SUMS OF CANTOR SETS WHOSE SUM OF DIMENSIONS IS CLOSE TO ONE

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Abstract

We prove that the arithmetic sum of two Cantor sets whose sum of Hausdorff dimensions is slightly bigger than one is generically an M-Cantorval. We conclude that, in most cases, homoclinic bifurcations in a surface associated to non-trivial basic sets with Hausdorff dimension slightly bigger than one present hyperbolicity and stable tangencies for positive values of the parameter, both with positive density at the initial bifurcation parameter.

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1 Introduction

The study of arithmetic sums of regular Cantor sets plays a special role in Dynamical Systems, mainly in the study of homoclinic bifurcations, as well as in certain aspects of Number Theory related with the study of Markov and Lagrange spectra (see [**Mo2**]). In this article, we study sums of Cantor sets whose sum of Hausdorff dimensions is slightly bigger than one. More precisely, we will study families at one parameter of pairs of regular Cantor sets in the real line (K_μ, K'_μ) with $HD\,(K_0) + HD\,(K'_0) = 1$ (where HD denotes the Hausdorff dimension), and we shall characterize topologically sums of the type:

$$K_\mu + \lambda K'_\mu = \{x + \lambda y \,:\, x \in K_\mu,\, y \in K'_\mu\}$$

with $\lambda \in \mathbb{R}^*$ at values of μ close to 0 for whose $HD\,(K_\mu) + HD\left(K'_\mu\right) > 1$ (we recall that, under these conditions, if $HD\,(K_\mu) + HD\left(K'_\mu\right) < 1$ then for all $\lambda \in \mathbb{R}$, $K_\mu + \lambda K'_\mu$ is a Cantor set with Hausdorff dimension less than one, hence of zero Lebesgue measure - see [**PT**], chapter IV).

Newhouse introduced the concept of thickness in order to give persistent examples of pairs of regular Cantor sets whose arithmetic difference (or sum) is a finite union of intervals (see [**PT**]). It was proved later in [**MY**] that arithmetic sums of generic pairs of regular Cantor sets whose sum of Hausdorff dimensions is bigger than one have dense interior (see Section 2), solving a conjecture by Palis.

In [**MMR**] were exhibited distinct open sets of pairs of regular Cantor sets, where the arithmetic sum presents different topological properties. One of them, introduced by Mendes and Oliveira, see [**MO**]**,** was of an M-Cantorval: we say that a compact set $C \subset \mathbb{R}$, is an *M-Cantorval* if it has dense interior (i.e., $C = \overline{int(C)}$) and any *gap* (connected component of the complement) of C has its extremes accumulated by others gaps of C. It is known that all M-Cantorvals are homeomorphic (see [**MO**]).

We are going to prove that for families, as were described above, typically, the sum of two regular Cantor sets whose sum of Hausdorff dimensions is slightly greater than one is an M-Cantorval (see Theorem 1 below).

Furthermore, in section 3, we use this result to conclude that for homoclinic bifurcations on surfaces associated to horseshoes with Hausdorff dimension slightly bigger than one, there are open sets of positive values of the parameter corresponding, respectively, to hyperbolicity and to stable tangencies, both having positive density at the initial parameter of bifurcation.

2 Families of pairs of Cantor sets

We will work with families $\{(K_\mu, K'_\mu) : \mu \in (-1,1)\}$ of pairs of regular Cantor sets of class C^∞ that depend (i.e., whose dynamics depend) C^∞ on the parameter μ. Let $d(\mu) = HD(K_\mu) + HD(K'_\mu)$ and let us suppose that $d(0) = 1$. Then we have that d is differentiable (see [**Ma**]) and, generically, $d'(0) \neq 0$. Besides, we will suppose, without loss of generality, that $d'(0) > 0$. Hence, if $\mu < 0$ and $|\mu|$ is small then $d(\mu) < 1$, and therefore $K_\mu + \lambda K'_\mu$ is a Cantor set of zero Lebesgue measure (with Hausdorff dimension less than one), for all $\lambda \in \mathbb{R}$ in this case.

