A dichotomy for three-dimensional vector fields

C. A. Morales and M. J. Pacifico

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1 Abstract

We prove that a generic C^1 vector field on a closed 3-manifold either has infinitely many sinks or sources or else is singular Axiom A without cycles. Singular Axiom A means that the non-wandering set of the vector field has a decomposition into compact invariant sets, each one being either a hyperbolic basic set or a singular hyperbolic attractor (like the Lorenz-like ones) or a singular hyperbolic repeller. An attractor is a transitive set which attracts all nearby future orbits, and a repeller is an attractor for the time-reversed flow. Our result implies that generic C^1 vector fields on closed 3-manifolds do exhibit either attractors or repellers.

2 Introduction

Let M be a closed 3-manifold and $\mathcal{X}^1(M)$ be the set of C^1 vector fields on M endowed with the C^1 topology. Given an open subset $\mathcal{A}$ of $\mathcal{X}^1(M)$ a subset $\mathcal{R} \subset \mathcal{A}$ is residual if it coincides with a countable intersection of open-dense subsets of $\mathcal{A}$. We say that *a generic vector field in $\mathcal{A}$ satisfies a property (P)* if there is a residual subset $\mathcal{R}$ of $\mathcal{A}$ such that (P) holds for every $X \in \mathcal{R}$. A vector field $X \in \mathcal{X}^1(M)$ is non-singular if it has no singularities, i.e., $X(p) \neq 0$ for all $p \in M$. A compact invariant set Λ is *transitive* if $\Lambda = \omega_X(q)$ for some $q \in \Lambda$, and *attracting* if $\Lambda = \cap_{t \geq 0} X_t(U)$ for some neighborhood U of Λ satisfying $X_t(U) \subset U$, $\forall t \geq 0$. An *attractor* of X is a transitive attracting set of X and a *repeller* is an attractor for $-X$. A *sink* of X is a trivial attractor of X, namely it reduces to a single orbit of X. A *source* of X is a trivial repeller of X. A regular orbit contained in the intersection of the stable and unstable manifold associated to a hyperbolic singularity is called a *homoclinic loop*.

It is known that a generic non-singular vector field $X \in \mathcal{X}^1(M)$ either has infinitely many sinks or sources or else is Axiom A without cycles [12], [11] (this is not true in general [6]). Allowing singularities such a result can be improved as follows. Let $\mathcal{V}^1(M) \subset \mathcal{X}^1(M)$ be the set of vector fields *that cannot be C^1 approximated by homoclinic loops.* By the Connecting Lemma [7], every $X \in \mathcal{V}^1(M)$ has singularities separated from the non-wandering set. So, using [21] and [7], we conclude that a generic vector field in $\mathcal{V}^1(M)$ either has infinitely many sinks or sources or else it is Axiom A without cycles. Note that this kind of result does not describe the dynamics when the vector field is approximated by homoclinic loops.

In this paper we shall consider a class of vector fields including both the Axiom A ones and the geometric Lorenz attractors. The elements of this class are called singular Axiom A and are characterized by the fact that its nonwandering set can be written as a finite number of compact invariant sets, each one being either a hyperbolic basic set or a singular hyperbolic attractor or a singular hyperbolic repeller. This definition differs slightly from the one given in [16]. Our main result is:

Theorem A . *A generic vector field $X \in \mathcal{X}^1(M)$ satisfies (only) one of the following properties:*

1. *X has infinitely many sinks or sources.*
2. *X is singular Axiom A without cycles.*

Let us decribe two immediate consequences of this theorem. The first one is related with the abundance of three-dimensional vector fields exhibiting either attractors or repellers.

Corollary 2.1. *A generic vector field in $\mathcal{X}^1(M)$ does exhibit either attractors or repellers.*

As observed in [12], a generic C^1 diffeomorphism in the 2-sphere S^2 does exhibit either sinks or sources. It is then natural to ask whether such a result is valid for C^1 vector fields in the 3-sphere S^3 instead of C^1 diffemorphisms in S^2. The answer is negative, as shows the following example. Write $S^3 = \mathbb{R}^3 \cup \{\infty\}$ and consider in $\mathbb{R}^3$ an unkotted two-torus T^2. Then the closure in S^3 of each connected component of $S^3 \setminus T^2$ is a solid two-torus. Now consider a Lorenz attractor in one of the solid two-torus and a Lorenz repeller in the other one. Since a fundamental domain for the Lorenz attractor (respectively repeller) is an unknotted solid two-torus, we can glue two solid two-torus through the unknotted one obtaining a flow in S^3 which exhibits as nonwandering set one Lorenz attractor and one Lorenz repeller. Such a flow is singular Axiom A but it can not be approximated by vector fields with either sinks or sources.

The second one is related with a conjecture by Palis [18], asserting the denseness of vector fields exhibiting a finite number of attractors whose basin of attraction form a full Lebesgue measure set. Indeed, Theorem A gives an approach to this conjecture in the (open) set $\mathcal{N}^1(M)$ of C^1 vector fields on a closed 3-manifold M which cannot be C^1 approximated by ones exhibiting infinitely many sinks or sources.

Corollary 2.2. *A generic vector field in $\mathcal{N}^1(M)$ exhibits a finite number of attractors whose basins of attraction form an open-dense set.*

We observe that it was proved recently that every C^1 diffeomorphism on a closed surface can be C^1 approximated by a diffeomorphism exhibiting a homoclinic tangency or by an Axiom A diffemorphism without cycles [20]. This result and Theorem A lead us to formulate the following conjecture.

Conjecture 2.3. *Every C^1 vector field on a closed 3-manifold can be C^1 approximated by a vector field exhibiting a homoclinic tangency or by a singular Axiom A one without cycles.*

We are thankful to M. Viana for a simplification of one of our proofs.

3 Preliminaries

In this section we recall some definitions, and some results proved elsewhere that will be used here. We shall also fix some notations.

If $X \in \mathcal{X}^1(M)$, we denote by X_t the flow generated by X. We say that $p \in M$ is *nonwandering* for X if for every $T > 0$ and every neighborhood U of p there is $t > T$ such that $X_t(U) \cap U \neq \emptyset$. The set of nonwandering points of X is denoted by $\Omega(X)$. If $q \in M$, we define $\omega_X(q)$ as the set of accumulation points of the positive orbit $\{X_t(q) : t \geq 0\}$ of q. We also define $\alpha_X(q) = \omega_{-X}$, where $-X$ is the time reversed vector field X. It is immediate that $\omega_X(q) \cup \alpha_X(q) \subset \Omega(X)$ for every $q \in M$.

An *orbit* of X is a set $O = O_X(q) = \{X_t(q) : t \in I\!R\}$ for some $q \in M$. A *singularity* of X is a point $\sigma \in M$ such that $X(\sigma) = 0$ (equivalently $O_X(\sigma) = \{\sigma\}$). A *periodic orbit* of X is an orbit $O = O_X(p)$ such that $X_T(p) = p$ for some minimal $T > 0$ (equivalently O is compact and $O \neq \{p\}$).

A subset Λ of M is *invariant* for X if $X_t(\Lambda) = \Lambda$, $\forall t \in I\!R$. For every compact invariant set Λ of X, we define

$$W^s_X(\Lambda) = \{q \in M : \omega_X(q) \subset \Lambda\},$$

and

$$W^u_X(\Lambda) = \{q \in M : \alpha_X(q) \subset \Lambda\}.$$

Recall that a compact invariant set Λ is *transitive* if $\Lambda = \omega_X(q)$ for some $q \in \Lambda$, and *attracting* if $\Lambda = \cap_{t \geq 0} X_t(U)$ for some neighborhood U of Λ satisfying

$X_t(U) \subset U$, $\forall t \geq 0$. An *attractor* of X is a transitive attracting set of X and a *repeller* is an attractor for $-X$. A *sink* of X is a trivial attractor of X, namely it reduces to a single orbit of X. A *source* of X is a trivial repeller of X.

Denote by $m(T) = \inf_{\|v\|=1} \|T(v)\|$ the *minimum norm* of a linear operator T [8]. A compact invariant set H of X is *hyperbolic* if it exhibits a continuous invariant tangent bundle decomposition $T_H M = E^s_H \oplus E^X_H \oplus E^u_H$ and there are constants $\lambda, K > 0$ such that

- E^s_H *is* (K, λ)*-contracting*, i.e.

$$\|DX_t(x)/E^s_x\| \leq K^{-1} e^{-\lambda t}, \quad \forall x \in H, \quad \forall\, t \geq 0.$$

- E^u_H *is* (K, λ)*-expanding*, i.e.

$$m(DX_t/E^u) \geq K e^{\lambda t}, \quad \forall x \in H, \quad \forall\, t \geq 0.$$

- E^X_x is the subspace in $T_x M$ generated by $X(x)$.

By the Invariant Manifold Theory [8] it follows that for every $p \in H$ the sets

$$U_X(p) = \{q \in M : \mathrm{dist}(X_t(q), X_t(p)) \to 0, t \to \infty\}$$

and

$$S_X(p) = \{q \in M : \mathrm{dist}(X_t(q), X_t(p)) \to 0, t \to -\infty\}$$

are invariant C^r-manifolds tangent to E^s_p and E^u_p respectively. Here dist is the distance on M. If $O = O_X(p) \subset H$ is an orbit of X one has that

$$W^s_X(O) = \cup_{t \in I\!R} S_X(X_t(p))$$

and

$$W^u_X(O) = \cup_{t \in I\!R} U_X(X_t(p))$$

are invariant C^r-manifolds tangent to $E^s_p \oplus E^X_p$ and $E^X_p \oplus E^u_p$ respectively. We shall denote $W^s_X(p) = W^s_X(O_X(p))$ and $W^u_X(p) = W^u_X(O_X(p))$ for the sake of simplicity.

A singularity (resp. periodic orbit) of X is hyperbolic if its orbit is a hyperbolic set of X. Note that $S_X(\sigma) = W^s_X(\sigma)$ and $U_X(\sigma) = W^u_X(\sigma)$ for every hyperbolic singularity σ of X. A hyperbolic set H of X is called *basic* if it is transitive and *isolated*, namely $H = \cap_{t \in I\!R} X_t(U)$ for some neighborhood U of H. It follows from the Shadowing Lemma [17] that every hyperbolic basic set of X either reduces to a singularity or else has no singularities and is the closure of its periodic orbits.

We say that X is *Axiom A* if $\Omega(X)$ is both hyperbolic and the closure of its periodic orbits and singularities. The Spectral Decomposition Theorem asserts that if X is Axiom A, then there is a disjoint decomposition

$$\Omega(X) = H_1 \cup \cdots \cup H_k,$$

where each H_i is a hyperbolic basic set of X, $i = 1, \cdots, k$.

On the other hand, a compact invariant set Λ of X is *partial hyperbolic* if there are a continuous invariant tangent bundle decomposition $T_\Lambda M = E^s_\Lambda \oplus E^c_\Lambda$ and constants $\lambda, K > 0$ such that

- E^s_Λ is (K,λ)-contracting.
- E^s_Λ (K,λ)-*dominates* E^c_Λ, i.e.

$$\|DX_t(x)/E^s_x\| \leq K^{-1}e^{-\lambda t} m(DX_t(x)/E^c_x),$$

  $\forall x \in \Lambda$, $\forall t \geq 0$.

We shall say that $T_\Lambda M = E^s_\Lambda \oplus E^c_\Lambda$ is a (K,λ)-*splitting* for short. For $x \in \Lambda$ and $t \in I\!R$ we let $J^c_t(x)$ be the absolute value of the determinant of the linear map $DX_t(x)/E^c_x : E^c_x \to E^c_{X_t(x)}$. We say that the subbundle E^c_Λ of the partial hyperbolic set Λ is *volume expanding* if

$$J^c_t(x) \geq Ke^{\lambda t},$$

for every $x \in \Lambda$ and $t \geq 0$ (in this case we say that E^c_Λ is (K,λ)-*volume expanding* to indicate the dependence on K, λ). It follows that a nonsingular partial hyperbolic set with volume expanding central direction is hyperbolic [16].

