made such assumptions on $C^\pm_{\eta_0}$. See Figure 1.1. On the other hand, we are still working in the problem of finding a polynomial vector field realizing the unfolding described in this paper.

Note that (A5) in [Rob1] implies $B > 1$ but not conversely. The assumption (A5) is open.

(A6) *There is a one-to-one resonance between the unstable and weak stable eigenvalue for* η_0*:*

$$\lambda_u(\eta_0) + \lambda_s(\eta_0) = 0.$$

Observe that condition (A6) means $\alpha(\eta) = 1$. This condition is needed to have (A5) satisfied, see [Rob1]. This resonance condition is a co-dimension one condition; in total, the conditions on η_0 are co-dimension three, (two conditions are from the double homoclinic connection and resonance gives the third and final co-dimension).

The final assumption is related to the unfolding of the bifurcation. We assume that the parameter space is large enough in order to break the double homoclinic loop in a correct way.

(A7) *Let* $\mathcal{N} \subset \mathcal{X}^1(I\!R^3)$ *be the* 3*-submanifold defined by conditions (A1)-(A6). We assume that the family* $\{X_\eta\}$ *is transverse to* $\mathcal{N}$ *at* η_0.

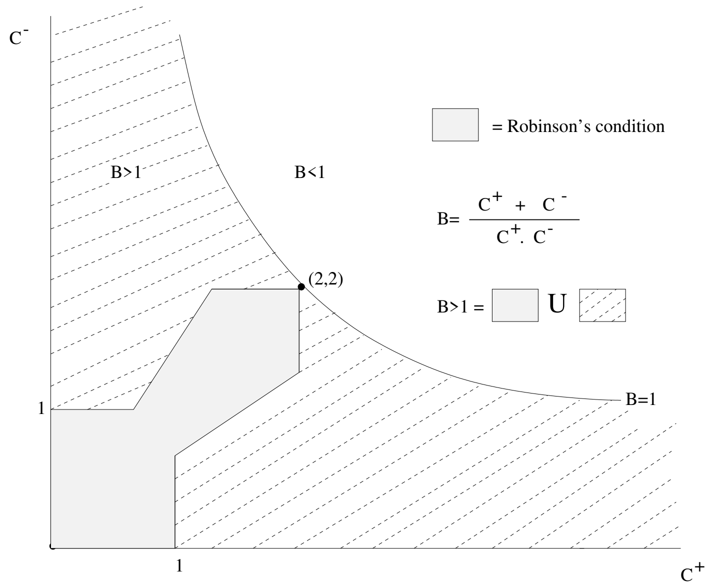

FIG. 1.1.

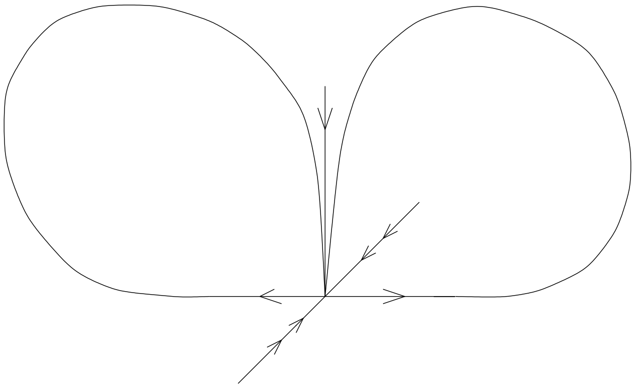

FIG. 1.2. *Vector field* X_{η_0}

1.3. The main result. It is now possible to announce our main result. Given a A a subset in the parameter space we set $Cl(A)$ for the closure of A.

THEOREM 1.1. *Let* $\{X_\eta\}$ *be a* C^k*-parametrized family of* C^r*-vector fields* $(r, k \geq 3)$ *satisfying (A1)-(A7). Then, there is an open set* $\mathcal{O}$ *in the parameter space with* $\eta_0 \in Cl(\mathcal{O})$ *such that* X_η *has an expanding Lorenz attractor for all* $\eta \in \mathcal{O}$.

The tools used in the proof are reduction of the dynamics to a one-dimensional Poincaré map and the existence of a suitable rescaling for such maps with a well defined limit dynamics. Rescaling techniques in the unfolding of homoclinic loops were used in [N] to obtain convergence to the Henon map. Here we use such techniques to obtain convergence to non-continuous maps. Convergence to non-continuous maps via rescaling was already considered in [MPu] for the study of certain heteroclinic connections involving saddle-node periodic orbits. See the work in progress [MSV] for more about rescaling techniques for singular cycles.

1.4. Sketch of the proof. Let us present the idea of the proofs. As in [Rob1], we observe that (A1)-(A3) imply the existence of a strong stable invariant foliation close to the loop. By (A4), the C^r Section Theorem [S] implies that such a foliation is C^1 and varies C^1 with the parameters. As usual we consider the Poincaré map along the homoclinic loop. Using the strong stable foliation we reduce the dynamics of the return map to a one-dimensional map $f_\eta(\tau)$. Following [Rob2] we denote α the order of $f_\eta(\tau)$. Clearly α depends on η. At this point the proof in [Rob1] requires to solve certain inequalities allowing trapping regions for suitable parameter values. Recall that a compact interval J is a trapping region for f_η if $f_\eta(J) \subset int(J)$, where $int(J)$ means the interior of J. Instead, we use a different approach in which we consider α as a parameter. In Lemma 2.2 we fix α and prove the existence of *good parameters values*, i.e., parameters for which the critical values $f_\eta(0^\pm)$ of f_η are either fixed or pre-fixed or periodic (with period 2) expanding points. Such parameters are solutions of certain equations that can be solved only for $\alpha < 1$ because of (A5). Using the critical values we construct, for those good parameters, a f_η-invariant closed interval $[p, q]$ containing $\tau = 0$. We also get $| f'_\eta(\tau) | > 1$ uniformly for $\tau \in [p, q]$. Afterward we use rescaling techniques [PT]: we take a suitable parameter-depending change of coordinates in a neighborhood of $[p, q]$ and, at the same time, we normalize the parameter space in a small neighborhood of those good parameters. This yields a new family $g_\alpha(\mu, \nu, \cdot)$ and new good parameters $(\mu(\alpha), \nu(\alpha))$. In Lemma 3.3 we show the existence of bounds for the derivative of $g_\alpha(\nu, \mu, \cdot)$. This is used to prove that $g_\alpha(\mu, \nu, \cdot)$ converges (in a C^1-sense to be defined below) to a map $g(\mu, \nu, \cdot)$ as $\alpha \to 1^-$. The limit map $g(\mu, \nu, \cdot)$ is piece-wise linear expanding that looks like the one in Figure 3.1. In the same lemma we show that $\lim_{\alpha \to 1^\pm} (\mu(\alpha), \nu(\alpha)) = (\mu(1), \nu(1))$ exists.

The maps $g(\mu, \nu, \cdot)$ above do not have trapping regions, but they can be approximated in our family by ones having them. Indeed, by Theorem 4.1, we have that $g_\alpha(\mu, \nu, \cdot)$ has trapping regions for $\alpha < 1$ close to 1 and for (μ, ν) close to $(\mu(\alpha), \nu(\alpha))$. That is, for each $\alpha < 1$ close to 1 there is a compact interval J_α such that $g_\alpha(\mu, \nu, J_\alpha) \subset int(J_\alpha)$. Then for all $\alpha < 1$ and close to 1 the flow associated with $g_\alpha(\mu, \nu, \cdot)$ has a trapping region. Thus a trapping region for the flow do exist in an open set $\mathcal{O}$ of parameters accumulating $\eta = 0$. To prove Theorem 1.1 we use the closeness of $g_\alpha(\mu, \nu, \cdot)$ to $g(\mu(1), \nu(1), \cdot)$. Such a closeness and Theorem 5.1 imply both transitiveness and expansiveness of $g_\alpha(\mu, \nu, \cdot)$ for the parameters with trapping regions, see also [Rob4, Theorem A].

2. One-dimensional reduction and good parameters. To start with, consider a cross-section Σ of X_{η_0} close to Q_{η_0} transversal $W^s(Q_{\eta_0})$ intersecting both branches of $W^u(Q_{\eta_0})$. There is a neighborhood V of $\Sigma \cap W^s(Q_{\eta_0})$ in Σ such that the positive orbit of every point at $V \setminus W^s(Q_{\eta_0})$ intersects Σ for every parameter η near enough η_0, defining in this way a Poincaré map $F_\eta : V \setminus W^s(Q_{\eta_0}) \subset \Sigma \to \Sigma$.

As X_{η_0} satisfies conditions (A1)-(A4), the standard stable manifold theory applies to show the existence of a C^1 stable foliation in a small neighborhood (that for convenience we assume equals to V) of $W^s(Q_\eta)$ varying C^1 with the parameter. As in [Rob2] the existence of a C^r stable foliation ($r \geq 1$) depends on the relation

$$C_3 e^{T(\lambda_{ss}(\eta_0) - \lambda_s(\eta_0))} (e^{T\lambda_u(\eta_0)})^r < 1 \, .$$

By the first eigenvalue inequality in (A4) we have the above relation for $r = 1$. Then, we can use the C^r-Section Theorem ([S, Theorem 5.18]) in the same way as in [Rob2].

Thus, via projection along the leaves of the strong stable foliation, the problem can be reduced to a one-dimensional Poincaré map $f_\eta : V' \setminus \{c_\eta\} \subset [-1,1] \to [-1,1]$. Here c_η is the projection of $W^s(Q_{\eta_0}) \cap V$ onto V'. We assume $c_\eta = 0$ for every η. Denote $a^\pm_\eta = \lim_{\tau \to \pm 0} f_\eta(\tau)$, $\tau \in [-1,1]$. Recall the description of the coefficients $\nu^\pm$ in the assumption (A3).

LEMMA 2.1. *There is an interval J, $0 \in J$ such that for every η sufficiently near to η_0, the map $f_\eta : J \subset [-1,1] \to [-1,1]$ has the following form:*

$$f_\eta(\tau) = \begin{cases} a^+_\eta + \nu^+ C^+_\eta |\tau|^{\alpha_\eta} + O_{\eta,1}(|\tau|^{\alpha_\eta}) & \text{if} \quad \tau > 0 \\ a^-_\eta - \nu^- C^-_\eta |\tau|^{\alpha_\eta} + O_{\eta,2}(|\tau|^{\alpha_\eta}) & \text{if} \quad \tau < 0, \end{cases}$$

where $O_{\eta,i}$ are C^1, varying C^1 with respect to η, and $\lim_{x\to 0} \frac{O_{\eta,i}(x)}{x} = 0$ *uniformly on η. Moreover, $C^\pm_\eta$ depends C^1 on η.*

This result was proved in [Rob2]. See Figure 2.1.

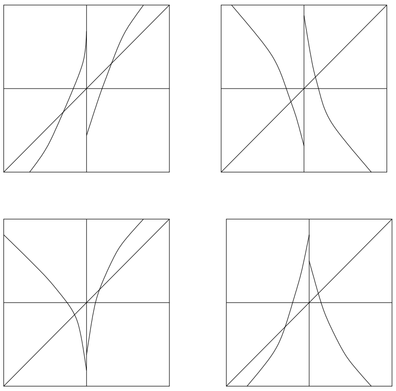

FIG. 2.1. *Possible f_η*

From now on we assume that X_η is a three-parameter family for which there is an open set $U \subset I\!R^3$ such that for $\eta \in U$, X_η satisfies conditions (A1)-(A7).

It follows from (A7) that the map $\eta \mapsto (\alpha_\eta, a^+_\eta, a^-_\eta)$ is a diffeomorphism from a neighborhood of η_0 onto a neighborhood of $(1,0,0)$. So, we can reparametrize η by $(\alpha, a^+, a^-) \mapsto \eta(\alpha, a^+, a^-)$ in such a way that $\alpha_{\eta(\alpha,a^+,a^-)} = \alpha$, $a^+_{\eta(\alpha,a^+,a^-)} = a^+$ and $a^-_{\eta(\alpha,a^+,a^-)} = a^-$.