On the other hand, it was proved in [**MY**] that, if

$$\Omega = \{(K, K') : K, K' \text{ are } C^\infty\text{-Cantor sets}, \ HD(K) + HD(K') > 1\},$$

equiped with the C^∞ topology on the expansive functions that define K and K', then, for most pairs $(K, K') \in \Omega$, $K + \lambda K'$ has dense interior for all $\lambda \in \mathbb{R}^*$. More precisely, for given $(K, K') \in \Omega$, we say that (K, K') have *stable intersection* if there exists a neighbourhood U_0 of (K, K') in Ω such that for $(\widetilde{K}, \widetilde{K}') \in U_0 \Longrightarrow \widetilde{K} \cap \widetilde{K}' \neq \emptyset$. For given $(K, K') \in \Omega$, we define

$$I_s\ (K,K') = \{t \in \mathbb{R}\ :\ (K, K'+t) \text{ has stable intersection}\}.$$

It is not difficult to see that $I_s\ (K,K')$ is an open set which is a subset of $K - K' = \{t \in \mathbb{R}\ : K \cap (K'+t) \neq \emptyset\} = \{x - y\ :\ x \in K,\ y \in K'\}$.

The main result of [**MY**] is that there exists $\mathcal{U} \subset \Omega$ open and dense (and in certain sense, of "total measure") such that $(K,K') \in \mathcal{U}$ implies that, for all $\lambda \in \mathbb{R}^*$, $I_s\ (K, \lambda K')$ is dense in $K - \lambda K'$ and, furthermore, $HD\left((K - \lambda K')\setminus I_s(K,\lambda K')\right) < 1$.

This result implies that for a residual set of families (K_μ, K'_μ) as above, for $\mu > 0$ small, it is true that $K_\mu + \lambda K'_\mu$ has dense interior (i.e., $K_\mu + \lambda K'_\mu = \overline{int(K_\mu + \lambda K'_\mu)}$) for all $\lambda \in \mathbb{R}^*$.

Let $\mathcal{H} = \{(K,K')\ :\ K, K' \text{ are } C^\infty\text{-Cantor sets, } HD\,(K) + HD\,(K') = 1\}$. Then, our main result is the following one.

Theorem 1. *Let $\mathcal{F}$ be the set of smooth families $\{(K_\mu, K'_\mu)\ :\ \mu \in (-1,1)\}$ of pairs of regular Cantor sets such that $(K_0, K'_0) \in \mathcal{H}$ and $d'(0) > 0$. Then there is a residual set $\mathcal{R} \subset \mathcal{F}$ such that, for $(K_0, K'_0) \in \mathcal{R}$, there exits $\delta > 0$ such that $0 < \mu < \delta$ implies that $K_\mu + K'_\mu$ is an M-Cantorval.*

The proof of this theorem is based on the following result due to Besicovitch (see [**F**]).

Theorem 2. *Let $\Lambda \subset \mathbb{R}^2$ be an irregular set of finite Hausdorff measure of dimension 1. Then $\pi_\theta\,(\Lambda)$ has zero Lebesgue measure for almost all $\theta \in (0,\pi)$, where $\pi_\theta : \mathbb{R}^2 \to \mathbb{R}$ is the orthogonal projection on the line that makes an angle θ with the $\overline{OX}$ axis.*

In our context, if K and K' are regular Cantor sets with $HD\,(K) + HD\,(K') = 1$, then $K \times K' \subset \mathbb{R}^2$ is a set of Hausdorff dimension 1 and finite measure of dimension 1. Moreover, $K \times K'$ is an irregular set because of the following criterion (Corollary 6.14 of [**F**] - see also [**PSS**]): a 1-set $E \subset \mathbb{R}^2$ is irregular if and only if it has projections of Lebesgue measure 0 in two distinct directions (in this case $K \times K'$ has the two zero-measure projections K and K'). Since $\pi_\theta\,(K \times K')$ is equivalent to $K + \lambda K'$ where $\lambda = \tan\theta$, we obtain the following corollary.