A partial hyperbolic set is *singular hyperbolic* if it has singularities (all of them hyperbolic) and it has volume expanding central direction [15]. A singular hyperbolic attractor is a singular hyperbolic set which is an attractor as well. A *singular hyperbolic repeller* of X is a singular hyperbolic attractor of $-X$. The most representative example of a singular hyperbolic attractor is the Lorenz attractor.

A vector field $X \in \mathcal{X}^1(M)$ is called *singular Axiom A* if there is a finite disjoint decomposition

$$\Omega(X) = \Lambda_1 \cup \cdots \cup \Lambda_n,$$

where each Λ_i is either a hyperbolic basic set or a singular hyperbolic attractor or a singular hyperbolic repeller, $i = 1, \cdots, n$. This definition differs slightly from the one given in [16]. It follows from the Spectral Decomposition Theorem mentioned above that every Axiom A vector field is singular Axiom A.

A *cycle* of a singular Axiom A vector field X is a subcollection $\{\Lambda_{i_0}, \cdots, \Lambda_{i_k}\}$ of $\{\Lambda_1, \cdots, \Lambda_n\}$ such that $i_0 = i_k$ and $W^u_X(\Lambda_{i_j}) \cap W^s_X(\Lambda_{i_{j+1}}) \neq \emptyset$, $\forall 0 \leq j \leq k-1$.

It is clear that every Axion A vector field without cycles is singular Axiom A without cycles.

Recall that a compact set $L \subset M$ is called *Lyapunov stable* for $X \in \mathcal{X}^1(M)$ if for every neighborhood U of L there is a neighborhood $V \subset U$ of L such that $X_t(V) \subset U$, $\forall t \geq 0$. Every attractor is a transitive Lyapunov stable set but not conversely.

The following lemmas summarize some classical properties of Lyapunov stable sets (see Chapter V in [1]).

Lemma 3.1. *Let Λ^+ be a Lyapunov stable set of X. Then,*

1. *If $x_n \in M$ and $t_n \geq 0$ satisfy $x_n \to x \in \Lambda^+$ and $X_{t_n}(x_n) \to y$, then $y \in \Lambda^+$;*
2. $W^u_X(\Lambda^+) \subset \Lambda^+$;
3. *if Γ is a transitive set of X and $\Gamma \cap \Lambda^+ \neq \emptyset$, then $\Gamma \subset \Lambda^+$.*

Lemma 3.2. *A Lyapunov stable set Λ of a vector field X is an attractor of X if and only if there is a neighborhood U of Λ such that $\omega_X(x) \subset \Lambda$, $\forall\, x \in U$.*

4 Proof of Theorem A

Denote by $\mathcal{G}^r(M)$ the set of vector fields $X \in \mathcal{X}^r(M)$ such that every periodic orbit and singularity of every $Y \in \mathcal{X}^r(M)$ C^r-close to X is hyperbolic.

Theorem B . *Generic vector fields in $\mathcal{G}^1(M)$ are singular Axiom A without cycles.*

As in [12], Theorem A follows from the result below. Indeed, let $\mathcal{S}^1(M)$ be the set of vector fields $X \in \mathcal{X}^1(M)$ such that every singularity of X is hyperbolic. It follows that $\mathcal{S}^1(M)$ is open-dense in $\mathcal{X}^1(M)$. If $X \in \mathcal{S}^1(M)$ let $\Lambda(X)$ be the set of periodic orbits and singularities of X that are sinks or sources, and let $\overline{\Lambda}(X)$ its closure. It is clear that the set valued function $X \to \overline{\Lambda}(X)$ is lower semicontinuous. By well known properties about set-valued maps, there is a residual subset $\mathcal{O} \subset \mathcal{S}^1(M)$ such that every $X \in \mathcal{O}$ is a point of upper-semicontinuity of $\overline{\Lambda}$. As in [12] we have that every $X \in \mathcal{O}$ non satisfying (1) of Theorem A is in $\mathcal{G}^1(M)$. Now, by Theorem B there is $\mathcal{R} \subset \mathcal{G}^1(M)$ residual such that every vector field in $\mathcal{R}$ is singular Axiom A without cycles. Define

$$\mathcal{O}' = \big(\mathcal{O} \cap (\mathcal{S}^1(M) \setminus \mathrm{Cl}(\mathcal{G}^1(M)))\big) \cup (\mathcal{O} \cap \mathcal{R}),$$

where $\mathrm{Cl}(\mathcal{G}^1(M))$ is the closure of $\mathcal{G}^1(M)$ in $\mathcal{X}^1(M)$. It follows that $\mathcal{O}'$ is residual in $\mathcal{S}^1(M)$. If $X \in \mathcal{O}'$ does not satisfy the option (1) of Theorem A , then

$X \in \mathcal{G}^1(M)$ since $X \in \mathcal{O}$. It follows that $X \in \mathcal{O} \cap \mathcal{R}$ since $X \in \mathcal{O}'$. Thus X is singular Axiom A without cycles by the definition of $\mathcal{R}$ and the proof follows.

We prove Theorem B in the next section using Theorems C and Theorem D below. Before we announce these theorems, let us recall that it was proved in [14] that for generic $X \in \mathcal{G}^1(M)$ every nontrivial attractor with singularities of X is singular hyperbolic. Theorem C asserts that the same holds replacing attractor by transitive Lyapunov stable set. The proof is the same as in [14] and we include it in the Appendix for completeness.

Theorem C . *For generic vector fields $X \in \mathcal{G}^1(M)$ every nontrivial transitive Lyapunov stable set with singularities of X is singular hyperbolic for X.*

Now, it is interesting the question whether a Lyapunov stable singular hyperbolic transitive set is an attractor. A positive answer for this question is given as follows. Let Λ be a compact invariant set with singularities (all of them hyperbolic) of $X \in \mathcal{X}^1(M)$. We say that Λ *has dense singular unstable branches* if for every singularity σ of X in Λ one has

$$\Lambda = \omega_X(q), \quad \forall q \in W^u_X(\sigma) \setminus \{\sigma\}.$$

Obviously a compact invariant set with dense singular unstable branches is a transitive set.

Theorem D . *Every Lyapunov stable singular hyperbolic set with dense singular unstable branches of $X \in \mathcal{X}^1(M)$ is an attractor of X.*

We prove Theorem D in Section 6.

5 Proof of Theorem B

In this section we shall prove Theorem B using Theorems C and D . For this we shall use the following notation. If A is a subset of M we denote $\mathrm{Cl}(A)$ the closure of A, and $\mathrm{int}(A)$ the interior of A. When A is a submanifold we denote by $\dim(A)$ the dimension of A.

Given $X \in \mathcal{X}^1(M)$ we denote by $\mathrm{Sing}(X)$ the set of singularities of X and by $\mathrm{Per}(X)$ the union of the periodic orbits of X. If A is invariant we denote $\mathrm{Per}_X(A) = \mathrm{Per}(X) \cap A$ and $\mathrm{Sing}_X(A) = \mathrm{Sing}(X) \cap A$.

Observe that if $\sigma \in \mathrm{Sing}(X) \cap \mathrm{Cl}(\mathrm{Per}(X))$ then σ is neither a sink nor a source of X. Note also that as $\dim M = 3$, if σ is hyperbolic then either $\dim(W^u_X(\sigma)) = 1$ or $\dim(W^s_X(\sigma)) = 1$.

Let us recall some properties for generic vector fields $X \in \mathcal{X}^1(M)$ proved elsewhere.

(a) X is *Kupka-Smale*, i.e. every periodic orbit and singularity of X is hyperbolic and the corresponding invariant manifolds intersect transversally [4]. In particular, $\mathrm{Sing}(X)$ is a finite set.

(b) $\Omega(X) = \mathrm{Cl}(\mathrm{Per}(X) \cup \mathrm{Sing}(X))$ (see [19]).

(c) $\mathrm{Cl}(W^u_X(\sigma))$ is Lyapunov stable for X, $\forall \sigma \in \mathrm{Sing}(X)$ (see [3]).

(d) $\mathrm{Cl}(W^s_X(\sigma))$ is Lyapunov stable for $-X$, $\forall \sigma \in Sing(X)$ (see [3]).

(e) If $\sigma \in \mathrm{Sing}(X)$ and $\dim(W^u_X(\sigma)) = 1$ then $\omega_X(q)$ is Lyapunov stable for X, $\forall q \in W^u_X(\sigma) \setminus \{\sigma\}$ (see [3]).

(f) If $\sigma \in \mathrm{Sing}(X)$ and $\dim(W^s_X(\sigma)) = 1$ then $\alpha_X(q)$ is Lyapunov stable for $-X$, $\forall q \in W^s_X(\sigma) \setminus \{\sigma\}$ (see [3]).

Lemma 5.1. *For generic vector fields $X \in \mathcal{X}^1(M)$, a transitive Lyapunov stable set with singularities Λ of X satisfying*

$$\dim(W^u_X(\sigma)) = 1, \quad \forall \sigma \in \mathrm{Sing}_X(\Lambda)$$

has dense singular unstable branches.

Proof. As X is generic we can assume that X satisfies the properties (a)-(f) above (in particular X is Kupka-Smale). Let Λ be a transitive Lyapunov stable set of X, $\sigma \in \mathrm{Sing}_X(\Lambda)$ and $q \in W^u_X(\sigma) \setminus \{\sigma\}$. As Λ is Lyapunov stable we have that $W^u_X(\sigma) \subset \Lambda$. In particular, $\omega_X(q) \subset \Lambda$. On the other hand, we have that $\dim(W^u_X(\sigma)) = 1$ by assumption. Then, $\omega_X(q)$ is Lyapunov stable by Property (e). As $\Lambda \cap \omega_X(q) \neq \emptyset$ and Λ is transitive, we conclude, by Lemma 3.1(3), that $\Lambda \subset \omega_X(q)$. Then $\omega_X(q) = \Lambda$ proving that Λ has dense singular unstable branches as desired.

□

Another preliminary lemma is given below.

Lemma 5.2. *Let $X \in \mathcal{X}^1(M)$ and $\sigma \in \mathrm{Sing}(X) \cap \mathrm{Cl}(\mathrm{Per}(X))$ be such that*

$$\dim(W^u_X(\sigma)) = 1.$$

If $\omega_X(q)$ is Lyapunov stable for X, $\forall q \in W^u_X(\sigma) \setminus \{\sigma\}$, then $\mathrm{Cl}(W^u_X(\sigma))$ is transitive.

Proof. As $\dim(W^u_X(\sigma)) = 1$ we have that

$$W^u_X(\sigma) \setminus \{\sigma\} = O_X(q_1) \cup O_X(q_2)$$

for every q_1, q_2 belonging to different connected components of $W^u_X(\sigma) \setminus \{\sigma\}$. As $\sigma \in \mathrm{Cl}(\mathrm{Per}(X))$ we can assume that $q_1 \in \mathrm{Cl}(\mathrm{Per}(X))$ without loss of generality. On the other hand, $\omega_X(q_1)$ is Lyapunov stable for X by assumption. This fact and $q_1 \in \mathrm{Cl}(\mathrm{Per}(X))$ imply that $\sigma \in \omega_X(q_1)$. Thus, $W^u_X(\sigma) \subset \omega_X(q_1)$ by the Lyapunov stability of $\omega_X(q_1)$ once more. In particular, $q_1, q_2 \in \omega_X(q_1)$ and so $W^u_X(\sigma) \subset \omega_X(q_1)$. Taking closure we conclude $\mathrm{Cl}(W^u_X(\sigma)) \subset \omega_X(q_1)$ proving that $\mathrm{Cl}(W^u_X(\sigma)) = \omega_X(q_1)$ and so $\mathrm{Cl}(W^u_X(\sigma))$ is transitive.

□

The following theorem improves the main result of [13].