LEMMA 2.2. *There are $\Lambda > 0$, an open dense set $O \subset (1-\Lambda, 1)$, and C^1 maps $a^+(\cdot), a^-(\cdot), p(\cdot), q(\cdot) : O \to I\!R$ such that $p(\alpha) < 0 < q(\alpha)$ and for $\eta = \eta(\alpha, a^+(\alpha), a^-(\alpha))$ the following hold:*

(a) if $\nu^+ = \nu^- = 1$ then $f_\eta(p(\alpha)) = p(\alpha)$, $f_\eta(q(\alpha)) = q(\alpha)$, $f_\eta(0^+) = p(\alpha)$, and $f_\eta(0^-) = q(\alpha)$,

(b) if $\nu^+ = \nu^- = -1$ then $f_\eta(p(\alpha)) = q(\alpha)$, $f_\eta(q(\alpha)) = p(\alpha)$, $f_\eta(0^+) = q(\alpha)$, and $f_\eta(0^-) = p(\alpha)$,

(c) if $\nu^+ = -\nu^- = 1$ then $f_\eta(p(\alpha)) = q(\alpha)$, $f_\eta(q(\alpha)) = q(\alpha)$, and $f_\eta(0^+) = f_\eta(0^-) = p(\alpha)$,

(d) if $\nu^+ = -\nu^- = -1$ then $f_\eta(p(\alpha)) = p(\alpha)$, $f_\eta(q(\alpha)) = p(\alpha)$, and $f_\eta(0^+) = f_\eta(0^-) = q(\alpha)$

In any case, $\lim_{\alpha\to 1^-} |p(\alpha)|/q(\alpha) = C^+_{\eta_0}/C^-_{\eta_0}$, $\lim_{\alpha \to 1^-} q(\alpha) = 0 = \lim_{\alpha\to 1} p(\alpha)$, *and* $\lim_{\alpha\to 1^-} q(\alpha)^{\alpha-1} = \lim_{\alpha \to 1^-} |p(\alpha)|^{\alpha-1} = B$, *where B is the number given in (A5).*

Proof. We prove (a). Define, for $\alpha < 1$, $p < 0$, and $q > 0$, the functions h, $\tilde{h}$, ℓ, and $\tilde{\ell}$ by

$$h(\alpha,p,q) = q^{\alpha}\left(\frac{1+\left(\frac{C_\eta^+ q^\alpha + O_{\eta,1}(|q|^\alpha)}{C_\eta^- |p|^\alpha - O_{\eta,2}(|p|^\alpha)}\frac{|p|^\alpha}{|q|^\alpha}\right)^{\frac{1}{\alpha}}}{C_\eta^+ q^\alpha + O_{\eta,1}(|q|^\alpha)}\right)$$

$$\tilde{h}(\alpha,p,q) = |p|^{\alpha}\left(\frac{1+\left(\frac{C_\eta^- |p|^\alpha - O_{\eta,2}(|p|^\alpha)}{C_\eta^+ |q|^\alpha + O_{\eta,1}(|q|^\alpha)}\frac{|q|^\alpha}{|p|^\alpha}\right)^{\frac{1}{\alpha}}}{C_\eta^- |p|^\alpha - O_{\eta,2}(|p|^\alpha)}\right),$$

$$\ell(\alpha,p,q) = q^{\alpha-1}, \quad \text{and} \quad \tilde{\ell}(\alpha,p,q) = |p|^{\alpha-1}$$

where $\eta = \eta(\alpha,p,q)$.

Observe that h and $\tilde{h}$ are differentiable except for $D = \{(1,p,0),(1,0,q),\forall p,q\}$, and they extend continuously to D. Indeed,

$$\lim_{(\alpha,p,q)\to(1,0,0)} h(\alpha,p,q) = B = \lim_{(\alpha,p,q)\to(1,0,0)} \tilde{h}(\alpha,p,q),$$

and so we define $h(1,0,0) = B = \tilde{h}(1,0,0)$. Furthermore,

$$(2.1)\quad h(\alpha,p,q)\left(\frac{C_\eta^+ q^\alpha + O_{\eta,1}(|q|^\alpha)}{q^\alpha}\right)^{\frac{\alpha-1}{\alpha}} = \tilde{h}(\alpha,p,q)\left(\frac{C_\eta^- |p|^\alpha - O_{\eta,2}(|p|^\alpha)}{|p|^\alpha}\right)^{\frac{\alpha-1}{\alpha}}.$$

Let us choose $\Lambda > 0$. As $B > 1$, by (A5) there is $\epsilon > 0$ such that $1 \notin [B-\epsilon, B+\epsilon]$. Fix δ_0 and Λ such that if $q, |p| \in (0,\delta_0)$ and $\alpha \in (1-\Lambda, 1)$ then

$$(2.2)\qquad B-\epsilon < h(\alpha,p,q), \tilde{h}(\alpha,p,q) < B+\epsilon\,.$$

Shrinking Λ if necessary, we can further assume $\delta_0^{\alpha-1} < B-\epsilon$.

For the proof of Lemma 2.2-(a) we shall use the following result which states that for all $\alpha \in (1-\Lambda, 1)$, the projection on the plane pq of $graph(h(\alpha,\cdot,\cdot)) \cap graph(\ell(\alpha,\cdot,\cdot))$ contains a regular curve C_α joining the vertical sides of $[-\delta_0, 0]\times[0,\delta_0]$.

Let $\mathcal{C}^1([0,1])$ be the set of C^1 maps $\gamma : [0,1] \to [-\delta_0,0]\times[0,\delta_0]$ endowed with the C^1 topology.

Lemma 2.3. *There are an open dense set $O \subset (1-\Lambda, 1)$ and a C^1 map*

$$\Gamma : O\times[0,1] \mapsto \mathcal{C}^1([0,1]), \quad (\alpha,t)\mapsto C_\alpha(t),$$

where

$$C_\alpha(t) = (p_\alpha(t), q_\alpha(t))$$

is such that $p_\alpha(0) = 0$, $p_\alpha(1) = -\delta_0$, $h(\alpha, p_\alpha(t), q_\alpha(t)) = [q_\alpha(t)]^{\alpha-1}$, and $0 < q_\alpha(t) < \delta_0$ $\forall\, t \in (0,1)$.

Proof. We first prove that for all α small enough, $graph(h(\alpha,\cdot,\cdot)) \cap graph(\ell(\alpha,\cdot,\cdot))$ is transversal. For this we proceed as follows.

By the definition of $h(\alpha,,p,q)$ above we have

$$h(\alpha,p,q)=\frac{1+\left(\frac{C_\eta^+ + O_{\eta,1}(|q|^\alpha)/q^\alpha}{C_\eta^- - O_{\eta,2}(|p|^\alpha)}/\mid p\mid^\alpha\right)^{\frac{1}{\alpha}}}{C_\eta^+ q^\alpha + O_{\eta,1}(|q|^\alpha)}.$$

Recall that $\eta=\eta(\alpha,p,q)$.

Assume that $O_{\eta,i}=0$ for $i=1,2$. Derivating the expression of h above with respect to q we obtain

$$\partial_q h(\alpha,p,q)=\frac{(1/\alpha)(C_\eta^+/C_\eta^-)^{1/\alpha-1}\cdot(\partial(C_\eta^+)\cdot C_\eta^- - C_\eta^+\partial_q(C_\eta^-))\cdot C_\eta^-)^{-2}\cdot C_\eta^+}{(C_\eta^+)^2}.$$

From this we have that $\lim_{(\alpha,p,q)\to(1,0,0)}\partial_q h(\alpha,p,q)$ exists, and so $\partial_q h(\alpha,p,q)$ has a lower bound $K>-\infty$. By the definition of $\ell(\alpha,p,q)$ one has

$$\partial_q\ell(\alpha,p,q)=(\alpha-1)q^{(\alpha-2)}.$$

If (α,p,q) satisfies $h(\alpha,p,q)=l(\alpha,p,q)$ then the definition of $\ell(\alpha,\cdot,\cdot)$ with (2.2) imply that

$$q^{\alpha-1}>B-\epsilon.$$

So, for such points (α,p,q) one has

$$\partial_q h(\alpha,p,q)-\partial_q l(\alpha,p,q)>K-(\alpha-1)q^{\alpha-2}>K+\frac{ln(B-\epsilon)\cdot(B-\epsilon)}{\mid ln(q)\mid\cdot q}. \tag{2.3}$$

Because $B-\epsilon>1$ it follows that the right-hand term of (2.3) goes to ∞ as $q\to 0^+$. In particular, the left-hand term of (2.3) is non-zero and so $graph(h(\alpha,\cdot,\cdot))\cap graph(\ell(\alpha,\cdot,\cdot))$ is transversal.

The general case follows from similar computations using that for (α,p,q) such that $h(\alpha,p,q)=l(\alpha,p,q)$ we have $B-\epsilon<q^{(\alpha-1)}<B+\epsilon$ and

$$\lim_{(\alpha,p,q)\to(1,0,0)}\frac{O_{\eta,1}(q^\alpha)}{q^\alpha}=\lim_{(\alpha,p,q)\to(1,0,0)}\partial_q\left(\frac{O_{\eta,1}(q^\alpha)}{q^\alpha}\right)=0.$$

Returning to the proof of Lemma 2.3, since $graph(h(\alpha,\cdot,\cdot))\cap graph(\ell(\alpha,\cdot,\cdot))$ is transversal, we have that this intersection is a compact 1-manifold. Although $graph(h(\alpha,\cdot,\cdot))\cap graph(\ell(\alpha,\cdot,\cdot))$ may be non-connected, the number of connected components of this intersection is finite. On the other hand, it follows from the Intermediate Value Theorem that $graph(\ell(\alpha,\cdot,\cdot))$ intersects $h(\alpha,p(q),q)$ for any curve $q\to p(q)$ with $p(-\delta)=\delta, p(0)=0$. Then, for all $\alpha\in(1-\Lambda,1)$, the projection onto the plane pq of one of the connected components of $graph(h(\alpha,\cdot,\cdot))\cap graph(\ell(\alpha,\cdot,\cdot))$ is a regular curve C_α joining the vertical sides of $[-\delta_0,0]\times[0,\delta_0]$. That is,

$$C_\alpha:[0,1]\to[-\delta_0,0]\times[0,\delta_0],\quad t\mapsto(p_\alpha(t),q_\alpha(t))$$

such that $p_\alpha(0)=0$, $p_\alpha(1)=-\delta_0$, $h(\alpha,p_\alpha(t),q_\alpha(t))=[q_\alpha(t)]^{\alpha-1}$, and $0<q_\alpha(t)<\delta_0\ \forall\, t\in(0,1)$.

Now observe that by usual transversality arguments, for all α' there is $I_{\alpha'}$ such that for all $\alpha\in I_{\alpha'}$ the curve C_α obtained above can be chosen in such way that the

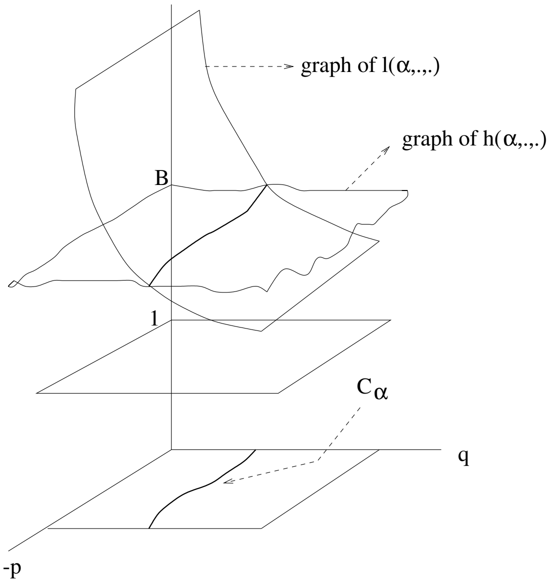

FIG. 2.2. *graph($h(\alpha,.,.)$) $\cap$ graph($l(\alpha,.,.)$)*

map $(\alpha,t) \in I_{\alpha'} \times [0,1] \mapsto C_\alpha(t) \in [-\delta_0,0] \times [0,\delta_0]$ is C^1. Hence we can choose an open dense set O in $(1-\Lambda,1)$ such that the map $(\alpha,t) \in O \times [0,1] \mapsto C_\alpha(t)$ is C^1. All of this together concludes the proof of Lemma 2.3. $\square$

Returning to the proof of Lemma 2.2, observe that we have the following inequalities:

$$\lim_{t\to 0} |p_\alpha(t)|^{\alpha-1} > B+\epsilon \geq \tilde{h}(\alpha,p_\alpha(t),q_\alpha(t)) \geq B-\epsilon > \delta_0^{\alpha-1} = |p_\alpha(1)|^{\alpha-1}.$$

These inequalities and the Intermediate Value Theorem imply that there is $t = t(\alpha)$ such that

$$\tilde{h}(\alpha,p_\alpha(t),q_\alpha(t)) = |p_\alpha(t)|^{\alpha-1}.$$

By Lemma 2.3, $C_\alpha(t) = (p_\alpha(t),q_\alpha(t))$ depends C^1 on (α,t) and so, by the Implicit Function Theorem we get that the map $\alpha \mapsto t(\alpha)$ is C^1.

