LYAPUNOV EXPONENTS:
HOW FREQUENTLY ARE DYNAMICAL SYSTEMS HYPERBOLIC ?

JAIRO BOCHI AND MARCELO VIANA

ABSTRACT. Lyapunov exponents measure the asymptotic behavior of tangent vectors under iteration, positive exponents corresponding to exponential growth and negative exponents corresponding to exponential decay of the norm. Assuming *hyperbolicity*, that is, that no Lyapunov exponents are zero, Pesin theory provides detailed geometric information about the system, that is at the basis of several deep results on the dynamics of hyperbolic systems. Thus, the question in the title is central to the whole theory.

Here we survey and sketch the proofs of several recent results on genericity of vanishing and non-vanishing Lyapunov exponents. Genericity is meant in both topological and measure-theoretical sense. The results are for dynamical systems (diffeomorphisms) and for linear cocycles, a natural generalization of the tangent map which has an important role in Dynamics as well as in several other areas of Mathematics and its applications.

The first section contains statements and a detailed discussion of main results. Outlines of proofs follow. In the last section and the appendices we prove a few useful related results.

1. INTRODUCTION

Let M be a compact manifold with dimension $d \geq 1$, and $f\colon M \to M$ be a C^r diffeomorphism, $r \geq 1$. The Oseledets theorem [Ose68] says that, relative to any f-invariant probability μ, almost every point admits a splitting of the tangent space

$$T_x M = E_x^1 \oplus \cdots \oplus E_x^k\,, \quad k = k(x), \tag{1}$$

and real numbers $\lambda_1(f,x) > \cdots > \lambda_k(f,x)$ such that

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \|Df^n(x) v_i\| = \lambda_i(f,x) \quad \text{for every nonzero } v_i \in E_x^i\,.$$

These objects are uniquely defined and they vary measurably with the point x. Moreover, the Lyapunov exponents $\lambda_i(f,x)$ are constant on orbits, hence they are constant μ-almost everywhere if μ is ergodic.

Assuming hyperbolicity, that is, that no Lyapunov exponents are zero, Pesin theory provides detailed geometric information about the system, including existence of stable and unstable sets that are smooth embedded disks at almost every point [Pes76, Rue81, FHY83, PS89]. This theory requires the diffeomorphism to have Hölder continuous derivative (see [Pu84]). Such geometric structure is at the basis of several deep results on the dynamics of hyperbolic systems, like [Pes77, Kat80, Led84, LY85, BPS99, SW00]. This makes the following problem central to the whole theory:

How often are dynamical systems hyperbolic ?

More precisely, consider the space $\mathrm{Diff}_\mu^r(M)$ of C^r, $r \geq 1$ diffeomorphisms that preserve a given probability μ, endowed with the corresponding C^r topology. Then the question is to be understood both in topological terms – dense, residual, or even open dense subsets – and in terms of Lebesgue

Date: September 26, 2003.

measure inside generic finite-dimensional submanifolds, or parameterized families, of $\mathrm{Diff}^r_\mu(M)$. The most interesting case [1] is when μ is Lebesgue measure in the manifold.

As we are going to see in section 1.1, systems with *zero* Lyapunov exponents are abundant among C^1 volume-preserving diffeomorphisms. But other results in section 1.3 below strongly suggest predominance of hyperbolicity among C^r systems with $r > 1$.

1.1. **A dichotomy for conservative systems.** Let μ be normalized Lebesgue measure on a compact manifold M.

Theorem 1 ([BV02, BVa])**.** *There exists a residual subset $\mathcal{R}$ of $\mathrm{Diff}^1_\mu(M)$ such that, for every $f \in \mathcal{R}$ and μ-almost every point x,*

(a) *either all Lyapunov exponents $\lambda_i(f,x) = 0$ for $1 \le i \le d$,*
(b) *or the Oseledets splitting of f is dominated on the orbit of x.*

The second case means there exists $m \ge 1$ such that for any y in the orbit of x

$$\frac{\|Df^m(y)v_i\|}{\|v_i\|} \ge 2\,\frac{\|Df^m(y)v_j\|}{\|v_j\|} \tag{2}$$

for any nonzero $v_i \in E^i_y$, $v_j \in E^j_y$ corresponding to Lyapunov exponents $\lambda_i > \lambda_j$. In other words, the fact that Df^n will eventually expand E^i_y more than E^j_y can be observed in finite time, *uniform over the orbit.* This also implies that the angles between the Oseledets subspaces E^i_y are bounded away from zero along the orbit, in fact *the Oseledets splitting extends to a dominated splitting over the closure of the orbit.*

In many situations (for instance, if the transformation f is ergodic) the conclusion gets a more global form: either (a) all exponents vanish at μ-almost every point or (b) the Oseledets splitting extends to a dominated splitting on the whole ambient manifold. The latter means that $m \ge 1$ as in (2) may be chosen uniformly over all of M.

It is easy to see that a dominated splitting into factors with constant dimensions is necessarily continuous. Now, existence of such a splitting is a very strong property that can often be excluded a priori. In any such case Theorem 1 is saying that generic systems must satisfy alternative (a).

A first example of this phenomenon is the 2-dimensional version of Theorem 1, proved by Bochi in 2000, partially based on a strategy proposed by Mañé in the early eighties [Mañ96].

Theorem 2 ([Boc02])**.** *For a residual subset $\mathcal{R}$ of C^1 area preserving diffeomorphisms on any surface, either*

(a) *the Lyapunov exponents vanish almost everywhere or*
(b) *the diffeomorphism is uniformly hyperbolic (Anosov) on the whole M.*

Alternative (b) can only occur if M is the torus; so, C^1 *generic area preserving diffeomorphisms on any other surface have zero Lyapunov exponents almost everywhere.*

It is an interesting question whether the theorem can always be formulated in this more global form. Here is a partial positive answer, for symplectic diffeomorphisms on any symplectic manifold (M, ω):

[1] But the problem is just as important for general dissipative diffeomorphisms, that is, without a priori knowledge of invariant measures. E.g. [ABV00] uses hyperbolicity type properties at Lebesgue almost every point to *construct* invariant Sinai-Ruelle-Bowen measures.

Theorem 3 ([BVa]). *There exists a residual set $\mathcal{R} \subset \mathrm{Sympl}^1_\omega(M)$ such that for every $f \in \mathcal{R}$ either the diffeomorphism f is Anosov or Lebesgue almost every point has zero as Lyapunov exponent, with multiplicity ≥ 2.*

Remark 1.1. *For r sufficiently large, KAM theory yields C^r-open sets of symplectic maps which are not hyperbolic, due to the presence of invariant Lagrangian tori restricted to which the map is conjugate to rotations and whose union has positive volume. Moreover, Cheng, Sun* [CS90], *Herman* [Yoc92, § 4.6], *Xia* [Xi92] *have constructed C^r-open sets (large r) of volume-preserving diffeomorphisms exhibiting positive volume invariant sets consisting of codimension-*1 *invariant tori. In all these latter examples where hyperbolicity fails,* all *Lyapunov exponents actually vanish: the dynamics on each torus is conjugate to a (diophantine) rotation; then, since the map is volume-preserving, the transverse Lyapunov exponent must also be zero.*

1.2. **Deterministic products of matrices.** Let $f\colon M \to M$ be a continuous transformation on a compact metric space M. A *linear cocycle* over f is a vector bundle automorphism $F\colon \mathcal{E} \to \mathcal{E}$ covering f, where $\pi\colon \mathcal{E} \to M$ is a finite-dimensional vector bundle over M. This means that

$$\pi \circ F = f \circ \pi$$

and F acts as a linear isomorphism on every fiber. The quintessential example is the derivative $F = Df$ of a diffeomorphism on a manifold (*dynamical cocycle*).

For simplicity, we focus on the case when the vector bundle is trivial $\mathcal{E} = M \times \mathbb{R}^d$, although this is not strictly necessary for what follows. Then the cocycle has the form

$$F(x, v) = (f(x), A(x)v) \quad \text{for some } A\colon M \to \mathrm{GL}(d, \mathbb{R}).$$

It is no real restriction to suppose that A takes values in $\mathrm{SL}(d, \mathbb{R})$. Moreover, we assume that A is at least continuous. Note that $F^n(x, v) = (f^n(x), A^n(x)v)$ for $n \in \mathbb{Z}$, with

$$A^j(x) = A(f^{j-1}(x)) \cdots A(f(x))\, A(x) \quad \text{and} \quad A^{-j}(x) = \text{ inverse of } A^j(f^{-j}(x)).$$

The theorem of Oseledets extends to linear cocycles: Given any f-invariant probability μ, then at μ-almost every point x there exists a filtration

$$\{x\} \times \mathbb{R}^d = F^0_x > F^1_x > \cdots > F^{k-1}_x > F^k_x = \{0\}$$

and real numbers $\lambda_1(A, x) > \cdots > \lambda_k(A, x)$ such that

$$\lim_{n \to +\infty} \frac{1}{n} \log \|A^n(x) v_i\| = \lambda_i(A, x)$$

for every $v_i \in F^{i-1}_x \setminus F^i_x$. If f is invertible there even exists an invariant splitting

$$\{x\} \times \mathbb{R}^d = E^1_x \oplus \cdots \oplus E^k_x$$

such that

$$\lim_{n \to \pm\infty} \frac{1}{n} \log \|A^n(x) v_i\| = \lambda_i(A, x)$$

for every $v_i \in E^i_x \setminus \{0\}$. It relates to the filtration by $F^j_x = \oplus_{i>j} E^i_x$.

In either case, the largest Lyapunov exponent $\lambda(A, x) = \lambda_1(A, x)$ describes the exponential rate of growth of the norm

$$\lambda(A, x) = \lim_{n \to +\infty} \frac{1}{n} \log \|A^n(x)\|.$$

If μ is an ergodic probability, the exponents are constant μ-almost everywhere. We represent by $\lambda_j(A, \mu)$ and $\lambda(A, \mu)$ these constants.

Example 1.2. *(Products of i.i.d. random matrices) Let ν be a probability on $\mathrm{SL}(d,\mathbb{R})$ with compact support. Define M to be the space $(\operatorname{supp}\nu)^{\mathbb{Z}}$ of sequences $(\alpha_j)_j$ in the support of ν, and let $\mu=\nu^{\mathbb{Z}}$. Let $f\colon M\to M$ be the shift map, and $A\colon M\to \mathrm{SL}(d,\mathbb{R})$ be the projection to the 0th coordinate: $A((\alpha_j)_j)=\alpha_0$. Then*

$$A^n\big((\alpha_j)_j\big)=\alpha_{n-1}\cdots\alpha_1\,\alpha_0\,.$$

A classical theory, initiated by Furstenberg and Kesten [FK60, Fur63]*, states that the largest Lyapunov exponent $\lambda(A,\mu)$ of the cocycle (A,μ) is positive, as long as the support of the probability ν is rich enough: it suffices that there be no probability on the projective space $\mathbb{RP}^{d-1}$ invariant under the action of* all *the matrices in $\operatorname{supp}\nu$. Also with great generality, one even has that the Lyapunov spectrum is simple: all Oseledets subspaces have dimension 1. See Guivarch, Raugi* [GR86] *and Gold'sheid, Margulis* [GM89].

Theorem 1 also extends to linear cocycles over any transformation. We state the ergodic invertible case:

Theorem 4 ([Boc02, BV02])**.** *Assume $f\colon (M,\mu)\to(M,\mu)$ is invertible and ergodic. Let $G\subset \mathrm{SL}(d,\mathbb{R})$ be any subgroup acting transitively on the projective space $\mathbb{RP}^{d-1}$. Then there exists a residual subset $\mathcal{R}$ of maps $A\in C^0(M,G)$ for which either the Lyapunov exponents $\lambda_i(A,\mu)$ are all zero at μ-almost every point, or the Oseledets splitting of A extends to a dominated splitting over the support of μ.*

The next couple of examples describe two simple mechanisms that exclude a priori the dominated splitting alternative in the dichotomy:

Example 1.3. *Let $f\colon M\to M$ and $A\colon M\to\mathrm{SL}(d,\mathbb{R})$ be such that for every $1\le i<d$ there exists a periodic point p_i in the support of μ, with period q_i, such that the eigenvalues $\{\beta^i_j\colon 1\le j\le d\}$ of $A^{q_i}(p_i)$ satisfy*

$$|\beta^i_1|\ge\cdots\ge|\beta^i_{i-1}|>|\beta^i_i|=|\beta^{i+1}_i|>|\beta^i_{i+2}|\ge\cdots\ge|\beta^i_d| \tag{3}$$

and β^i_i, β^i_{i+1} are complex conjugate (not real). Such an A may be found, for instance, starting with a constant cocycle and deforming it on disjoint neighborhoods of the periodic orbits. Property (3) *remains valid for every B in a C^0 neighborhood $\mathcal{U}$ of A. It implies that no B admits an invariant dominated splitting over the support of μ: if such a splitting $E\oplus F$ existed then, at every periodic point, the $\dim E$ largest eigenvalues would be strictly larger than the other eigenvalues, which is incompatible with* (3)*. It follows, by Theorem 4, that every cocycle in a residual subset $\mathcal{U}\cap\mathcal{R}$ of the neighborhood has all the Lyapunov exponents equal to zero.*

Example 1.4. *Let $f\colon S^1\to S^1$ be a homeomorphism and μ be any invariant ergodic measure with $\operatorname{supp}\mu=S^1$. Let $\mathcal{N}$ be the set of all continuous $A\colon S^1\to\mathrm{SL}(2,\mathbb{R})$ non-homotopic to a constant. For a residual subset of $\mathcal{N}$, the Lyapunov exponents of the corresponding cocycle over (f,μ) are zero. That is because the cocycle has no invariant continuous subbundle if A is non-homotopic to a constant (this may be shown by the same kind of arguments as in example 3.4 below).*

Remark 1.5. *Theorem 4 also carries over to the space $L^\infty(X,\mathrm{SL}(d,\mathbb{R}))$ of measurable bounded cocycles, still with the uniform topology. We also mention that in weaker topologies, cocycles having a dominated splitting may cease to constitute an open set. In fact, for $1\le p<\infty$,* generic L^p cocycles have all exponents equal, *see Arnold, Cong* [AC97] *and Arbieto, Bochi* [AB].

1.3. **Prevalence of nonzero exponents.** We are now going to see that the conclusions of the previous section change radically if one considers linear cocycles which are better than just continuous: assuming the base dynamics is hyperbolic, the *overwhelming majority of Hölder continuous or differentiable cocycles admit nonzero Lyapunov exponents.*

Let G be any subgroup of $\mathrm{SL}(d,\mathbb{R})$. For $0<\nu\leq\infty$ denote by $C^\nu(M,G)$ the space of C^ν maps from M to G endowed with the C^ν norm. When $\nu\geq 1$ it is implicit that M has a smooth structure. For integer ν the notation is slightly ambiguous: C^ν means either that f is ν times differentiable with continuous νth derivative, or that it is $\nu-1$ times differentiable with Lipschitz continuous derivative. All the statements are meant for both interpretations.

Let $f\colon M\to M$ be a C^1 diffeomorphism with Hölder continuous derivative. An f-invariant probability measure μ is *hyperbolic* if every $\lambda_i(f,x)$ is different from zero at μ-almost every point. The notion of measure with local product structure is recalled at the end of this section, and we also observe that this class contains most interesting invariant measures.