Theorem 5.3. *For generic $X \in \mathcal{G}^1(M)$ every $\sigma \in \mathrm{Sing}(X) \cap \mathrm{Cl}(\mathrm{Per}(X))$ belongs to a singular hyperbolic attractor or a singular hyperbolic repeller of X.*

Proof. Let $\sigma_0 \in \mathrm{Sing}(X) \cap \mathrm{Cl}(\mathrm{Per}(X))$ of a generic $X \in \mathcal{G}^1(M)$. By the Property (a) σ_0 is hyperbolic, and so, either $\dim(W^u_X(\sigma_0)) = 1$ or $\dim(W^s_X(\sigma_0)) = 1$. Assuming that $\dim(W^u_X(\sigma_0)) = 1$ we shall prove that $\Lambda = \mathrm{Cl}(W^u_X(\sigma_0))$ is a singular hyperbolic attractor of X. Similar arguments (reversing the flow direction of X) show that $\mathrm{Cl}(W^s_X(\sigma_0))$ is a singular hyperbolic repeller when $\dim(W^s_X(\sigma_0)) = 1$.

Thus, suppose that $\dim(W^u_X(\sigma_0)) = 1$ and define $\Lambda = \mathrm{Cl}(W^u_X(\sigma_0))$. By the Property (c) we have that Λ is Lyapunov stable for X. As $\sigma_0 \in \mathrm{Sing}_X(\Lambda) \cap \mathrm{Cl}(\mathrm{Per}(X))$ and $\dim(W^u_X(\sigma_0)) = 1$, Lemma 5.2 and Property (e) imply that Λ is transitive. Clearly Λ is nontrivial since σ_0 is neither a sink nor a source. In conclusion, Λ is a nontrivial transitive Lyapunov stable set of X. As X is generic, Theorem C implies that Λ is singular hyperbolic. By [16] (see also Theorem 6.4) we have $\dim(W^u_X(\sigma)) = 1$ for every $\sigma \in \mathrm{Sing}_X(\Lambda)$. We conclude that Λ has dense singular unstable branches by Lemma 5.1 since X is generic. Then, Λ is an attractor by Theorem D and the proof follows.

□

Proof of Theorem B : For $X \in \mathcal{G}^1(M)$ we denote

$$\mathrm{Sing}^*(X) = \mathrm{Sing}(X) \cap \mathrm{Cl}(\mathrm{Per}(X)) = \{\sigma_1 \cdots, \sigma_k\}.$$

Assume that X is generic. Theorem (5.3) implies that for every $i = 1, \cdots, k$ there is a compact invariant set Λ_i of X such that $\sigma_i \in \Lambda_i$ and Λ_i is either a singular hyperbolic attractor or a singular hyperbolic repeller of X. Denote

$$H^* = \Omega(X) \setminus \cup_{i=1}^k \Lambda_i.$$

We claim that H^* splits into a finite disjoint union of hyperbolic basic sets. Indeed, we have that $H^* \setminus \mathrm{Sing}(X)$ is closed in M for, otherwise, there would exist $\sigma \in \mathrm{Sing}(X)$ and a sequence $x_n \in H^* \setminus \mathrm{Sing}(X)$ converging to σ. By the

Property (b) we would have $\sigma \in \mathrm{Cl}(\mathrm{Per}(X))$ since $\mathrm{Sing}(X)$ is a finite set. In particular, $\sigma \in \mathrm{Sing}^*(X)$ and so $\sigma \in \Lambda_i$ for some $i = 1, \cdots, k$. As Λ_i is either attractor or repeller we would have that $x_n \in \Lambda_i$ for n large, a contradiction since $x_n \in H^* = \Omega(X) \setminus \cup_{i=1}^k \Lambda_i$ for all n. This proves that $H^* \setminus \mathrm{Sing}(X)$ is closed in M. Applying [21] we obtain that H^* is a hyperbolic set of X. On the other hand, the Property (b) implies $H^* = \mathrm{Cl}(\mathrm{Per}_X(H^*)) \cup \mathrm{Sing}_X(H^*)$. Then, the claim follows from the Spectral Decomposition Theorem (see [17]). This proves the claim.

By the claim $\Omega(X)$ splits into a disjoint union of compact invariant sets each one being either a hyperbolic basic set or Λ_i for some i. As each Λ_i is either a singular hyperbolic attractor or a singular hyperbolic repeller $\forall i$ we conclude that X is singular Axiom A. As X is Kupka-Smale (Property (a)) it follows that X has no cycles. This proves the result.

6 Proof of Theorem D

Theorem D follows from the following propositions.

Proposition 6.1. *Let Λ be a Lyapunov stable singular hyperbolic set with dense singular unstable branches of $X \in \mathcal{X}^r(M)$. If there is $q \in \Lambda$ such that $\omega_X(q)$ has no singularities, then Λ is an attractor of X.*

Proposition 6.2. *Let Λ be a Lyapunov stable singular hyperbolic set with dense singular unstable branches of $X \in \mathcal{X}^r(M)$. Then, either Λ is an attractor of X or else there is $q \in \Lambda$ such that $\omega_X(q)$ has no singularities.*

6.1 Proof of Proposition 6.1

In this section we shall prove Proposition 6.1. Before we start with the proof, let us recall some results and notations given elsewhere.

The next definition is in [15].

Definition 6.3. *A singularity σ of X is* Lorenz-like *if σ has three real eigenvalues $\lambda_1, \lambda_2, \lambda_3$ satisfying*

$$\lambda_2 < \lambda_3 < 0 < -\lambda_3 < \lambda_1.$$

By the Invariant Manifold Theory it follows that the eigenspace associated to λ_2 is tangent to a one-dimensional invariant manifold of X denoted by $W^{ss}_X(\sigma)$. This manifold is called *strong stable manifold* of σ. As usual we denote by $W^s_{loc}(\sigma)$ a small ball in $W^s_X(\sigma)$ centered at σ. Similarly we define $W^u_{loc}(\sigma)$.

In the remainder of this section we let Λ be a transitive singular hyperbolic set of X.

Theorem 6.4. *([15]) Every singularity σ of X in Λ is Lorenz-like and satisfies*

$$\Lambda \cap W_X^{ss}(\sigma) = \{\sigma\}.$$

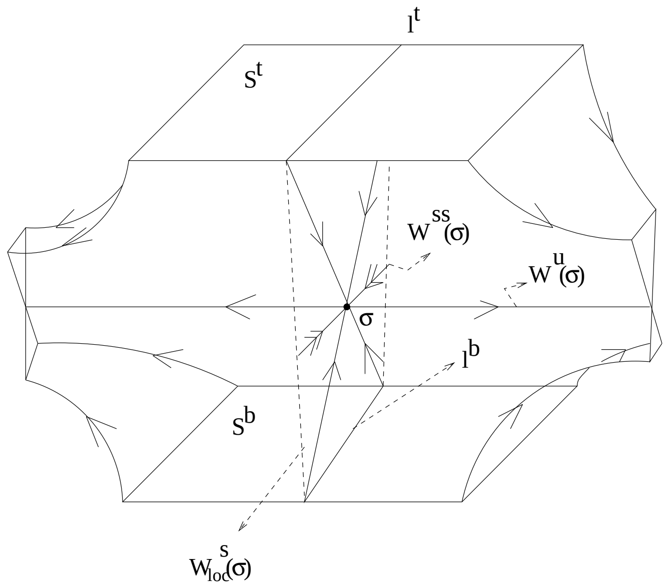

Figure 1: Lorenz-like singularity.

This result and the Hartman Grobman Theorem [4] allow us to describe the flow of X around a singularity $\sigma \in \Lambda$ (see Figure 1). Indeed, σ has a two-dimensional stable manifold $W_X^s(\sigma)$, a strong stable manifold $W_X^{ss}(\sigma)$, and an unstable manifold $W_X^u(\sigma)$ both one-dimensional.

It follows that $W_X^{ss}(\sigma)$ separates $W_{loc}^s(\sigma)$ in two connected components denoted the top and the bottom respectively. In one of these components, say the top one, we consider a cross-section S^t together with a curve l^t as in Figure 1. Similarly we consider a cross-section S^b and a curve l^b located in the bottom component of $W_{loc}^s(\sigma)$. See Figure 1. Both S^t and S^b are homeomorphic to $[0,1] \times [0,1]$. S^t and S^b are chosen in a way that l^t and l^b are contained in $W_X^s(\sigma) \setminus W_X^{ss}(\sigma)$. The positive flow lines of X starting at $S^t \cup S^b \setminus (l^t \cup l^b)$ exit a small neighborhood of σ passing through the cusp region as indicated in Figure 1. The positive orbits starting at $l^t \cup l^b$ goes directly to σ. We note that the boundary of S^t is formed by four curves, two of them being transverse to l^t. We denote by $\partial^h S^t$ the union of such curves. Similarly we define $\partial^h S^b$.

Definition 6.5. *Let σ be a Lorenz-like singularity of X. The cross-sections S^t, S^b introduced above are called* singular cross-sections *associated to σ. The curves l^t, l^b are called* singular curves *of S^t, S^b respectively.*

Remark 6.6. *A first consequence of Theorem 6.4 is the following. Let U a neighborhood of Λ and $\sigma \in \Lambda$ be a singularity of X. Then, there are singular cross-sections S^t, S^b associated to σ such that $S^t \cup S^b \subset U$ and*

$$\Lambda \cap (\partial^h S^t \cup \partial^h S^b) = \emptyset.$$

Another property to be used later is given below. A hyperbolic set is *saddle-type* if $\dim E^s = \dim E^u = 1$.

Lemma 6.7. *([15]) Every compact invariant nonsingular set of X in Λ is a saddle-type hyperbolic set.*

Next we shall assume that Λ is Lyapunov stable. Denote by $T_\Lambda M = E^s_\Lambda \oplus E^c_\Lambda$ the splitting of Λ. It follows that E^s_Λ extends to an invariant contracting splitting $E^s_{U(\Lambda)}$ on a neighborhood $U(\Lambda)$ of Λ and $E^s_{U(\Lambda)}$ is integrable in $U(\Lambda)$ ([8]). We extend E^c_Λ continuously to a subbundle $E^c_{U(\Lambda)}$ of $T_{U(\Lambda)}M$. *In what follows we fix such a neighborhood $U(\Lambda)$ of Λ.*

We denote $\mathcal{F}^s$ the foliation obtained by integrating $E^s_{U(\Lambda)}$ and by $\mathcal{F}^s(x)$ the leave of $\mathcal{F}^s$ passing through x. Denote D^2 the two-disc in $\mathbb{R}^2$. If S is contained in an embedding $D^2 \to M$ transverse to X, we denote by $\mathcal{F}^s_S$ the foliation of S obtained projecting $\mathcal{F}^s(x)$ into S along the flow of X.

The next definition follows closely [2].

Definition 6.8. *A* s-rectangle *of Λ is a closed set $S \subset M$, contained in the image of an embedding $D^2 \to M$ transverse to X, such that there is a homeomorphism $h : [0,1] \times [0,1] \to S$ mapping $\{s\} \times [0,1]$ $\forall s \in [0,1]$ into leaves of $\mathcal{F}^s_S$.*

If $S = h([0,1] \times [0,1])$ is a s-rectangle, we define the *interior* and the *boundary* of S by $\mathrm{int}(S) = h((0,1) \times (0,1))$ and $\partial S = h([0,1] \times \{0,1\} \cup \{0,1\} \times [0,1])$ respectively. It follows that ∂S is formed by four segments

$$\partial^h S = [0,1] \times \{0,1\}, \quad \partial^v S = \{0,1\} \times [0,1].$$

A s-rectangle is always assumed to be contained in $U(\Lambda)$. We define $\mathrm{diam}(S)$, the diameter of S, by

$$\mathrm{diam}(S) = \sup_{x \in \partial^v S} \inf\{\mathrm{dist}(x,y) : x, y \text{ are not in the same component of } \partial^v S\},$$

where dist is the distance in M.

Lemma 6.9. *Let Λ be a Lyapunov stable singular hyperbolic set with dense singular unstable branches of X. If Λ exhibits a s-rectangle S satisfying*

$$\mathrm{int}(S) \cap \mathcal{F}^s_S(x) \cap \Lambda \neq \emptyset, \quad \forall x \in \mathrm{int}(S), \tag{1}$$

then Λ is an attractor of X.