From now on O is the open dense set given by Lemma 2.3.

Let $a^+(\cdot)$, $a^-(\cdot)$, $p(\cdot)$, $q(\cdot) : O \to \mathbb{R}$ be given by $p(\alpha) = p_\alpha(t_0(\alpha))$, $q(\alpha) = q_\alpha(t_0(\alpha))$, $a^+(\alpha) = p(\alpha)$, and $a^-(\alpha) = q(\alpha)$. These maps are C^1 and satisfy

$$h(\alpha,p(\alpha),q(\alpha)) = (q(\alpha))^{\alpha-1}, \quad \tilde{h}(\alpha,p(\alpha),q(\alpha)) = |p(\alpha)|^{\alpha-1}. \tag{2.4}$$

From this it follows that $(B-\epsilon)^{\frac{1}{\alpha-1}} < q(\alpha) < (B+\epsilon)^{\frac{1}{\alpha-1}}$, and hence we obtain $\lim_{\alpha\to 1^-} q(\alpha) = 0$. Similarly for $p(\alpha)$. Applying these facts in (2.4) we get $\lim_{\alpha\to 1^-} q(\alpha)^{\alpha-1} = B$ and $\lim_{\alpha\to 1^-} |p(\alpha)|^{\alpha-1} = B$.

We claim that $p = p(\alpha)$, $q = q(\alpha)$, $a^+ = a^+(\alpha)$, and $a^- = a^-(\alpha)$ chosen as above satisfy (a) in Lemma 2.2.

Indeed, since $h(\alpha,p,q) = q^{\alpha-1}$ and $\tilde{h}(\alpha,p,q) = |p|^{\alpha-1}$, (2.1) implies that

$$q^{\alpha-1}\left(\frac{C_\eta^+ q^\alpha + O_{\eta,1}(q^\alpha)}{q^\alpha}\right)^{\frac{\alpha-1}{\alpha}} = |p|^{\alpha-1}\left(\frac{C_\eta^- |p|^\alpha - O_{\eta,2}(|p|^\alpha)}{|p|^\alpha}\right)^{\frac{\alpha-1}{\alpha}}.$$

Hence

$$C_\eta^+ q^\alpha + O_{\eta,1}(q^\alpha) = C_\eta^- |p|^\alpha - O_{\eta,2}(|p|^\alpha)\,. \tag{2.5}$$

On the other hand, from the definition of h we get

$$q^{\alpha-1} = q^\alpha \left(\frac{1 + \left(\frac{C_\eta^+ q^\alpha + O_{\eta,1}(q^\alpha)}{C_\eta^- |p|^\alpha - O_{\eta,2}(|p|^\alpha)} \frac{|p|^\alpha}{q^\alpha} \right)^{\frac{1}{\alpha}}}{C_\eta^+ q^\alpha + O_{\eta,1}(q^\alpha)} \right). \tag{2.6}$$

From (2.5) and (2.6) we get $C_\eta^+ q^\alpha + O_{\eta,1}(q^\alpha) = q - p$, implying that $f_\eta(q) = q$ and $f_\eta(p) = p$.

Note that (2.5) implies

$$q^\alpha (C_\eta^+ + \frac{O_{\eta,1}(q^\alpha)}{q^\alpha}) = |p|^\alpha (C_\eta^- - \frac{O_{\eta,2}(|p|^\alpha)}{|p|^\alpha})$$

and so $\lim_{\alpha\to 1^-} \frac{|p(\alpha)|}{q(\alpha)} = \frac{C_{\eta_0}^+}{C_{\eta_0}^-}$.

All of this together finishes the proof of Lemma 2.2-(a). The remaining cases are similar and are left to the reader. □

Note 1: Let O be as in Lemma 2.2 and for $\alpha \in O$, let $a^+(\alpha)$ and $a^-(\alpha)$ be the functions defined above. We shall call $\eta = \eta(\alpha, a^+(\alpha), a^-(\alpha))$ for $\alpha \in O$ the *good parameters* of X_η.

3. Rescaling. In this section we perform rescaling techniques [PT]. Keeping the notation p, q in Lemma 2.2 we take suitable parameter-depending change of coordinates in a neighborhood of $[p, q]$ and, at the same time, we normalize the parameter space in a small neighborhood of the good parameters in Notation 2. This yields a new family $g_\alpha(\mu, \nu, \cdot)$ and new good parameters $(\mu(\alpha), \nu(\alpha))$. The goal of this section is to prove that $g_\alpha(\mu, \nu, \cdot)$ converges to a map $g(\mu, \nu, \cdot)$ in the sense to be described below.

To start we consider the parametrized family $\{X_\eta\}$ as in Section 2. Recall the notation in such a section.

Given $\alpha \in O$ let $a^-(\alpha)$, $a^+(\alpha)$, $p(\alpha)$ and $q(\alpha)$ be as in Lemma 2.2. Define $(\mu(\alpha), \nu(\alpha)) = (\frac{a^+(\alpha)}{q(\alpha)}, \frac{a^-(\alpha)}{q(\alpha)})$ and $(\mu, \nu) = (\frac{a^+}{q(\alpha)}, \frac{a^-}{q(\alpha)})$ in a neighborhood of $(a^+(\alpha), a^-(\alpha))$ onto a neighborhood of $(\mu(\alpha), \nu(\alpha))$, and the family of maps

$$g_\alpha(\mu, \nu, x) = \frac{1}{q(\alpha)} f_\eta(q(\alpha)x), \tag{3.1}$$

where $\eta = \eta(\alpha, q(\alpha)\mu, q(\alpha)\nu)$. We set $\mathrm{Dom}(g_\alpha)$ for the domain of g_α.

Note 2: Observe that for each fixed α, this change of variables renormalizes the parameters $a^\pm$ with $a^+ = q(\alpha)\mu$ and $a^- = q(\alpha)\nu$. Moreover, by Lemma 2.2, $\lim_{\alpha\to 1^-} \mu(\alpha)$ and $\lim_{\alpha\to 1^-} \nu(\alpha)$ exist and we denote them by $\mu(1)$ and $\nu(1)$ respectively.

Definition 3.1. *Let $g : \mathbb{R}^2 \times (\mathbb{R} \setminus \{0\}) \to \mathbb{R}$. We say that $g_\alpha \to g$ in the C^0 topology in compact sets of $\mathbb{R}^3$ as $\alpha \to 1^-$ if*

(a) $Dom(g_\alpha) \to \mathbb{R}^2 \times (\mathbb{R} \setminus \{0\})$ as $\alpha \to 1^-$, that is, for all $R > 0$ there is $0 < \alpha_0 < 1$ such that if $\alpha_0 < \alpha < 1$ then $B_R(0) \cap (\mathbb{R}^2 \times (\mathbb{R} \setminus \{0\}) \subset Dom(g_\alpha)$, where $B_R(0)$ is the ball of radius R centered at $(0,0,0)$.

(b) for every compact set $K \subset I\!R^3$ and every $\epsilon > 0$ there is $\delta > 0$ such that if $|\alpha - 1| < \delta$ then

$$\sup_{y \in K \cap (I\!R^2 \times (I\!R \setminus \{0\}))} |g_\alpha(y) - g(y)| < \epsilon.$$

DEFINITION 3.2. *Let $g : I\!R^2 \times (I\!R \setminus \{0\}) \to I\!R$. We say that $g_\alpha \to g$ in the C^1 topology in compact sets of $I\!R^2 \times (I\!R \setminus \{0\})$ if*

(a) $Dom(g_\alpha) \to I\!R^2 \times (I\!R \setminus \{0\})$ as $\alpha \to 1^-$,

(b) for every compact set $K \subset I\!R^2 \times (I\!R \setminus \{0\})$ and every $\epsilon > 0$ there is $\delta > 0$ such that if $|\alpha - 1| < \delta$ then

$$\sup_{i \in \{0,1\}, y \in K} |D^i g_\alpha(y) - D^i g(y)| < \epsilon.$$

Note that with these notions, C^1 convergence does not imply C^0 convergence.

We have the following result.

LEMMA 3.3. *Let g_α be as in (3.1) and define*

$$g(\mu, \nu, x) = \begin{cases} \mu + \nu^+ C^+_{\eta_0} Bx & \text{if} \quad x > 0 \\ \nu + \nu^- C^-_{\eta_0} Bx & \text{if} \quad x < 0 \end{cases}$$

where $\eta_0 = \eta(1, 0, 0)$. Then

(i) $g_\alpha \to g$ in the C^0 topology in compact sets of $I\!R^3$ as $\alpha \to 1^-$, $\alpha \in O$,

(ii) $g_\alpha \to g$ in the C^1 topology in compact sets of $I\!R^2 \times (I\!R \setminus \{0\})$ as $\alpha \to 1^-$, $\alpha \in O$. Moreover, for any $c > \max\{1, C^+_{\eta_0}/C^-_{\eta_0}\}$, there are constants $\Delta_0 > 0$, $0 < K_1 < K_2$ such that for $\alpha \in O \cap [1 - \Delta_0, 1]$ we have

(a) $[-c, c]^2 \times ([-c, c] \setminus \{0\}) \subset Dom(g_\alpha)$,

(b) $K_1 |x|^{\alpha - 1} \le \left|\frac{\partial}{\partial x} g_\alpha(\mu, \nu; x)\right| \le K_2 |x|^{\alpha - 1}$, $\quad \forall (\mu, \nu, x) \in [-c, c]^2 \times ([-c, c] \setminus \{0\})$.

Proof. First, let us prove that $\mathrm{Dom}(g_\alpha) \to I\!R^2 \times (I\!R \setminus \{0\})$. For this, fix $R > 0$. We shall prove that

$$B_R(0) \cap (I\!R^2 \times (I\!R \setminus \{0\})) \subset \mathrm{Dom}(g_\alpha)$$

for all α close to 1.

For simplicity set $k = q(\alpha)$, where $q(\alpha)$ is given by the Lemma 3.2. Recall $k \to 0$ as $\alpha \to 1^-$.

Let $0 < \epsilon_0$, ϵ_1 so that $(1 - \epsilon_1, 1 + \epsilon_1) \times (-\epsilon_0, \epsilon_0) \times (-\epsilon_0, \epsilon_0) \subset \mathrm{Dom}(\eta)$.

Given $(\mu, \nu, x) \in B_R(0)$ and α with $|\alpha - 1| < \epsilon_1$, we have $|k\mu|, |k\nu| \le kR < \epsilon_0$ for ϵ_1 small enough. Thus, $(\alpha, k\mu, k\nu) \in \mathrm{Dom}(\eta)$. In a similar way we have that $kx \in \mathrm{Dom} f_{\eta(\alpha, k\mu, k\nu)}$. This proves (a) at Definitions 3.1 and 3.2.

To verify (b) we proceed as follows. First observe that, by definition, see Lemma 2.1, we have

$$g_\alpha(\mu, \nu, x) = k^{-1} f_\eta(kx) = \begin{cases} k^{-1}[k\mu + \nu^+ C^+_\eta |kx|^\alpha + O_{\eta,1}(|k_\alpha x|^\alpha)] & x > 0 \\ k^{-1}[k\nu - \nu^- C^-_\eta |kx|^\alpha + O_{\eta,2}(|kx|^\alpha)] & x < 0 \end{cases}$$

$$(3.2) \qquad = \begin{cases} \mu + \nu^+ C^+_\eta k^{\alpha-1} |x|^\alpha + \frac{O_{\eta,1}(|kx|^\alpha)}{k} & x > 0 \\ \nu - \nu^- C^-_\eta k^{\alpha-1} |x|^\alpha + \frac{O_{\eta,2}(|kx|^\alpha)}{k} & x < 0. \end{cases}$$

As $k^{\alpha-1} \to B$, $k \to 0$, and $\eta \to \eta_0$ when $\alpha \to 1$, we obtain $\nu^\pm C^\pm_\eta k^{\alpha-1} |x|^\alpha \to \nu^\pm C^\pm_{\eta_0} B|x|$ as $\alpha \to 1^-$. On the other hand, by Lemma 2.1 we have $\lim_{x \to 0} \frac{O_{\eta,i}(x)}{|x|} = 0$

uniformly on η. So, $\frac{O_{\eta,i}(|kx|^\alpha)}{k} = \frac{\mathcal{O}_{\eta,i}(|kx|^\alpha)}{|kx|^\alpha} k^{\alpha-1}|x|^\alpha \to 0$ as $\alpha \to 1^-$ for $|x| < R$. All of this together imply (b) at Definition 3.1. So we finish the proof of (i).