Corollary 3. *If K and K' are regular Cantor sets, such that $HD(K) + HD(K') = 1$, then for almost all $\lambda \in \mathbb{R}$, $K + \lambda K'$ is a Cantor set with zero Lebesgue measure.*

The idea of the proof for the Theorem 1 is to combine the Corollary above with an extension of Theorem IV.1 of **[Mo1]** as we describe next. First of all, we will make some considerations about the geometry of regular Cantor sets near the extremes of the gaps. Let K be a regular Cantor set defined by an expanding differentiable function $\Psi : I_1 \cup I_2 \cup ... \cup I_r \overset{C^{1+\epsilon}}{\longrightarrow} I$, where I is the convex hull of the $I_1 \cup I_2 \cup ... \cup I_r$ and, for all $j \leq r$, $\Psi(I_j)$ is the convex hull of some union of the intervals I_s.For $I_j = [a_j, b_j]$, $1 \leq j \leq r$, we define $B(K) = \cup_{j=1}^{r} \{a_j, b_j\}$, the set of the extreme points of K. Since $\Psi(B(K)) \subset B(K)$ and $B(K)$ is a finite set, then for given $p \in B(K)$ there exist integers k_p and l_p, $k_p \geq 0$, $l_p > 0$ such that $\Psi^{l_p}(\Psi^{k_p}(p)) = \Psi^{k_p}(p)$, (where it had been choosen the smallest l_p and k_p) Let $P(K) = \{\Psi^{l}(\Psi^{k_p}(p)) : 0 \leq l < l_p,\ p \in B(K)\}$ be the set of periodic points of $B(K)$. Thus, for a given $p \in P(K)$ there exists a diffeomorphism $g_p : I \longrightarrow I$ such that, if $\Psi_p = g_p \circ \Psi \circ g_p^{-1}$ then $\Psi_p^{l_p} = g_p \circ \Psi^{l_p} \circ g_p^{-1}$ is linear on I_{j_p}, where j_p is such that $p \in I_{j_p}$. Let $K_p = g_p\left(K \cap I_{j_p}\right)$, then we say that K_p is the *linearization of K relative to p.* Next, for given $q \in B(K)$ and $n \in \mathbb{N}$ we denote by $I_{q,n}$ the connected component of $\Psi^{-n}(I)$ that contains q (i.e., the interval in the n-step at the construction of K that contains q). Let $\alpha_n : I_{q,n} \longrightarrow g_{\widetilde{q}}\left(I_{j_{\widetilde{q}}}\right)$ an affine surjective homeomorphism, where $\widetilde{q} = \Psi^{k_q}(q)$ is the point of $P(K)$ corresponding to q. Therefore, we have that $\alpha_n(K \cap I_{q,n})$ converges exponentially in the Hausdorff's metric to $K_{\widetilde{q}} = g_{\widetilde{q}}\left(K \cap I_{j_{\widetilde{q}}}\right)$. In fact, $K_{\widetilde{q}}$ is the limit geometry of K associated to the (periodic) orbit of $\widetilde{q}$.

In order to guarantee that certain sums of Cantor sets are M-Cantorvals, our criterium will be the following one: we recall that an interval $J_1 = [a, b]$ *is left-linked with* $J_2 = [c, d]$ if $a < c$, $b < d$ and $c < b$. Let K and K' be regular Cantor sets defined by Ψ and $\widetilde{\Psi}$, respectively, and suppose that for any $p \in P(K)$ and $p' \in P(K')$ periodic extremes of K and K', we have:

i) $\log \left|(\Psi^{l_p})'(p)\right| \Big/ \log \left|(\widetilde{\Psi}^{l_{p'}})'(p')\right| \notin \mathbb{Q}$.