Proof. By Lemma 3.2 it suffices to prove that if $x_n \in M$ is a sequence converging to $p \in \Lambda$, then $\omega_X(x_n) \subset \Lambda$ for every n large. By contradiction assume that $\omega_X(x_n)$ is not contained in Λ. We have two cases :

Case 1: $\omega_X(p)$ has no singularities. Then $\omega_X(p)$ is a saddle-type hyperbolic set by Lemma 6.7. Choose $y \in \omega_X(p)$. Then, $W^u_X(y)$ is well defined and has dimension two. Let S_y be a cross-section of X containing y. As Λ is Lyapunov stable we have that $W^u_X(y) \subset \Lambda$. Choosing S_y small enough, there is $\epsilon > 0$ such that $\forall\, x \in V_y = X_{(0,\epsilon)}S_y$, there is $z \in W^u_X(y) \subset \Lambda$ so that $x \in W^s_X(z)$. As $y \in \omega_X(p)$ we have that the positive orbit of p intersects V_y. This implies that the positive orbit of x_n also intersects V_y for n large because $x_n \to p$. In particular, for every n large there is $z_n \in \Lambda$ such that $x_n \in W^s_X(z_n)$. Thus, $\omega_X(x_n) = \omega_X(z_n) \subset \Lambda$, contradicting the assumption that $\omega_X(x_n)$ is not contained in Λ.

Case 2: $\omega_X(p)$ contains a singularity σ of X. As $p \in \Lambda$ we have that $\sigma \in \Lambda$. As $x_n \to p$ and $\omega_X(x_n)$ is not contained in Λ, there is $q \in W^u_X(\sigma) \setminus \{\sigma\}$ such that the positive orbit of x_n passes close to q for n large. As Λ has dense singular unstable branches we have $\Lambda = \omega_X(q)$, and so, the positive orbit of q intersects $\mathrm{int}(S)$. Since S is a s-rectangle of Λ satisfying (1), we conclude as before that for every n large there is $z_n \in \Lambda$ such that $x_n \in W^s_X(z_n)$. This contradicts once more the assumption that $\omega_X(x_n)$ is not contained in Λ. This completes the proof.

□

Corollary 6.10. *Let Λ be a Lyapunov stable singular hyperbolic set with dense singular unstable branches. Suppose that Λ is not an attractor. If S^t_0, S^b_0 are singular cross-sections associated to a singularity $\sigma \in \Lambda$ and*

$$\Lambda \cap (\partial^h S^t_0 \cup \partial^h S^b_0) = \emptyset,$$

then there are singular cross-sections S^t, S^b of σ contained in S^t_0, S^b_0 respectively, such that

$$\Lambda \cap (\partial S^t \cup \partial S^b) = \emptyset.$$

Proof. Let Λ a singular hyperbolic set of X and S^t_0, S^b_0 be singular cross-sections associated to $\sigma \in \Lambda$ as in the statement. Let F be a connected component of $\partial^v S^t_0$. If there is no $F^t \in \mathcal{F}^s_S$ close to F satisfying $\Lambda \cap F^t = \emptyset$, then one could construct a s-rectangle $S \subset S^t_0$ close to F satisfying (1). As Λ has dense singular unstable branches it would follow from Lemma 6.9 that Λ is an attractor. This is a contradiction because we are assuming that Λ is not an attractor. This proves the existence of a leave F^t of $\mathcal{F}^s_S$ arbitrarily close to F such that $\Lambda \cap F^t = \emptyset$. Similarly, if G denotes the another connected component of $\partial^v S^t_0$, there is a leave $G^t \in \mathcal{F}^s_S$ close to G such that $\Lambda \cap G^t = \emptyset$. Let S^t be the s-rectangle of Λ such that $S^t \subset S^t_0$, $\partial^h S^t \subset \partial^h S^t_0$ and $\partial^v S^t = F^t \cup G^t$. Then, S^t satisfies the required properties. We construct S^b in a similar way. This finishes the proof.

□

Proof of Proposition 6.1: Let Λ be a Lypunov stable singular hyperbolic set with dense singular unstable branches of X. Suppose that there is $q \in \Lambda$ such that $\omega_X(q) \cap \mathrm{Sing}_X(\Lambda) = \emptyset$. Then, by Lemma 6.7 $H = \omega_X(q)$ is a saddle-type hyperbolic set of X. Choose $y \in H$. As H is saddle-type we have that y has a nontrivial two-dimensional unstable manifold $W^u_X(y)$. As Λ is Lyapunov stable we have $W^u_X(y) \subset \Lambda$. Using $W^u_X(y)$ as in the proof of Lemma 6.9 (Case 1), we construct a s-rectangle $S = S_y$ of Λ satisfying (1). So, Λ is an attractor by Lemma 6.9 and the result follows. □

7 Proof of Proposition 6.2

Let us introduce some definitions and preliminary facts before the proof.

Notation 7.1. *Let X be a C^r vector field and Λ a transitive Lyapunov stable singular hyperbolic set of X. Let $\mathcal{S} = \{R_1, \cdots, R_r\}$ be a finite disjoint collection of s-rectangles of Λ. We shall use the following notation :*

- $\mathcal{S} = \cup_{j=1}^r R_j$.
- $\sigma_{\mathcal{S},*} = \{q \in \Lambda : X_{I\!R}(q) \cap \mathcal{S} = \emptyset\}$.
- $l_{\mathcal{S},*} = \{q \in \Lambda \cap \mathcal{S} : X_t(q) \notin \mathcal{S}, \forall t > 0\}$.
- $\mathrm{int}(\mathcal{S}) = \cup_{j=1}^r \mathrm{int}(R_j)$.
- $\partial \mathcal{S} = \cup_{j=1}^r \partial R_j$, $\partial^v \mathcal{S} = \cup_{j=1}^r \partial^v R_j$, *and* $\partial^h \mathcal{S} = \cup_{j=1}^r \partial^h R_j$.

We shall denote
$$\mathrm{Sing}_X(\Lambda) = \{\sigma_1, \cdots, \sigma_k\}$$
the set of singularities of X in Λ.

Definition 7.2. *Given $X \in \mathcal{X}^r(M)$ and a transitive Lyapunov stable singular hyperbolic set Λ of X, a* singular partition *of Λ is a disjoint collection*
$$\mathcal{S} = \{S_1^t, S_1^b, \cdots, S_k^t, S_k^b\}$$
of s-rectangles of Λ satisfying (see Figure 2):

1. *For every $i = 1, \cdots, k$ the s-rectangles S_i^t, S_i^b are singular cross-sections associated to σ_i (recall Definition 6.5).*
2. *$\Lambda \cap \partial \mathcal{S} = \emptyset$.*

3. $\sigma_{\mathcal{S},*} = \mathrm{Sing}_X(\Lambda)$.

Remark 7.3. *Note that Properties 1, 2, and 3 above imply*

4. For every $i = 1, \cdots, k$ *if* l_i^t, l_i^b *denotes the singular curves of* S_i^t, S_i^b *respectively, then* l_i^t, l_i^b *are leaves of* $\mathcal{F}^s_{\mathcal{S}}$ *and*

$$l_{\mathcal{S},\Lambda} = \Lambda \cap l,$$

where $l = l_{\mathcal{S}} = \cup_{i=1}^k (l_i^t \cup l_i^b)$.

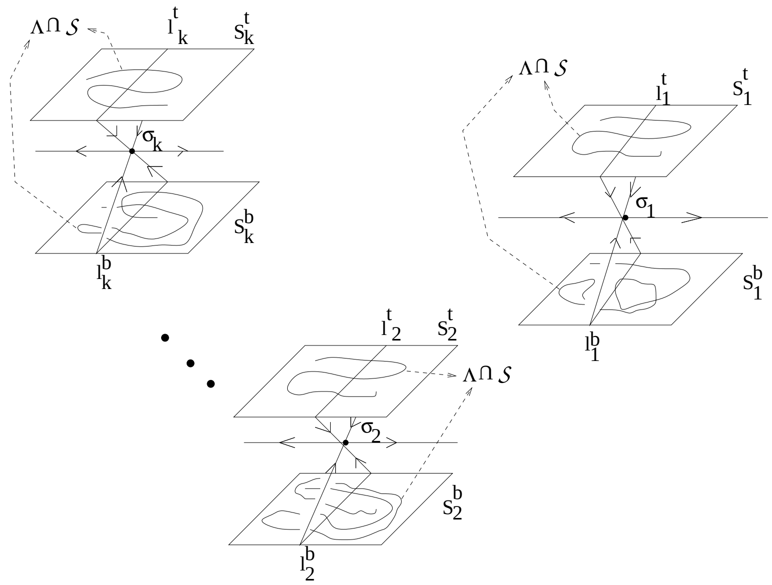

Figure 2: Singular partition.

Given a singular partition $\mathcal{S}$ we set

$$\mathrm{diam}(\mathcal{S}) = \sup\{\mathrm{diam}(S) : S \in \mathcal{S}\}.$$

A singular partition $\mathcal{S}$ of Λ,defines a return map

$$\Pi : \mathrm{Dom}(\Pi) \subset \mathcal{S} \to \mathcal{S}$$

by

$$\Pi(x) = X_{t(x)}(x),$$

where $\mathrm{Dom}(\Pi)$ denotes the domain of $\mathcal{S}$ and $t(x)$ denotes the return time of x. Clearly Π preserves $\mathcal{F}^s_{\mathcal{S}}$ and $\mathrm{Dom}(\Pi) \cap l = \emptyset$.

Lemma 7.4. *Let Λ a Lyapunov stable singular hyperbolic set of X and $\mathcal{S}$ be a singular partition of Λ with return map Π. Then, there is a neighborhood O of $\Lambda \cap \mathcal{S}$ in $\mathcal{S}$ such that if $B \subset O$ and $n \in I\!N$ satisfy*

$$B, \Pi(B), \Pi^2(B), \cdots, \Pi^{n-1}(B) \subset \mathrm{Dom}(\Pi),$$

then $\Pi^n(B) \setminus \mathrm{Dom}(\Pi) \subset l$.

Proof. By Property 4 in Remark 7.3 we have $(\Lambda \cap \mathcal{S}) \setminus l \subset \mathrm{Dom}(\Pi)$ and $\Pi((\Lambda \cap \mathcal{S}) \setminus l) \subset \mathrm{int}(\mathcal{S})$.

Definition 7.2 (2),(3) imply that we can choose for every $i \in \{1, \cdots, k\}$ and $* = t, s$ a s-rectangle $V_i^* \subset \mathrm{int}(S_i^*)$ satisfying

- $l \cap V_i^*$ is a leave of $\mathcal{F}^s_{V_i^*}$;
- $l \cap \partial^v V_i^* = \emptyset$;
- $V_i^* \setminus l \subset \mathrm{Dom}(\Pi)$.

It follows that

$$\Lambda_0 = (\Lambda \cap \mathcal{S}) \setminus \cup_{i \in \{1,\cdots,k\}, *=t,b} \mathrm{int}(V_i^*)$$

is a compact subset of $\mathcal{S}$ contained in $(\Lambda \cap \mathcal{S}) \setminus l$. Thus,

$$\Pi(\Lambda_0) \subset \Pi(\Lambda \setminus l) \subset \mathrm{int}(\mathcal{S}).$$

As $\Lambda_0 \cap \partial \mathcal{S} = \emptyset$, the Tubular Flow Box Theorem [4] implies that $\forall x \in \Lambda_0$ there is a neighborhood $V_x \subset \mathrm{int}(\mathcal{S})$ of x such that $V_x \subset \mathrm{Dom}(\Pi)$ and $\Pi(V_x) \subset \mathrm{int}(\mathcal{S})$. The family $\{V_x : x \in \Lambda_0\}$ is an open covering of Λ_0 and so it has a finite subcovering $\{V_{x_1}, \cdots, V_{x_r}\}$ by compactness.