To prove (ii) we proceed as follows. We have, for $x \neq 0$,

$$(3.3) \quad \partial_x g_\alpha(\mu,\nu,x) = \begin{cases} \nu^+ C_\eta^+ \alpha k^{\alpha-1}|x|^{\alpha-1} + O'_{\eta,1}(|kx|^\alpha)\alpha|kx|^{\alpha-1} & x > 0 \\ \nu^- C_\eta^- \alpha k^{\alpha-1}|x|^{\alpha-1} + O'_{\eta,2}(|kx|^\alpha)\alpha|kx|^{\alpha-1} & x < 0 \end{cases}.$$

Now fix a compact set $K \subset I\!R^2 \times (I\!R \setminus \{0\})$. Then there are $0 < C_1 = C_1(K) < C_2 = C_2(K)$ such that for all $(\mu,\nu,x) \in K$ we have $C_1 < |x| < C_2$, implying that $|x|^{\alpha-1} \to 1$ as $\alpha \to 1^-$. On the other hand, by Lemma 2.1, $O'_{\eta,i}(0) = 0$, for $i = 1,2$. Since $k \to 0$ as $\alpha \to 1^-$, $C_1 < |x| < C_2$, we get $kx \to 0$ as $\alpha \to 1$. So, $O'_{\eta,i}(|kx|^\alpha)\alpha|x|^{\alpha-1} \to 0$ as $\alpha \to 1$. Thus $\partial_x g_\alpha(\mu,\nu,x) \to \partial_x g(\mu,\nu,x)$ as $\alpha \to 1^-$ in compact sets of $I\!R^2 \times (I\!R \setminus \{0\})$.

On the other hand, since $k = q(\alpha)$, we have $a^+ = k\mu$ and $a^- = k\nu$, see Note 2, and so

$$(3.4) \qquad \partial_\mu g_\alpha(\mu,\nu,x) = \begin{cases} 1 + \nu^+ \partial_{a^+} C_\eta^+ k^\alpha |x|^\alpha + \partial_{a^+}\mathcal{O}_{\eta,1}(|kx|^\alpha) & x > 0 \\ -\nu^- \partial_{a^+} C_\eta^- k^\alpha |x|^\alpha + \partial_{a^+}\mathcal{O}_{\eta,2}(|kx|^\alpha) & x < 0 \end{cases}.$$

$$(3.5) \qquad \partial_\nu g_\alpha(\mu,\nu,x) = \begin{cases} \nu^+ \partial_{a^-} C_\eta^+ k^\alpha |x|^\alpha + \partial_{a^-}\mathcal{O}_{\eta,1}(|kx|^\alpha) & x > 0 \\ 1 - \nu^- \partial_{a^-} C_\eta^- k^\alpha |x|^\alpha + \partial_{a^-}\mathcal{O}_{\eta,2}(|kx|^\alpha) & x < 0 \end{cases}.$$

As $C_\eta^\pm$ is C^1, we obtain that $|\partial_{a^+} C_\eta^\pm|$ uniformly bounded on K. Since $k^{\alpha-1} \to B$ and $k \to 0$ as $\alpha \to 1^-$ we get $k^\alpha |x|^\alpha = k^{\alpha-1} k|x| \to 0$ as $\alpha \to 1^-$ on K. Moreover, $\mathcal{O}_{\eta,i}(0) = 0$ and so $\partial_{a^+}\mathcal{O}_{\eta,i}(|kx|^\alpha) \to 0$ as $\alpha \to 1^-$. Replacing these bounds on (3.4) and (3.5) respectively we obtain that $\partial_\mu g_\alpha(\mu,\nu,x) \to \partial_\mu g(\mu,\nu,x)$ and $\partial_\nu g_\alpha(\mu,\nu,x) \to \partial_\nu g(\mu,\nu,x)$ for $(\mu,\nu,x) \in K$. This finishes the proof. $\square$

Note 3: Observe that $|g'| = |C_{\eta_0}^\pm B|$. Since $|\nu^\pm C_{\eta_0}^\pm B| > 1$ we obtain that g is an expanding map.

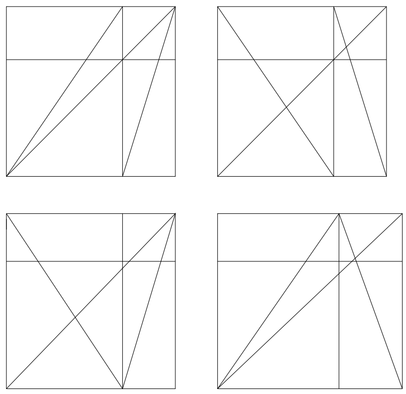

FIG. 3.1. *Possible* $g(\mu(1),\nu(1),.)$

4. Trapping region. The main goal of this section is to prove Theorem 4.1, which asserts the existence of trapping regions for $g_\alpha(\mu,\nu,\cdot)$, for $\alpha \in O$ close to 1 and (μ,ν) close to $(\mu(\alpha),\nu(\alpha))$ (recall the notation in Section 3). A trapping region for $g_\alpha(\mu,\nu,.)$ is as closed interval J such that $g_\alpha(\mu,\nu,J) \subset Int(J)$, where $Int(J)$ stands for the interior of J. To prove the existence of such a trapping region we define an auxiliary function $F_\alpha : I\!R^2 \to I\!R^2$, which will be used to find them. This function involves the critical and the fixed points of $g_\alpha(\mu, \nu, \cdot)$. We shall prove that F_α is locally one-to-one, i.e. $\det(DF_\alpha(\mu(\alpha), \nu(\alpha))) \neq 0$. Once we have that, the parameters corresponding to maps with trapping regions will be the ones in the inverse image $F_\alpha^{-1}(W_\alpha)$ for certain open set W_α to be defined below, proving Theorem 4.1.

THEOREM 4.1. *There is an open set $\mathcal{O}^- \subset I\!R^3$ such that the properties below hold:*

1. $(\alpha, \mu(\alpha), \nu(\alpha)) \in Cl(\mathcal{O}^-)$ *for all* $\alpha \in O$;
2. *If* $\theta = (\alpha, \mu, \nu) \in \mathcal{O}^-$ *then there is a closed interval* $I_\theta \subset I\!R$ *with* $0 \in Int(I_\theta)$ *such that* $g_\alpha(\mu, \nu, x) \subset Int(I_\theta)$ *for all* $x \in I_\theta$.

To prove the theorem we need some terminology. Let O be the set given in the last section. For $\alpha \in O$ let $\mu(\alpha) = \frac{a^+(\alpha)}{q(\alpha)}$, $\nu(\alpha) = \frac{a^-(\alpha)}{q(\alpha)}$, $x(\alpha) = \frac{p(\alpha)}{q(\alpha)}$ and $y(\alpha) = \frac{q(\alpha)}{q(\alpha)} = 1$. Fix $c > \max\{1, C^+_{\eta_0}/C^-_{\eta_0}\}$. Recall that Lemma 2.2 gives us $\lim_{\alpha\to 1} \frac{|p(\alpha|}{q(\alpha)} = \frac{C^+_{\eta_0}}{C^-_{\eta_0}}$. Again, by Lemma 2.2, we have that if $\nu^+ = \nu^- = 1$ then $a^+(\alpha) = p(\alpha)$, $a^-(\alpha) = q(\alpha)$ and so $\mu(\alpha) = \frac{p(\alpha)}{q(\alpha)}$, $\nu(\alpha) = 1$. Then $\lim_{\alpha\to 1} \mu(\alpha) = \lim_{\alpha\to 1} x(\alpha) = \frac{-C^+_{\eta_0}}{C^-_{\eta_0}}$. This implies that $(\mu(\alpha), \nu(\alpha), x(\alpha))$ and $(\mu(\alpha), \nu(\alpha), 1)$ belong to $[-c, c]^2 \times ([-c, c] \setminus \{0\})$ and we conclude, by Lemma 3.3-(a) that $(\mu(\alpha), \nu(\alpha), x(\alpha))$ and $(\mu(\alpha), \nu(\alpha), 1) \in \mathrm{Dom}(g_\alpha)$ for all α close to 1.

In a similar way, for the remaining cases $\nu^+ = \nu^- = -1$, $\nu^+ = -\nu^- = 1$, and $\nu^+ = -\nu^- = -1$, we also obtain $(\mu(\alpha), \nu(\alpha), x(\alpha))$ and $(\mu(\alpha), \nu(\alpha), 1) \in \mathrm{Dom}(g_\alpha)$ for all α close to 1.

From now on we assume that $\nu^+ = \nu^- = 1$. In this case

$$g_\alpha(\mu(\alpha), \nu(\alpha), x(\alpha)) = x(\alpha), \quad g_\alpha(\mu(\alpha), \nu(\alpha), 1) = 1,$$

$$\partial_x g_\alpha(\mu(\alpha), \nu(\alpha), x(\alpha)) > 1, \quad \partial_x g_\alpha(\mu(\alpha), \nu(\alpha), 1) > 1.$$

Next, set $x(\alpha, \mu, \nu)$ and $y(\alpha, \mu, \nu)$ for the C^1 continuations of $x(\alpha)$ and $y(\alpha)$ for (μ, ν) close to $(\mu(\alpha), \nu(\alpha))$ in the (μ, ν) parameter space, i. e., $g_\alpha(x(\alpha, \mu, \nu)) = x(\alpha, \mu, \nu)$ and $g_\alpha(y(\alpha, \mu, \nu)) = y(\alpha, \mu, \nu)$. Observe that $x(\alpha, \mu(\alpha), \nu(\alpha)) = x(\alpha)$, and $y(\alpha, \mu(\alpha), \nu(\alpha)) = y(\alpha) = 1$ for all $\alpha \in O$. As before, set $k = q(\alpha)$.

Define

$$F_\alpha(\mu, \nu) = (\mu - x(\alpha, \mu, \nu), \nu - y(\alpha, \mu, \nu)).$$

We shall prove the following theorem:

THEOREM 4.2. $\det DF_\alpha(u(\alpha), \nu(\alpha)) \neq 0$ *for all* $\alpha \in O$ *close to* 1.

Assuming Theorem 4.2, let us show how to obtain Theorem 4.1.

Observe that Theorem 4.2 implies that F_α is a local diffeomorphism in a neighborhood V_α of $(\mu(\alpha), \nu(\alpha))$ onto a neighborhood of $F_\alpha(\mu(\alpha), \nu(\alpha)) = (0, 0)$. Let $W_\alpha = F_\alpha(V_\alpha) \cap A$ where $A = \{(x, y), x > 0, y < 0\}$.

Consider $\theta = (\alpha, \mu, \nu)$ such that $\alpha \in O$ and $(\mu, \nu) \in F_\alpha^{-1}(W_\alpha)$.

We claim that there is a closed interval I_θ satisfying (2) in Theorem 4.1. Indeed, we have the following inequalities:

$$x(\alpha, \mu, \nu) < \mu, \quad \text{and} \quad \nu < y(\alpha, \mu, \nu), \text{ since } (\mu, \nu) \in F_\alpha^{-1}(W_\alpha). \tag{4.1}$$

Moreover, by (3.2),

$$\nu = g_\alpha(\mu,\nu,0^-) \quad \text{and} \quad \mu = g_\alpha(\mu,\nu,0^+).$$

Furthermore, since $\nu^+ = \nu^- = 1$, we have that

$$g_\alpha(\mu,\nu,.) \text{ is monotonic in } [x(\alpha,\mu,\nu),0) \text{ and in } \ (0,y(\alpha,\mu,\nu)], \tag{4.2}$$

and

$$\partial_x g_\alpha(\mu,\nu,\sigma) > 1, \sigma \in \{x(\alpha,\mu,\nu), y(\alpha,\mu,\nu)\}. \tag{4.3}$$

Now, (4.1) and (4.3) imply that there is $\epsilon > 0$ (depending on μ and ν), small such that

$$\nu < y(\alpha,\mu,\nu) - \epsilon \quad \text{and} \quad g_\alpha(\mu,\nu,x(\alpha,\mu,\nu)+\epsilon) > x(\alpha,\mu,\nu)+\epsilon. \tag{4.4}$$

$$x(\alpha,\mu,\nu)+\epsilon < \mu \text{ and } g_\alpha(\mu,\nu,y(\alpha,\mu,\nu)-\epsilon) < y(\alpha,\mu,\nu)-\epsilon. \tag{4.5}$$

Define $I_\theta^-(\epsilon) = [x(\alpha,\mu,\nu)+\epsilon,0)$ and $I_\theta^+(\epsilon) = (0,y(\alpha,\mu,\nu)-\epsilon]$ and $I_\theta(\epsilon) = I_\theta^- \cup I_\theta^+$.