Theorem 5 ([Via]). *Assume $f\colon(M,\mu)\to(M,\mu)$ is ergodic and hyperbolic with local product structure. Then, for every $\nu>0$, the set of cocycles A with largest Lyapunov exponent $\lambda(A,x)>0$ at μ-almost every point contains an open dense subset $\mathcal{A}$ of $C^\nu(M,\mathrm{SL}(d,\mathbb{R}))$. Moreover, its complement has infinite codimension.*

The last property means that the set of cocycles with vanishing exponents is locally contained inside finite unions of closed submanifolds of $C^\nu(M,\mathrm{SL}(d,\mathbb{R}))$ with arbitrary codimension. Thus, generic parameterized families of cocycles do not intersect this exceptional set at all!

Now suppose $f\colon M\to M$ is uniformly hyperbolic, for instance, a two-sided shift of finite type, or an Axiom A diffeomorphism restricted to a hyperbolic basic set. Then every invariant measure is hyperbolic. The main novelty is that the set $\mathcal{A}$ may be taken the same for all invariant measures with local product structure.

Theorem 6 ([BGMV, Via]). *Assume $f\colon M\to M$ is a uniformly hyperbolic homeomorphism. Then, for every $\nu>0$, the set of cocycles A whose largest Lyapunov exponent $\lambda(A,x)$ is positive at μ-almost every point for every invariant measure with local product structure contains an open dense subset $\mathcal{A}$ of $C^\nu(M,\mathrm{SL}(d,\mathbb{R}))$. Moreover, its complement has infinite codimension.*

Theorem 6 was first proved in [BGMV], under an additional hypothesis called domination. Under this additional hypothesis [BVb] gets a stronger conclusion: all Lyapunov exponents have multiplicity 1, in other words, the Oseledets subspaces E^i are one-dimensional. We expect this to extend to full generality:

Conjecture. *Theorems 5 and 6 should remain true if one replaces $\lambda(A,x)>0$ by all Lyapunov exponents $\lambda_i(A,x)$ having multiplicity 1.*

Theorems 5 and 6 extend to cocycles over non-invertible transformations, respectively, local diffeomorphisms equipped with invariant non-uniformly expanding probabilities (all Lyapunov exponents positive), and uniformly expanding continuous maps, like one-sided shifts of finite type, or smooth expanding maps. Moreover, both theorems remain true if we replace $\mathrm{SL}(d,\mathbb{R})$ by any subgroup G such that

$$G\ni B\mapsto(B\xi_1,\dots,B\xi_d)\in(\mathbb{RP}^{d-1})^d,$$

is a submersion, for any linearly independent $\{\xi_1,\dots,\xi_d\}\subset\mathbb{RP}^{d-1}$. In particular, this holds for the symplectic group.

Motivated by results to be presented in sections 2 and 3, we ask

Problem. *What are the continuity points of Lyapunov exponents as functions of the cocycle in $C^\nu(M, \mathrm{SL}(d,\mathbb{R}))$, when $\nu > 0$? It may help to assume that the base system (f,μ) is hyperbolic.*

Finally, we recall the notion of *local product structure* for invariant measures. Let μ be a hyperbolic measure. We also assume that μ has no atoms. By Pesin's stable manifold theorem [Pes76], μ-almost every $x \in M$ has a local stable set $W^s_{loc}(x)$ and a local unstable set $W^u_{loc}(x)$ which are C^1 embedded disks. Moreover, these disks vary in a measurable fashion with the point. So, for every $\varepsilon > 0$ we may find $M_\varepsilon \subset M$ with $\mu(M_\varepsilon) > 1-\varepsilon$ such that $W^s_{loc}(x)$ and $W^u_{loc}(x)$ vary continuously with $x \in M_\varepsilon$ and, in particular, their sizes are uniformly bounded away from zero. Thus for any $x \in M_\varepsilon$ we may construct sets $\mathcal{H}(x,\delta)$ with arbitrarily small diameter δ, such that (i) $\mathcal{H}(x,\delta)$ contains a neighborhood of x inside M_ε , (ii) every point of $\mathcal{H}(x,\delta)$ is in the local stable manifold and in the local unstable manifold of some pair of points in M_ε , and (iii) given y, z in $\mathcal{H}(x,\delta)$ the unique point in $W^s(y) \cap W^u(z)$ is also in $\mathcal{H}(x,\delta)$. Then we say that μ has a local product structure if $\mu \mid \mathcal{H}(x,\delta)$ is equivalent to $\mu^u \times \mu^s$, where μ^u (resp. μ^s) is the projection of $\mu \mid \mathcal{H}(x,\delta)$ onto $W^u(x)$ (resp. $W^s(x)$).

Lebesgue measure has local product structure if it is hyperbolic; this follows from the absolute continuity of Pesin's stable and unstable foliations [Pes76]. The same is true, more generally, for any hyperbolic probability having absolutely continuous conditional measures along unstable manifolds or along stable manifolds. Also, in the uniformly hyperbolic case, every equilibrium state of a Hölder continuous potential [Bow75] has local product structure.

2. PROVING ABUNDANCE OF VANISHING EXPONENTS

We shall sketch the proofs of Theorems 1 and 3, given in [BV02].

Let $f \in \mathrm{Diff}^1_\mu(M)$ and Γ be an invariant set We say that an invariant splitting $T_\Gamma = E \oplus F$ is m*-dominated*, for some $m \in \mathbb{N}$, if for all $x \in \Gamma$

$$\frac{\|Df^m_x|_{F_x}\|}{\mathbf{m}(Df^m_x|_{E_x})} < \frac{1}{2},$$

where $\mathbf{m}(A) = \|A^{-1}\|^{-1}$. We call $E \oplus F$ a *dominated splitting* if it is m-dominated for some m.

2.1. Volume-preserving diffeomorphisms.

Given $f \in \mathrm{Diff}^1_\mu(M)$ and $1 \le p \le d$, we write

$$\Lambda_p(f,x) = \lambda_1(f,x) + \cdots + \lambda_p(f,x) \quad \text{and} \quad \mathrm{LE}_p(f) = \int_M \Lambda_p(f,x)\, d\mu(x).$$

As f preserves volume, $\Lambda_d(f,x) \equiv 0$. It is a well-known fact that the functions $f \in \mathrm{Diff}^1_\mu(M) \mapsto \mathrm{LE}_p(f)$ are upper semi-continuous. Continuity of these functions is much more delicate:

Theorem 7. *Let $f_0 \in \mathrm{Diff}^1_\mu(M)$ be such that the map*

$$\mathrm{Diff}^1_\mu(M) \ni f \mapsto \big(\mathrm{LE}_1(f), \ldots, \mathrm{LE}_{d-1}(f)\big) \in \mathbb{R}^{d-1}$$

is continuous at $f = f_0$. Then for almost every $x \in M$, the Oseledets splitting of f_0 is either dominated or trivial (all $\lambda_p(f,x) = 0$) along the orbit of x.

Since the set of points of continuity of a upper semi-continuous function is always a residual set, we see that Theorem 1 is an immediate corollary of Theorem 7. Also, Theorem 7 remains valid for linear cocycles, and in this setting the necessary condition is also sufficient.

Example 2.1. *Let $f\colon S^1 \to S^1$ be an irrational rotation, μ be Lebesgue measure, and $A\colon S^1 \to \mathrm{SL}(2,\mathbb{R})$ be given by*

$$A = \begin{pmatrix} E - V(\theta) & -1 \\ 1 & 0 \end{pmatrix}$$

for some $E \in \mathbb{R}$ and $V\colon S^1 \to \mathbb{R}$ continuous. Then A is a point of discontinuity for the Lyapunov exponents, among all continuous cocycles over (f,μ), if and only if the exponents are nonzero and E is in the spectrum of the associated Schrödinger operator. Compare [BJ]. *This is because E is in the complement of the spectrum if and only if the cocycle is uniformly hyperbolic, which for $\mathrm{SL}(2,\mathbb{R})$ cocycles is equivalent to domination.*

We shall explain the main steps in the proof of Theorem 7.

2.2. **First step: Mixing directions along an orbit segment.** The following notion, introduced in [Boc02], is crucial to the proofs of our theorems. It captures the idea of sequence of linear transformations that can be (almost) realized *on subsets with large relative measure* as tangent maps of diffeomorphisms close to the original one.

Definition 2.2. *Given $f \in \mathrm{Diff}^1_\mu(M)$ or $f \in \mathrm{Sympl}^1_\mu(M)$, a neighborhood $\mathcal{U}$ of f in $\mathrm{Diff}^1_\mu(M)$ or $\mathrm{Sympl}^1_\mu(M)$, $0 < \kappa < 1$, and a nonperiodic point $x \in M$, we call a sequence of (volume-preserving or symplectic) linear maps*

$$T_xM \xrightarrow{L_0} T_{fx}M \xrightarrow{L_1} \cdots \xrightarrow{L_{n-1}} T_{f^nx}M$$

an $(\mathcal{U},\kappa)$-realizable sequence of length n at x *if the following holds:*

For every $\gamma > 0$ there is $r > 0$ such that the iterates $f^j(\overline{B}_r(x))$ are pairwise disjoint for $0 \le j \le n$, and given any nonempty open set $U \subset B_r(x)$, there are $g \in \mathcal{U}$ and a measurable set $K \subset U$ such that

(i) *g equals f outside the disjoint union $\bigsqcup_{j=0}^{n-1} f^j(\overline{U})$;*
(ii) *$\mu(K) > (1-\kappa)\mu(U)$;*
(iii) *if $y \in K$ then $\left\| Dg_{g^jy} - L_j \right\| < \gamma$ for every $0 \le j \le n-1$.*

To make the definition clear, let us show (informally) that if v, $w \in T_xM$ are two unit vectors with $\sphericalangle(v,w)$ sufficiently small then there exists a realizable sequence $\{L_0\}$ of length 1 at x such that $L_0(v) = Df_x(w)$.

Indeed, let $R\colon T_xM \to T_xM$ be a rotation of angle $\sphericalangle(v,w)$ along the plane P generated by v and w, with axis $P^\perp$. We take $L_0 = Df_xR$. In order to show that $\{L_0\}$ is a realizable sequence we must, for any sufficiently small neighborhood U of x, find a perturbation g of f and a subset $K \subset U$ such that conditions (i)-(iii) in definition 2.2 are satisfied. Since this is a local problem, we may suppose, for simplicity, that $M = \mathbb{R}^d = T_xM$. First assume U is a cylinder $B \times B'$, where B and B' are balls centered at x and contained in P and $P^\perp$, respectively. We also assume that $\operatorname{diam} B \ll \operatorname{diam} B' \ll 1$. Define $K \subset U$ as a slightly shrunk cylinder also centered at x, so condition (ii) in definition 2.2 holds. Then there is a volume-preserving diffeomorphism h such that h equals the rotation R inside the cylinder K and equals the identity outside U. Moreover, the conditions $\theta \ll 1$ and $\operatorname{diam} B \ll \operatorname{diam} B'$ permit us to take h C^1-close to the identity. Define $g = f \circ h$; then condition (iii) also holds.

This deals with the case where U is a thin cylinder. Now if U is any small neighborhood of x then we only have to cover μ-most of it with disjoint thin cylinders and rotate (as above) each one of them. This “shows” that $\{L_0 = Df_yR\}$ is a realizable sequence.

Our first proposition towards the proof of Theorem 7 says that if a splitting $E \oplus F$ is *not* dominated then one can find a realizable sequence that sends one direction from E to F.

Proposition 2.3. *Given $f \in \mathrm{Diff}^1_\mu(M)$, a neighborhood $\mathcal{U} \ni f$, and $0 < \kappa < 1$, let $m \in \mathbb{N}$ be large. Given a nontrivial splitting $T_{\mathrm{orb}(y)}M = E \oplus F$ along the orbit of a nonperiodic $y \in M$, satisfying the "non-dominance" condition*

$$\frac{\|Df^m_y|_F\|}{\mathbf{m}(Df^m_y|_E)} \geq \frac{1}{2}\,, \tag{4}$$

there exists a $(\mathcal{U}, \kappa)$-realizable sequence $\{L_0, \dots, L_{m-1}\}$ at y of length m and a nonzero vector $v \in E_y$ such that $L_{m-1} \cdots L_0(v) \in F_{f^m y}$.

Let us explain how the sequence is constructed, at least in the simplest case. Assume that $\sphericalangle(E_{f^i y}, F_{f^i y})$ is very small for some $i = 1, \dots, m-1$. We take unit vectors $v_i \in E_{f^i y}$, $w_i \in F_{f^i y}$ such that $\sphericalangle(v_i, w_i)$ is small. As we have explained before, there is a realizable sequence $\{L_i\}$ of length 1 at $f^i x$ such that $L_i(v_i) = w_i$. We define $L_j = Df_{f^j x}$ for $j \neq i$; then $\{L_0, \dots, L_{m-1}\}$ is the desired realizable sequence.

The construction of the sequence is more difficult when $\sphericalangle(E, F)$ is not small, because several rotations may be necessary.

2.3. **Second step: Lowering the norm.** Let us recall some facts from linear algebra. Given a vector space V and a non-negative integer p, let $\bigwedge^p(V)$ be the pth exterior power of V. This is a vector space of dimension $\binom{d}{p}$, whose elements are called *p-vectors*. It is generated by the p-vectors of the form $v_1 \wedge \cdots \wedge v_p$ with $v_j \in V$, called the *decomposable p-vectors*. We take the norm $\|\cdot\|$ in $\bigwedge^p(V)$ such that if $\mathbf{v} = v_1 \wedge \cdots \wedge v_p$ then $\|\mathbf{v}\|$ is the p-dimensional volume of the parallelepiped with edges $v_1, \dots, v_p$. A linear map $L \colon V \to W$ induces a linear map $\bigwedge^p(L) \colon \bigwedge^p(V) \to \bigwedge^p(W)$ such that

$$\bigwedge^p(L)(v_1 \wedge \cdots \wedge v_p) = L(v_1) \wedge \cdots \wedge L(v_p)$$

Let $f \in \mathrm{Diff}^1_\mu(M)$ be fixed from now on. Although it is not necessary, we shall assume for simplicity that f is *aperiodic*, that is, the set of periodic points of f has zero measure.

Given $f \in \mathrm{Diff}^1_\mu(M)$ and $p \in \{1, \dots, d-1\}$, we have, for almost every x,

$$\frac{1}{n} \log \|\bigwedge^p(Df^n_x)\| \to \Lambda_p(f, x) \quad \text{as } n \to \infty.$$

Suppose the Oseledets splitting along the orbit of a point x is not dominated. Our next task (Proposition 2.4) is to construct long realizable sequences $\{\widehat{L}_0, \dots, \widehat{L}_{n-1}\}$ at x such that

$$\frac{1}{n} \log \|\bigwedge^p(\widehat{L}_{n-1} \cdots \widehat{L}_0)\|$$

is smaller then the expected value $\Lambda_p(f, x)$.

Given p and $m \in \mathbb{N}$, we define $\Gamma_p(f, m)$ as the set of points x such that if $T_{\mathrm{orb}(x)}M = E \oplus F$ is an invariant splitting along the orbit, with $\dim E = p$, then it is not m-dominated. It follows from basic properties of dominated splittings (see section 4.1) that $\Gamma_p(f, m)$ is an open set. Of course, it is also invariant.

Proposition 2.4. *Let $\mathcal{U} \subset \mathrm{Diff}^1_\mu(M)$ be a neighborhood of f, $0 < \kappa < 1$, $\delta > 0$ and $p \in \{1, \dots, d-1\}$. Let $m \in \mathbb{N}$ be large. Then for μ-almost every point $x \in \Gamma_p(f, m)$, there exists an integer $N(x)$ such that for every $n \geq N(x)$ there exists a $(\mathcal{U}, \kappa)$-realizable sequence*

$$\{\widehat{L}_0, \dots, \widehat{L}_{n-1}\} = \{\widehat{L}^{(x,n)}_0, \dots, \widehat{L}^{(x,n)}_{n-1}\}$$

at x of length n such that

$$\frac{1}{n}\log\|\bigwedge^p(\widehat{L}_{n-1}\cdots\widehat{L}_0)\| \leq \frac{\Lambda_{p-1}(x)+\Lambda_{p+1}(x)}{2}+\delta. \tag{5}$$

Moreover, the function $N\colon \Gamma_p(f,m)\to\mathbb{N}$ is measurable.