Define

$$O_0 = \left(\cup_{j=1}^r V_{x_j}\right) \cup \left(\cup\{\mathrm{int}(V_{x_i}^*) : i \in \{1, \cdots, k\}, * = t, b\}\right).$$

Then, O_0 is a neighborhood of $\Lambda \cap \mathcal{S}$ in $\mathcal{S}$ satisfying $O_0 \setminus l \subset \mathrm{Dom}(\Pi)$ and

$$O_0 \setminus \mathrm{Dom}(\Pi) = \{(O_0 \setminus l) \cup l)\} \setminus \mathrm{Dom}(\Pi) =$$

$$= \{(O_0 \setminus l) \setminus \mathrm{Dom}(\Pi)\} \cup \{l \setminus \mathrm{Dom}(\Pi)\} \subset$$

$$\subset \{\mathrm{Dom}(\Pi) \setminus \mathrm{Dom}(\Pi)\} \cup \{l \setminus \mathrm{Dom}(\Pi)\} \subset \emptyset \cup l = l.$$

So,

$$O_0 \setminus \mathrm{Dom}(\Pi) \subset l.$$

On the other hand, as Λ is Lyapunov stable, there is a neighborhood $O \subset O_0$ of $\Lambda \cap \mathcal{S}$ such that if $x \in O$ and $t > 0$ satisfy $X_t(x) \in \mathcal{S}$, then $X_t(x) \in O_0$. We claim that O satisfies the required properties.

Indeed, let $B \subset O$ and $n \in I\!N$ satisfying

$$B, \Pi(B), \Pi^2(B), \cdots, \Pi^{n-1}(B) \subset \mathrm{Dom}(\Pi),$$

and $y \in \Pi^n(B) \setminus \mathrm{Dom}(\Pi)$. It follows that $y = \Pi^n(x)$ for some $x \in B$. Denoting $t(\cdot)$ the return time we have

$$\Pi^n(x) = X_{t(x)+t(\Pi(x))+\cdots+t(\Pi^{n-1}(x))}(x).$$

We conclude that $y \in O_0$ since $t = t(x) + t(\Pi(x)) + \cdots + t(\Pi^{n-1}(x)) > 0$. As $y \notin \mathrm{Dom}(\Pi)$ we conclude that $y \in O_0 \setminus \mathrm{Dom}(\Pi)$. Thus, $\Pi^n(B) \setminus \mathrm{Dom}(\Pi) \subset O_0 \setminus \mathrm{Dom}(\Pi) \subset l$ as desired.

□

We shall denote by TM the tangent space of M. It follows that TM is endowed with a metric $|| \cdot ||$ induced by an inner product $<,>$.

If $x \in M$, the angle between $v_x, w_x \in T_xM$ is denoted by $\angle(v_x, w_x)$. If L_x is a linear subspace of T_xM, we define

$$\angle(v_x, L_x) = \inf_{w_x \in L_x} \angle(v_x, w_x).$$

Given $\alpha > 0$ we define

$$C_\alpha(L_x) = \{v_x \in T\mathcal{S}_x : \angle(v_x, L_x) \leq \alpha\}.$$

We shall use the Linear Poincaré Flow [5]. If x is a regular point of X (i.e. $X(x) \neq 0$), we denote by $N_x = \{v_x \in T_xM :< v_x, X(x) >= 0\}$ the orthogonal complement of $X(x)$ in T_xM. Denote $O_x : T_xM \to N_x$ the orthogonal projection onto N_x. For every $t \in I\!R$ we define $P_x^t : N_x \to N_{X_t(x)}$ by

$$P_x^t = O_{X_t(x)} \circ DX_t(x).$$

It follows that $P = \{P_x^t : t \in I\!R, X(x) \neq 0\}$ satisfies the cocycle relation

$$P_x^{s+t} = P_{X_t(x)}^s \circ P_x^s,$$

for every $t, s \in I\!R$. The parametrized family P is called the *Linear Poincaré Flow* of X.

We denote by $\mathrm{vol}(v_x, w_x)$ the area of the parallelogram in T_xM generated by $v_x, w_x \in T_xM$. As M is a compact manifold, there is a constant $V \geq 1$ such that

$V^{-1} \leq \mathrm{vol}(v_x, w_x) \leq V$, $\forall x \in M$, $\forall v_x, w_x \in T_xM$ satisfying $\|v_x\| = \|w_x\| = 1$ and $< v_x, w_x >= 1$. For simplicity we shall assume that $V = 1$. In other words,

$$\mathrm{vol}(v_x, w_x) = \|v_x\| \cdot \|w_x\|,$$

$\forall x \in M$, $\forall v_x, w_x \in T_xM$ with $< v_x, w_x >= 0$.

In addition,

$$\mathrm{vol}(v_x, X(x)) = \|O_x v_x\| \cdot \|X(x)\|,$$

$\forall x \in M$ regular, $\forall v_x \in T_xM$. In particular,

$$\mathrm{vol}(DX_t(x)v_x, X(X_t(x))) = \|P_x^t(v_x)\| \cdot \|X(X_t(x))\|, \tag{2}$$

$\forall x \in M$ regular, $\forall t \in I\!R$, $\forall v_x \in N_x$.

Recall that if Λ is a singular hyperbolic set of X with (K, λ)-splitting $T_\Lambda M = E^s_\Lambda \oplus E^c_\Lambda$, then

$$J^c_t(x) \geq K e^{\lambda t},$$

$\forall x \in \Lambda$, $\forall t \geq 0$, where $J^c_t(x)$ is the absolute value of the determinant of the linear map $DX_t(x)/E^c_x$. So, if Λ is a singular hyperbolic set as above one has

$$\mathrm{vol}(DX_t(x)v^c_x, DX_t(x)w^c_x) \geq K e^{\lambda t}\, \mathrm{vol}(v^c_x, w^c_x),$$

$\forall x \in \Lambda$, $\forall t \geq 0$, $\forall v^c_x, w^c_x \in E^c_x$.

Lemma 7.5. *Let Λ be a Lyapunov stable singular hyperbolic set of X. Then for every $\alpha \in (0, 1]$ there are a neighborhood $U_\alpha \subset U(\Lambda)$ of Λ and constants $T_\alpha, K_\alpha, \lambda_\alpha > 0$ such that :*

1. *If $x \in U_\alpha$ and $t \geq T_\alpha$, then*

$$DX_t(x)(C_\alpha(E^c_x)) \subset C_{\frac{\alpha}{2}}(E^c_{X_t(x)}).$$

2. *If $x \in U_\alpha$ is regular, $X(x) \in C_\alpha(E^c_x)$, $t \geq T_\alpha$ and $v_x \in C_\alpha(E_x) \cap N_x$, then*

$$\|P^t_x(v_x)\| \cdot \|X(X_t(x))\| \geq K_\alpha e^{\lambda_\alpha t} \cdot \|v_x\| \cdot \|X(x)\|.$$

Proof. Let Λ and $\alpha \in (0, 1]$ be as in the statement. Recall that $T_{U(\Lambda)}M = E^s_{U(\Lambda)} \oplus E^c_{U(\Lambda)}$ denotes the extension of the (K, λ)-splitting $T_\Lambda M = E^s_\Lambda \oplus E^c_\Lambda$ of Λ to a neighborhood $U(\Lambda)$ of Λ. Let π^s the projection of T_Λ on E^s_Λ, and π^c be the projection of T_Λ on E^c_Λ. Denote $v_x = v^s_x + v^c_x \in E^s_x \oplus E^c_x = T_xM$ $\forall x \in U(\Lambda)$, $\forall v_x \in T_xM$. In another words, $v^s_x = \pi^s(v_x)$ and $v^c_x = \pi^c(v_x)$

As E^s_Λ (K, λ)-dominates E^c_Λ we have that

$$\|DX_t(x)/E^s_x\| \leq K^{-1} e^{-\lambda t} m(DX_t(x)/E^c_x), \tag{3}$$

$\forall x \in \Lambda$, $\forall t \geq 0$.

Fix $R > 4$ such that
$$\frac{K}{R} < 1. \tag{4}$$
Choosing $T^1 = T^1_\alpha > 0$ large enough one has
$$\|DX_{T^1}(x)/E^s_x\| \leq \frac{K}{2R} m(DX_t(x)/E^c_x), \tag{5}$$
$\forall x \in \Lambda$, $\forall t \geq 0$.

Since $E^s_\Lambda \oplus E^c_\Lambda$ is invariant we have $\pi^{s(c)}_{X_t(x)} \circ DX_t(x) = DX_t(x) \circ \pi^{s(c)}_x$, and so
$$\angle(DX_t(x)v_x, E^c_{X_t(x)}) = \frac{\|DX_t(x)v^s_x\|}{\|DX_t(x)v^c_x\|}, \quad \forall x \in \Lambda \quad \forall t \geq 0. \tag{6}$$

The inequality (5) and the last equality imply
$$\angle(DX_{T^1}(x)v_x, E^c_{X_{T^1}(x)}) \leq \frac{K\alpha}{2R}, \quad \forall x \in \Lambda, \ \forall v_x \in C_\alpha(E^c_x).$$
So,
$$DX_{T^1}(x)(C_\alpha(E^c_x)) \subset C_{\frac{K\alpha}{2R}}(E^c_{X_{T^1}}(x)), \quad \forall x \in \Lambda.$$

Choose a neighborhood $U^1 = U^1_\alpha \subset U(\Lambda)$ of Λ sufficiently close to Λ such that
$$DX_{T^1}(x)(C_\alpha(E^c_x)) \subset C_{\frac{K\alpha}{R}}(E^c_{X_{T^1}}(x)), \quad \forall x \in U^1. \tag{7}$$

On the other hand, using (3) we get
$$\frac{\|DX_t(x)v^s_x\|}{\|v^s_x\|} \leq K^{-1}e^{-\lambda t}\frac{\|DX_t(x)v^c_x\|}{\|v^c_x\|}$$
and so,
$$\frac{\|DX_t(x)v^s_x\|}{\|DX_t(x)v^c_x\|} \leq K^{-1}e^{-\lambda t}\frac{\|v^s_x\|}{\|v^c_x\|} = K^{-1}e^{-\lambda t}\angle(v_x, E^c_x).$$
So, by (6), we get
$$\angle(DX_r(x)v_x, E^c_{X_r(x)}) \leq K^{-1}e^{-\lambda r}\angle(v_x, E^c_x) \leq K^{-1}\angle(v_x, E^c_x),$$
$\forall x \in \Lambda$, $\forall r \in [0, T^1]$, $\forall v_x \in T_xM$. This implies
$$DX_r(x)\left(C_{\frac{K\alpha}{R}}(E^c_x)\right) \subset C_{\frac{\alpha}{R}}(E^c_{X_r(x)}), \quad \forall x \in \Lambda \quad \forall r \in [0, T^1].$$
Choose a neighborhood $V^2 = V^2_\alpha \subset U^1$ of Λ sufficiently close to Λ such that
$$DX_r(x)\left(C_{\frac{K\alpha}{R}}(E^c_x)\right) \subset C_{\frac{2\alpha}{R}}(E^c_{X_r(x)}), \ \forall x \in V^2, \ \forall r \in [0, T^1]. \tag{8}$$

As Λ is Lyapunov stable there is a neighborhood $U^2 \subset V^2$ of Λ such that

$$X_t(U^2) \subset V^2, \quad \forall t \geq 0.$$

Now, let $x \in U^2$ and $t \geq T^1$ be given. Then, $t = nT^1 + r$ for some integer $n \geq 1$ and some $r \in [0, T^1]$. Thus,

$$DX_t(x)(C_\alpha(E^c_x)) = DX_{nT^1+r}(x)(C_\alpha(E^c_x)) =$$

$$= DX_r(X_{nT^1}(x))(DX_{nT^1}(x)(C_\alpha(E^c_x))). \tag{9}$$

Using (7) and (9) recursively, and that $\frac{K\alpha}{R} < \alpha$, and $n \geq 1$, we obtain

$$DX_{nT^1}(x)(C_\alpha(E^c_x)) = DX_{(n-1)T^1}(X_{T^1}(x))(DX_{T^1}(x)C_\alpha(E^c_x)) \subset$$

$$\subset DX_{(n-1)T^1}(X_{T^1}(x))\left(C_{\frac{K\alpha}{R}}(E^c_{X_{T^1}(x)})\right) \subset DX_{(n-1)T^1}(X_{T^1}(x))(C_\alpha(E^c_{X_{T^1}(x)})) \subset$$

$$\subset \cdots \subset DX_{T^1}(X_{(n-1)T^1}(x))(C_\alpha(E^c_{X_{(n-1)T^1}(x)})) \subset C_{\frac{K\alpha}{R}}(E^c_{X_{nT^1}(x)}).$$

Henceforth

$$DX_{nT^1}(x)(C_\alpha(E^c_x)) \subset C_{\frac{K\alpha}{R}}(E^c_{X_{nT^1}(x)}).$$

Applying $DX_r(X_{nT^1}(x))$ to both sides of the last expression, replacing in (9) and using (8) we obtain

$$DX_t(x)(C_\alpha(E^c_x)) \subset DX_r(X_{nT^1}(x))\left(C_{\frac{K\alpha}{R}}(E^c_{X_{nT^1}(x)})\right) \subset C_{\frac{2\alpha}{R}}(E^c_{X_t(x)}).$$

As $R > 4$ we have $\frac{2\alpha}{R} < \frac{\alpha}{2}$ and so

$$DX_t(x)(C_\alpha(E^c_x)) \subset C_{\frac{\alpha}{2}}(E^c_{X_t(x)}).$$

$\forall x \in U^2$, $\forall t \geq T^1$, proving (1) of Lemma 7.5.

