SINGULAR-HYPERBOLIC ATTRACTORS WITH HANDLEBODY BASINS

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Abstract. A *handlebody of genus* $n \in I\!N$ (or cube with n-handles) is a compact 3-manifold V containing a disjoint collection of n properly embedded 2-cells such that the result of cutting V along these disks is a 3-cell ([4] p. 15). An *attractor* of a vector field X with generating flow X_t is a transitive set of it equals to $\cap_{t>0} X_t(U)$ for some neighborhood U called *isolating block*. A partially hyperbolic attractor is *singular-hyperbolic* if it contains singularities (all hyperbolic) and has volume expanding central direction [11]. A *singular-hyperbolic repeller* is a singular-hyperbolic attractor for the time-reversed vector field. We show that every orientable handlebody of genus $n \geq 2$ can be realized as the isolating block of a singular-hyperbolic attractor with $n-1$ singularities. Hence every closed orientable 3-manifold supports a vector field whose nonwandering set consists of a singular-hyperbolic attractor and a singular-hyperbolic repeller. In particular, there are open sets of C^r vector fields without hyperbolic attractors or hyperbolic repellers on *every* closed 3-manifold, $r \geq 1$.

1. Introduction

A *handlebody of genus* $n \in I\!N$ (or cube with n-handles) is a compact 3-manifold V containing a disjoint collection of n properly embedded 2-cells such that the result of cutting V along these disks is a 3-cell ([4] p. 15). An *attractor* of a vector field X with generating flow X_t is a transitive set of it equals to $\cap_{t>0} X_t(U)$ for some neighborhood U called *isolating block*. A partially hyperbolic attractor is *singular-hyperbolic* if it contains singularities (all hyperbolic) and has volume expanding central direction [11]. A *singular-hyperbolic repeller* is a singular-hyperbolic attractor for the time-reversed vector field. In this paper we shall prove the following.

Theorem A. *Every orientable handlebody of genus $n \geq 2$ can be realized as the isolating block of a singular-hyperbolic attractor with $n-1$ singularities.*

The attractors obtained in Theorem A are robust as defined in [2]. Theorem A implies that for every $n \geq 1$ there is a C^∞ vector field in $I\!R^3$ exhibiting a singular-hyperbolic attractor with just n singularities. This answers positively a

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question possed to me for some authors. Another corollaries of Theorem A are the following.

Corollary 1.1. *All closed orientable* 3*-manifolds support vector fields whose nonwandering set consist of a singular-hyperbolic attractor and a singular-hyperbolic repeller.*

Corollary 1.2. *There are open sets of* C^r *vector fields without hyperbolic attractors and hyperbolic repellers on* every *closed* 3*-manifold,* $r \geq 1$.

Corollary 1.2 generalizes an example in [8] where it is shown that the existence of open sets of C^1 vector fields without hyperbolic attractors and hyperbolic repellers in S^3. The example [8] was given as a counterexample for a possible generalization (to the three-dimensional vector field context) of a the result [6] concerning the C^1 denseness of diffeomorphisms with hyperbolic attractors or hyperbolic repellers on every closed surface.

2. Proof

First we give a brief description of some terminology used in the Introduction. Hereafter M is a closed 3-manifold and X is a vector field with generating flow X_t, $t \in \mathbb{R}$. Given $p \in M$ we define $\omega(p)$, the *ω-limit set of p*, as the set of points $x \in M$ such that $x = \lim_{n\to\infty} X_{t_n}(p)$ for some real number sequence $t_n \to \infty$. The *nonwandering set* of X is the set $\Omega(X)$ of $p \in M$ such that for every neighborhood U of p and every $T > 0$ there is $t > T$ such that $X_t(U) \cap U \neq \emptyset$. A compact invariant set of X is *attracting* if it realizes as $\cap_{t>0} X_t(U)$ for some compact neighborhood U of it. Such a neighborhood is called *isolating block*. An *attractor* is an attracting set which is *transitive* i.e. $\omega(p)$ for some point p on it. A compact invariant set Λ of X is *hyperbolic* it it exhibits a continuous tangent bundle decomposition $E^s \oplus E^X \oplus E^u$ such that E^s is contracting, E^X is the flow direction and E^u is expanding. A *hyperbolic attractor* is an attractor which is simultaneously a hyperbolic set. A *hyperbolic repeller* is a hyperbolic attractor for the time-reversed vector field. A *partially hyperbolic set* of a vector field X in M is a compact invariant set Λ of X exhibiting a continuous invariant tangent bundle decomposition $T_\Lambda M = E^s \oplus E^c$ such that E^s is contracting and dominates E^u. The central direction E^c is *volume expanding* if the derivative of the flow expands the volume along E^c. A compact invariant set of X is *singular-hyperbolic* if it has singularities (all hyperbolic) and is partially hyperbolic with volume expanding central direction (see p. 105 in [2] or [11]). A *singular-hyperbolic attractor* is an attractor which is simultaneously a singular-hyperbolic set. A *singular-hyperbolic repeller* is a singular-hyperbolic attractor for the time reversed vector field. The singular-hyperbolic attractors were introduced in [11] as a generalization of the geometric Lorenz attractor [5]. They have been taken some attention because of their rich dynamics resembling both hyperbolic and geometric Lorenz attractors (see [2] p.104).