The proof of the proposition may be sketched as follows. Given $x\in\Gamma_p(f,m)$, we may assume $\lambda_p(x)>\lambda_{p+1}(x)$, otherwise we can take the trivial sequence $\widehat{L}_j = Df_{f^j x}$ and there is nothing to prove. Then we can consider the splitting $T_xM = E_x\oplus F_x$, where E_x (resp. F_x) is the sum of the Oseledets spaces corresponding to the exponents $\lambda_1(x)$, …, $\lambda_p(x)$ (resp. $\lambda_{p+1}(x)$, …, $\lambda_d(x)$). By assumption, the splitting $E\oplus F$ is not m-dominated along the orbit of x, that is, there exists $\ell\geq 0$ such that

$$y=f^\ell(x) \;\Rightarrow\; \frac{\|Df^m_y|_{F_y}\|}{\mathbf{m}(Df^m_y|_{E_y})}\geq\frac{1}{2}.$$

By Poincaré recurrence, there are infinitely many integers $\ell\geq 0$ such that the above relation is satisfied (for almost every x). Moreover, it can be shown, using Birkhoff's theorem, that for all large enough n, that is, for every $n\geq N(x)$, we can find $\ell\approx n/2$ such that the inequality above holds for ℓ. Here $\ell\approx n/2$ means that $|\frac{\ell}{n}-\frac{1}{2}|<$ const.δ.

Fix x, $n\geq N(x)$, ℓ as above, $y=f^\ell(x)$ and $z=f^\ell(y)$. Proposition 2.3 gives a $(\mathcal{U},\kappa)$-realizable sequence $\{L_0,\dots,L_{m-1}\}$, such that there is a nonzero vector $v_0\in E_y$ for which

$$L_{m-1}\dots L_0(v_0)\in F_z \tag{6}$$

We form the sequence $\{\widehat{L}_0,\dots,\widehat{L}_{n-1}\}$ of length n by concatenating

$$\{Df_{f^i(x)};\ 0\leq i<\ell\},\quad \{L_0,\dots,L_{m-1}\},\quad \{Df_{f^i(x)};\ \ell+m\leq i<m\}.$$

It is not difficult to show that the concatenation is a $(\mathcal{U},\kappa)$-realizable sequence at x.

We shall give some informal indication why relation (5) is true. Let $\mathbf{v}\in\bigwedge^p(T_xM)$ be a p-vector with $\|\mathbf{v}\|=1$, and let $\mathbf{v}'=\bigwedge^p(L_{m-1}\cdots L_0 Df^\ell_x)(\mathbf{v})\in\bigwedge^p(T_zM)$. Since $m\ll n$, and L_0, …, L_{m-1} are bounded, we have

$$\frac{1}{n}\log\|\bigwedge^p(\widehat{L}_{n-1}\cdots\widehat{L}_0)\mathbf{v}\| \lesssim \frac{1}{n}\log\|\bigwedge^p(Df_z^{n-\ell-m})\mathbf{v}'\|+\frac{1}{n}\log\|\bigwedge^p(Df^\ell_x)\mathbf{v}\|. \tag{7}$$

To fix ideas, suppose $\mathbf{v}$ is a decomposable p-vector belonging to the subspace $\bigwedge^p(E_x)$. Then

$$\frac{1}{\ell}\log\|\bigwedge^p(Df^\ell_x)\mathbf{v}\|\simeq\Lambda_p(f,x). \tag{8}$$

If we imagine decomposable p-vectors as p-parallelepipeds then, by (6), the parallelepiped $\mathbf{v}'$ contains a direction in F_z. This direction is expanded by the derivative with exponent at most $\lambda_{p+1}(z)=\lambda_{p+1}(x)$. On the other hand, the $(p-1)$-volume of every $(p-1)$-parallelepiped in T_zM grows with exponent at most $\Lambda_{p-1}(x)$. This "shows" that

$$\frac{1}{n-\ell-m}\log\|\bigwedge^p(Df_z^{n-\ell-m})\mathbf{v}'\|\lesssim\lambda_{p+1}(x)+\Lambda_{p-1}(x). \tag{9}$$

Substituting (8) and (9) in (7), and using that $\ell\approx n-\ell-m\approx n/2$, we obtain

$$\frac{1}{n}\log\|\bigwedge^p(\widehat{L}_{n-1}\cdots\widehat{L}_0)\mathbf{v}\|\lesssim\frac{\lambda_{p+1}(x)+\Lambda_{p-1}(x)}{2}+\frac{\Lambda_p(x)}{2}=\frac{\Lambda_{p+1}(x)+\Lambda_{p-1}(x)}{2}.$$

So the bound from (5) holds at least for p-vectors $\mathbf{v}$ in $\bigwedge^p(E_x)$. Similar arguments carry over to all $\bigwedge^p(T_xM)$.

2.4. **Third step: Globalization.** The following results renders global the construction of Proposition 2.4.

Proposition 2.5. *Let a neighborhood $\mathcal{U} \ni f$, $p \in \{1, \dots, d-1\}$ and $\delta > 0$ be given. Then there exist $m \in \mathbb{N}$ and a diffeomorphism $g \in \mathcal{U}$ that equals f outside the open set $\Gamma_p(f,m)$ and such that*

$$\int_{\Gamma_p(f,m)} \Lambda_p(g,x)\, d\mu(x) < \delta + \int_{\Gamma_p(f,m)} \frac{\Lambda_{p-1}(f,x) + \Lambda_{p+1}(f,x)}{2}\, d\mu(x). \tag{10}$$

The proof goes as follows. Let $m \in \mathbb{N}$ be large and let $N \colon \Gamma_p(f,m) \to \mathbb{N}$ be the function given by Proposition 2.4 with $\kappa = \delta^2$. For almost every $x \in \Gamma_p(f,m)$ and every $n \geq N(x)$, the proposition provides a realizable sequence $\{\widehat{L}_i\}$ of length n at x satisfying (5). "Realizing" this sequence (see definition 2.2), we obtain a perturbation g of f supported in a small neighborhood of the segment of orbit $\{x, \dots, f^n(x)\}$, which is a tower $U \sqcup \cdots \sqcup f^n(U)$. Since the set $\Gamma_p(f,m)$ is open and invariant, these towers can always be taken inside it. Each tower $U \sqcup \cdots \sqcup f^n(U) = U \sqcup \cdots \sqcup g^n(U)$ contains a *sub-tower* $K \sqcup \cdots \sqcup f^n(K)$ where the perturbed derivatives are very close to the maps $\widehat{L}_i$. Hence if we choose U small enough then (5) will imply

$$\frac{1}{n} \log \|{\textstyle\bigwedge}^p Dg^n_y\| < \frac{\Lambda_{p-1}(x) + \Lambda_{p+1}(x)}{2} + 2\delta, \qquad \forall\, y \in K. \tag{11}$$

To construct the perturbation g globally, we cover all $\Gamma_p(f,m)$ but a subset of small measure with a (large) finite number of *disjoint* towers as above. Moreover, the towers can be chosen so that they have approximately the same heights (more precisely, all heights are between H and $3H$, where H is a constant). Then we glue all the perturbations (each one supported in a tower) and obtain a C^1 perturbation g of f. Let S be the support of the perturbation, i.e., the disjoint union of the towers. Let $S' \subset S$ be union of the corresponding sub-towers; then $\mu(S \setminus S') < \kappa\mu(S) \leq \delta^2$. Moreover, if $y \in S'$ is in the first floor of a sub-tower of height n then (11) holds.

To bound the integral in the left hand side of (10), we want to use the elementary fact (notice $\Gamma_p(f,m)$ is also g-invariant):

$$\int_{\Gamma_p(f,m)} \Lambda_p(g,x)\, d\mu(x) \leq \frac{1}{n} \int_{\Gamma_p(f,m)} \log \|{\textstyle\bigwedge}^p (Dg^n_x)\|\, d\mu(x) \quad \text{for all } n \in \mathbb{N}. \tag{12}$$

Let $n_0 = H/\delta$. Here comes a major step in the proof: To show that *most points (up to a set of measure of order of δ) in $\Gamma_p(f,m)$ are in S' and its positive iterates stay inside S' for at least n_0 iterates.* Intuitively, this is true by the following reason: The set S' is a g-castle[2], whose towers have heights $\approx H$. Therefore a segment of orbit of length $n_0 = \delta^{-1}H$, if it is contained in S', "winds" $\approx \delta^{-1}$ times around S'. Since S' is a castle, there are only δ^{-1} opportunities for the orbit to leave S'. In each opportunity, the probability of leave is of order of δ^2 (the measure of the complementary $\Gamma_p(f,m) \setminus S'$). Therefore the probability of leave S' in n_0 iterates is $\approx \delta^{-1}\delta^2 = \delta$.

Using the fact above, one shows that the right hand side of (12) with $n = n_0$ is bounded by the left hand side of (10), completing the proof of the proposition.

2.5. **Conclusion of the proof.** Let $\Gamma_p(f,\infty)$ be the set of points where there is no dominated splitting of index p, that is, $\Gamma_p(f,\infty) = \bigcap_{m \in \mathbb{N}} \Gamma_p(f,m)$.

The following is an easy consequence of Proposition 2.5.

[2]That is, a union of disjoint g-towers.

Proposition 2.6. *Given $f \in \mathrm{Diff}^1_\mu(M)$ and $p \in \{1, \dots, d-1\}$, let*

$$J_p(f) = \int_{\Gamma_p(f,\infty)} \frac{\lambda_p(f,x) - \lambda_{p+1}(f,x)}{2}\, d\mu(x).$$

Then for every $\mathcal{U} \ni f$ and $\delta > 0$, there exists a diffeomorphism $g \in \mathcal{U}$ such that

$$\int_M \Lambda_p(g,x)\, d\mu(x) < \int_M \Lambda_p(f,x)\, d\mu(x) - J_p(f) + \delta.$$

Using the proposition we can give the:

Proof of Theorem 7. Let $f \in \mathrm{Diff}^1_\mu(M)$ be a point of continuity of all maps $\mathrm{LE}_p(\cdot)$, $p = 1, \dots, d-1$. Then $J_p(f) = 0$ for every p. This means that $\lambda_p(f,x) = \lambda_{p+1}(f,x)$ for almost every x in the set $\Gamma_p(f,\infty)$.

Let $x \in M$ be an Oseledets regular point. If all Lyapunov exponents of f at x vanish, there is nothing to do.

For each p such that $\lambda_p(f,x) > \lambda_{p+1}(f,x)$, we have (if we exclude a zero measure set of x) $x \notin \Gamma_p(f,\infty)$. This means that there is a dominated splitting of index p, $T_{f^n x}M = E_n \oplus F_n$ along the orbit of x. It is not hard to see that E_n is necessarily the sum of the Oseledets spaces of f, at the point $f^n x$, associated to the Lyapunov exponents $\lambda_1(f,x), \dots, \lambda_p(f,x)$, and F_n is the sum of the spaces associated to the other exponents. This shows that the Oseledets splitting is dominated along the orbit of x. $\square$

2.6. **Symplectic diffeomorphisms.** Now let (M,ω) be a compact symplectic manifold without boundary, of dimension $\dim M = 2q$.

The Lyapunov exponents of symplectic diffeomorphisms have a symmetry property: $\lambda_j(f,x) = -\lambda_{2q-j+1}(f,x)$ for all $1 \le j \le q$. In particular, $\lambda_q(x) \ge 0$ and $\mathrm{LE}_q(f)$ is the integral of the sum of all non-negative exponents. Consider the splitting

$$T_x M = E^+_x \oplus E^0_x \oplus E^-_x,$$

where E^+_x, E^0_x, and E^-_x are the sums of all Oseledets spaces associated to positive, zero, and negative Lyapunov exponents, respectively. Then $\dim E^+_x = \dim E^-_x$ and $\dim E^0_x$ is even.

Theorem 8. *Let $f_0 \in \mathrm{Sympl}^1_\mu(M)$ be such that the map*

$$f \in \mathrm{Sympl}^1_\mu(M) \mapsto \mathrm{LE}_q(f) \in \mathbb{R}$$

is continuous at $f = f_0$. Then for μ-almost every $x \in M$, either $\dim E^0_x \ge 2$ or the splitting $T_x M = E^+_x \oplus E^-_x$ is uniformly hyperbolic along the orbit of x.

In the second alternative, what we actually prove is that the splitting is dominated at x. This is enough because, for symplectic diffeomorphisms, dominated splittings into two subspaces of the same dimension are uniformly hyperbolic. See section 4.

Theorem 3 follows from Theorem 8: As in the volume-preserving case, the function $f \mapsto \mathrm{LE}_q(f)$ is continuous on a residual subset $\mathcal{R}_1$ of $\mathrm{Sympl}^1_\mu(M)$. Also (see appendix B), there is a residual subset $\mathcal{R}_2 \subset \mathrm{Sympl}^1_\mu(M)$ such that for every $f \in \mathcal{R}_2$ either f is an Anosov diffeomorphism or all its hyperbolic sets have zero measure. The residual set of Theorem 3 is $\mathcal{R} = \mathcal{R}_1 \cap \mathcal{R}_2$.

The proof of Theorem 8 is similar to that of Theorem 7. Actually the only difference is in the first step. In the symplectic analogue of Proposition 2.3, we have to suppose that the spaces E and F are Lagrangian[3].

3. PROVING PREVALENCE OF NONZERO EXPONENTS

We discuss some main ingredients in the proofs of Theorems 5 and 6, focusing on the case when the base dynamics $f\colon M \to M$ is uniformly expanding, and μ is ergodic with $\operatorname{supp}\mu = M$. The general cases of the theorems follow from a more local version of similar arguments.

Notice that it is no restriction to consider $\nu \geq 1$: the Hölder cases $0 < \nu < 1$ are immediately reduced to the Lipschitz one $\nu = 1$ by replacing the metric $\operatorname{dist}(x,y)$ in M by $\operatorname{dist}(x,y)^\nu$.

3.1. **Bundle-free cocycles: genericity.**

Definition 3.1. $A\colon M \to \mathrm{SL}(d,\mathbb{R})$ *is called* bundle-free *if it admits no finite-valued Lipschitz continuous invariant line bundle: in other words, given any $\eta \geq 1$, there exists no Lipschitz continuous map $\psi\colon x \mapsto \{v_1(x),\dots,v_\eta(x)\}$ assigning to each $x \in M$ a subset of $\mathbb{RP}^{d-1}$ with exactly η elements, such that*

$$A(x)\big(\{v_1(x),\dots,v_\eta(x)\}\big) = \{v_1(f(x)),\dots,v_\eta(f(x))\} \quad \textit{for all } x \in M.$$

A is called stably bundle-free *if all Lipschitz maps in a neighborhood are bundle-free.*

The case $\eta = 1$ means that the cocycle has no invariant *Lipschitz* subbundles. The regularity requirement is crucial in view of the next theorem: invariant Lipschitz subbundles are exceptional, whereas Hölder invariant subbundles with poor Hölder constants are often robust! The following example illustrates these issues.