Throughout we fix the neighborhood U^2 of Λ and the constant $T^1 > 0$ obtained above.

As E^c_Λ is (K, λ)-volume expanding we have

$$\mathrm{vol}(DX_t(x)v^c_x, DX_t(x)w^c_x) \geq Ke^{\lambda t}\,\mathrm{vol}(v^c_x, w^c_x),$$

$\forall x \in \Lambda$, $\forall t \geq 0$, $\forall v^c_x, w^c_x \in E^c_x$.

Clearly there is $L > 1$ such that

$$L^{-1} \cdot \mathrm{vol}(v^c_x, w^c_x) \leq \mathrm{vol}(v_x, w_x) \leq L \cdot \mathrm{vol}(v^c_x, w^c_x),$$

$\forall x \in \Lambda$, $\forall v_x, w_x \in C_\alpha(E^c_x)$, $\forall \alpha \in (0, 1]$. Applying the last two relations and the invariance of $E^s_\Lambda \oplus E^c_\Lambda$ we obtain

$$\mathrm{vol}(DX_t(x)v_x, DX_t(x)w_x) \geq L^{-1}\,\mathrm{vol}(DX_t(x)v^c_x, DX_t(x)w^c_x) \geq$$

$$\geq L^{-1}Ke^{\lambda t}\operatorname{vol}(v_x^c, w_x^c) \geq L^{-2}Ke^{\lambda t}\operatorname{vol}(v_x, w_x),$$

$\forall x \in \Lambda$, $\forall t \geq T^1$, $\forall v_x, w_x \in C_\alpha(E_x^c)$ (note that $DX_t(x)v_x, DX_t(x)w_x \in C_\alpha(E^c_{X_t(x)})$ since $t \geq T^1$).

Choose $S > 0$ large so that

$$\frac{S}{L^{-2}K} > 1. \tag{10}$$

It follows that there is $T^2 = T^2_\alpha > T^1$ such that

$$\operatorname{vol}(DX_{T^2}(x)v_x, DX_{T^2}(x)w_x) \geq \frac{2S}{L^{-2}K}\operatorname{vol}(v_x, w_x),$$

$\forall x \in \Lambda$, $\forall v_x, w_x \in C_\alpha(E_x^c)$. In particular,

$$\inf\{\operatorname{vol}(DX_{T^2}(x)v_x, DX_{T^2}(x)w_x) : x \in \Lambda,$$

$$v_x, w_x \in C_\alpha(E_x^c), \|v_x\| = \|w_x\| = 1, < v_x, w_x >= 0\} \geq \frac{2S}{L^{-2}K}.$$

Since Λ is compact there is a neighborhood $V^3 = V^3_\alpha \subset U^2$ of Λ so that

$$\inf\{\operatorname{vol}(DX_{T^2}(x)v_x, DX_{T^2}(x)w_x) : x \in U^3,$$

$$v_x, w_x \in C_\alpha(E_x^c), \|v_x\| = \|w_x\| = 1, < v_x, w_x >= 0\} \geq \frac{S}{L^{-2}K}.$$

Then,

$$\operatorname{vol}(DX_{T^2}(x)v_x, DX_{T^2}(x)w_x) \geq \frac{S}{L^{-2}K}\operatorname{vol}(v_x, w_x), \tag{11}$$

$\forall x \in U^3$, $\forall v_x, w_x \in C_\alpha(E_x^c)$ with $< v_x, w_x >= 0$. As Λ is Lyapunov stable there is a neighborhood $U^3 \subset V^3$ of Λ such that

$$X_t(U^3) \subset V^3, \quad \forall t \geq 0.$$

We have

$$\|P_x^{nT^2}v_x\|\ \|X(X_{nT^2}(x))\| = \|P^{T^2}_{X_{(n-1)T^2}(x)}(P_x^{(n-1)T^2}v_x\|\ \|X(X_{T^2}(X_{(n-1)T^2}(x))\|.$$

Call $z = X_{(n-1)T^2}(x)$, and $v_z = P_x^{(n-1)T^2}v_x$. From the last equality, using that $X(X_{nT^2}(x)) = DX_{nT^2}(x)(X(x))$, v_z is orthogonal to z, and combining (2) with (11) we get

$$\|P_x^{nT^2}v_x\|\ \|X(X_{nT^2}(x))\| = \|P_z^{T^2}v_z\|\ \|X(X_{T^2}(z))\| =$$

$$\operatorname{vol}(DX_{T^2}(z)v_z, X(X_{T^2}(z))) = \operatorname{vol}(DX_{T^2}(z)v_z, DX_{T^2}(z)(X(z)) \geq$$

$$\frac{S}{L^{-2}K}\operatorname{vol}(v_z, X(z)) = \frac{S}{L^{-2}K}\operatorname{vol}(P_x^{(n-1)T^2} v_x, X(X_{(n-1)T^2}(x))).$$

Thus,

$$\|P_x^{nT^2} v_x\| \cdot \|X(X_{nT^2}(x))\| \geq \left\{ \frac{S}{L^{-2}K} \right\}^n \|v_x\| \cdot \|X(x)\|, \tag{12}$$

$\forall x \in U^3$ regular with $X(x) \in C_\alpha(E_x^c)$, $\forall n \in I\!N$, $\forall v_x \in C_\alpha(E_x^c) \cap N_x$ (recall that N_x denotes the orthogonal complement of $X(x)$ in T_xM).

On the other hand

$$\operatorname{vol}(DX_r(x)v_x, DX_r(x)w_x) \geq L^{-2}K \cdot \operatorname{vol}(v_x, w_x),$$

$\forall x \in \Lambda, \quad \forall r \in [0, T^2], \quad \forall v_x, w_x \in C_\alpha(E_x^c)$.

As before there is a neighborhood $V^4 = V_\alpha^4 \subset U^3$ of Λ such that

$$\operatorname{vol}(DX_r(x)v_x, DX_r(x)w_x) \geq \frac{L^{-2}K}{2}\operatorname{vol}(v_x, w_x), \tag{13}$$

$\forall x \in V^4$, $\forall v_x, w_x \in C_\alpha(E_x^c)$ with $< v_x, w_x >= 0$, $r \in [0, T^2]$. As Λ is Lyapunov stable there is a neighborhood $U^4 \subset V^4$ of Λ such that

$$X_t(U_4) \subset V^4, \quad \forall t \geq 0.$$

Now, let $x \in U^4$ regular with $X(x) \in C_\alpha(E_x^c)$, $t \geq T^2$ and $v_x \in C_\alpha(E_x^c) \cap N_x$. Then, $t = nT^2 + r$ for some integer $n \geq 1$ and some $r \in [0, T^2]$.

Applying (12), (13), and using (2) and (10) we obtain

$$\|P_x^t v_x\| \cdot \|X(X_t(x))\| = \|P^r_{X_{nT^2}(x)} P_x^{nT^2} v_x\| \cdot \|DX_r(X_{nT^2}(x))X(X_{nT^2}(x))\| =$$

$$= \operatorname{vol}\left(DX_r(X_{nT^2}(x))P_x^{nT^2} v_x, DX_r(X_{nT^2}(x))X(X_{nT^2}(x))\right) \geq$$

$$\geq \frac{L^{-2}K}{2}\operatorname{vol}(P_x^{nT^2} v_x, X(X_{nT^2}(x))) = \frac{L^{-2}K}{2}\|P_x^{nT^2} v_x\| \cdot \|X(X_{nT^2}(x))\| \geq$$

$$\geq \left(\frac{L^{-2}K}{2}\right) \cdot \left\{ \frac{SN}{2L^{-2}K} \right\}^n \cdot \|v_x\| \cdot \|X(x)\| =$$

$$= \left(\frac{L^{-2}K}{2}\right) \cdot \left(\frac{S}{L^{-2}K}\right)^{-\frac{r}{T^2}} \left\{ \left(\frac{S}{L^{-2}K}\right)^{\frac{1}{T^2}} \right\}^t \cdot \|v_x\| \cdot \|X(x)\| \geq$$

$$\geq \left(\frac{L^{-4}K^2}{2S}\right) \cdot \left\{ \left(\frac{S}{L^{-2}K}\right)^{\frac{1}{T^2}} \right\}^t \cdot \|v_x\| \cdot \|X(x)\|.$$

Thus, choosing $U_\alpha = U^4$, $T_\alpha = T^2$, $K_\alpha = \frac{L^{-4}K^2}{2S}$ and $\lambda_\alpha = \ln\left(\frac{S}{L^{-2}K}\right)^{\frac{1}{T^2}} > 0$ we obtain (2) of Lemma 7.5.

$\square$

Lemma 7.6. *Let Λ a Lyapunov stable singular hyperbolic set of a vector field X, $\alpha > 0$ and $U_\alpha \subset U(\Lambda)$ be a neighborhood of Λ. Let $T_\Lambda M = E^s_\Lambda \oplus E^c_\Lambda$ the splitting of Λ and $E^c_{U_\alpha}$ be a continuous extension of E^c_Λ to U_α. Then, for every singularity σ of X in Λ there are singular cross-sections S^t, S^b associated to σ such that $S^t \cup S^b \subset U_\alpha$,*

$$\Lambda \cap (\partial^h S^t \cup \partial^h S^b) = \emptyset, \quad \textit{and} \quad X(x) \in C_\alpha(E^c_x),$$

$\forall x \in S^t \cup S^b$.

Proof. Let Λ, α be as in the statement. By Remark 6.6 applied to $U = U_\alpha$ there are singular cross-sections S^t_0, S^b_0 associated to σ such that $S^t_0, S^b_0 \subset U(\Lambda)$ and $\Lambda \cap (\partial^h S^t_0 \cup \partial^h S^b_0) = \emptyset$. Recall that $U(\Lambda)$ is the neighborhood of Λ where the (K, λ)-spliting $T_\Lambda M = E^s_\Lambda \oplus E^c_\Lambda$ has an extension to $T_{U(\Lambda)}M = E^s_{U(\Lambda)} \oplus E^c_{U(\Lambda)}$, with $E^s_{U(\Lambda)}$ invariant and contracting. We denote by l^t_0, l^b_0 the singular curves of S^t_0, S^b_0 respectively (Definition 6.5).

Choose two sequences of singular cross-sections $S^t_n \subset S^t_0, S^b_n \subset S^b_0$ associated to σ satisfying

(a) $\Lambda \cap (\partial^h S^t_n \cup \partial^h S^b_n) = \emptyset$;

(b) $\mathrm{diam}(S^t_n), \mathrm{diam}(S^b_n) \to 0$ as $n \to \infty$;

(c) the singular curves l^t_n, l^b_n of S^t_n, S^b_0 satisfy $l^t_n = l^t_0$, $l^b_n = l^b_0$, $\forall n$.