A map $f : [0,1] \to [0,1]$ is called *n-Lorenz map* if there are $n \in I\!N$ and $C = \{c_0, \cdots, c_n\}$ with

$$0 < c_n < c_{n-1}, \cdots, c_1 < c_0 = 1/2$$

such that the following properties hold.

1. C is the set of discontinuity points of f.
2. $\lim_{x \to c_i^-} f(x) = 1$ for all $i = \{1, \cdots, n\}$.
3. $\lim_{x \to c_i^+} f(x) = r_i^+$ exist for all $i = \{1, \cdots, n\}$.
4. f is C^1 on $[0,1] \setminus C$ and there is $\lambda > \sqrt{2}$ such that $f'(x) \geq \lambda$ if $x \in [0,1] \setminus C$.

In Figure 1 we describe the graph of f.

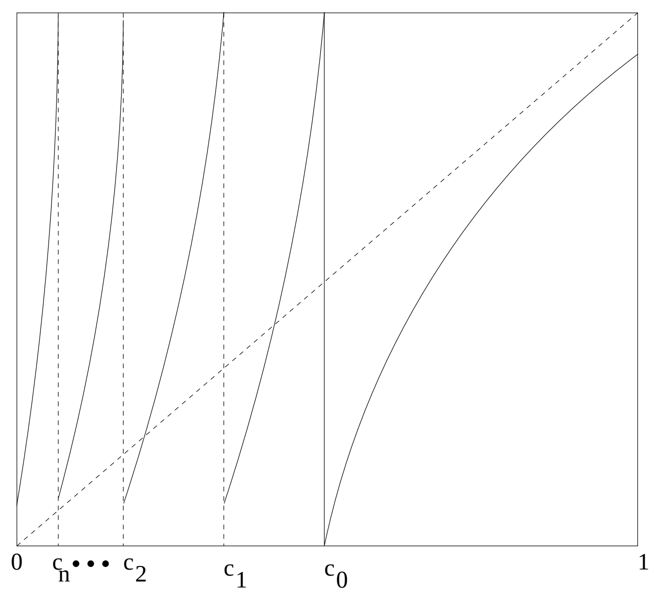

FIGURE 1. n-Lorenz map

The n-Lorenz map is a direct generalization of the classical Lorenz maps in the interval $[0,1]$ (which corresponds to 0-Lorenz maps). Note that a n-Lorenz map has just $n+1$ discontinuity points. For every n-Lorenz map f we consider the $(n+1)$-vector $(f(0), r_1^+, \cdots, r_n^+)$. The *norm* of f is the euclidian norm of such a vector. As usual we say that f is *transitive* if it has a dense forward orbit. The following lemma is an obvious extension of the classical Guckenheimer-Williams conditions for transitivity of Lorenz maps [5].

Lemma 2.1. *All n-Lorenz maps with sufficiently small norm are transitive.*

The proof of Corollary 1.2 will use the following standard lemma in 3-manifold topology. To state it we need some short definitions. A *Heegaard Splitting* of a closed 3-manifold M is a decomposition $M = V_1 \cup V_2$ where $V_1, V_2 \subset M$ are handlebodies sush that $V_1 \cap V_2 = \partial V_1 = \partial V_2$ (see [He, p. 17]). The *genus* of the Heegaard Splitting $M = V_1 \cup V_2$ is the genus of V_1 (or V_1). It is well known result that every closed 3-manifold M has a Heegaard Splitting [4]. We shall denote by ST_n the orientable handlebody of genus n.