Example 3.2. *Let $G\colon S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}$, $G(\theta,x) = (f(\theta), g(\theta,x))$ be a smooth map with*

$$\sigma_1 \geq |f'| \geq \sigma_2 > \sigma_3 > |\partial_x g| > \sigma_4 > 1.$$

Let θ_0 be a fixed point of f and x_0 be the fixed point of $g(\theta_0,\cdot)$. Then

(1) *The set of points whose forward orbit is bounded is the graph of a continuous function $u\colon S^1 \to \mathbb{R}$ with $u(\theta_0) = x_0$. This function is ν-Hölder for any $\nu < \log\sigma_4/\log\sigma_1$. Typically it is not Lipschitz:*
(2) *The fixed point $p_0 = (\theta_0, x_0)$ has a strong-unstable set $W^{uu}(p_0)$ invariant under G and which is locally a Lipschitz graph over S^1. If u is Lipschitz then its graph must coincide with $W^{uu}(p_0)$.*
(3) *However, for an open dense subset of choices of g the strong-unstable set is not globally a graph: it intersects vertical lines at infinitely many points.*

Theorem 9. *Suppose $A \in C^\nu(M,\mathrm{SL}(d,\mathbb{R}))$ has $\lambda(A,x) = 0$ with positive probability, for some invariant measure μ. Then A is approximated in $C^\nu(M,\mathrm{SL}(d,\mathbb{R}))$ by stably bundle-free maps.*

Here is a sketch of the proof. The first step is to deduce from the hypothesis

$$\lim_{n\to\infty} \frac{1}{n}\log\|A^n(x)\| = 0 \quad \text{for } \mu\text{−almost all } x$$

[3]A subspace E of a symplectic vector space (V,ω) is called *Lagrangian* when $\dim E = \frac{1}{2}\dim V$ and $\omega(v_1,v_2) = 0\ \forall\, v_1,v_2 \in E$.

that Birkhoff averages of $\log\|A^i\|$ are also small: given δ there is $N \geq 1$ such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \frac{1}{N} \log\|A^N(f^{jN}(x))\| < \delta \quad \text{for } \mu\text{−almost all } x. \tag{13}$$

Using the shadowing lemma, one finds periodic points $p \in M$ satisfying (13) with δ replaced by 2δ. This implies that the eigenvalues β_j of $A^q(p)$, $q = \mathrm{per}(p)$ are all close to 1:

$$2(1-d)\delta < \frac{1}{q} \log|\beta_j| < 2\delta \quad \text{for all } j = 1, \dots, d.$$

We may take all the norms $|\beta_j|$ to be distinct. Now the argument is very much inspired by example 3.2. The eigenspaces of $A^q(p)$, seen as periodic points of the cocycle acting in the projective space, have strong-unstable sets that are *locally* Lipschitz graphs over M. Any Lipschitz continuous invariant line bundle ψ as in definition 3.1 has to coincide with the strong-unstable sets. But a simple transversality argument shows that *globally* the strong-unstable sets are not graphs (not even up to finite covering), if certain configurations with positive codimension are avoided.

3.2. **A geometric criterion for nonzero exponents.** Another key ingredient is the following result, which may be thought of as a geometric version of a classical result of Furstenberg [Fur63] about products of i.i.d. random matrices:

Theorem 10. *Suppose $A \in C^\nu(M, \mathrm{SL}(d,\mathbb{R}))$ is bundle-free and there exists some periodic point $p \in M$ of f such that the norms of the eigenvalues of A over the orbit of p are all distinct. Then $\lambda(A,\mu) > 0$ for any ergodic measure μ with local product structure and $\mathrm{supp}\,\mu = M$.*

The condition on the existence of some periodic point over which the cocycle has all eigenvalues with different norm is satisfied by an open and dense subset of $C^\nu(M, \mathrm{SL}(d,\mathbb{R}))$, that we denote SP. See the last section of [BVb]. We also denote by BF the subset of bundle-free maps. The proof of Theorem 10 may be sketched as follows.

Let $\hat{f}\colon \hat{M} \to \hat{M}$ be the natural extension of f, and $\hat{\mu}$ be the lift of μ to $\hat{M}$. Let $\hat{f}_A\colon \hat{M} \times \mathbb{RP}^{d-1} \to \hat{M} \times \mathbb{RP}^{d-1}$ be the projective cocycle induced by A over $\hat{f}$. Let us suppose that $\lambda(A,\mu) = 0$, and conclude that A is not bundle-free.

The first step is to prove that all points in the projective fiber of $\hat{\mu}$-almost every $\hat{x} \in \hat{M}$ have strong-stable and strong-unstable sets for $\hat{f}_A$ that are Lipschitz graphs over the stable manifold and the unstable manifold of $\hat{x}$ for $\hat{f}$. This follows from (13) and the corresponding fact for negative iterates. The strong-stable sets are locally horizontal: by definition, the cocycle is constant over local stable sets of the natural extension $\hat{f}$.

Next, one considers invariant probability measures m on $\hat{M} \times \mathbb{RP}^{d-1}$, invariant under $\hat{f}_A$ and projecting down to μ. One constructs such a measure admitting a family of conditional probabilities $\{m_{\hat{x}}\colon \hat{x} \in \hat{M}\}$ that is invariant under strong-unstable holonomies. Using the hypothesis $\lambda(A,\mu) = 0$ and a theorem of Ledrappier [Led86], one proves that the conditional measures are constant on local stable leaves (in other words, invariant under strong-stable holonomies), restricted to a full $\hat{\mu}$-measure subset of $\hat{M}$. Using local product structure and $\mathrm{supp}\,\hat{\mu} = \hat{M}$, one concludes that m admits some family of conditional measures $\{\tilde{m}_{\hat{x}}\colon \hat{x} \in \hat{M}\}$ that vary continuously with the point $\hat{x}$ on M and are invariant by both strong-stable and strong-unstable holonomies.

Finally, one considers a periodic point $\hat{p}$ of $\hat{f}$ such that the norms of the eigenvalues of $A^q(\hat{p})$, $q = \mathrm{per}(p)$ are all distinct. Then the probability $\tilde{m}_p$ is a convex combination of Dirac measures supported on the eigenspaces. Using the strong-stable and strong-unstable holonomies one propagates the support of $\tilde{m}_p$ over the whole M. This defines an invariant map ψ as in definition 3.1, with

$\eta \leq \# \operatorname{supp} \tilde{m}_p$. This map is Lipschitz, because strong-stable and strong-unstable holonomies are Lipschitz. Thus, A is not bundle-free.

3.3. **Conclusion of the argument and further comments.** Finally, we explain how to obtain Theorem 6, in the special case we are considering, from the two previous theorems. Let ZE be the subset of $A \in C^\nu(M, SL(d,\mathbb{R}))$ such that $\lambda(A,\mu) = 0$ for some ergodic measure with local product structure and $\operatorname{supp} = M$. Theorem 9 implies that any $A \in \mathrm{ZE}$ is approximated by the interior of BF. Since SP is open and dense, A is also approximated by the interior of $\mathrm{BF} \cap \mathrm{SP}$. By Theorem 10, the latter is contained in the complement of ZE. This proves that the interior of $C^\nu \setminus \mathrm{ZE}$ is dense in ZE, and so it is dense in the whole $C^\nu(M, SL(d,\mathbb{R}))$, as claimed. To get the infinite codimension statement observe that it suffices to avoid the positive codimension configuration mentioned before for *some* of infinitely many periodic points of f.

The following couple of examples help understand the significance of Theorem 10.

Example 3.3. *Let $M = S^1$, $f\colon M \to M$ be given by $f(x) = kx \bmod \mathbb{Z}$, for some $k \geq 2$, and μ be Lebesgue measure on M. Let*

$$A\colon M \to \mathrm{SL}(2,\mathbb{R}), \qquad A(x) = \begin{pmatrix} \beta(x) & 0 \\ 0 & 1/\beta(x) \end{pmatrix}$$

for some smooth function β such that $\int \log \beta \, d\mu = 0$. It is easy to ensure that the set $\beta^{-1}(1)$ is finite and does not contain $x = 0$. Then $A \in \mathrm{SP}$ and indeed the matrix A "looks hyperbolic" at most points. Nevertheless, the Lyapunov exponent $\lambda(A,\mu) = \int \log \beta \, d\mu = 0$. Notice that A is not bundle-free.

Hence the following heuristic principle: assuming there is a source of hyperbolicity somewhere in M (here the fact that $A \in \mathrm{SP}$), the only way Lyapunov exponents may happen to vanish is by having expanding directions mapped *exactly* onto contracting directions, thus causing hyperbolic behavior to be "wasted away".

Putting Theorems 4 and 10 together we may give a sharp account of Lyapunov exponents for a whole C^0 open set of cocycles. This construction contains the main result of [You93]. It also shows that the present results are in some sense optimal.

Example 3.4. *Let $f\colon S^1 \to S^1$ be C^2 uniformly expanding with $f(0) = 0$, and μ be any invariant probability with $\operatorname{supp} \mu = S^1$. Take $A\colon S^1 \to \mathrm{SL}(2,\mathbb{R})$ of the form*

$$A(x) = R_{\alpha(x)} A_0$$

where A_0 is some hyperbolic matrix, $\alpha\colon S^1 \to S^1$ is a continuous function with $\alpha(0) = 0$, and $R_{\alpha(x)}$ denotes the rotation of angle $\alpha(x)$. Let $\deg(\,\cdot\,)$ represent topological degree.

Corollary 3.5. *Assume $2\deg(\alpha)$* is not *a multiple of $\deg(f) - 1$. Then there exists a C^0 neighborhood $\mathcal{U}$ of A such that*

(1) *for B in a residual subset $\mathcal{R} \cap \mathcal{U}$ we have $\lambda(B,\mu) = 0$;*
(2) *for every $B \in C^\nu(S^1, \mathrm{SL}(2,\mathbb{R}))$, $\nu > 0$, we have $\lambda(B,\mu) > 0$.*

Proof. Start by taking $\mathcal{U}$ to be the isotopy class of A in the space of continuous maps from M to $\mathrm{SL}(2,\mathbb{R})$. We claim that, given any $B \in \mathcal{U}$, there is no *continuous* B-invariant map

$$\psi\colon M \ni x \mapsto \{\psi_1(x), \dots, \psi_\eta(x)\}$$

assigning a constant number $\eta \geq 1$ of elements of $\mathbb{RP}^1$ to each point $x \in M$. The proof is by contradiction. Suppose there exists such a map and the graph

$$G = \{(x, \psi_i(x)) \in S^1 \times \mathbb{RP}^1 \colon x \in S^1 \text{ and } 1 \leq i \leq \eta\}$$

is connected. Then G represents some element (η, ζ) of the fundamental group $\pi_1(S^1 \times \mathbb{RP}^1) = \mathbb{Z} \oplus \mathbb{Z}$. Because B is isotopic to A, the image of G under the cocycle must represent $(\eta \deg(f), \zeta + 2\deg(\alpha)) \in \pi_1(S^1 \times \mathbb{RP}^1)$; here the factor 2 comes from the fact that S^1 is the 2-fold covering of $\mathbb{RP}^1$. By the invariance of ψ we get

$$\zeta + 2\deg(\alpha) = \deg(f)\zeta$$

which contradicts the hypothesis that $\deg(f) - 1$ does not divide $2\deg(\alpha)$. If the graph G is not connected, consider the connected components instead. Since connected components are pairwise disjoint, they all represent elements with the same direction in the fundamental group. Then the same type of argument as before proves the claim in full generality.

Now let $\mathcal{R}$ be the residual subset in Theorem 4. The previous observation implies that no $B \in \mathcal{R} \cap \mathcal{U}$ may have an invariant dominated splitting. Then B must have all Lyapunov exponents equal to zero as claimed in (1). Similarly, that observation ensures that every $B \in \mathcal{U} \cap C^\nu$ is bundle-free. It is clear that A is in SP, and so is any map C^0 close to it. Thus, reducing $\mathcal{U}$ if necessary, we may apply Theorem 10 to conclude that $\lambda(B, \mu) > 0$. This proves (2). $\square$

4. PROJECTIVE *versus* PARTIAL HYPERBOLICITY

Here we prove that for symplectic cocycles existence of a dominated splitting implies partial hyperbolicity. This was pointed out by Mañé in [Mañ84]. A proof in dimension 4 was given by Arnaud [Arn].

Let $F\colon \mathcal{E} \to \mathcal{E}$ be a bundle automorphism covering a map $f\colon M \to M$. Let us recall some definitions. If $\Gamma \subset M$ is an f-invariant measurable set, we say that an F-invariant splitting $E^1 \oplus E^2$ over Γ is m-dominated if

$$\frac{\|F_x^m \mid E_x^2\|}{\mathbf{m}(F_x^m \mid E_x^1)} \leq \frac{1}{2}. \tag{14}$$

We call $E^1 \oplus E^2$ a dominated splitting if it is m-dominated for some $m \in \mathbb{N}$. Then we write $E^1 \succ E^2$. More generally, we say that a splitting $E^1 \oplus \cdots \oplus E^k$, into any number of sub-bundles, is dominated if

$$E^1 \oplus \cdots \oplus E^j \succ E^{j+1} \oplus \cdots \oplus E^k \quad \text{for every } 1 \leq j < k.$$

We are most interested in the case where the vector bundle $\mathcal{E}$ is endowed with a symplectic form, that is, a nondegenerate antisymmetric 2-form $\omega = (\omega_x)_{x \in M}$ varying continuously with the base point x. For this $\mathcal{E}$ must have even dimension. We say that F is a *symplectic cocycle*, when it preserves the symplectic form:

$$\omega_{f(x)}(F_x v, F_x w) = \omega_x(v, w) \quad \text{for every } x \in M \text{ and } v, w \in \mathcal{E}_x$$

Our aim in this section is to establish the following result:

Theorem 11. *Suppose F is a symplectic cocycle. Let Γ be an f-invariant set and $\mathcal{E}_\Gamma = E^+ \oplus E^{c-}$ be a dominated splitting of F such that* $\dim E^+ \leq \dim E^{c-}$.

(1) *Then E^{c-} splits invariantly as $E^{c-} = E^c \oplus E^-$, with* $\dim E^- = \dim E^+$.
(2) *$T_\Gamma M = E^+ \oplus E^c \oplus E^-$ is a dominated splitting.*
(3) *E^+ is uniformly expanding and E^- is uniformly contracting.*

Note that $E^c = 0$ if $\dim E^+ = \dim E^{c-}$.

4.1. **General properties of dominated splittings.** The angle $\sphericalangle(E^1, E^2)$ between two subbundles E^1 and E^2 on a set Γ is the infimum of $\sphericalangle(E^1_x, E^2_x)$ over all $x \in \Gamma$.

Transversality: *If $\mathcal{E}_\Gamma = E^1 \oplus E^2$ is a dominated splitting then $\sphericalangle(E^1, E^2) > 0$.*

Indeed, let $v_1 \in E^1_x$ and $v_2 \in E^2_x$ be arbitrary unit vectors. Condition (14) gives

$$2\|F^m_x v_2\| \le \|F^m_x v_1\| \le \|F^m_x v_2\| + \|F^m_x (v_1 - v_2)\|,$$

and this implies $\|v_1 - v_2\| \ge \|F^m_x\|^{-1}\, \mathbf{m}(F^m_x)$. By continuity of F and compactness of M, the last expression is bounded away from zero.