On one hand, (4.4) together with (4.2) imply that $g_\alpha(\mu,\nu,I_\theta^-(\epsilon)) \subset \mathrm{Int}(I_\theta(\epsilon))$. On the other hand, (4.5) together with (4.2) imply $g_\alpha(\mu,\nu,I_\theta^+(\epsilon)) \subset \mathrm{Int}(I_\theta(\epsilon))$. Thus, $g_\alpha(\mu,\nu,I_\theta(\epsilon)) \subset \mathrm{Int}(I_\theta(\epsilon))$, proving the claim with $I_\theta = I_\theta(\epsilon)$.

The proof for the remaining cases is similar.

As the existence of a trapping region is an open condition, we conclude that there is $\delta_\theta > 0$ such that if $\|\zeta - \theta\| < \delta_\theta$ then $I_\zeta = I_\theta$ is such that $g_\alpha(I_\zeta) \subset \mathrm{Int}(I_\zeta)$.

Now define

$$\mathcal{O}_0^- = \{\theta = (\alpha,\mu,\nu); \alpha \in O,\ (\mu,\nu) \in F_\alpha^{-1}(W_\alpha)\},$$

and

$$\mathcal{O}^- = \cup_{\theta \in \mathcal{O}_0^-} B(\theta,\delta_\theta),$$

where $B(\theta,\delta_\theta)$ is the ball centered at θ with radius δ_θ. Clearly, $\mathcal{O}^-$ satisfies the required properties. This finishes the proof of Theorem 4.1. □

To prove Theorem 4.2 we shall use the following lemmas.

Lemma 4.3.

(a) $\lim_{\alpha\to 1^-} \frac{q(\alpha)}{\alpha-1} = \lim_{\alpha\to 1^-} \frac{p(\alpha)}{\alpha-1} = 0$

(b) $\lim_{\alpha\to 1^-} \frac{q(\alpha)^\alpha}{\alpha-1} = \lim_{\alpha\to 1^-} \frac{|p(\alpha)|^\alpha}{\alpha-1} = 0$

Proof. **(a)** As part of the proof of the existence of $q(\alpha)$ and $p(\alpha)$, we obtain that

$$B - \varepsilon < q(\alpha)^{\alpha-1} < B + \varepsilon,$$

and from this it follows

$$\frac{(B-\varepsilon)^{\frac{1}{\alpha-1}}}{|\alpha-1|} < \frac{q(\alpha)}{|\alpha-1|} < \frac{(B+\varepsilon)^{\frac{1}{\alpha-1}}}{|\alpha-1|}.$$

Now recall that $q(\alpha)$ and $p(\alpha)$ are defined for $\alpha < 1$. Moreover, since $B-\epsilon > 1$, we get

$$\lim_{\alpha\to 1^-} \frac{(B-\varepsilon)^{\frac{1}{\alpha-1}}}{|\alpha-1|} = \lim_{\alpha\to 1^-} \frac{(B+\varepsilon)^{\frac{1}{\alpha-1}}}{|\alpha-1|} = 0\,.$$

So $\lim_{\alpha\to 1^-} \frac{q(\alpha)}{|\alpha-1|} = 0$.

The same argument implies $\lim_{\alpha\to 1^-} \frac{p(\alpha)}{|\alpha-1|} = 0$.

(b) $\lim_{\alpha\to 1^-} \frac{q(\alpha)^\alpha}{|\alpha-1|} = \lim_{\alpha\to 1^-} \frac{q(\alpha)}{|\alpha-1|} \cdot q(\alpha)^{\alpha-1} = 0 \cdot B = 0.$ □

LEMMA 4.4. *Set* $\eta = \eta(\alpha, p(\alpha), q(\alpha))$. *Then*

$$\lim_{\alpha\to 1^-} \frac{1}{\alpha-1}\left[C_\eta^+ C_\eta^- \cdot |p(\alpha)|^{\alpha-1} \cdot q(\alpha)^{\alpha-1} - C_\eta^- \; |p(\alpha)|^{\alpha-1} - C_\eta^+ \; q(\alpha)^{\alpha-1}\right] = 0.$$

Proof.

Equation (2.6) in Lemma 2.2 implies

$$q(\alpha)^{\alpha-1} = \frac{1 + \sqrt[\alpha]{\dfrac{C_\eta^+ + \frac{O_{n,1}(q(\alpha)^\alpha)}{q(\alpha)^\alpha}}{C_\eta^- - \frac{O_{n,2}(|p(\alpha)|^\alpha)}{|p(\alpha)|^\alpha}}}}{C_\eta^+ + \frac{O_{n,1}(q(\alpha)^\alpha)}{q(\alpha)^\alpha}}.$$

In a similar way we obtain

$$|p(\alpha)|^{\alpha-1} = \frac{1 + \sqrt[\alpha]{\dfrac{C_\eta^- - \frac{O_{n,2}(|p(\alpha)|^\alpha)}{|p(\alpha)|^\alpha}}{C_\eta^+ + \frac{O_{n,1}(q(\alpha)^\alpha)}{q(\alpha)^\alpha}}}}{C_\eta^- - \frac{O_{n,2}(|p(\alpha)|^\alpha)}{|p(\alpha)|^\alpha}}.$$

¿From these equalities it follows that

$$C_\eta^+ q(\alpha)^{\alpha-1} = 1 + \sqrt[\alpha]{\frac{C_\eta^+ + \frac{O_{n,1}(q(\alpha)^\alpha)}{q(\alpha)^\alpha}}{C_\eta^- - \frac{O_{n,2}(|p(\alpha)|^\alpha)}{|p(\alpha)|^\alpha}}} - \frac{O_{n,1}\left(q(\alpha)^\alpha\right)}{q(\alpha)}$$

and

$$C_\eta^- \; |p(\alpha)|^{\alpha-1} = 1 + \sqrt[\alpha]{\frac{C_\eta^- - \frac{O_{n,2}(|p(\alpha)|^\alpha)}{|p(\alpha)|^\alpha}}{C_\eta^+ + \frac{O_{n,1}(q(\alpha)^\alpha)}{q(\alpha)^\alpha}}} + \frac{O_{n,2}\left(|p(\alpha)|^\alpha\right)}{|p(\alpha)|}\,.$$

In this way we obtain that

$$C_\eta^+ \, C_\eta^- \; |p(\alpha)|^{\alpha-1} q(\alpha)^{\alpha-1} - C_\eta^- \, |p(\alpha)|^{\alpha-1} - C_\eta^+ q(\alpha)^{\alpha-1} = \tag{4.6}$$

$$- \sqrt[\alpha]{\frac{C_\eta^- - \frac{O_{n,2}(|p(\alpha)|^\alpha)}{|p(\alpha)|^\alpha}}{C_\eta^+ + \frac{O_{n,1}(q(\alpha)^\alpha)}{q(\alpha)^\alpha}}} \cdot \frac{O_{n,1}\left(q(\alpha)^\alpha\right)}{q(\alpha)} +$$

$$\sqrt[\alpha]{\frac{C_\eta^+ + \frac{O_{n,1}(q(\alpha)^\alpha)}{q(\alpha)^\alpha}}{C_\eta^- - \frac{O_{n,2}(|p(\alpha)|^\alpha)}{|p(\alpha)|^\alpha}}} \cdot \frac{O_{n,2}\left(|p\left(\alpha\right)|^\alpha\right)}{p\left(\alpha\right)} -$$

$$\frac{O_{n,1}\left(q\left(\alpha\right)^\alpha\right)}{q\left(\alpha\right)} \cdot \frac{O_{n,2}\left(|p\left(\alpha\right)|\right)^\alpha}{|p\left(\alpha\right)|}.$$

But

$$\left|\frac{O_{n,1}\left(q\left(\alpha\right)^\alpha\right)}{q\left(\alpha\right)}\right| = \left|O'_{n,1}\left(x\right)\right| \cdot q\left(\alpha\right)^{\alpha-1} \; C \cdot |x| \cdot q\left(\alpha\right)^{\alpha-1} \;,$$

where $0 < x < q\left(\alpha\right)^\alpha$ and C is a positive constant (Mean Value Theorem).

Thus, by Lemma 4.3

$$\lim\nolimits_{\alpha\to 1^-} \left|\frac{O_{n,1}\left(q\left(\alpha\right)^\alpha\right)}{\left(\alpha-1\right)\cdot q\left(\alpha\right)}\right| \; C \cdot \lim\nolimits_{\alpha\to 1^-} \frac{q(\alpha)^\alpha}{\alpha-1} \cdot \lim\nolimits_{\alpha\to 1} q\left(\alpha\right)^{\alpha-1} = 0\,. \tag{4.7}$$

In the same way we obtain

$$\lim\nolimits_{\alpha\to 1^-} \left|\frac{O_{n,2}\left(\left|p_{(\alpha)}\right|^\alpha\right)}{\left(\alpha-1\right)\cdot p\left(\alpha\right)}\right| = 0\,. \tag{4.8}$$

In addition,

$$\lim_{\alpha\to 1^-} \sqrt[\alpha]{\frac{C_\eta^- - \frac{O_{n,2}(|p(\alpha)|^\alpha)}{|p(\alpha)|^\alpha}}{C_\eta^+ + \frac{O_{n,1}(q(\alpha)^\alpha)}{q(\alpha)^\alpha}}} = \frac{C_{\eta_0}^-}{C_{\eta_0}^+},$$

and

$$\lim_{\alpha\to 1^-} \sqrt[\alpha]{\frac{C_\eta^+ + \frac{O_{n,1}(q(\alpha)^\alpha)}{q(\alpha)^\alpha}}{C_\eta^- - \frac{O_{n,2}(|p(\alpha)|^\alpha)}{|p(\alpha)|^\alpha}}} = \frac{C_{\eta_0}^+}{C_{\eta_0}^-}.$$

Now, taking limit as $\alpha \to 1^-$ and replacing (4.7) and (4.8) in (4.6) we conclude the proof of Lemma 4.4. □

Now we prove Theorem 4.2. For this, recall that $x\left(\alpha,\mu,\nu\right)$ and $y\left(\alpha,\mu,\nu\right)$ are the continuations of $x\left(\alpha\right)$ and $y\left(\alpha\right)$ for $\left(\mu,\nu\right)$ close to $\left(\mu\left(\alpha\right),\nu\left(\alpha\right)\right)$ in the $\left(\mu,\nu\right)$ −parameter space. Observe that $x\left(\alpha,\mu\left(\alpha\right),\nu\left(\alpha\right)\right) = x\left(\alpha\right)$ and $y\left(\alpha,\mu\left(\alpha\right),\nu\left(\alpha\right)\right) = y\left(\alpha\right)$.

Since $F_\alpha\left(\mu,\nu\right) = \left(\mu - x\left(\alpha,\mu,\nu\right), \nu - y\left(\alpha,\mu,\nu\right)\right)$ we have that

$$DF_\alpha\left(u\left(\alpha\right),\nu\left(\alpha\right)\right) = \begin{pmatrix} 1-\partial_\mu x\left(\alpha,\mu,\nu\right) & -\partial_\nu x\left(\alpha,\mu,\nu\right) \\ -\partial_\mu y\left(\alpha,\mu,\nu\right) & 1-\partial_\nu y\left(\alpha,\mu,\nu\right) \end{pmatrix}$$

So, $\det DF_\alpha\left(u\left(\alpha\right),\nu\left(\alpha\right)\right) = \left(1-\partial_\mu x\left(\alpha,\mu\left(\alpha\right),\nu\left(\alpha\right)\right)\right)\cdot\left(1-\partial_\nu y\left(\alpha,\mu\left(\alpha\right),\nu\left(\alpha\right)\right)\right)$
$-\partial_\mu y\left(\alpha,\mu\left(\alpha\right),\nu\left(\alpha\right)\right)\cdot\partial_\nu x\left(\alpha,\mu\left(\alpha\right),\nu\left(\alpha\right)\right)$.