Lemma 2.2. *Every closed* 3*-manifold with a genus* one *Heegaard Splitting has a genus* two *Heegaard Splitting.*

Next we describe a deformation resembling [2] p. 114. It was used to deform hyperbolic attractors in the proof of Theorem 4 in [10] p. 1321 (the details are in [9]). By a *Cherry flow box* we mean a three-dimensional flow box whose phase portrait is represented in Figure 2-(b). See [3] for the description of the Cherry flows.

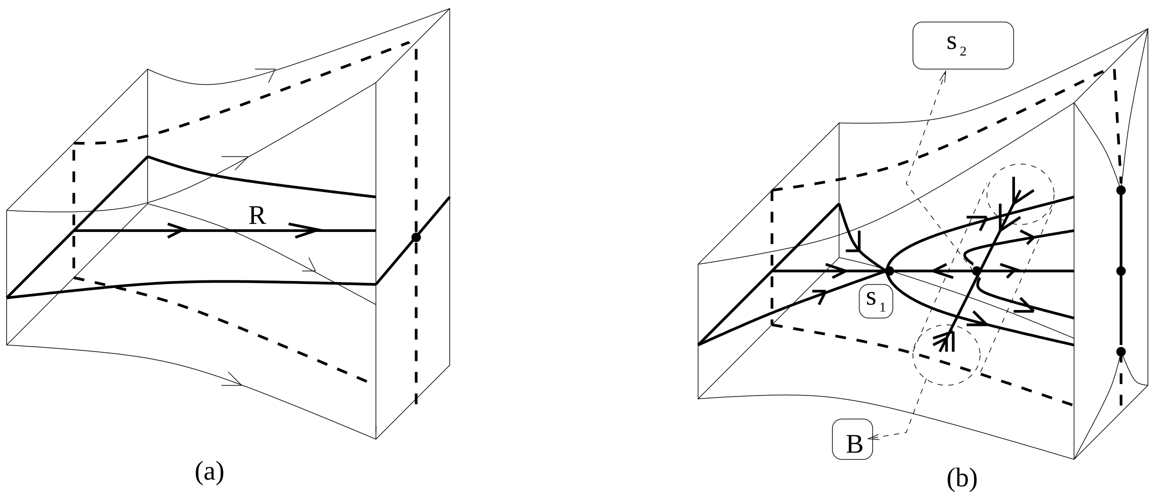

FIGURE 2. (a) regular flow box. (b) Cherry flow box

Observe that the Cherry flow-box has two singularities of saddle type: one with one-dimensional unstable manifold (say s_1) and another with two-dimensional stable manifold (say s_2). We can construct a three-dimensional Cherry flow box by deforming a tubular flow box around a regular orbit path R which we call *regular flow box* in Figure 2-(a). As a matter of fact we observe that the 3-ball like regular flow box in Figure 2-(a) is transformed in the Cherry flow box in Figure 2-(b). This last flow box is the solid bitorus obtained by removing the cylinder-like 3-ball B centered at s_2 in Figure 2-(b) from the regular flow box.

Proof of Theorem A: Let Λ^2 be the geometric Lorenz attractor with associated vector field X^2 described in [5]. This attractor realizes the solid bitorus ST_2 as isolating block (recall that ST_n denotes the orientable handlebody of genus n). We deform Λ^2 by inserting a Cherry flow box as described in Figure 3.

More precisely, we choose a path R contained in a regular orbit of Λ^2 as indicated in Figure 3-(a). By deforming a regular flow box around an orbit path R by a Cherry flow box we get a vector field as described in Figure 3-(b). We note that such a deformation produces a positively invariant handlebody ST_3 which is obtained by removing a 3-ball B centered at s_2 from the original isolating block