The properties (b) and (c) imply that $\forall n$ large there is $T = T_n > 0$ such that

$$\angle(X(X_T(x)), E^c_\sigma) < \frac{\alpha}{2},$$

$\forall x \in S^t_n \cup S^b_n$. As $E^c_{U_\alpha}$ is a continuous extension of E^c_Λ, we have that $E^c_{X(X_T(x))}$ is close to E^c_σ, $\forall x \in S^t_n \cup S^b_n$. Then,

$$X_T(x) \in U_\alpha, \text{ and } X(X_T(x)) \in C_\alpha(E^c_{X(X_T(x))}),$$

$\forall n$ large, $\forall x \in S^t_n \cup S^b_n$.

By the Property (a) and the above relation we have that for n large enough $S^t = X_T(S^t_n)$ and $S^b = X_T(S^b_n)$ are singular cross-sections associated to σ satisfying the required properties.

$\square$

Definition 7.7. *Let $\epsilon > 0$ and Λ be a Lyapunov stable transitive singular hyperbolic set. A* ϵ-singular partition *of Λ is a singular partition $\mathcal{S}$ of Λ for which there is $\alpha > 0$ such that the family $\mathcal{C} = \{C_\alpha(E^c_x) \cap T\mathcal{S}_x : x \in \mathcal{S}\}$ satisfies :*

1. *If $x \in \mathrm{Dom}(\Pi)$, then*

$$D\Pi(x)(C_\alpha(E^c_x) \cap T\mathcal{S}_x) \subset \mathrm{int}(C_{\frac{\alpha}{2}}(E^c_{\Pi(x)}) \cap T\mathcal{S}_{\Pi(x)}).$$

2. *If $x \in \mathrm{Dom}(\Pi)$ and $v_x \in C_\alpha(E^c_x) \cap T\mathcal{S}_x$, then*

$$\|D\Pi(x)(v_x)\| \geq \epsilon^{-1}\|v_x\|.$$

3.

$$\inf\{\angle(v_x, E^s_x) : x \in \mathcal{S}, v_x \in C_\alpha(E^c_x) \cap T\mathcal{S}_x\} > 0.$$

We have the following lemma.

Lemma 7.8. *Let Λ be a Lyapunov stable singular hyperbolic set with dense singular unstable branches of X. Suppose that Λ is not an attractor of X and*

$$\omega_X(q) \cap \mathrm{Sing}_X(\Lambda) \neq \emptyset,$$

$\forall q \in \Lambda$. Then, Λ exhibits a ϵ-singular partition for every $\epsilon > 0$.

Proof. Let Λ be a Lyapunov stable singular hyperbolic set of X as in the statement and $\epsilon > 0$ be fixed. As already mentioned the splitting $T_\Lambda M = E^s_\Lambda \oplus E^c_\Lambda$ of Λ extends to a splitting $T_{U(\Lambda)}M = E^s_{U(\Lambda)} \oplus E^c_{U(\Lambda)}$ on a neighborhood $U(\Lambda)$ of Λ in a way that $E^s_{U(\Lambda)}$ is invariant and contracting. Recall that $k = \#\,\mathrm{Sing}_X(\Lambda)$.

Choose $\alpha \in (0, 1]$ such that

$$\inf\{\angle(v_x, E^s_x) : x \in U(\Lambda), v_x \in C_\alpha(E^c_x) \cap T\mathcal{S}_x\} > 0.$$

This α exists because $T_{U(\Lambda)}M = E^s_{U(\Lambda)} \oplus E^c_{U(\Lambda)}$.

Let $U_\alpha \subset U(\Lambda)$ the neighborhood of Λ and $T_\alpha, K_\alpha, \lambda_\alpha > 0$ be given by Lemma 7.5 for that α.

By Lemma 7.6 we can choose for every $\sigma_i \in \mathrm{Sing}_X(\Lambda)$ two singular cross-sections $S^{t,1}_i, S^{b,1}_i \subset U(\Lambda)$ associated to σ_i satisfying

$$\Lambda \cap (\partial^h S^{t,1}_i \cup \partial^h S^{b,1}_i) = \emptyset, \text{ and } X(x) \in C_\alpha(E^c_x),$$

$\forall x \in S^{t,1}_i \cup S^{b,1}_i$, $\forall i = 1, \cdots, k$. Define

$$\mathcal{S}^1 = \{S^{t,1}_i, S^{b,1}_i\}_{i \in \{1,\cdots,k\}}.$$

We denote by $l^{t,1}_i, l^{b,1}_i$ the singular curves of $S^{t,1}_i, S^{b,1}_i$ (recall Definition 6.5). For simplicity we shall assume that

$$T\mathcal{S}^1_x = N_x, \quad \forall x \in \mathcal{S}^1.$$

Let $D > 0$ (depending on $\mathcal{S}^1$ only) be such that

$$\frac{\|X(x)\|}{\|X(y)\|} \geq D, \quad \forall x, y \in \mathcal{S}^1. \tag{14}$$

Choose $T_\epsilon > T_\alpha$ large enough so that

$$K_\alpha e^{\lambda_\alpha t} \cdot D > \epsilon^{-1}, \quad \forall t \geq T_\epsilon. \tag{15}$$

Next we consider, for every $\delta > 0$ small, $i \in \{1, \cdots, k\}$ and $* = t, b$, a s-rectangle $S_i^{*,\delta} \subset S_i^{*,1}$ containing $l_i^{*,1}$ such that

$$l_i^{*,1} \cap \partial^v S_i^{*,\delta} = \emptyset$$

and

$$\operatorname{diam}(S_i^{*,\delta}) \leq 2\delta.$$

By Corollary 6.10, as Λ is not an attractor, we can assume that

$$\Lambda \cap (\partial S_i^{t,\delta} \cup \partial S_i^{b,\delta}) = \emptyset.$$

In particular, the collection

$$\mathcal{S} = \mathcal{S}^\delta = \{S_i^{t,\delta}, S_i^{b,\delta}\}_{i \in \{1, \cdots, k\}}$$

satisfies the conditions (1) and (2) of Definition 7.2.

By hypothesis

$$\omega_X(q) \cap \operatorname{Sing}_X(\Lambda) \neq \emptyset, \quad \forall q \in \Lambda.$$

This implies condition (3) of Definition 7.2 for $\mathcal{S} = \mathcal{S}^\delta$. Thus, $\mathcal{S} = \mathcal{S}^\delta$ is a singular partition of Λ, $\forall \delta > 0$.

Next we show that for $\delta > 0$ small the family

$$\mathcal{C} = \{C_\alpha(E_x^c) \cap T\mathcal{S}_x : s \in \mathcal{S} = \mathcal{S}^\delta\}$$

satisfies the properties (1),(2),(3) of Definition 7.7.

Indeed, as $T\mathcal{S}_x^1 = N_x$ for every $x \in \mathcal{S}^1$ one has

$$D\Pi(x) = P_x^{t_\delta(x)}, \ \forall x \in \operatorname{Dom}(\Pi), \tag{16}$$

where Π is the return map of $\mathcal{S} = \mathcal{S}^\delta$, $\operatorname{Dom}(\Pi)$ is the domain of Π and $t_\delta(x)$ is the return time of $x \in \operatorname{Dom}(\Pi)$. Note that $t_\delta(x) \to \infty$ uniformly as $\delta \to 0^+$. So, there is $\delta > 0$ such that $t_\delta(x) \geq T_\epsilon$, $\forall x \in \operatorname{Dom}(\Pi)$.

By (16) we have

$$D\Pi(x)(C_\alpha(E_x^c) \cap T\mathcal{S}_x) = P_x^{t_\delta(x)}(C_\alpha(E_x^c) \cap N_x).$$

Then, by Lemma 7.5(1) one has

$$D\Pi(x)(C_\alpha(E^c_x)\cap T\mathcal{S}_x)\subset C_{\frac{\alpha}{2}}(E^c_{\Pi(x)})\cap T\mathcal{S}_{\Pi(x)},$$

$\forall x\in \mathrm{Dom}(\Pi)$ because $t_\delta(x)\geq T_\epsilon > T_\alpha$. This proves Property (1) of Definition 7.7.

By Lemma 7.5(2), (14), (15), and the definition of T_ϵ one has

$$\|D\Pi(x)(v_x)\| = \|P^{t_\delta(x)}_x(v_x)\| =$$

$$= \|P^{t_\delta(x)}_x(v_x)\|\cdot\|X(X_{t_\delta(x)}(x))\|\cdot\|X(X_{t_\delta(x)}(x))\|^{-1}\geq$$

$$\geq K_\alpha e^{\lambda_\alpha t_\delta(x)}\cdot\frac{\|X(x)\|}{\|X(X_{t_\delta(x)}(x))\|}\cdot\|v_x\|\geq K_\alpha e^{\lambda_\alpha t_\delta(x)}\cdot D\cdot\|v_x\|\geq \epsilon^{-1}\|v_x\|,$$

$\forall x\in \mathrm{Dom}(\Pi)$, $\forall v_x\in C_\alpha(E^c_x)\cap T_x\mathcal{S}$ because $X(x)\in C_\alpha(E^c_x)$ $\forall x\in\mathcal{S}^1$ and $t_\delta(x)>T_\alpha$ $\forall x\in\mathrm{Dom}(\Pi)$. This proves Property (2) of Definition 7.7. The Property (3) follows from the choice of α. This finishes the proof.

□

Proof of Proposition 6.2: Let Λ be a Lyapunov stable singular hyperbolic set with dense singular unstable branches of a C^r vector field X in M. By contradiction suppose that Λ is not an attractor and

$$\omega_X(q)\cap \mathrm{Sing}_X(\Lambda)\neq\emptyset,$$

$\forall q\in\Lambda$. Then, by the previous lemma there is a ϵ-singular partition $\mathcal{S}$ of Λ with $\epsilon\in(0,1/2)$. Recall that $\mathcal{S}$ consists of a finite family of singular cross-sections equipped with singular curves $l=\cup_{i=1}^k(l^t_i\cup l^b_i)$ and a family $\mathcal{C}=\{C_\alpha(E^c_x)\cap T\mathcal{S}_x : x\in\mathcal{S}\}$ (for some $\alpha>0$) given by Definition 6.5. As before Π denotes the return map associated to $\mathcal{S}$.

A curve in $\mathcal{S}$ will be a C^1 map $c: I\to\mathcal{S}$ for some interval $I\subset I\!R$. We shall identify c with its image $c(I)$. In particular, c is a connected subset of $\mathcal{S}$. The length of c with respect to the metric of $\mathcal{S}$ is denoted by $L(c)$. A $\mathcal{C}^u$-*curve* is a curve c in $\mathcal{S}$ such that T_xc, the tangent vector of c at $x\in c$, satisfies $Tc_x\in C_\alpha(E^c_x)\cap T\mathcal{S}_x$ for all $x\in c$. We have the following properties of $\mathcal{C}^u$-curves c of $\mathcal{S}$:

(i). If $c\subset\mathrm{Dom}(\Pi)$, then $\Pi(c)$ is a $\mathcal{C}^u$-curve in $\mathcal{S}$. This follows from Definition 7.7(1).

(ii). If $c\cap l\neq\emptyset$, then $c\cap l$ consists of a single point x_c. This follows from Definition 7.7(3). In this case we denote by c^+ the largest connected component of $c\setminus\{x_c\}$. Clearly

$$L(c^+)\geq\frac{L(c)}{2}.$$

Let O be the neighborhood of $\Lambda \cap \mathcal{S}$ in $\mathcal{S}$ given by Lemma 7.4. Fix a $\mathcal{C}^u$-curve c_1 in $\mathcal{S}$ contained in O. As $\mathcal{S}$ is an ϵ-singular partition with $\epsilon > 1$ and c_1 is a $\mathcal{C}^u$-curve, we have that $L(\Pi^n(c_1))$ growth exponentially as $n \to \infty$. Then, as $\mathcal{S}$ has finite diameter, there is a first iterate $\Pi^{n_1}(c_1)$ of c_1 such that $\Pi^{n_1}(c_1) \setminus \mathrm{Dom}(\Pi) \neq \emptyset$. Therefore, by Lemma 7.4 applied to $B = c_1$ we have that $\Pi^{n_1}(c_1) \cap l \neq \emptyset$. Set $\beta_1 = \Pi^{n_1}(c_1)$. Then, $\beta_1 \cap l \neq \emptyset$ and so β_1^+ is well defined by (ii).