By definition $g_\alpha\left(\mu,\nu,x\left(\alpha,\mu,\nu\right)\right) = x\left(\alpha,\mu,\nu\right)$ and $g_\alpha\left(\mu,\nu,y\left(\alpha,\mu,\nu\right)\right) = y\left(\alpha,\mu,\nu\right)$. Then, by the Implicit Function Theorem we get

$$\partial_\mu x\left(\alpha,\mu\left(\alpha\right),\nu\left(\alpha\right)\right) = \tfrac{\partial_\mu g_\alpha(u(\alpha),\nu(\alpha),x(\alpha))}{1-\partial_x g_\alpha(\mu(\alpha),\nu(\alpha),x(\alpha))}\,,$$

$\partial_\nu x\left(\alpha,\mu\left(\alpha\right),\nu\left(\alpha\right)\right)=\frac{\partial_\nu g_\alpha\left(u(\alpha),\nu(\alpha),x(\alpha)\right)}{1-\partial_x g_\alpha\left(\mu(\alpha),\nu(\alpha),x(\alpha)\right)}$,

$\partial_\mu y\left(\alpha,\mu\left(\alpha\right),\nu\left(\alpha\right)\right)=\frac{\partial_\mu g_\alpha\left(u(\alpha),\nu(\alpha),y(\alpha)\right)}{1-\partial_x g_\alpha\left(\mu(\alpha),\nu(\alpha),y(\alpha)\right)}$,

$\partial_\nu y\left(\alpha,\mu\left(\alpha\right),\nu\left(\alpha\right)\right)=\frac{\partial_\nu g_\alpha\left(u(\alpha),\nu(\alpha),y(\alpha)\right)}{1-\partial_x g_\alpha\left(\mu(\alpha),\nu(\alpha),y(\alpha)\right)}$.

Thus,
$\det DF_\alpha\left(u\left(\alpha\right),\nu\left(\alpha\right)\right)=$

$$=\frac{1-\partial_x g_\alpha\left(u\left(\alpha\right),\nu\left(\alpha\right),x\left(\alpha\right)\right)-\partial_\mu g_\alpha\left(u\left(\alpha\right),\nu\left(\alpha\right),x\left(\alpha\right)\right)}{1-\partial_x g_\alpha\left(u\left(\alpha\right),\nu\left(\alpha\right),x\left(\alpha\right)\right)}\cdot$$

$$\frac{\left(1-\partial_x g_\alpha\left(u\left(\alpha\right),\nu\left(\alpha\right),y\left(\alpha\right)\right)-\partial_\nu g_\alpha\left(u\left(\alpha\right),\nu\left(\alpha\right),y\left(\alpha\right)\right)\right)}{1-\partial_x g_\alpha\left(\mu\left(\alpha\right),\nu\left(\alpha\right),y\left(\alpha\right)\right)}$$

$$-\frac{\partial_\mu g_\alpha\left(u\left(\alpha\right),\nu\left(\alpha\right),y\left(\alpha\right)\right)}{1-\partial_x g_\alpha\left(u\left(\alpha\right),\nu\left(\alpha\right),x\left(\alpha\right)\right)}\cdot\frac{\partial_\nu g_\alpha\left(u\left(\alpha\right),\nu\left(\alpha\right),y\left(\alpha\right)\right)}{1-\partial_x g_\alpha\left(u\left(\alpha\right),\nu\left(\alpha\right),y\left(\alpha\right)\right)}\,.$$

Now we will prove that

$$\lim_{\alpha\to1^-}\frac{\det DF_\alpha\left(\mu(\alpha),\nu(\alpha)\right)}{\alpha-1}=2+\frac{C^+_{\eta_0}}{C^-_{\eta_0}}+\frac{C^-_{\eta_0}}{C^+_{\eta_0}}\,.$$

In fact, note that
$\det DF_\alpha\left(u\left(\alpha\right),\nu\left(\alpha\right)\right)=$

$$\frac{E}{\left(1-\partial_x g_\alpha\left(u\left(\alpha\right),\nu\left(\alpha\right),x\left(\alpha\right)\right)\right)\left(1-\partial_x g_\alpha\left(u\left(\alpha\right),\nu\left(\alpha\right),y\left(\alpha\right)\right)\right)},\tag{4.9}$$

where $E=\left[1-\partial_x g_\alpha\left(u\left(\alpha\right),\nu\left(\alpha\right),y\left(\alpha\right)\right)-\partial_\mu g_\alpha\left(u\left(\alpha\right),\nu\left(\alpha\right),y\left(\alpha\right)\right)\right]\cdot$
$\left[1-\partial_x g_\alpha\left(u\left(\alpha\right),\nu\left(\alpha\right),y\left(\alpha\right)\right)-\partial_\nu g_\alpha\left(\partial_x g_\alpha\left(u\left(\alpha\right),\nu\left(\alpha\right),y\left(\alpha\right)\right)\right)\right]-$
$\partial_\mu g_\alpha\left(u\left(\alpha\right),\nu\left(\alpha\right),y\left(\alpha\right)\right)\cdot\partial_\nu g_\alpha\left(u\left(\alpha\right),\nu\left(\alpha\right),y\left(\alpha\right)\right)$.

Using the definitions of g_α we obtain

$$E=\Big[1-\alpha C^-_\eta k^{\alpha-1}\left|x\left(\alpha\right)\right|^{\alpha-1}+O^{'}_{n,2}\left(\left|kx\left(\alpha\right)\right|^{\alpha}\right)\left|kx\left(\alpha\right)\right|^{\alpha-1}+$$

$$\partial_{a^+}C^-_\eta\;\left|kx\left(\alpha\right)\right|^{\alpha}-\partial_{a^+}O_{n,2}\left(\left|kx\left(\alpha\right)\right|^{\alpha}\right)\Big]\cdot$$

$$\Big[1-\alpha C^+_\eta k^{\alpha-1}y\left(\alpha\right)^{\alpha-1}-O^{'}_{n,1}\left(\left|ky\left(\alpha\right)\right|^{\alpha}\right)\left|ky\left(\alpha\right)\right|^{\alpha-1}-$$

$$\partial_{a^-}C^-_\eta\left|ky\left(\alpha\right)\right|^{\alpha}-\partial_{a^-}O_{n,1}\left(\left|ky\left(\alpha\right)\right|^{\alpha}\right)\Big]-$$

$$- \left[1 + \partial_{a^+} C_\eta^+ \ |ky(\alpha)|^\alpha + \partial_{a^+} O_{n,1}(|ky(\alpha)|^\alpha)\right] \cdot$$

$$\left[1 - \partial_{a^-} C_\eta^- \ |kx(\alpha)|^\alpha + \partial_{a_-} \ O_{n,2}(|kx(\alpha)|^\alpha)\right] \ .$$

Arranging terms we get

$$E = \left[1 - \alpha C_\eta^- \ |kx(\alpha)|^{\alpha-1} + A_1 \ |kx(\alpha)|^\alpha\right] \cdot \left[1 - \alpha C_\eta^+ \ |ky(\alpha)|^{\alpha-1} + A_2 \ |ky(\alpha)|^\alpha\right]$$

$$- \left[1 + \partial_{a^+} C_\eta^+ \ |ky(\alpha)|^{\alpha-1} + A_3\right] \cdot \left[1 - \partial_{a^-} C'_\eta \ |kx(\alpha)|^\alpha + A_4\right] \ ,$$

where

$A_1 = \frac{O'_{n,2}(|kx(\alpha)|^\alpha)}{|kx(\alpha)|} + \partial_{a^+} C_\eta^- - \frac{\partial_{a^+} O_{n,2}(|kx(\alpha)|^\alpha)}{|kx(\alpha)|^\alpha}$,

$A_2 = -\frac{O'_{n,1}(|ky(\alpha)|^\alpha)}{|ky(\alpha)|} + \partial_{a_-} C_\eta^+ - \frac{\partial_{a^+} O_{n,1}(|ky(\alpha)|^\alpha)}{|ky(\alpha)|^\alpha}$,

$A_3 = \partial_{a^+} \ O_{n,1}(|ky(\alpha)|^\alpha)$,

and $A_4 = \partial_{a^-} \ O_{n,2}(|kx(\alpha)|^\alpha)$.

Note that $\lim_{\alpha\to 1^-} A_i$ exists and is a finite number for $1 \le i \le 4$.

Recall that $k\,x(\alpha) = p(\alpha)$ and $k\,y(\alpha) = q(\alpha)$, then

$$\begin{aligned}
E = & \left(1 - \alpha C_\eta^- \ |p(\alpha)|^{\alpha-1}\right) \cdot \left(1 - \alpha C_\eta^+ q(\alpha)^{\alpha-1}\right) \\
& + \left(1 - \alpha C_\eta^- \ |p(\alpha)|^{\alpha-1}\right) A_2 \ q(\alpha)^\alpha \\
& + \left(1 - \alpha C_\eta^+ q(\alpha)^{\alpha-1}\right) A_1 \ |p(\alpha)|^\alpha + A_1 \ A_2 \ q(\alpha)^\alpha \ |p(\alpha)|^\alpha \\
& -1 - \partial_{a^+} C_\eta^+ \ q(\alpha)^\alpha - A_3 + \partial_{a^-} \ C_\eta^- \ |p(\alpha)|^\alpha - \partial_{a^+} C_\eta^+ \ q(\alpha)^\alpha \ \partial_{a^-} C_\eta^- \ |p(\alpha)|^\alpha \\
& -A_3 \ \partial_{a^-} C_\eta^- \ |p(\alpha)|^\alpha - A_4 - \partial_{a^+} C_\eta^+ \ q(\alpha)^\alpha \ A_4 - A_3 \ A_4 \ .
\end{aligned}$$

So,

$$\begin{aligned}
E = & -\alpha \ C_\eta^- \ |p(\alpha)|^{\alpha-1} - \alpha \ C_\eta^+ \ q(\alpha)^{\alpha-1} + \alpha^2 \ C_\eta^- \ C_\eta^+ \ |p(\alpha)|^{\alpha-1} \ q(\alpha)^{\alpha-1} \\
& + A_5 \ q(\alpha)^\alpha + A_6 \ |p(\alpha)|^\alpha + A_7 \ q(\alpha)^\alpha \ |p(\alpha)|^\alpha ,
\end{aligned}$$

where

$A_5 = \left(1 - \alpha \ C_\eta^- \ |p(\alpha)|^{\alpha-1}\right) A_2 - \partial_{a^+} \ C_\eta^+ - \frac{A_3}{q(\alpha)^\alpha} - \partial_{a^+} \ C_\eta^+ \ A_4$,

$A_6 = \left(1 - \alpha \ C_\eta^+ \ q(\alpha)^{\alpha-1}\right) A_1 + \partial_{a^-} \ C_\eta^- - A_3 \ \partial_{a^-} \ C_\eta^- - \frac{A_4}{|p(\alpha)|^\alpha}$,

and $A_7 = A_1 \ A_2 - \partial_{a^+} \ C_\eta^+ \ \partial_{a^-} \ C_\eta^- - \frac{A_3 \ A_4}{q(\alpha)^\alpha \ |p(\alpha)|^\alpha}$.

Note that $\lim_{\alpha\to 1^-} A_i$ exists and is a finite number for $5 \le i \le 7$.