ii) There exist $\lambda_1, \lambda_2, \widetilde{\lambda}_1, \widetilde{\lambda}_2 \in \mathbb{R}^*$, $t_1, t_2 \in \mathbb{R}$ (with $\lambda_1\lambda_2 < 0, \widetilde{\lambda}_1\widetilde{\lambda}_2 < 0$) such that $\lambda_1\left(g_p\left(I_{j_p}\right)\right) = [a_1, \lambda_1 g_p(p)]$ is left-linked with $\widetilde{\lambda}_1 g_{p'}\left(I'_{j_{p'}}\right) + t_1 = [\widetilde{\lambda}_1(g_{p'}(p')) + t_1, d_1]$, $\widetilde{\lambda}_2 g_{p'}\left(I'_{j_{p'}}\right) + t_2 = [a_2, \widetilde{\lambda}_2(g_{p'}(p')) + t_2]$ is left-linked with $\lambda_2\left(g_p\left(I_{j_p}\right)\right) = [\lambda_2(g_p(p)), d_2]$ and $\lambda_j \cdot K_p \cap \left(\widetilde{\lambda}_j K'_{p'} + t_j\right) = \emptyset$, $j = 1, 2$.

Then, for every $\lambda \in \mathbb{R}^*$, the infinite gaps of $K + \lambda K'$ (i.e., the extreme points of the convex hull of $K + \lambda K'$) are accumulated by gaps of $K + \lambda K'$ (i.e., $K + \lambda K'$ does not begin nor end by intervals). The proof is analogue to that of Theorem IV.1 of [**Mo1**].

We can use this fact (as in Proposition 1 of [**MMR**]) to show the following: suppose that $\lambda \in \mathbb{R}^*$ is such that, if a_1 and a_2 are extremes of gaps of K, b_1 and b_2 are extremes of gaps of K', then $a_1 + \lambda b_1 \neq a_2 + \lambda b_2$. Then every gap of $K + \lambda K'$ is accumulated on both sides by gaps of $K + \lambda K'$. In particular, if $K + \lambda K'$ has dense interior, then $K + \lambda K'$ is an M-Cantorval.

Note that if $HD(K) + HD(K') = 1$ then $HD(K_p) + HD\left(K'_{p'}\right) = 1$ for all $p \in P(K), p' \in P(K')$ (since $HD(K) = HD(K_p)$ and $HD(K') = HD\left(K'_{p'}\right)$; in fact, diffeomorphisms preserve the Hausdorff's dimensions). Therefore, in this case by the Besicovitch Theorem, for given $p \in P(K), p' \in P(K')$ and $\mu \in \mathbb{R}^*$ we have that for almost all $\lambda \in \mathbb{R}$, $\mu K_p + \lambda K'_{p'}$ has zero Lebesgue measure. Since $P(K) \times P(K')$ is a finite set, then for almost all $(\mu, \lambda) \in \mathbb{R}^2$, $\mu K_p + \lambda K'_{p'}$ has zero Lebesgue measure for all $(p, q) \in P(K) \times P(K')$. This implies that the condition *ii)* above is fullfilled inmediatly for the case $HD(K) + HD(K') = 1$ (and indeed in a neighbourhood, since the condition *ii)* is an open one). Because the condition *i)* is clearly generic, we conclude that for generic pairs of Cantor sets (K, K') with $HD(K) + HD(K')$ slightly bigger than one, $K + K'$ is an M-Cantorval.

Unfortunately, given a generic family (K_μ, K'_μ) of pairs of regular Cantor sets, we have a non-empty (countable) set of paramenters μ for which there are a_1,a_2 extremes of gaps of K_μ and b_1, b_2 extremes of gaps of K'_μ with $a_1 + b_1 = a_2 + b_2$. However, for generic one-parameter families (K_μ, K'_μ) of pairs of regular Cantor sets (contained in the complement of a countable union of codimension 2 submanifolds), we never have $a_1+b_1 = a_2+b_2 = a_3+b_3$ for a_1,a_2, a_3 distinct extremes of gaps of K_μ and b_1, b_2, b_3 distinct extremes of gaps of K'_μ, $\forall \mu \in (-1,1)$.