Observe that

$$L(\beta_1^+) \geq \frac{L(\beta_1)}{2} \geq \frac{L(c_1)}{2}.$$

Moreover, $\beta_1^+ \subset \mathrm{Dom}(\Pi)$ for otherwise β_1 would intersect l twice by Lemma 7.4, contradicting (ii). So, $\Pi(\beta_1^+)$ is well defined and

$$L(\Pi(\beta_1^+)) \geq \epsilon^{-1} L(\beta_1^+) \geq R \cdot L(c_1),$$

where $R = (2\epsilon)^{-1} > 1$. Note that $R > 1$ because $\epsilon \in (0, 1/2)$.

Define $c_2 = \Pi(\beta_1^+)$. Then c_2 is a $\mathcal{C}^u$-curve in $\mathcal{S}$. As before there is a first iterate $\Pi^{n_2}(c_2)$ of c_2 such that $\Pi^{n-2}(c_2) \cap l \neq \emptyset$. Set $\beta_2 = \Pi^{n_2}(c_2)$. Then, $\beta_2 \cap l \neq \emptyset$ and so β_2^+ is well defined by (ii). As before we have

$$L(\Pi(\beta_2^+)) \geq \epsilon^{-1} L(\beta_2^+) \geq R^2 \cdot L(c_1).$$

In this way we can construct a sequence of $\mathcal{C}^u$-curves $c_1, c_2, \cdots, c_j, \cdots$, in $\mathcal{S}$ such that $L(c_j) \geq R^j \cdot L(c_1)$. This is a contradiction since $R > 1$ and $\mathcal{S}$ has finite diameter. The proof is completed. □

8 Appendix

In this Appendix we shall prove Theorem C . The proof of this theorem is essentially the same as the one of Theorem D in [14], where a similar result was proved for nontrivial attractors instead of nontrivial transitive Lyapunov stable sets.

We start by giving some notations, definitions and recall some results proved elsewhere, that will be used to obtain Theorem C . Consider a vector field $X \in \mathcal{X}^r(M)$ and a hyperbolic singularity σ of X, $r \geq 1$. Assume that the eigenvalues λ_1, λ_2, λ_3 of σ are real and satisfy $\lambda_2 < \lambda_3 < 0 < \lambda_1$. Recall that $W_X^{ss}(\sigma)$ denote the unique invariant manifold which is tangent to the eigenspace associated to λ_2. By [8] there is a *center-unstable invariant manifold* $W_X^{cu}(\sigma)$ of class C^1 at least, non unique and tangent to the eigenspace associated to $\{\lambda_1, \lambda_3\}$. A regular orbit $\Gamma \in W_X^s(\sigma) \cap W_X^u(\sigma)$ is called a *homoclinic loop* of σ. We have the following generic properties of a homoclinic loop Γ associated to σ:

(G1) For any centre-unstable $W_X^{cu}(\sigma)$ manifold of X at σ containing Γ we have that $W_X^{cu}(\sigma)$ is transverse to $W_X^s(\sigma)$ along Γ.

(G2) $\Gamma \notin W^{ss}_X(\sigma)$, the strong-unstable manifold associated to σ.

We say that Γ is *inclination-flip* (respectively *orbit-flip*) if (G1) (respectively (G2)) above fails [9].

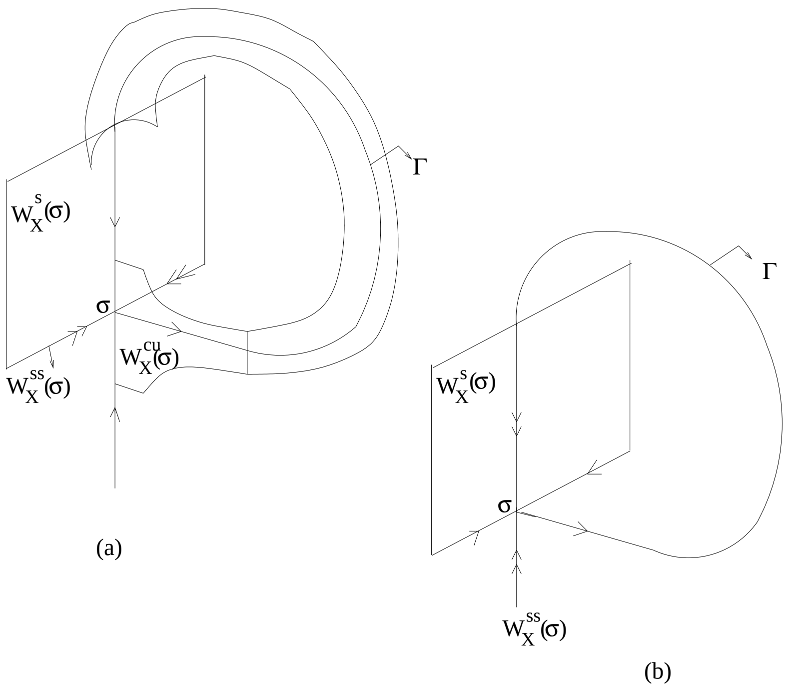

Figure 3: (a) Inclination-flip; (b) orbit-flip.

The following result is a consequence of Theorem 4.1 in [10], see Theorem 7.1 in [14].

Theorem 8.1. *There is no $Z \in \mathcal{G}^1(M)$ exhibiting an inclination-flip homoclinic loop.*

The following result and its collorary are in [14], see Theorem 7.2 in [14]:

Theorem 8.2. *Let $X \in \mathcal{X}^1(M)$ and σ be a singularity of X exhibiting an orbit-flip homoclinic loop. Suppose*

(H) There are neighborhoods $\mathcal{U}_0 \subset \mathcal{X}^1(M)$ and $U_0 \subset M$ of X and σ (resp.) such that the Z-orbit of every sink of $Z \in \mathcal{U}_0$ does not intersect U_0.

Then, for every neighborhood $\mathcal{U} \subset \mathcal{X}^1(M)$ of X there is $Y \in \mathcal{U}$ such that $\sigma \in$ $\mathrm{Sing}(Y)$ has an inclination-flip homoclinic loop.

Corollary 8.3. *Let $X \in \mathcal{G}^1(M)$ and $\sigma \in \mathrm{Sing}(X)$ with eigenvalues $\lambda_2 < \lambda_3 < 0 < \lambda_1$. If $W^{ss}_X(\sigma) \cap \Omega(X) \neq \{\sigma\}$, then (H) does not hold for X and σ.*

Lemma 8.4. *There is a residual subset $\mathcal{R} \subset \mathcal{G}^1(M)$ such that if $X \in \mathcal{R}$ and $\sigma \in \mathrm{Sing}(X)$ belongs to a nontrivial transitive Lyapunov stable set Λ of X, then σ is Lorenz-like for X and*

$$W^{ss}_X(\sigma) \cap \Lambda = \{\sigma\}.$$

Proof. For every $Z \in \mathcal{X}^1(M)$ let

$$\Phi_s(X) = \mathrm{Cl}\left(\{q \in M : \sigma \text{ is a sink of } Z\}\right).$$

Similarly we define $\Phi_u(Z)$ with source instead of sink.

Clearly

$$\Phi_s, \Phi_u : \mathcal{X}^1(M) \to 2^M_c$$

are lower semi-continuous functions. So, there is a residual subset $\mathcal{R}^*$ of $\mathcal{X}^1(M)$ such that Φ_s and Φ_u are semi-continuous at $X \in \mathcal{R}^*$.

Setting $\mathcal{R} = \mathcal{R}^* \cap \mathcal{G}^1(M)$ we have that $\mathcal{R}$ is residual in $\mathcal{G}^1(M)$.

Let us prove that $\mathcal{R}$ satisfies the required property. Since $\mathcal{R} \subset \mathcal{G}^1(M)$ we have that every $X \in \mathcal{R}$ has finitely many sinks and sources [10].

Let $\sigma \in \mathrm{Sing}(X) \cap \Lambda$ for some nontrivial transitive Lyapunov stable set Λ of $X \in \mathcal{R}$. We claim that the eigenvalues of σ are real.

Indeed, suppose by contradiction, that σ has a complex eigenvalue λ. Let λ' denote the real eigenvalue. We can suppose, without loss of generality, that λ' is positive.

We have that $\frac{\lambda'}{-Re(\lambda)}$ is either > 1 or ≤ 1. In the former case we can C^1 approximate X by a flow Z exhibiting a sink close to σ. On the other hand, $\sigma \notin \Phi^s(X) \cup \Phi^u(X)$, of Φ^s at X. This proves the claim in the former case. In the later case, we can C^1 approximate X by a flow Z exhibiting a source close to σ contradicting now the upper semi-continuity of Φ^u at X, as before. This proves the claim. Let $\lambda_1, \lambda_2, \lambda_3 \in I\!R$ the eigenvalues of σ. We have either

(I) $\lambda_2 \leq \lambda_3 < 0 < \lambda_1$ or

(II) $\lambda_2 < 0 < \lambda_3 \leq \lambda_1$.

Let us prove that (II) cannot occur. Indeed, suppose by contradiction, that (II) holds. We can assume that $\lambda_3 < \lambda_1$ without loss of generality. Thus, $W^{uu}_X(\sigma)$ is well defined. As $W^u_X(\sigma) \subset \Lambda$, recall Λ is Lyapunov stable, we have $W^{uu}_X(\sigma) \subset \Lambda$. Since Λ is non trivial, applying the Connecting Lemma, we obtain an orbit-flip homoclinic loop associated to a singularity $\sigma(Z)$ for a vector field Z close to $-X$. Note that $\sigma(Z)$ is close to σ.

Applying Corollary 8.3 to the vector field Z (which is obviously in $\mathcal{G}^1(M)$), we have that there is Z' C^1 close to Z exhibiting a sink arbitrarily close to $\sigma(Z)$, and so, arbitrarily close to σ. As $-Z'$ is close to X, this contradicts the upper semi-continuity of Φ^u. So, (I) always holds. A similar argument implies $\lambda_2 < \lambda_3$.

Thus, we have proved

$$\lambda_2 < \lambda_3 < 0 < \lambda_1.$$

To prove that σ is Lorenz-like, we just need to prove the inequality $-\lambda_3 < \lambda_1$. If this were not true, we apply the Connecting Lemma to obtain X' C^1 close to X exhibiting an homoclinic loop associated to $\sigma(X')$. Hence, as before, we can approximate X' by Z exhibiting a sink close to $\sigma(X')$, and so to σ, contradicting the semi-continuity of Φ^s once more. Thus, σ is Lorenz-like.

It remains to prove $W^{ss}_X(\sigma) \cap \Lambda = \{\sigma\}$. But this equality follows from Corollary 8.3 since $\Lambda \subset \Omega(X)$. This proves the lemma.

□

Proof of Theorem C : Let $\mathcal{R}$ be given by Lemma 8.4. By construction, $\mathcal{R}$ has the following property :

(**) If $X \in \mathcal{R}$ and Λ is a nontrivial transitive Lyapunov stable set of X, then there are neighborhoods $\mathcal{U} \subset \mathcal{X}^1(M)$ and $U \subset M$ of X and Λ (respectively) such that $Z \in \mathcal{U}$ exhibits neither a sink nor a source whose orbit intersects U.

Now choose a vector field X and a nontrivial transitive Lyapunov stable set Λ of X with $\mathrm{Sing}_X(\Lambda) \neq \emptyset$. If $\sigma \in Sin_X(\Lambda)$, Lemma 8.4 implies that every $\sigma \in \Lambda \cap \mathrm{Sing}(X)$ is Lorenz-like and satisfy

$$W^{ss}_X(\sigma) \cap \Lambda = \{\sigma\}.$$

Then, replacing Lemma 1 in [15] by (**) and using the arguments there we conclude that Λ is singular hyperbolic for X. This finishes the proof.

□

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C. A. Morales, M. J. Pacifico
Instituto de Matemática
Universidade Federal do Rio de Janeiro
C. P. 68.530, CEP 21.945-970
Rio de Janeiro, R. J. , Brazil
e-mail: morales@impa.br, pacifico@impa.br