Thus,

$$E = \left(\alpha^2 - \alpha\right)\left[C_\eta^- \ |p(\alpha)|^{\alpha-1} + C_\eta^+ \ q(\alpha)^{\alpha-1}\right] +$$

$$\alpha^2\left[C_\eta^+ \ C_\eta^- \ |p(\alpha)|^{\alpha-1} q(\alpha)^{\alpha-1} - C_\eta^- \ |p(\alpha)|^{\alpha-1} - C_\eta^+ q(\alpha)^{\alpha-1}\right] +$$

$$A_5 \ q(\alpha)^\alpha + A_6 \, |p(\alpha)|^\alpha + A_7 \ q(\alpha)^\alpha \, |p(\alpha)|^\alpha \ .$$

Now, taking limits and using Lemmas 4.3 and 4.4 we obtain that

$$\lim_{\alpha\to 1^-} \frac{E}{\alpha - 1} = \lim_{\alpha\to 1}\alpha\left[C_\eta^- \ |p(\alpha)|^{\alpha-1} + C_\eta^+ \ q(\alpha)^{\alpha-1}\right] +$$

$$\lim_{\alpha\to 1^-}\frac{\alpha^2}{\alpha-1}\left[C_\eta^+ \ C_\eta^- \ |p(\alpha)|^{\alpha-1} \ q(\alpha)^{\alpha-1} - C_\eta^- \ |p(\alpha)|^{\alpha-1} + C_\eta^+ \ q(\alpha)^{\alpha-1}\right] +$$

$$\lim_{\alpha\to 1^-}\left[A_5 \ \frac{q(\alpha)^\alpha}{\alpha-1} + A_6 \ \frac{|p(\alpha)|^\alpha}{\alpha-1} + A_7 \ \frac{q(\alpha)^\alpha}{\alpha-1} \ |p(\alpha)|^\alpha\right] =$$

$$C_{\eta_0}^- \ B + C_{\eta_0}^+ \ B = 2 + \frac{C_{\eta_0}^+}{C_{\eta_0}^-} + \frac{C_{\eta_0}^-}{C_{\eta_0}^+} \, .$$

To finish, we have that

$$\begin{aligned} &\lim_{\alpha\to 1^-}\left(1 - \partial_x g_\alpha\left(\mu(\alpha), \nu(\alpha), x(\alpha)\right)\right)\cdot\left(1 - \partial_x g_\alpha\left(\mu(\alpha), \nu(\alpha), y(\alpha)\right)\right) \\ &= \left(1 - C_{\eta_0}^- B\right)\left(1 - C_{\eta_0}^+ \ B\right) = 1 \, . \end{aligned}$$

So, replacing this last equality in (4.9) we get

$$\lim_{\alpha\to 1^-}\frac{\det DF_\alpha\left(\mu_{(\alpha)}, \nu_{(\alpha)}\right)}{\alpha - 1} = 2 + \frac{C_{\eta_0}^+}{C_{\eta_0}^-} + \frac{C_{\eta_0}^-}{C_{\eta_0}^+} \, .$$

All of these facts together conclude the proof of Theorem 4.2. □

5. One-dimensional analysis. In this section we will proof that the trapping region obtained in the previous section is contained in the basin of a transitive attractor of $g_\alpha(\mu,\nu)$. Depending on the geometry of the original vector field we can obtain a different kind of such attractor. When $\nu^+ = \nu^- = 1$, the attractor is the interval $[\mu,\nu]$. If $\nu^+ = \nu^- = -1$ we have three alternatives: (1) the attractor is given by the interval $[\nu, g_\alpha(\mu,\nu,\mu)]$ or (2) $[\nu,\mu]$ or (3) $[g_\alpha(\mu,\nu,\mu),\mu]$, depending on $\nu < g_\alpha(\mu,\nu,\mu) < \mu \leq g_\alpha(\mu,\nu,\nu)$, or $\nu < g_\alpha(\mu,\nu,\mu) < g_\alpha(\mu,\nu,\nu) < \mu$ or $g_\alpha(\mu,\nu,\mu) \leq \nu < g_\alpha(\mu,\nu,\nu) < \mu$ respectively. For $\nu^+ = -\nu^- = 1$ we have two possibilities: (1) $[\mu, g_\alpha(\mu,\nu,\mu)]$ when $\mu \leq \nu$ and (2) $[\nu, g_\alpha(\mu,\nu,\nu)]$ when $\nu < \mu$. In

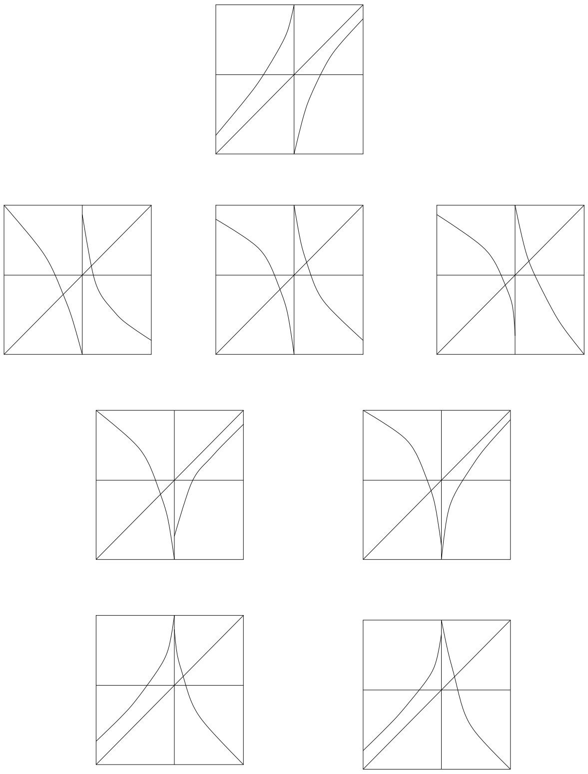

FIG. 5.1. *Possible attractors for* $g_\alpha(\mu, \nu, .)$

the same way, for $\nu^+ = -\nu^- = -1$, we have: (1) $[g_\alpha(\mu, \nu, \mu), \nu]$ or (2) $[g_\alpha(\mu, \nu, \nu), \mu]$, when $\mu \leq \nu$ or $\nu < \mu$ respectively (see Figure 5.1).

The proof bellow is similar to the one given in [Rob4, Theorem A] and we are doing it here for completeness.

THEOREM 5.1. *Fix* $\lambda > 1$ *and let* $n_0 \in I\!N$ *be such that*

$$\frac{\lambda^{n_0}}{2} > 1.$$

Let $a < 0 < b$ *and* $f : [a, b]^* \to [a, b]$, $[a, b]^* = [a, b] \setminus \{0\}$ *a map such that* $f(0_-) = \mu$, $f(0_+) = \nu$. *Assume also that the following conditions hold:*
(H0) $[a, b] = convexhull\{\mu, \nu, f(\mu), f(\nu)\}$
(H1) f *has a single singularity at* $x = 0$,
(H2) f *is differentiable in* $[a, b]^*$ *and* $|f'(x)| \geq \lambda$ *for all* $x \in [a, b]^*$,
(H3) there are $x < 0 < y$ *such that* $f(x) = f(y) = 0$,
(H4) there is $\epsilon > 0$ *such that if* $f(0_+) = \nu$ *and* $f(0_-) = \mu$ *then* $\{f^i(\mu), f^i(\nu), f^i(a), f^i(b) : 0 \leq i \leq n_0\} \subset [a, a+\epsilon] \cup [b - \epsilon, b]$ *for all* $1 \leq i \leq n_0$.
Then, for ϵ *sufficently small* f *is transitive.*

*Note 5:*For $n_0 = 2$ this lemma reduces to a well known result in [W].

Proof. First of all, hypothesis $(H0)$ shows that $f([x, 0))$ or $f((0, y])$ must be contain $(a, 0]$ or $[0, b)$. Now, fix and open interval $I \subset [a, b]$. We claim that one of the alternatives below hold.

(A) $f(I)$ is an interval and $|f(I)| \geq \lambda |I|$,

(B) there is $n \in I\!N$ such that $f^n(I)$ contains $(a, 0]$ or $[0, b)$,

(C) there is an interval $J \subset f^{n_0}(I)$ such that $|J| > \frac{\lambda^{n_0}}{2} |I|$.

Indeed, if $0 \notin I$ then (A) holds. Thus, we can assume that that $0 \in I$. Then $I^* = I \setminus \{0\}$ splits into $I^* = I^+ \cup I^-$ where $I^+ = I \cap (0, b]$ and $I^- = I \cap [a, 0)$.

Let $\tilde{I} = I^+$ or I^- such that $|f(\tilde{I}| \geq \lambda/2 \ |I|$.

We have either

(a) $f(\tilde{I}) \supset (a, 0]$ or

(b) $f(\tilde{I}) \supset [0, b)$ or

(c) $f(\tilde{I}) = (x, \delta)$ with $0 < x < \delta$ where $\delta \in \{\mu, \nu\}$ or

(d) $f(\tilde{I}) = (\delta, x)$ with $\delta < x < 0$ where $\delta \in \{\mu, \nu\}$.

In cases (a) and (b) we have (B) for I with $n = 1$.

Assume case (c).

In this case we have two possibilities, namely,

(c1) there is $1 \leq i = i(I) < n_0$ such that i is the first iterate such that $0 \in f^i(f(\tilde{I}))$.

(c2) $0 \notin f^i(f(\tilde{I})$ for all $1 \leq i \leq n_0$.

If (c2) holds then

$$|f^{n_0}(\tilde{I})| = |f^{n_0 - 1}(f(\tilde{I}))| \geq \lambda^{n_0 - 1} \ |f(\tilde{I})| \geq \lambda^{n_0 - 1} \ \lambda/2 \ |I| \geq \lambda^{n_0}/2 \ |I|.$$

If (c1) holds then $f^i(f(\tilde{I}))$ is an interval containing both 0 and $f^i(\delta)$.

As $f^i(f(\tilde{I}))$ is an interval we conclude that $f^i(f(\tilde{I}))$ contains an interval containing 0 and $f^i(\delta))$ and this implies that contains one of the intervals $[x, 0]$ or $[0, y]$. ¿From here, $f^{i+1}(f(\tilde{I}))$ contains both $[x, 0]$ and $[0, y]$. Hence,

$$f^{i+2}(f(\tilde{I})) = f(f^{i+1}(f(\tilde{I}))) \supset (a, 0].$$

or

$$f^{i+2}(f(\tilde{I})) = f(f^{i+1}(f(\tilde{I}))) \supset [0, b).$$

Thus, (B) holds with $n = i + 2$.

Finally, assume (d). The proof is similar to the previous case.

To finish the proof we proceed as follows.

First we claim that for all open interval $I \subset [a, b]$ there is $N \in I\!N$ such that $f^N(I)$ contains either $(a, 0)$ or $(0, b)$.

The proof goes by contradiction. Assume that such a N does not exist. Then, either (A) or (C) holds.

Fix $I_1 = I$. Then there is $N_1 \in I\!N$ such that $f^{N_1}(I_1)$ contains an interval I_2 such that $|I_2| \geq \min\{\lambda, \frac{\lambda^{n_0}}{2}\} \ |I_1|$. Set $k = \min\{\lambda, \frac{\lambda^{n_0}}{2}\}$. Note that (B) is not true for I_2 (because $I_2 \subset f^{N_1}(I_2)$). Then, there is $N_2 \in I\!N$ such that $f^{N_2}(I_2))$ contains an interval I_3 such that $|I_3| \geq k \ |I_2| \geq k^2 \ |I_1|$.

In this way we construct a sequence of intervals $I_1, I_2, I_3, \cdots$ such that

$$|I_i| \geq k^i \ |I_1|.$$

This yields a contradiction since $k > 1$. This proves the claim.

Now we finish the proof arguing in the following way:

if $f^N(I) \supset (a, 0)$ or $f^N(I) \supset (0, b)$ then $f^N(I) \cup f^{N+1}(I) \supset (a, b)$. This implies that f is transitive and the theorem follows. $\square$

Note 6: It is not difficult to prove, using (H4), that the map f in Theorem 5.1 is *leo* (locally eventually onto), i.e., for any interval $J \subset I$, there is $n \geq 1$ such that $f^n(J) = I$. Clearly a leo map is transitive.

Proof of Theorem 1.1.

Let X_η satisfying (A1)-(A7), and f_η be the one parameter family associated to X_η given by Lemma 2.1 in Section 2. Recall that $\eta = \eta(\alpha, a^+, a^-)$. Next, as in Section 3 we obtain $g_\alpha(\mu, \nu, x)$ which is a renormalization of f_η.

¿From now on we work with $g_\alpha(\mu, \nu, x)$.

By (A5), we can apply Theorem 4.1. Let $\mathcal{O}^- \subset I\!\!R^3$ be the open set given by Theorem 4.1. For each $\alpha \in O$ close enough 1, where O is given by Lemma 2.2, consider $(\alpha, \mu(\alpha), \nu(\alpha))$, with $(\mu(\alpha), \nu(\alpha))$ were defined at the beginning of Section 3.

By the definition of $\mathcal{O}^-$ we have that $(\alpha, \mu(\alpha), \nu(\alpha)) \in \mathrm{Cl}(\mathcal{O}^-)$.