Uniqueness: *If $\mathcal{E}_\Gamma = E^1 \oplus E^2$ and $\mathcal{E}_\Gamma = \hat{E}^1 \oplus \hat{E}^2$ are dominated splittings with $\dim E^i = \dim \hat{E}^i$ then $E^i = \hat{E}^i$ for $i = 1, 2$.*

To see this, suppose there is $x \in \Gamma$ such that $E^2_x \ne \hat{E}^2_x$. Then there exist $E^2_x \ni v_2 = \hat{v}_1 + \hat{v}_2 \in \hat{E}^1_x \oplus \hat{E}^2_x$ with $\hat{v}_1 \ne 0$. By domination, $\|F^n_x \hat{v}_1\|$ is much larger than $\|F^n_x \hat{v}_2\|$ and so $\|F^n_x v_2\|$ is comparable to $\|F^n_x \hat{v}_1\|$, when n is large. Since $\|F^n_x \hat{v}_1\| \ge \mathbf{m}(F^n_x \mid_{\hat{E}^1_x}) \|\hat{v}_1\|$, this proves that

$$\frac{\|F^n_x \mid_{E^2_x}\|}{\mathbf{m}(F^n_x \mid_{\hat{E}^1_x})} \ge C_1 \quad \text{and} \quad \frac{\|F^n_x \mid_{\hat{E}^2_x}\|}{\mathbf{m}(F^n_x \mid_{E^1_x})} \ge C_2$$

(the second inequality follows by symmetry) for constants $C_1 > 0$ and $C_2 > 0$ independent of n. In particular,

$$\frac{\|F^n_x \mid_{E^2_x}\|}{\mathbf{m}(F^n_x \mid_{E^1_x})} \frac{\|F^n_x \mid_{\hat{E}^2_x}\|}{\mathbf{m}(F^n_x \mid_{\hat{E}^1_x})} \ge C_1 C_2 > 0.$$

On the other hand, the domination condition (14) implies that the left hand side converges to zero as $n \to +\infty$. This contradiction proves that $E^2 = \hat{E}^2$. A similar argument, iterating backwards, proves that $E^1 = \hat{E}^1$.

Continuity: *A dominated splitting $\mathcal{E}_\Gamma = E^1 \oplus E^2$ is continuous, and extends continuously to a dominated splitting over the closure of Γ.*

Indeed, let $(x_j)_j$ be any sequence in Γ converging to some $x \in M$. Taking subsequences if necessary, each $(E^i_{x_j})_j$ converges to a subspace $\hat{E}^i_x$ with the same dimension, when $j \to \infty$. By transversality, $T_x M = \hat{E}^1_x \oplus \hat{E}^2_x$. For each $n \in \mathbb{Z}$ and $i = 1, 2$, the sequence $E^i_{f^n(x_j)}$ converges to $\hat{E}^i_{f^n(x)} = F^n_x(\hat{E}^i_x)$ when $j \to \infty$. Taking m as in (14), by continuity we have

$$\frac{\|F^m_y \mid_{\hat{E}^2_y}\|}{\mathbf{m}(F^m_y \mid_{\hat{E}^1_y})} \le \frac{1}{2} \quad \text{for all } y = f^n(x),\, n \in \mathbb{Z}.$$

This means that $\hat{E}^1 \oplus \hat{E}^2$ is a dominated splitting over the orbit of x. By uniqueness, $\hat{E}^1 \oplus \hat{E}^2$ does not depend on the choice of the sequence $(x_j)_j$, and it coincides with $E^1 \oplus E^2$ if $x \in \Gamma$. This proves continuity, and continuous extension to the closure.

Lemma 4.1. *Let Γ be a measurable f-invariant subset of M, and E^1, E^2, E^3 be sub-bundles of $\mathcal{E}$ restricted to Γ.*

(1) *If $E^1 \succ E^2$, $E^1 \succ E^3$ and $\sphericalangle(E^2, E^3) > 0$ then $E^1 \succ E^2 \oplus E^3$.*
(2) *If $E^1 \succ E^3$, $E^2 \succ E^3$ and $\sphericalangle(E^1, E^2) > 0$ then $E^1 \oplus E^2 \succ E^3$.*
(3) *If $E^1 \succ E^2$ and $E^2 \succ E^3$ then $E^1 \succ E^2 \oplus E^3$ and $E^1 \oplus E^2 \succ E^3$.*

Proof. Suppose $E^1 \succ E^2$ and $E^1 \succ E^3$, with $\sphericalangle(E^2,E^3) > 0$. The last condition implies that there exists $c > 0$ such that $\|v_2 \oplus v_3\| \geq c\big(\|v_2\| + \|v_3\|\big)$, for every $v_2 + v_3 \in E^2_x \oplus E^3_x$ and $x \in \Gamma$. The first two imply

$$\frac{\|F_x^{km} v_2\|}{\|v_2\|} \leq \frac{1}{2^k} \frac{\|F_x^{km} v_1\|}{\|v_1\|} \quad \text{and} \quad \frac{\|F_x^{km} v_3\|}{\|v_3\|} \leq \frac{1}{2^k} \frac{\|F_x^{km} v_1\|}{\|v_1\|}, \tag{15}$$

for $k \geq 1$, $x \in \Gamma$, and any nonzero $v_i \in E^i_x$, $i = 1,2,3$. Fix $k \geq 1$ large enough so that $c2^k > 2$. Then

$$\frac{\|F_x^{km}(v_2+v_3)\|}{\|v_2+v_3\|} \leq \frac{1}{c} \frac{\|F_x^{km} v_2\| + \|F_x^{km} v_3\|}{\|v_2\| + \|v_3\|} \leq \frac{1}{c2^k} \frac{\|F_x^{km} v_1\|}{\|v_1\|} \leq \frac{1}{2} \frac{\|F_x^{km} v_1\|}{\|v_1\|}$$

for all nonzero $v_1 \in E^1_x$ and $\|v_2 + v_3\| \in E^2_x \oplus E^3_x$. This proves claim 1. The proof of claim 2 is analogous.

Statement 3 is a consequence of the previous two. Indeed, by section 4.1, $E^2 \succ E^3$ implies $\sphericalangle(E^2,E^3) > 0$. Moreover, the hypotheses give

$$\frac{\|F_x^{km} v_2\|}{\|v_2\|} \leq \frac{1}{2^k} \frac{\|F_x^{km} v_1\|}{\|v_1\|} \quad \text{and} \quad \frac{\|F_x^{km} v_3\|}{\|v_3\|} \leq \frac{1}{2^k} \frac{\|F_x^{km} v_2\|}{\|v_2\|},$$

which is stronger than (15). So, $E^1 \succ E^2 \oplus E^3$ follows just as in 1. Similarly, $E^1 \oplus E^2 \succ E^3$ is proved in the same way as in 2. □

Remark 4.2. *One may have $E^1 \succ E^3$ and $E^2 \succ E^3$ but $E^1 \oplus E^2 \not\succ E^3$. Similarly, $E^1 \succ E^2$ and $E^1 \succ E^3$ does not imply $E^1 \succ E^2 \oplus E^3$.*

4.2. **Partial hyperbolicity.** We recall a few elementary facts; see [Arn78, § 43] for more information. Given a symplectic vector space (V_0, ω_0), the *skew-orthogonal complement* H^ω of a subspace $H \subset V_0$ is defined by

$$H^\omega = \{v \in V_0;\ \omega_0(v,h) = 0 \text{ for all } h \in H\}.$$

The subspace H is called *null* (or *isotopic*) if $H \subset H^\omega$, that is, if the symplectic form vanishes in $H \times H$.

For any chosen scalar product $\cdot$ in V_0, let $J_0 \colon V_0 \to V_0$ be the antisymmetric isomorphism defined by $\omega_0(u,u') = J_0 u \cdot u'$. Then H^ω is the orthogonal complement of $J_0(H)$. In particular, it has $\dim H^\omega = \dim V_0 - \dim H$.

The following simple consequences of compactness and continuity are used in the proofs that follow: there exists a constant $C_0 > 0$ such that

$$|\omega(u,u')| \leq C_0 \,\|u\|\,\|u'\| \quad \text{and} \quad C_0^{-1}\|u\| \leq \|Ju\| \leq C_0\|u\|$$

for all vectors u and u', where $J \colon \mathcal{E} \to \mathcal{E}$ is defined by $\omega(u,u') = Ju \cdot u'$.

Now we prove Theorem 11. Up to partitioning Γ into finitely many invariant subsets, we may suppose that the dimensions of E^+ and E^{c-} are constant, and we do so. The first step is to show that E^+ is uniformly expanding. Let $2n$ be the dimension of the bundle.

Lemma 4.3. *Let Γ be an f-invariant set and E^1, E^2 be invariant subbundles of $\mathcal{E}_\Gamma$, such that $E^1 \succ E^2$ and $\dim E^2 \geq n$. Then E^1 is uniformly expanding and, consequently, the space E^1 is null.*

Proof. Assume the splitting $E^1 \oplus E^2$ is m-dominated. Fix any $x \in \Gamma$ and $v_1 \in E^1_x$ with $\|v_1\| = 1$. The space $H = \mathbb{R}v_1 \oplus E^2_x$ has dimension $\dim H > n$, therefore the intersection $H \cap J_x(H)$ is nontrivial: there exists some nonzero vector $u \in H$ such that $u' = J_x^{-1}(u) \in H$. Assume $\|u\| = 1$;

then $\|u'\| \le C_0$. Write $u = av_1 + w_2$, with $a \in \mathbb{R}$ and $w_2 \in E^2$. Since $\sphericalangle(E^1, E^2) > 0$ (see section 4.1), there exists a constant $c > 0$, independent of x, such that $|a|$, $\|w_2\| \le c$. Analogously, writing $u' = a'v_1 + w_2'$, with $a' \in \mathbb{R}$ and $w_2' \in E^2$, we have $|a'|$, $\|w_2'\| \le c\|u'\| \le C_0 c$. Now

$$1 = \|u\|^2 = \omega(u', u) \le |\omega(w_2, a'v_1)| + |\omega(w_2', av_1)| + |\omega(w_2, w_2')| \tag{16}$$

Let $k \in \mathbb{N}$. Then, by domination, $\|F_x^{mk} w\| \le 2^{-k}\|F_x^{mk} v_1\|\,\|w\|$ for all $w \in E_x^2$. We are going to use this fact and the invariance of ω to bound terms in the right hand side of (16). First,

$$\begin{aligned}|\omega(w_2, a'v_1)| = |a'|\,|\omega(F_x^{mk} w_2, F_x^{mk} v_1)| &\le C_0^2 c\,\|F_x^{mk} w_2\|\,\|F_x^{mk} v_1\| \\ &\le C_0^2 c\,2^{-k}\|w_2\|\,\|F_x^{mk} v_1\|^2 \le C_0^2 c^2 2^{-k}\|F_x^{mk} v_1\|^2.\end{aligned}$$

In an analogous way, we see that the right hand side is also an upper bound for $|\omega(w_2', av_1)|$. Also,

$$\begin{aligned}|\omega(w_2, w_2')| = |\omega(F_x^{mk} w_2, F_x^{mk} w_2')| &\le C_0\|F_x^{mk} w_2\|\,\|F_x^{mk} w_2'\| \\ &\le C_0 2^{-2k}\|w_2\|\,\|w_2'\|\,\|F_x^{mk} v_1\|^2 \le C_0^2 c^2\, 2^{-2k}\|F_x^{mk} v_1\|^2.\end{aligned}$$

Substituting these estimates in (16), we conclude that

$$1 \le C_0^2 c^2 (2^{-2k} + 2^{-k+1})\|F_x^{mk} v_1\|^2,$$

That is $\|F_x^{mk} v_1\| \ge 2$, if $k = k(C_0, c)$ is chosen large enough. This estimate holds for any x and any unit vector $v_1 \in E_x^1$, so E^1 is uniformly expanding.

Let $x \in \Gamma$ and v_1, w_1 be any vectors in E_x^1. By uniform expansion,

$$\|F_x^{-mkj} v_1\| \le 2^{-j}\|v_1\| \quad \text{and} \quad \|F_x^{-mkj} w_1\| \le 2^{-j}\|w_1\|$$

and so

$$|\omega(v_1, w_1)| = |\omega(F_x^{-mkj} v_1, F_x^{-mkj} w_1)| \le C_0 2^{-2j}\|v_1\|\,\|w_1\|$$

for all $j \ge 1$. This implies $\omega(v_1, w_1) = 0$. $\square$

The next lemma does not require the domination condition:

Lemma 4.4. *Let $\mathcal{E}_\Gamma = E^1 \oplus E^{23}$ be an invariant and continuous splitting such that the spaces E^1 are null and $\dim E^1 < \dim E^{23}$. Then E^{23} splits invariantly and continuously as $E^{23} = E^2 \oplus E^3$, with $\dim E^1 = \dim E^3$. Moreover, $(E^1)^\omega = E^1 \oplus E^2$ and $(E^2)^\omega = E^1 \oplus E^3$.*

In the proof we shall use the following simple properties of the skew-orthogonal complement. If H, $G \subset \mathbb{R}^{2n}$ are vector spaces then:

$$\dim H + \dim H^\omega = 2n, \qquad (H^\omega)^\omega = H, \quad \text{and} \quad (H+G)^\omega = H^\omega \cap G^\omega.$$

Proof of Lemma 4.4. Define the following subbundles:

$$E^2 = (E^1)^\omega \cap E^{23}, \quad E^{13} = (E^2)^\omega, \quad E^3 = E^{13} \cap E^{23}.$$

All these subbundles are continuous and invariant under F, because E^1, E^{23} and the symplectic form ω are continuous and invariant.

We first check that $E^2 \cap E^{13} = \{0\}$. Let $v \in E^2 \cap E^{13}$. We have $E^{13} = \big((E^1)^\omega \cap E^{23}\big)^\omega = E^1 + (E^{23})^\omega$, so we can write $v = u_1 + w$, with $u_1 \in E^1$ and $w \in (E^{23})^\omega$. Since E^1 is null and $v \in (E^1)^\omega$, we have $\omega(w, v_1) = \omega(u_1 + w, v_1) = 0$ for all $v_1 \in E^1$. That is, $w \in (E^1)^\omega$. But $(E^1)^\omega \cap (E^{23})^\omega = (\mathbb{R}^{2n})^\omega = \{0\}$, so $w = 0$. Thus $v = u_1 \in E^1 \cap E^{23}$ and $v = 0$.

Denote $u = \dim E^1$. It is easy to see that $E^1 \cap (E^{23})^\omega = \{0\}$ and thus $E^{13} = E^1 \oplus (E^{23})^\omega$. It follows that $\dim E^{13} = 2u$ and $\dim E^2 = 2n - 2u$. Also,

$$E^3 = E^{13} \cap E^{23} \Rightarrow \dim E^3 \ge \dim E^{13} + \dim E^{23} - 2n = u.$$

But $E^2 + E^3 \subset E^{23}$, therefore $\dim E^3 = u$ and $E^{23} = E^2 \oplus E^3$.