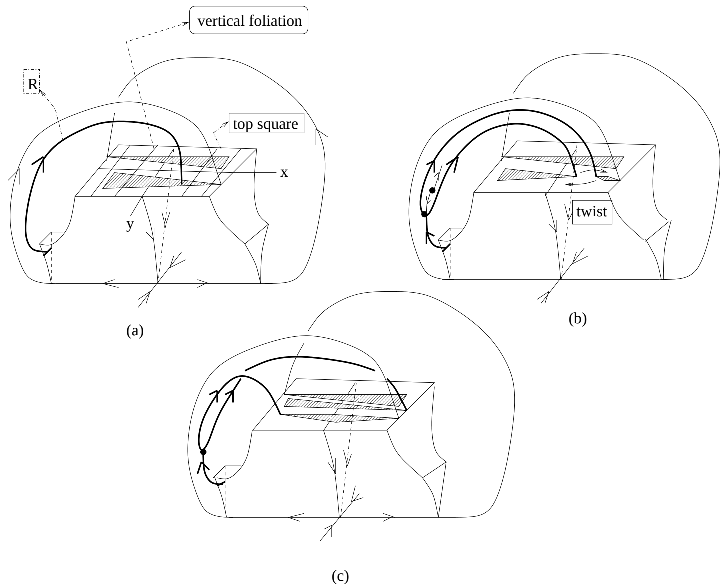

FIGURE 3. Deforming the Lorenz attractor

ST_2 of Λ^2. Afterward we twist the cusp points of the resulting vector field as explained in Figure 3-(b). This produces the vector field X^3 described in Figure 3-(c). We observe that ST_3 is still a positively invariant compact neighborhood of X^3. The deformation is done in a way that the top square in Figure 3-(a) realizes as a cross section of X^3. The corresponding Poincaré map $\Pi(x, y)$ in the top square preserves the vertical foliation $\{x = cnt\}$, contracts such a foliation and induces a 2-Lorenz map with small norm in the verticals. In other words

$$\Pi(x, y) = (f(x), g(x, y)),$$

where f is a 2-Lorenz map with small norm and $\mid \partial_y g(x, y) \mid$ is uniformly small. Next we define

$$\Lambda^3 = \cap_{t>0} X_t^3(ST_3).$$

Since ST_3 is positively invariant for X^3 we have that Λ^3 is an attracting set. Such an attracting set is an attractor for X^3 since f is transitive and the vertical foliation in the top square is contracted by Π. We can show that Λ^3 is singular-hyperbolic for X^3 as in the Lorenz attractor's case (see the expository paper [1] and also Section 3 p. 3399 in [12] for an idea of the proof). By repeating the proceeding n times we get a vector field X^{n+2} having a singular-hyperbolic attracting set Λ^{n+2} with $n+1$ singularities and the genus $n+2$ handlebody ST_{n+3} as isolating block. We still have that the top square is a cross section of X^3 whose return map has the form $(f(x), g(x, y))$ where f is a n-Lorenz-like map and $\mid \partial_y g \mid$ is small. By applying Lemma 2.1 as before we have that Λ^{n+2} is transitive. Hence Λ^{n+2} is a singular-hyperbolic attractor with $n + 1$ singularities. This finishes the proof. $\square$

Proof of corollaries 1.1 and 1.2: Let M be a closed oriantable 3-manifold. As already mentioned M has a Heegaard Splitting [4]. If M has a genus one Heegaard Splitting, then M also has a genus two Heegaard Splitting by Lemma 2.2. Hence we can assume that M has a genus ≥ 2 Heegaard Splitting, i.e. $M = V_1 \cup V_2$ where V_i is a genus $n \geq 2$ handlebody $(i = 1, 2)$. Since M is orientable we have that V_i does $(i = 1, 2)$. So, $V_i = ST_n$ $(i = 1, 2)$. By Theorem A we have that V_1 can be realized as the isolating block of a singular-hyperbolic attractor. By considering the flow-reversed vector field we have that V_2 can be realized as the isolating block of a singular-hyperbolic repeller. Gluing V_1, V_2 to obtain M we can see that M is equipped with a C^∞ vector field X whose nonwandering set consists of a singular-hyperbolic attractor with isolating block V_1 and a singular-hyperbolic repeller with isolating block V_2. This proves Corollary 1.1. Corollary 1.2 follows from Corollary 1.1 (by passing to the orientable double cover if necessary) and the result in [7]. $\square$

Final Remarks: There is an alternative argument to prove Corollary 1.2 which uses Morse Theory instead of Heegaard Splitting. The idea is to replace a small 3-ball neighborhood of an attracting (resp. a repelling) singularity of a gradient vector field by one exhibiting a geometric Lorenz attractor (resp. geometric Lorenz repeller) in its interior. In this way we can destroy all the hyperbolic attractors (resp. repellers) of the gradient vector field yielding the vector field required in the conclusion of the corollary. However, this argument cannot be used to prove Corollary 1.1 whose proof needs Theorem A and the Heegaard Splitting.

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