By Theorem 4.1-(2) we have that for all $\theta \in \mathcal{O}^-$ there is I_θ such that $g_\alpha(\mu, \nu, I_\theta) \subset \mathrm{Int}(I_\theta)$. Set $\tilde{I}_\theta \subset I_\theta$ the $convexhull\{\mu, \nu, g_\alpha(\mu, \nu, \mu), g_\alpha(\mu, \nu, \nu)\}$. Taking (μ, ν) close to $(\mu(\alpha), \nu(\alpha))$, the map $g_\alpha(\mu, \nu, \cdot)$ satisfies the hypotheses of Theorem 5.1. Indeed, (H0) follows from the definition of $\tilde{I}_\theta$, (H1) is straightforward by the definition of g_α, and (H2), (H3), and (H4) follow from Lemma 3.3 for (μ, ν) close to $(\mu(\alpha), \nu(\alpha))$. Hence, the restriction of $g_\alpha(\mu, \nu, .)$ to $\tilde{I}_\theta$ is transitive, and thus we conclude that $g_\alpha(\mu, \nu, .)$ is transitive in $\tilde{I}_\theta$. All of these facts together conclude the proof of Theorem 1.1. □

REFERENCES

[ABS] Afraimovich, V. S., Bykov, and L. P. Shil'nikov, *On the appearance and structure of the Lorenz attractor*, Dokl. Acad. Sci. USSR (1977), 234:336–339.

[ACL] Afraimovich, V.; Chow, S.; Liu, W., *Lorenz-type attractors from codimension one bifurcation*, J. Dynam. Differential Equations 7 (1995), no. 2, 375–407.

[AH] Afraimovich, V. S, Hsu, S-B, *Lectures on Chaotic Dynamical Systems*, AMS/IP, Studies in Advanced Mathematics, 28.

[AL] Alsedá, L., Llibre, B. D., *Wandering intervals for Lorenz maps with bounded nonlinearity*, Dynamical systemas and ergodic theory (Warsaw, 1986), 83–89, Banach Center Publ., 23, PWN, Warsaw, 1989.

[BPV] Bonatti, Ch.; Pumarino, A.; Viana, M., *Lorenz attractors with arbitrary expanding dimension.* C. R. Acad. Sci. Paris Ser. I Math. 325 (1997), no. 8, 883–888.

[BW] Birman, Joan S.; Williams, R. F. *Knotted periodic orbits in dynamical systems. I. Lorenz's equations*, Topology 22 (1983), no. 1, 47–82.

[BL] Bouichenko, V. A.; Leonov, G. A., *The Hausdorff dimension of attractors of the Lorenz system*, (Russian) Differentsialnye Uravneniya 25 (1989), no. 11, 1999–2000, 2023.

[Bu] Bunimovich, L. A. *Statistical properties of the Lorenz model*, (Russian) Izv. Vyssh. Uchebn. Zaved. Radiofiz. 28 (1985), no. 11, 1472–1473.

[CPRV] Costa, M. J., Pacifico, M. J., Rovella, A., Viana, M., *Persistence of global spiral attractors* in preparation.

[GW] Guckenheimer,J., Williams , A., *Structural stability of Lorenz attractors.*, Publ. IHES 50 (1979), 59–72.

[dMP] de Melo, W., Palis, J., *Geometric theory of dynamical systems. An introduction.*, Springer-Verlag, New York-Berlin, 1982.

[DY] Ding, Y., Fan, W., *The asymptotic periodicity of Lorenz maps*, Acta Math. Sci. (English Ed.) 19 (1999), no. 1, 114–120.

[DKO] Dumortier, F., Kokubu, H., Oka, H., *A degenerate singularity generating geometric Lorenz attractors*, Ergodic Theory Dynam. Systems 15 (1995), no. 5, 833–856.

[GH] Ghrist, R. W., Holmes, P. J., *An ODE whose solutions contain all knots and links*, Internat. J. Bifur. Chaos Appl. Sci. Engrg. 6 (1996), no. 5, 779–800.

[HS] Hubbard, J. H., Sparrow, C., *The classification of topologically expansive Lorenz maps*, Comm. Pure Appl. Math. 43 (1990), 4, 431–443.

[KS] Keller, G., St. Pierre, M., *Topological and measurable dynamics of Lorenz maps* Ergodic Theory, analysis and efficient simulations of dynamical systems, (2001) 333–361, Springer, Berlin.

[KKO] Kisaka, M.; Kokubu, H.; Oka, H. *Supplement to homoclinic doubling bifurcation in vector fields*, Dynamical systems (Santiago, 1990), 92–116, Pitman Res. Notes Math. Ser., 285, Longman Sci. Tech., Harlow, 1993.

[Kl] Klinshpont, N., *A topological invariant of the Lorenz attractor*, (Russian) Uspekhi Mat. Nauk 47 (1992), no. 2(284), 195–196; translation in Russian Math. Surveys 47 (1992), no. 2, 221–223.

[Ko1] Komuro, M., *Expansive properties of Lorenz attractors*, The theory of dynamical systems and its applications to nonlinear problems (Kyoto, 1984), 4–26, World Sci. Publishing, Singapore, 1984.

[Ko2] Komuro, M., *Lorenz attractors do not have the pseudo-orbit tracing property*, J. Math. Soc. Japan 37 (1985), no. 3, 489–514.

[LM1] Labarca, R., Moreira, C. , *Bifurcation of the essential dynamicas of Lorenz maps and applications to Lorenz-like flows: contributions to the study of the expanding case*, Bol. Soc. Mat. (N. S.) 32 (2001), 107–144.

[LM2] Labarca, R., Moreira, C. , *Bifurcation of the essential dynamicas of Lorenz maps on the real line and the bifurcation scenario for the linear family*, Sci. Ser. A. Math. Sci.(N. S.) 7 (2001), 13–29.

[Le1] Leonov, G. A., *Formulas for the Lyapunov dimension of H?non and Lorenz attractors*, (Russian) Algebra i Analiz 13 (2001), no. 3, 155–170; translation in St. Petersburg Math. J. 13 (2002), no. 3, 453–464.

[Le2] Leonov, G. A., *Asymptotic behavior of the solutions of the Lorenz system* (Russian) Differentsialnye Uravneniya 24 (1988), no. 5, 804–809, 917; translation in Differential Equations 24 (1988), no. 5, 527–531

[Lo] Lorenz, E. N., *Deterministic nonperiodic flow,* J. Atmosph. Sci., 20 (1963), 130–141, .

[Lo84] Lorenz, E. N., *The local structure of a chaotic attractor in four dimensions,* Phys. D 13 (1984), no. 1-2, 90–104.

[LV] Luzzatto, S., Viana, M. *Positive Lyapunov exponents for Lorenz-like families with criticalities*, Geomtrie complexe et systemes dynamiques (Orsay, 1995). Asterisque 261 (2000), xiii, 201–237.

[MdeM] Martens, M., de Melo, W., *Universal models for Lorenz maps* Ergodic Theory and Dynamical Systems, 21 (2001), 833-860.

[Me1] Metzger, R. J. *Stochastic stability for contracting Lorenz maps and flows*, Comm. Math. Phys. 212 (2000), no. 2, 277–296.

[Me2] Metzger, R. J. *Sinai-Ruelle-Bowen measures for contracting Lorenz maps and flows,* Ann. Inst. H. Poincarè Anal. Non Lineaire 17 (2000), no. 2, 247–276.

[Mo] Morales, C., *Lorenz attractors through saddle-node bifurcations*, Ann. IHP Non Lin. 13 (1996), 589–617.

[MPP1] Morales, C. A., Pacifico, M. J., Pujals, E. R., *Strange attractors across the boundary of hyperbolic systems*, Commun. Math. Phys., 211 (2000), 527–558.

[MPP2] Morales, C. A., Pacifico, M. J., Pujals, E. R., *On C^1 robust transitive sets for three dimensional flows*, C. R. Acad. Sci. Paris, 326 (1998), Série 1, 81–86.

[MPP3] Morales, C. A., Pacifico, M. J., Pujals, E. R., *Robust transitive singular sets for 3-flows are partially hyperbolic attractors or repellers*, To appear in the Annals of Mathematics.

[MPu] Morales, C. A., Pujals, E. R. *Singular strange attractors on the boundary of Morse-Smale systems.*, Ann. Sci. Ecole Norm. Sup. (4) 30 (1997), no. 6, 693-717.

[MSV] Munoz, E., San Martin, B., Vera, J. *Non hyperbolic persistent attractor near the Morse-Smale boundary.*, preprint.

[N] Naudot, V., *Strange attractor in the unfolding of an inclination-flip homoclinic orbit*, Ergodic Theory Dynam. Systems 16 (1996), no. 5, 1071–1086.

[PRV] Pacifico, M. J., Rovella, A., Viana, M., *Infinite-modal maps with global chaotic behavior* Annals of Math., 148 (1998), 441–484.

[PT] Palis, J., Takens,F., *Hyperbolicity and sensitive chaotic dynamics at homoclinic bifurcations. Fractal dimensions and infinitely many attractors.*, Cambridge Studies in Advanced Mathematics, 35. Cambridge University Press, Cambridge, 1993.

[P] Pesin, Ya., *Dynamical systems with generalized hyperbolic attractors: hyperbolic, ergodic and topological properties*, Ergod. Th. and Dynam. Sys., 12 (1992), 123–151.

[PS] Pesin, Ya. B.; Sinai, Ya. G., *Hyperbolicity and stochasticity of dynamical systems*, Mathematical physics reviews, Vol. 2, pp. 53–115, Soviet Sci. Rev. Sect. C: Math. Phys. Rev., 2, Harwood Academic, Chur, 1981.

[Rob1] Robinson, C., *Nonsymmetric Lorenz attractors from a homoclinic bifurcation*, SIAM J. Math.

Anal. 32 (2000), no. 1, 119–141.
[Rob2] Robinson, C., *Homoclinic bifurcation to a transitive attractor of Lorenz type. II.*, SIAM J. Math. Anal. 23 (1992), no. 5, 1255–1268.
[Rob3] Robinson, C., *Homoclinic bifurcation to a transitive attractor of Lorenz type.*, Nonlinearity 2 (1989), no. 4, 495–518.
[Rob4] Robinson, C., *Transitivity and invariant measures for the geometric model of the Lorenz equations.*, Ergodic Theory Dynam. Systems 4 (1984), no. 4, 605–611.
[Rov] Rovella, A., *The dynamics of perturbations of the contracting Lorenz attractor*, Bol. Soc. Brasil. Mat. (N.S.) 24 (1993), no. 2, 233–259.
[Ry] Rychlik, M. R., *Lorenz attractors through Silnikov-type bifurcation*, I. Ergodic Theory Dynam. Systems 10 (1990), no. 4, 793–821.
[S] Sataev, E. A., *In variant measures for hyperbolic maps with singularities*, Russ. Math. Surveys, 471 (1992), 191–251.
[Sp] Sparrow C., *The Lorenz equations: bifurcations, chaos and strange attractors*, (1982), 41, Applied Mathematical Sciences, Springer Verlag, Berlin.
[Sh] Shilnikov, Andrey L. *On bifurcations of the Lorenz attractor in the Shimizu-Morioka model*, Homoclinic chaos (Brussels, 1991). Phys. D 62 (1993), no. 1-4, 338–346.
[ST] Shilnikov, L. P., Turaev, D. V., *An example of a wild strange attractor.* (Russian) Mat. Sb. 189 (1998), no. 2, 137–160; translation in Sb. Math. 189 (1998), no. 1-2, 291–314.
[SM] Shimizu, T., Morioka, N., *On the bifurcation of a symmetric limit cycle to an asymmetric one in a simple model*, Phys. Lett. A76 (1980), 201–204.
[S] Shub, M., *Global stability of dynamical systems.*, Springer Verlag, 1987.
[Tu1] W. Tucker. *The Lorenz attractor exists.* C. R. Acad. Sci. Paris, 328 (1999), 1197–1202.
[Tu2] W. Tucker. *A rigorous ODE solver and Smale'sc14th problem.* Found. Comput. Math., 2 (2002) , 53–117.
[V] Viana, M., *What's new on Lorenz strange attractor ?* Math. Intelligencer 22 (2000) 3, 6–19.
[W] Williams, R., *The structure of Lorenz attractors*, Publ. IHES 50 (1979), 73–99.
[Ya] Young, L. S., *On the prevalence of horseshoes*, Trans. Amer. Math. Soc. 263 (1981), no. 1, 75–88.

B. San Martín
Departamento de Matemáticas
Universidad Católica del Norte
Casilla 1280, Antofagasta, Chile
e-mail: sanmarti@socompa.ucn.cl

C. A. Morales and M. J. Pacifico
Instituto de Matemática
Universidade Federal do Rio de Janeiro
C. P. 68.530, CEP 21.945-970
Rio de Janeiro, R. J. , Brazil
e-mail: morales@im.ufrj.br, pacifico@im.ufrj.br