Let us check the last two claims of the lemma. We have $E^1 \oplus E^2 \subset (E^1)^\omega$ and, by dimension counting, this inclusion is an equality. Analogously, we prove that $E^1 \oplus E^3 = (E^2)^\omega$. $\square$

Lemma 4.5. *In the setting of Lemma 4.4, there exists a constant $\gamma > 0$ such that for every $x \in \Gamma$,*

(1) *given $v_2 \in E^2_x \setminus \{0\}$ there is $w_2 \in E^2_x \setminus \{0\}$ with $\omega(v_2, w_2) \geq \gamma \|v_2\| \|w_2\|$;*
(2) *given $v_1 \in E^1_x \setminus \{0\}$ there is $w_3 \in E^3_x \setminus \{0\}$ with $\omega(v_1, w_3) \geq \gamma \|v_1\| \|w_3\|$.*
(3) *given $v_3 \in E^3_x \setminus \{0\}$ there is $w_1 \in E^1_x \setminus \{0\}$ with $\omega(v_3, w_1) \geq \gamma \|v_3\| \|w_1\|$;*

Proof. We first note that since $\mathcal{E}_\Gamma = E^1 \oplus E^2 \oplus E^3$ is a continuous splitting,

$$\sphericalangle(E^2 \oplus E^3, E^1) > 0, \quad \sphericalangle(E^1 \oplus E^3, E^2) > 0 \quad \text{and} \quad \sphericalangle(E^1 \oplus E^2, E^3) > 0.$$

Given $v_2 \in E^2_x \setminus \{0\}$, let $w = Jv_2$. Then

$$\omega(v_2, w) = \|Jv_2\| \|w\| \geq C_0^{-1} \|v_2\| \|w\|.$$

Write $w = w_{13} + w_2$, with $w_{13} \in E^1_x \oplus E^3_x$ and $w_2 \in E^2_x$. Since $\sphericalangle(E^1 \oplus E^3, E^2) > 0$, there exists a constant $\gamma_0 > 0$, independent of x, such that $\|w\| \geq \gamma_0 \|w_2\|$. On the other hand, $\omega(v_2, w_{13}) = 0$, because $E^1 \oplus E^3 = (E^2)^\omega$. Then $w_2 \neq 0$ and

$$\omega(v_2, w_2) = \omega(v_2, w) \geq C_0^{-1} \|v_2\| \|w\| \geq C_0^{-1} \gamma_0 \|v_2\| \|w_2\|.$$

This proves claim 1, with $\gamma = \gamma_0 / C_0$.

The proof of claim 2 is analogous. Given $v_1 \in E^2_x \setminus \{0\}$, let $w = Jv_1$. Then $\omega(v_1, w) \geq C_0^{-1} \|v_1\| \|w\|$. Write $w = w_{12} + w_3$, with $w_{12} \in E^1_x \oplus E^2_x$ and $w_3 \in E^2_x$. As $\sphericalangle(E^1 \oplus E^2, E^3) > 0$, there exists a uniform constant $\gamma_0 > 0$ such that $\|w\| \geq \gamma_0 \|w_3\|$. And since $E^1 \oplus E^2 = (E^1)^\omega$, $\omega(v_1, w_{12}) = 0$. Therefore $w_3 \neq 0$ and

$$\omega(v_1, w_3) = \omega(v_1, w) \geq C_0^{-1} \|v_1\| \|w\| \geq C_0^{-1} \gamma_0 \|v_1\| \|w_3\|.$$

To prove the last claim, notice that the map $L \colon v_1 \in E^1_x \mapsto w_3 \in E^3_x$ defined in the proof of claim 2 is linear and injective. Since $\dim E^1_x = \dim E^3_x$, L is an isomorphism. Now, given $v_3 \in E^3_x$, take $w_1 = L^{-1}(v_3)$. $\square$

Now we can complete the proof of Theorem 11:

Proof. Let $\mathcal{E}_\Gamma = E^+ \oplus E^{c-}$ be as in the assumption. By Lemma 4.3, E^+ is uniformly expanding. If E^+ and E^{c-} have the same dimension, we set $E^- = E^{c-}$. Applying Lemma 4.3 to the inverse cocycle, we conclude that E^- is uniformly contracting, completing the proof. From now on we assume $\dim E^+ < \dim E^{c-}$.

The symplectic form is identically zero on E^+, by Lemma 4.3. Then we may apply Lemma 4.4, with $E^1 = E^+$ and $E^{23} = E^{c-}$. Let $E^{c-} = E^c \oplus E^-$ be the invariant splitting provided by Lemma 4.4, that is, $E^c = E^2$ and $E^- = E^3$.

Claim 2 in the theorem means that $E^+ \succ E^c \oplus E^-$ and $E^+ \oplus E^c \succ E^-$. Since the former is part of the assumptions, we only have to prove the latter statement. Also by assumption, $E^+ \succ E^-$ and $\sphericalangle(E^+, E^c) > 0$. So, by part 2 of Lemma 4.1, it is enough to show that $E^c \succ E^-$.

Let $m \in \mathbb{N}$ be fixed such that E^+ m-dominates E^{c-}. Fix $k \in \mathbb{N}$ such that $2^{k-1} > C_0^2 \gamma^{-2}$. Let $x \in \Gamma$ and unit vectors $v^c \in E^c_x$ and $v^- \in E^-_x$ be given. By Lemma 4.5, there are unit vectors $w^c \in E^c_x$ and $w^+ \in E^+_x$ such that

$$\omega(w^c, v^c) \geq \gamma \quad \text{and} \quad \omega(F^{mk}_x w^+, F^{mk}_x v^-) \geq \gamma \|F^{mk}_x w^+\| \|F^{mk}_x v^-\|.$$

Then, for any $k \in \mathbb{N}$,

$$\|F_x^{mk} w^c\| \, \|F_x^{mk} v^c\| \geq C_0^{-1} |\omega(F_x^{mk} w^c, F_x^{mk} v^c)| \geq C_0^{-1}\gamma \quad \text{and}$$
$$\|F_x^{mk} w^+\| \, \|F_x^{mk} v^-\| \leq \gamma^{-1} |\omega(w^+, v^-)| \leq C_0 \gamma^{-1}.$$

The assumption $E^+ \succ E^{c-}$ implies that $\|F^{mk} w^+\| \geq 2^k \|F^{mk} w^c\|$. Therefore

$$\|F^{mk} v^c\| \geq \frac{C_0^{-1}\gamma}{\|F^{mk} w^c\|} \geq \frac{C_0^{-1}\gamma 2^k}{\|F^{mk} w^+\|} \geq C_0^{-2}\gamma^2 2^k \|F^{mk} v^-\| \geq 2\|F^{mk} v^-\|.$$

Thus E^c dominates E^- and part 2 of the theorem is proved.

Now we consider part 3. We already know, from Lemma 4.3, that E^u is uniformly expanding: there exists $m_1 \in \mathbb{N}$ such that $\|F_x^{m_1 j} v^+\| \geq 2^j \|v^+\|$ for all $x \in \Gamma$, $v^+ \in E_x^+$, and $j \in \mathbb{N}$. Fix $j \in \mathbb{N}$ such that $2^{j-1} > C_0 \gamma^{-1}$. The following argument proves that E^s is uniformly contracting. Given an unit vector $v^- \in E_x^-$, use part 2 of Lemma 4.5 to find an unit vector $v^+ \in E_x^+$ such that

$$\omega(F_x^{m_1 j} v^-, F_x^{m_1 j} v^+) \geq \gamma \|F_x^{m_1 j} v^-\| \, \|F_x^{m_1 j} v^+\| \, .$$

Then $\|F_x^{m_1 j} v^-\| \, \|F_x^{m_1 j} v^+\| \leq \gamma^{-1}\omega(v^-, v^+) \leq C_0 \gamma^{-1}$ and so

$$\|F_x^{m_1 j} v^-\| \leq \frac{C_0 \gamma^{-1}}{\|F_x^{m_1 j} v^+\|} \leq C_0 \gamma^{-1} 2^{-j} \leq \frac{1}{2} \, .$$

This proves that $F^{m_1 j}$ contracts every $v^- \in E_x^-$, with uniform rate of contraction. The proof of Theorem 11 is complete. □

Remark 4.6. *Uniform contraction implies that the symplectic form is identically zero also on E^-.*

APPENDIX A. $\mathrm{Aut}(\mathbb{D})$-COCYCLES AND THE OSELEDETS THEOREM

Here we are going to discuss cocycles with values in the group of isometries of the Poincaré disk. There is a natural notion of Lyapunov exponent for these cocycles, and we prove some of its properties in Theorem 12. In fact, we are going to show that Theorem 12 is equivalent to Oseledets theorem in the case when the vector bundle is 2-dimensional.

There are several proofs of Oseledets theorem in the literature, besides the original one. See for instance [Mañ87, Chapter 4]. Another proof of the 2-dimensional case may be found in [You95]. The same basic strategy as in here was used by Thieullen [Thi97] to prove a geometric reduction theorem for 2-dimensional cocycles, that we recall below.

Karlsson and Margulis [KM99] recently generalized Oseledets theorem to cocycles with values in much more general groups, satisfying some geometric assumptions.

A.1. **Automorphisms of the disk.** $\mathrm{Aut}(\mathbb{D})$ is the set of all *automorphisms* of the unit disk $\mathbb{D} = \{z \in \mathbb{C};\ |z| < 1\}$, that is, all conformal diffeomorphisms $f\colon \mathbb{D} \to \mathbb{D}$ (orientation-preserving or not). The *hyperbolic metric* on the disk is given by

$$d\rho = \frac{2\,|dz|}{1 - |z|^2} \, . \tag{17}$$

Straight lines through the origin are geodesics, and therefore

$$\rho(z, 0) = 2\int_0^{|z|} \frac{dr}{1 - r^2} = \log\frac{1 + |z|}{1 - |z|} = 2\,\mathrm{arctgh}\,|z| \, . \tag{18}$$

All automorphisms of the disk are isometries for the hyperbolic metric. Using this we may deduce the general expression of ρ:

$$\rho(z_1, z_2) = \rho\big(\frac{z_1 - z_2}{1 - z_1\overline{z}_2}\,, 0\big) = 2\,\mathrm{arctgh}\,\frac{|z_1 - z_2|}{|1 - z_1\overline{z}_2|}\,. \tag{19}$$

A.2. $\mathrm{Aut}(\mathbb{D})$**-cocycles and Lyapunov exponents.** Let (X, μ) be a probability space and let $T\colon X \to X$ be a μ-preserving invertible transformation. Let $f\colon X \to \mathrm{Aut}(\mathbb{D})$ be a measurable map, whose values we indicate by $x \mapsto f_x$. We also denote $f_x^0 = \mathrm{id}$, $f_x^n = f_{T^{n-1}x} \circ \cdots \circ f_x$ and $f_x^{-n} = (f_{T^n x})^{-1} \circ \cdots \circ (f_{T^{-1}x})^{-1}$, for each $x \in X$ and $n \in \mathbb{N}$.

Theorem 12. *Let $T\colon X \to X$ and let $f\colon X \to \mathrm{Aut}(\mathbb{D})$ be as above. Assume that*

$$\int_X \rho(f_x(0), 0)\, d\mu(x) < \infty. \tag{20}$$

Then there exists a measurable function $\lambda\colon X \to [0, \infty)$ such that

$$\lim_{n\to\infty} \frac{1}{n} \rho(f_x^n(0), 0) = 2\lambda(x) \quad \text{for } \mu\text{-almost every } x \in X. \tag{21}$$

Furthermore, if $\lambda(x) > 0$ there are $w^s(x)$, $w^u(x) \in \partial\mathbb{D}$ such that

$$\lim_{n\to+\infty} \frac{1}{n} \log |(f_x^n)'(z)| = \begin{cases} 2\lambda & \text{if } z = w^s(x) \\ -2\lambda & \text{if } z \in \overline{\mathbb{D}} \text{ with } z \neq w^s(x), \end{cases}$$

$$\lim_{n\to-\infty} \frac{1}{n} \log |(f_x^n)'(z)| = \begin{cases} -2\lambda & \text{if } z = w^u(x) \\ 2\lambda & \text{if } z \in \overline{\mathbb{D}} \text{ with } z \neq w^u(x). \end{cases}$$

If $\lambda(x) = 0$ then

$$\lim_{n\to\pm\infty} \frac{1}{n} \log |(f_x^n)'(z)| = 0 \text{ for all } z \in \overline{\mathbb{D}}.$$

Remark A.1. *In view of* (18)*, the relation* (21) *is equivalent to*

$$\lim_{n\to\infty} \frac{1}{n} \log\left(1 - |f_x^n(0)|\right) = 2\lambda \quad \text{for } \mu\text{-almost every } x.$$

Remark A.2. *The contents of* (20) *and* (21) *do not change if we replace the origin with any other point a in the open disk, because $\rho(f(a), a) \leq \rho(f(0), 0) + 2\rho(a, 0)$.*

We shall use Kingman's subadditive ergodic theorem in the following form:

Theorem 13 ([Kin68])**.** *If $(\varphi_n)_{n=1,2,\ldots}$ is a sequence of integrable functions such that $\inf_n \int \varphi_n > -\infty$ and $\varphi_{m+n} \leq \varphi_m + \varphi_n \circ T^m$ for all $m, n \geq 1$ then $\frac{1}{n}\varphi_n$ converges almost everywhere.*

Proof of Theorem 12. Define $\varphi_n(x) = \rho(f_x^n(0), 0)$. Then $\varphi_{m+n} \leq \varphi_m + \varphi_n \circ T^m$, by the triangle inequality. Using Theorem 13 we get that $\frac{1}{n}\varphi_n$ converges μ-almost everywhere to a function 2λ. Since $\varphi_n \geq 0$, $\lambda \geq 0$. This proves (21).

Define $w_n(x) = (f_x^n)^{-1}(0)$ for every integer n. Notice that, by the invariance of the hyperbolic metric, $\rho(w_n(x), 0) = \rho(f_x^n(0), 0)$. Using (18) we get, for almost every x,

$$\lim_{n\to+\infty} \frac{1}{n} \log(1 - |w_n(x)|) = -2\lambda(x). \tag{22}$$

If $\lambda(x) > 0$ then the hyperbolic distance from $w_n(x)$ to the origin goes to infinity, which means that $w_n(x)$ converges to the boundary of $\mathbb{D}$ as $n \to \infty$.

Lemma A.3. *We have $\limsup\limits_{n\to+\infty} \dfrac{1}{n} \log |w_{n+1}(x) - w_n(x)| \leq -2\lambda(x)$ for μ-almost every x.*

Proof. We shall write w_n for $w_n(x)$. Since the hyperbolic metric is invariant under automorphisms,

$$\begin{aligned}\rho(w_{n+1}, w_n) &= \rho\big((f_x^n)^{-1} \circ (f_{T^n x})^{-1}(0), (f_x^n)^{-1}(0)\big) \\ &= \rho((f_{T^n x})^{-1}(0), 0) = \rho(f_{T^n x}(0), 0).\end{aligned} \tag{23}$$

The idea of the proof is that if $\rho(f_{T^n x}(0), 0)$ is not too big, that is, if w_{n+1} and w_n are not too far away from each other in terms of the hyperbolic metric, then the Euclidean distance between w_{n+1} and w_n will have to be exponentially small, since $w_n \to \partial\mathbb{D}$ exponentially fast (assuming $\lambda(x) > 0$). Write $b_n(x) = f_{T^n(x)}(0)$, for simplicity. For almost every x, we have

$$\frac{1}{n}\rho(b_n(x), 0) \to 0\,. \tag{24}$$

This follows from Birkhoff's theorem applied to the function $\varphi(x) = \rho(f_x(0), 0)$, which, by assumption (20), is integrable. Fix x in the full measure set where (22) and (24) hold. In view of (18)–(19) the equality (23) implies

$$\frac{|w_{n+1} - w_n|}{|1 - w_n \overline{w}_{n+1}|} = |b_n|$$

or, equivalently,

$$\begin{aligned}|w_{n+1} - w_n| = |b_n|\,|1 - w_n\overline{w}_{n+1}| &= |b_n|\left|1 - |w_n|^2 + w_n(\overline{w}_n - \overline{w}_{n+1})\right| \\ &\le |b_n|\left(1 - |w_n|^2 + |w_n|\,|w_{n+1} - w_n|\right)\,.\end{aligned}$$

That is,

$$|w_{n+1} - w_n| \le \frac{|b_n|(1 - |w_n|^2)}{1 - |b_n|\,|w_n|}\,.$$

Since $|w_n| < 1$ and $|b_n| < 1$, the last inequality implies

$$|w_{n+1} - w_n| < \frac{1 - |w_n|^2}{1 - |b_n|} < \frac{2\,(1 - |w_n|)}{1 - |b_n|}.$$

The condition (24) is equivalent to $\frac{1}{n}\log(1 - |b_n|) \to 0$. Combining this with (22) and the inequality above, we conclude

$$\limsup_{n\to+\infty} \frac{1}{n}\log|w_{n+1} - w_n| \le -2\lambda(x).$$

This proves the lemma. $\square$

Assume $\lambda = \lambda(x) > 0$. Then the lemma implies that $w_n(x) = (f_x^n)^{-1}(0)$ is a Cauchy sequence, relative to the Euclidean metric, for almost every such x. Let $w^s(x) \in \partial\mathbb{D}$ be the limit. Let us show how to compute the growth rate of $\log|(f_x^n)'(z)|$ for $z \in \partial\mathbb{D}$. We are going to use the following formula, whose proof may be found in [Nic89, page 12] :

$$f \in \mathrm{Aut}(\mathbb{D}),\ z \in \partial\mathbb{D} \implies |f'(z)| = \frac{1 - |f^{-1}(0)|^2}{|z - f^{-1}(0)|^2}\,.$$

Therefore

$$|(f_x^n)'(z)| = \frac{1 - |f_x^n(0)|^2}{|z - (f_x^n)^{-1}(0)|^2} = (1 + |w_n|)\frac{1 - |w_n|}{|z - w_n|^2}.$$

Using (22) we deduce

$$\lim_{n\to+\infty} \frac{1}{n}\log|(f_x^n)'(z)| = -2\lambda - 2\lim_{n\to+\infty}\frac{1}{n}\log|z - w_n|\,. \tag{25}$$

For all $z \neq w^s(x)$ this gives that the limit is -2λ.

Now consider the case $z = w^s(x)$. Since $|w_n - w^s| \geq 1 - |w_n|$, we have $\liminf \frac{1}{n} \log |w^s - w_n| \geq -2\lambda$. On the other hand, take $0 < \varepsilon < 2\lambda$. By Lemma A.3, we have $|w_{j+1} - w_j| \leq e^{(-2\lambda+\varepsilon)j}$ if j is large enough. Hence

$$|w^s - w_n| \leq \sum_{j=n}^{\infty} |w_{j+1} - w_j| \leq \frac{e^{(-2\lambda+\varepsilon)n}}{1 - e^{-2\lambda+\varepsilon}}\,,$$

and so $\limsup \frac{1}{n} \log |w^s - w_n| \leq -2\lambda + \varepsilon$. This proves that $\lim \frac{1}{n} \log |w^s - w_n| = -2\lambda$. Substituting in (25), we get $\lim \frac{1}{n} \log |(f_x^n)'(z)| = -2\lambda$.

Now we do the corresponding calculation for $z \in \mathbb{D}$. By the invariance of the hyperbolic metric (17) under f_x^n, we have

$$|(f_x^n)'(z)| = \frac{1 - |f_x^n(z)|^2}{1 - |z|^2} = \frac{1 - |w_n|^2}{1 - |z|^2}\,. \tag{26}$$

It follows, using (22), that $\lim \frac{1}{n} \log |(f_x^n)'(z)| = 2\lambda$.

The statements about w^u follow by symmetry, considering the inverse cocycle.

At last, we consider the case $\lambda = 0$. If $z \in \partial\mathbb{D}$, then using (25) and $1 - |w_n| \leq |z - w_n| \leq 2$, we get $\lim \frac{1}{n} \log |(f_x^n)'(z)| = 0$. If $z \in \mathbb{D}$ then we simply use (26). $\square$

A.3. **Automorphisms *versus* matrices.** Next we relate automorphisms of the disk with 2×2 matrices.

For each $v \in \mathbb{R}^2 \setminus \{0\}$, let $[v] \in \mathbb{RP}^1$ denote the corresponding projective class. We define a homeomorphism $[v] \in \mathbb{RP}^1 \mapsto \xi_v \in \partial\mathbb{D}$ by $\xi_{(\cos\theta, \sin\theta)} = e^{2i\theta}$.

In the next proposition, whose proof we omit, $\|\cdot\|$ denotes Euclidean metric in $\mathbb{R}^2$ or $\mathrm{GL}(2,\mathbb{R})$.

Proposition A.4. *There exists a group isomorphism*

$$[A] \in \mathrm{PGL}(2,\mathbb{R}) \mapsto \phi_A \in \mathrm{Aut}(\mathbb{D})$$

such that for all $A \in \mathrm{GL}(2,\mathbb{R})$ with $\det A = \pm 1$, we have:

(1) $\phi_A(\xi_v) = \xi_{A(v)}$ *for all nonzero* $v \in \mathbb{R}^2$*;*
(2) $\|Av\| = |\phi'_A(\xi_v)|^{-1/2}$ *for all unit vectors* $v \in \mathbb{R}^2$*;*
(3) *if u, s are unit vectors in $\mathbb{R}^2$ such that $\|Au\| = \|A\|$ and $\|As\| = \|A\|^{-1}$ then* $\xi_u = -\dfrac{\phi_A^{-1}(0)}{|\phi_A^{-1}(0)|}$ *and* $\xi_s = \dfrac{\phi_A^{-1}(0)}{|\phi_A^{-1}(0)|}$*;*
(4) $\|A\| = \left(\dfrac{1 + |\phi_A(0)|}{1 - |\phi_A(0)|}\right)^{1/2} = \exp\left(\frac{1}{2}\rho(\phi_A(0), 0)\right)$.

We are going to deduce the Oseledets theorem, in the case when the vector bundle is 2-dimensional, from Theorem 12. For simplicity we also suppose that it is a trivial bundle, that is, $\mathcal{E} = X \times \mathbb{R}^2$. Then each F_x is given by a 2×2 matrix. We take these matrices to have determinant ± 1. This is not a restriction because the validity of the theorem is not affected if we multiply F by some nonzero function φ (as long as the integrability condition is preserved) : the Oseledets subspaces remain the same, and one adds the Birkhoff average of $\log |\varphi|$ to the Lyapunov exponents.

We use the isomorphism from Proposition A.4 to associate to F a $\mathrm{Aut}(\mathbb{D})$-cocycle f given by $f_x = \phi_{F_x}$. Let λ be as in Theorem 12. Then, by item (4) of Proposition A.4,

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \|F_x^n\| = \lim_{n\to\pm\infty} \frac{1}{2n} \rho(f_x^n(0), 0) = \lambda(x),$$

which is the first assertion in Oseledets' theorem. The others are also easily deduced.

A.4. **A geometric reduction theorem.** Exploring this strategy even further, Thieullen [Thi97] obtained the following classification of $\mathrm{SL}(2,\mathbb{R})$ cocycles, which may be seen as a geometric version of results of Zimmer [Zim76].

Let $f\colon (M,\mu) \to (M,\mu)$ be an ergodic system and $r > 0$. A function ϕ is *cohomologous to zero mod* r if there exists a measurable function u on M such that

$$\phi + u \circ f - u \in r\mathbb{Z} \quad \text{almost everywhere.}$$

Given a set $E \subset M$, we define $R_E\colon M \to \mathrm{SL}(2,\mathbb{R})$ by $R_E(x) =$ rotation of $\pi/2$ if $x \in E$ and $R_E(x) = \mathrm{id}$ otherwise.

Theorem 14 (Thieullen [Thi97])**.** *If $A\colon M \to \mathrm{SL}(2,\mathbb{R})$ is such that $\log \|A^{\pm 1}\|$ is integrable, then there exists $P\colon M \to \mathrm{SL}(2,\mathbb{R})$ such that, denoting $B(x) = P^{-1}(f(x)) \cdot A(x) \cdot P(x)$, one of the following cases holds μ-almost everywhere:*

(1) $B(x) = \begin{pmatrix} a(x) & 0 \\ 0 & 1/a(x) \end{pmatrix}$ *with* $\lambda = \int \log|a|\, d\mu > 0$.

(2) $B(x) = \begin{pmatrix} a(x) & b(x) \\ 0 & 1/a(x) \end{pmatrix}$ *with* $\int \log|a|\, d\mu = 0$ *and* $\log|b|$ *integrable.*

(3) $B(x) = \begin{pmatrix} \cos\theta(x) & -\sin\theta(x) \\ \sin\theta(x) & \cos\theta(x) \end{pmatrix}$ *with θ not cohomologous to zero mod π.*

(4) $B(x) = R_E(x) \begin{pmatrix} a(x) & 0 \\ 0 & 1/a(x) \end{pmatrix}$ *where $\log|a|$ is integrable and the characteristic function of $E \subset M$ is not cohomologous to zero mod 2.*

The Lyapunov exponents are $\pm\lambda$ in the first case, and zero in the other three. In all cases, the norms $\|P(f(x))P^{-1}(x)\|$ and $\|B(x)\|$ are bounded above by $\|A(x)\|$.

Appendix B. Hyperbolic sets of C^2 diffeomorphisms

We prove that the uniformly hyperbolic sets of every C^2 volume-preserving diffeomorphism have zero Lebesgue measure, unless they coincide with the whole ambient manifold (Anosov case). This fact, which is used in [Boc02] and [BVa], seems to be well-known but we could not find a proof in the literature. Notice that we *do not* assume Λ to be the maximal invariant set in a neighborhood.

Theorem 15. *Let M be a compact manifold, μ be normalized Lebesgue measure on M, f be a C^2 diffeomorphism preserving μ, and Λ be a compact hyperbolic set for f. Then either $\mu(\Lambda) = 0$ or $\Lambda = M$.*

Using Theorem 15 and a result of Zehnder [Zeh77] that says that every C^1 symplectic diffeomorphism can be approximated by a C^2 one, we deduced in [BVa]

Corollary B.1. *There is a residual subset $\mathcal{R}_2 \subset \mathrm{Sympl}^1_\omega(M)$ such that if $f \in \mathcal{R}_2$ then f is Anosov or every hyperbolic set of f has zero measure.*

Proof of Theorem 15. We show that if $\mu(\Lambda) > 0$ then $\Lambda = M$. It is no restriction to suppose that $\Lambda = \mathrm{supp}(\mu|\Lambda)$, replacing Λ by $\Lambda' = \mathrm{supp}(\mu|\Lambda)$ from the beginning, if necessary.

Let $\varepsilon > 0$ be given by the stable manifold theorem: for every $x \in \Lambda$ the sets

$$W^s_\varepsilon(x) = \{y \in M\colon d(f^n(x), f^n(y)) < \varepsilon \text{ for all } n \geq 0\}$$
$$W^u_\varepsilon(x) = \{y \in M\colon d(f^n(x), f^n(y)) < \varepsilon \text{ for all } n \leq 0\}$$

are embedded disks, contained in the (global) stable and unstable sets $W^s(x)$ and $W^u(x)$, respectively, and depending continuously on the point x.

We always suppose that the metric in M is adapted to the hyperbolic set Λ: there is $\lambda < 1$ such that

$$d(f(x), f(y)) \le \lambda d(x, y) \quad \text{and} \quad d(f^{-1}(x), f^{-1}(z)) \le \lambda d(x, z)$$

for all $x \in \Lambda$, $y \in W^s_\varepsilon(x)$, and $z \in W^u_\varepsilon(x)$. In particular, given any $x \in \Lambda$,

$$W^s_\delta(x) = W^s_\varepsilon(x) \cap B(x, \delta) \quad \text{and} \quad f(W^s_\delta(x)) \subset W^s_{\lambda\delta}(f(x))$$

for every $0 < \delta \le \varepsilon$, and

$$W^s_\varepsilon(x) = \bigcup_{\delta<\varepsilon} W^s_\delta(x).$$

Let μ_u denote u-dimensional Lebesgue measure along unstable manifolds.

Lemma B.2. *There exists $x \in \Lambda$ such that $\mu_u(W^u_\varepsilon(x) \cap \Lambda) > 0$.*

Proof. This follows from $\mu(\Lambda) > 0$ and absolute continuity of the unstable foliation (which uses the hypothesis $f \in C^2$). □

Lemma B.3. *There exist points $x_k \in \Lambda$ such that $\mu_u(W^u_\varepsilon(x_k) \setminus \Lambda) \to 0$ as $k \to \infty$.*

Proof. The proof is easier when $\dim W^u = 1$. Take x as in Lemma B.2 and let $y \in \Lambda$ be a density point for $W^u_\varepsilon(x) \cap \Lambda$. Define $x_k = f^k(y)$. Since $\operatorname{diam} f^{-k}(W^u_\varepsilon(x_k)) \to 0$,

$$\frac{\mu_u(f^{-k}(W^u_\varepsilon(x_k)) \setminus \Lambda)}{\mu_u(f^{-k}(W^u_\varepsilon(x_k)))} \to 0$$

as $k \to \infty$. Then, by bounded distortion (this uses $f \in C^2$ once more),

$$\frac{\mu_u(W^u_\varepsilon(x_k) \setminus \Lambda)}{\mu_u(W^u_\varepsilon(x_k))} \to 0$$

as $k \to \infty$. Since $\mu_u(W^u_\varepsilon(x_k))$ is bounded above and below, this gives the statement.

Now we treat the general case. As before, let $y \in \Lambda$ be a density point for $W^u_\varepsilon(x) \cap \Lambda$. The difficulty is that density points are defined in terms of balls but, in dimension > 1, iterates of balls need not be balls. This is handled using a trick from [BV00, Section 4].

Given $k \ge 1$ take a small ball $D_k \subset W^u_\varepsilon(x)$ around y, so that

$$\mu_u(D_k \setminus \Lambda) \le k^{-1} \mu_u(D_k).$$

For each large n, consider the embedded disk $f^n(D_k)$ endowed with the metric $d(\cdot, \cdot)$ associated to the induced Riemannian structure. Let $E \subset f^n(D_k)$ be a maximal set such that $d(e', e'') \ge \varepsilon$ for every $e', e'' \in E$. Define $V_e = f^{-n}(B(e, \varepsilon))$ and $W_e = f^{-n}(B(e, 2\varepsilon))$, for each $e \in E$: the V_e are pairwise disjoint, and the W_e cover D_k. We separate E into three subsets:

(I) $B(e, 2\varepsilon) \cap \Lambda \ne \emptyset$ and $B(e, 2\varepsilon) \cap \partial f^n(D_k) = \emptyset$
(II) $B(e, 2\varepsilon) \cap \Lambda \ne \emptyset$ and $B(e, 2\varepsilon) \cap \partial f^n(D_k) \ne \emptyset$
(III) $B(e, 2\varepsilon) \cap \Lambda = \emptyset$.

If $e \in (I) \cup (II)$ then $B(e, 2\varepsilon)$ is contained in a local unstable manifold (replace ε by $\varepsilon/10$ throughout). This implies that

1. $B(e, 2\varepsilon)$ and its backward iterates have uniformly bounded curvature.
2. the diameter of $f^{-j}(B(e, 2\varepsilon))$ decreases exponentially fast with j; in particular, $\operatorname{diam} W_e \le 4\varepsilon\lambda^n$.

(3) backward iterates f^{-j} have uniformly bounded volume distortion on the ball $B(e, 2\varepsilon)$ (again, this uses $f \in C^2$).

Property 2 implies that for $e \in (II)$ the set W_e is contained in the tubular neighborhood of width $4\varepsilon\lambda^n$ of ∂D_k. Thus, taking n large enough we ensure that

$$\mu_u\Big(\bigcup_{e\in(II)} W_e\Big) \le \frac{1}{2}\mu_u(D_k) \tag{27}$$

On the other hand, bounded distortion and bounded curvature, as in properties 1 and 3, imply that

$$\frac{\mu_u(V_e)}{\mu_u(W_e)} \approx \frac{\mu_u(B(e,\varepsilon))}{\mu_u(B(e,2\varepsilon))} \approx 1,$$

for every $e \in (I)$, where $\approx$ means equality up to a uniform factor. Therefore,

$$\mu_u(W_e) \le C\mu_u(V_e) \tag{28}$$

where C depends only on f and Λ.

Consider k much larger than C, and fix n as in (27). We claim that there exists $e \in (I)$ such that

$$\mu_u(V_e \setminus \Lambda) \le 4Ck^{-1}\mu_u(V_e). \tag{29}$$

The proof is by contradiction:

$$\begin{aligned}
\mu_u(D_k \setminus \Lambda) &\ge \sum_{e\in(I)} \mu_u(V_e\setminus\Lambda) + \mu_u\Big(\bigcup_{e\in(III)} W_e\Big) && \text{(the } V_e \text{ are pairwise disjoint)}\\
&\ge \sum_{e\in(I)} 4Ck^{-1}\mu_u(V_e) + \mu_u\Big(\bigcup_{e\in(III)} W_e\Big) && \text{(assuming (29) were false)}\\
&\ge \sum_{e\in(I)} 4k^{-1}\mu_u(W_e) + \mu_u\Big(\bigcup_{e\in(III)} W_e\Big) && \text{(by relation (27))}\\
&\ge 4k^{-1}\mu_u\Big(\bigcup_{e\in(I)\cup(III)} W_e\Big) && \text{(take } k \ge 4\text{)}.
\end{aligned}$$

Using relation (27) we conclude that $\mu_u(D_k \setminus \Lambda) \ge 2k^{-1}\mu_u(D_k)$, which contradicts the choice of D_k. This contradiction proves our claim.

Now, fix $e \in (I)$ as in (29). Let x_k be the center of the ε-ball $f^n(V_e)$. Using bounded distortion once more,

$$\mu_u(f^n(V_e) \setminus \Lambda) \le 4C^2k^{-1}\mu_u(f^n(V_e)).$$

Since C is independent of k, the last inequality proves the lemma. □

Lemma B.4. *There exists $x_0 \in \Lambda$ such that $W^u_\varepsilon(x_0) \subset \Lambda$.*

Proof. Let x_k be as in Lemma B.3. We may suppose that the sequence converges to some $x_0 \in \Lambda$. By continuity, the local unstable manifolds of the x_k converge to $W^u_\varepsilon(x_0)$. Since Λ is closed, Lemma B.3 implies that $W^u_\varepsilon(x_0) \subset \Lambda$. □

Lemma B.5. *There is a hyperbolic periodic point $p_0 \in \Lambda$ such that $W^u(p_0) \subset \Lambda$.*

Proof. Take x_0 as in Lemma B.4. Let $\beta > 0$ be small enough so that the maximal invariant set Λ_β inside the closed β-neighborhood U_β of Λ is hyperbolic. Let $\alpha > 0$ be much smaller than β (cf. condition below) and V_α be the closed α-neighborhood of x_0. Since $x_0 \in \Lambda = \operatorname{supp}(\mu|\Lambda)$, the compact $\Gamma = \Lambda \cap V_\alpha$ has positive measure.

Let $z \in \Gamma$ and $N \geq 1$ be such that $f^N(z) \in \Gamma$. Taking α small enough, we may use the shadowing lemma (version without local product structure, see [Shu87]), to find a point $p_0 \in M$ such that $f^N(p_0) = p_0$ and

$$d(f^j(p_0), f^j(z)) < \beta \quad \text{for all } 0 \leq j \leq N.$$

The periodic point p_0 is hyperbolic and its local stable and local unstable manifolds are close to the ones of x_0 and z, if α and β are small. That is because all three points belong to the hyperbolic set Λ_β. In particular, we may suppose that $W^s_{\varepsilon/2}(p_0)$ intersects $W^u_{\varepsilon/2}(x_0)$ transversely. Then, by the λ-lemma, the entire $W^u(p_0)$ is accumulated by the forward iterates of $W^u_{\varepsilon/2}(x_0)$. Using Lemma B.4 and the fact that Λ is invariant and closed, we conclude that $W^u(p_0) \subset \Lambda$. In particular, $p_0 \in \Lambda$. $\square$

Now let $\Lambda_0 \subset \Lambda$ be the closure of $W^u(p_0)$. Define

$$W^s_\varepsilon(\Lambda_0) = \bigcup_{x \in \Lambda_0} W^s_\varepsilon(x) \quad \text{and} \quad W^u_\varepsilon(\Lambda_0) = \bigcup_{x \in \Lambda_0} W^u_\varepsilon(x).$$

Lemma B.6. Λ_0 *consists of entire (global) unstable manifolds. Consequently,* $W^s_\varepsilon(\Lambda_0)$ *is an open neighborhood of* Λ_0.

Proof. Let $z \in \Lambda_0$. Then there exist $z_k \in W^u(p_0) \subset \Lambda_0$ accumulating on z. The ε-unstable manifolds of all z_k are contained in Λ_0, and they accumulate on the ε-unstable manifold of z. Since Λ_0 is closed, it follows that z is in the interior of $\Lambda_0 \cap W^u(z)$. This proves that the intersection of Λ_0 with the unstable manifold $W^u(z)$ of any of its points is an open subset. Since it is also closed, it must be the whole unstable manifold. This proves the first statement. The second one is a direct consequence, using the continuous dependence of local stable manifolds. $\square$

The next lemma is the only place where we use that f preserves volume.

Lemma B.7. $f(W^s_\varepsilon(\Lambda_0)) = W^s_\varepsilon(\Lambda_0)$.

Proof. For any $\delta \in (\lambda\varepsilon, \varepsilon)$, we have

$$f(W^s_\delta(\Lambda_0)) \subset f(\overline{W^s_\delta(\Lambda_0)}) \subset f(W^s_\varepsilon(\Lambda_0)) \subset W^s_{\lambda\varepsilon}(\Lambda_0) \subset W^s_\delta(\Lambda_0).$$

Since f preserves volume

$$\mu(W^s_\delta(\Lambda_0) \setminus f(\overline{W^s_\delta(\Lambda_0)})) \leq \mu(W^s_\delta(\Lambda_0) \setminus f(W^s_\delta(\Lambda_0))) = 0.$$

It follows that $W^s_\delta(\Lambda_0) \setminus f(\overline{W^s_\delta(\Lambda_0)}) = \emptyset$, because μ is positive on nonempty open sets. Then,

$$W^s_\delta(\Lambda_0) \setminus f(W^s_\varepsilon(\Lambda_0)) = \emptyset.$$

Taking the union over all $\delta < \varepsilon$, we get $W^s_\varepsilon(\Lambda_0) \setminus f(W^s_\varepsilon(\Lambda_0)) = \emptyset$, which proves the lemma. $\square$

It follows that $W^s_\varepsilon(\Lambda_0) = \bigcap_{n \geq 0} f^n(W^s_\varepsilon(\Lambda_0)) = \Lambda_0$. Since $W^s_\varepsilon(\Lambda_0)$ is open and Λ_0 is closed, we get that $\Lambda_0 = M$. Then, $\Lambda = M$ as claimed in Theorem 15. $\square$

Acknowledgement. We are grateful to Alexander Arbieto and Amie Wilkinson for comments on a previous version of this paper. This work was partially supported by Faperj and Pronex-Dynamical Systems, CNPq.

REFERENCES

[ABV00] J. F. Alves, C. Bonatti, and M. Viana. SRB measures for partially hyperbolic systems whose central direction is mostly expanding. *Invent. Math.*, 140:351–398, 2000.

[AB] A. Arbieto and J. Bochi. L^p-generic cocycles have one-point Lyapunov spectrum. *Stochastics & Dynamics*, 3:73–81, 2003.

[Arn] M.-C. Arnaud. The generic C^1 symplectic diffeomorphisms of symplectic 4-dimensional manifolds are hyperbolic, partially hyperbolic, or have a completely elliptic point. *Ergod. Th. & Dynam. Sys.*, 22:1621–1639, 2002.

[AC97] L. Arnold and N. D. Cong. On the simplicity of the Lyapunov spectrum of products of random matrices. *Ergod. Th. & Dynam. Sys.*, 17(5):1005–1025, 1997.

[Arn78] V. I. Arnold. *Mathematical methods of classical mechanics*. Springer Verlag, 1978.

[BPS99] L. Barreira, Ya. Pesin, and J. Schmeling. Dimension and product structure of hyperbolic measures. *Ann. of Math.*, 149:755–783, 1999.

[Boc02] J. Bochi. Genericity of zero Lyapunov exponents. *Ergod. Th. & Dynam. Sys.*, 22:1667–1696, 2002.

[BV02] J. Bochi and M. Viana. Uniform (projective) hyperbolicity or no hyperbolicity: a dichotomy for generic conservative maps. *Annales de l'Inst. Henri Poincaré - Analyse Non-linéaire*, 19:113–123, 2002.

[BVa] J. Bochi and M. Viana. The Lyapunov exponents of generic volume preserving and symplectic systems. Annals of Math. To appear.

[BGMV] C. Bonatti, X. Gomez-Mont, and M. Viana. Généricité d'exposants de Lyapunov non-nuls pour des produits déterministes de matrices. *Annales de l'Inst. Henri Poincaré - Analyse Non-linéaire*, 20:579–624, 2003.

[BVb] C. Bonatti and M. Viana. Lyapunov exponents with multiplicity 1 for deterministic products of matrices. *Ergod. Th. & Dynam. Sys.* Special volume in honor of M. Herman. To appear.

[BV00] C. Bonatti and M. Viana. SRB measures for partially hyperbolic systems whose central direction is mostly contracting. *Israel J. Math.*, 115:157–193, 2000.

[BJ] J. Bourgain and S. Jitomirskaya. Continuity of the Lyapunov exponent for quasiperiodic operators with analytic potential. *J. Statist. Phys.*, 108:1203–1218, 2002.

[Bow75] R. Bowen. *Equilibrium states and the ergodic theory of Anosov diffeomorphisms*, volume 470 of *Lect. Notes in Math.* Springer Verlag, 1975.

[CS90] C.-Q. Cheng and Y.-T. Sun. Existence of invariant tori in three dimensional measure-preserving maps. *Celest. Mech. Dynam. Astronom.*, 47:275–292, 1989/1990.

[FHY83] A. Fathi, M. Herman, and J.-C. Yoccoz. A proof of Pesin's stable manifold theorem. In *Geometric dynamics (Rio de Janeiro 1981)*, volume 1007 of *Lect. Notes in Math.*, pages 177–215. Springer Verlag, 1983.

[Fur63] H. Furstenberg. Non-commuting random products. *Trans. Amer. Math. Soc.*, 108:377–428, 1963.

[FK60] H. Furstenberg and H. Kesten. Products of random matrices. *Ann. Math. Statist.*, 31: 457–469, 1960.

[GM89] I. Ya. Gold'sheid and G. A. Margulis. Lyapunov indices of a product of random matrices. *Uspekhi Mat. Nauk.*, 44:13–60, 1989.

[GR86] Y. Guivarc'h and A. Raugi. Products of random matrices : convergence theorems. *Contemp. Math.*, 50:31–54, 1986.

[KM99] A. Karlsson and G. Margulis. A multiplicative ergodic theorem and nonpositively curved spaces. *Comm. Math. Phys*, 208:107–123, 1999.

[Kat80] A. Katok. Lyapunov exponents, entropy and periodic points of diffeomorphisms. *Publ. Math. IHES*, 51:137–173, 1980.

[Kin68] J. Kingman. The ergodic theorem of subadditive stochastic processes. *J. Royal Statist. Soc.*, 30:499–510, 1968.

[Led84] F. Ledrappier. Propriétés ergodiques des mesures de Sinaï. *Publ. Math. I.H.E.S.*, 59:163–188, 1984.

[Led86] F. Ledrappier. Positivity of the exponent for stationary sequences of matrices. In *Lyapunov exponents (Bremen, 1984)*, volume 1186 of *Lect. Notes Math.*, pages 56–73. Springer, 1986.

[LY85] F. Ledrappier and L.-S. Young. The metric entropy of diffeomorphisms. I. Characterization of measures satisfying Pesin's entropy formula. *Ann. of Math.*, 122:509–539, 1985.

[Mañ84] R. Mañé. Oseledec's theorem from the generic viewpoint. In *Procs. International Congress of Mathematicians, Vol. 1, 2 (Warsaw, 1983)*, pages 1269–1276, Warsaw, 1984. PWN Publ.

[Mañ87] R. Mañé. *Ergodic theory and differentiable dynamics*. Springer Verlag, 1987.

[Mañ96] R. Mañé. The Lyapunov exponents of generic area preserving diffeomorphisms. In *International Conference on Dynamical Systems (Montevideo, 1995)*, pages 110–119. Longman, 1996.

[Nic89] P. J. Nichols. *The Ergodic Theory of Discrete Groups*. Cambridge Univ. Press, 1989.

[Ose68] V. I. Oseledets. A multiplicative ergodic theorem: Lyapunov characteristic numbers for dynamical systems. *Trans. Moscow Math. Soc.*, 19:197–231, 1968.

[Pes76] Ya. B. Pesin. Families of invariant manifolds corresponding to non-zero characteristic exponents. *Math. USSR. Izv.*, 10:1261–1302, 1976.

[Pes77] Ya. B. Pesin. Characteristic Lyapunov exponents and smooth ergodic theory. *Russian Math. Surveys*, 324:55–114, 1977.

[Pu84] C. Pugh. The $C^{1+\alpha}$ hypothesis in Pesin theory. *Publ. Math. IHES*, 59:143–161, 1984.

[PS89] C. Pugh and M. Shub. Ergodic attractors. *Trans. Amer. Math. Soc.*, 312:1–54, 1989.

[Rue81] D. Ruelle. Ergodic theory of diferentiable dynamical systems. *Publ. Math. IHES*, 50:27–58, 1981.

[Shu87] M. Shub. *Global stability of dynamical systems*. Springer Verlag, 1987.

[SW00] M. Shub and A. Wilkinson. Pathological foliations and removable zero exponents. *Invent. Math.*, 139:495–508, 2000.

[Thi97] Ph. Thieullen. Ergodic reduction of random products of two-by-two matrices. *J. Anal. Math.*, 73:19–64, 1997.

[Via] M. Viana. Almost all cocycles over any hyperbolic system have non-zero Lyapunov exponents. Preprint IMPA 2003.

[Xi92] Z. Xia. Existence of invariant tori in volume-preserving diffeomorphisms. *Ergod. Th. & Dynam. Sys.*, 12:621–631, 1992.

[Yoc92] J.-C. Yoccoz. Travaux de Herman sur les tores invariants. *Astérisque*, 206:Exp. No. 754, 4, 311–344, 1992. Séminaire Bourbaki, Vol. 1991/92.

[You93] L.-S. Young. Some open sets of nonuniformly hyperbolic cocycles. *Ergod. Th. & Dynam. Sys.*, 13(2):409–415, 1993.

[You95] L.-S. Young. Ergodic theory of differentiable dynamical systems. In *Real and Complex Dynamical Systems*, volume NATO ASI Series, C-464, pages 293–336. Kluwer Acad. Publ., 1995.

[Zeh77] E. Zehnder. Note on smoothing symplectic and volume preserving diffeomorphisms. *Lect. Notes in Math.*, 597:828–854, 1977.

[Zim76] R. Zimmer. Ergodic actions with generalized spectrum. *Illinois J. Math.*, 20:555–588, 1976.

E-mail address: `bochi@impa.br`

IMPA, EST. D. CASTORINA 110, JARDIM BOTÂNICO, 22460-320 RIO DE JANEIRO, BRAZIL

E-mail address: `viana@impa.br`

IMPA, EST. D. CASTORINA 110, JARDIM BOTÂNICO, 22460-320 RIO DE JANEIRO, BRAZIL