For these families, we may apply a variation of the above criterium, and in order to do that, let us first make a definition.

Given a compact set X and a point $a \in X$, we define the *logarithmic density* $\widehat{d}(X,a)$ of X at a as follows: $\widehat{d}(X,a) = \limsup\limits_{M\to\infty} \frac{m(\{t\in[0,M]\,;\, a+e^{-t}\in X \text{ or } a-e^{-t}\in X\})}{M}$, where m denotes Lebesgue measure.

Example 4. *The set $X = \bigcup_{n=1}^{\infty}[\frac{1}{n!}, \frac{2}{n!}]$ does not have zero density at 0 but it has zero logarithmic density at 0.*

Now, let $K, \widetilde{K}$ be two regular Cantor sets with left extreme points $a, \widetilde{a}$, respectively, and invariant by contractions: there are $\lambda_1, \lambda_2 > 1$ such that, putting $g_1(x) = \lambda_1^{-1}(x-a)+a$ and $g_2(x) = \lambda_2^{-1}(x-\widetilde{a})+\widetilde{a}$ then $g_1(K)$ and $g_2(\widetilde{K})$ are respectively open and closed neighbourhoods of a and $\widetilde{a}$ in K and $\widetilde{K}$ which are intersections of K and $\widetilde{K}$ with intervals. We claim that, given $\epsilon > 0$, the set of pairs of regular Cantor sets $K, \widetilde{K}$ as before such that, for every $\lambda > 0, \widehat{d}(K+\lambda\widetilde{K}, a+\lambda\widetilde{a}) < \epsilon$ is open. Indeed, if d and $\widetilde{d}$ are the diameters of K and $\widetilde{K}$ and $\delta > 0$ is such that $\delta < \lambda_1^{-m}d$ and $\delta < \lambda_2^{-n}\widetilde{d}$ then $a+\lambda\widetilde{a}+e^{-t} \in K+\lambda\widetilde{K}$ if and only if $a+\lambda\lambda_1^m\lambda_2^{-n}\widetilde{a}+e^{m\log\lambda_1 - t} \in K+\lambda\lambda_1^m\lambda_2^{-n}\widetilde{K}$.

We may take $A = \max\{2\lambda_1, 2\lambda_2\}$ and $M \gg 1$ such that

$$\frac{m(\{t\in[0,M]\,;\, a+\lambda\widetilde{a}+e^{-t}\in K+\lambda\widetilde{K}\})}{M} < \epsilon,$$

$\forall \lambda \in [A^{-1}, A]$.

Since for every $\lambda > 0, m \in \mathbb{N}$ it is possible to find $n \in \mathbb{N}$ such that $\lambda\lambda_1^m\lambda_2^{-n} \in (A^{-1}, A)$, it is not difficult to conclude our claim.

Notice that we can replace left extremes by right extremes in our previous discussion.

Notice also that, if K and $\widetilde{K}$ are as before, $g, \widetilde{g}$ are orientation-preserving diffeomorphisms, and $\widehat{d}(K + \lambda\widetilde{K}, a + \lambda\widetilde{a}) < \epsilon$ for every $\lambda > 0$ then $\widehat{d}(g(K) + \widetilde{g}(\widetilde{K}), g(a) + \widetilde{g}(\widetilde{a})) < \epsilon$, by the self-similarity of K and $\widetilde{K}$, and the fact that rescaling of small neighbourhoods of $g(a)$ in $g(K)$ (resp. of $\widetilde{g}(\widetilde{a})$ in $\widetilde{g}(\widetilde{K})$) converge in the Hausdorff metric to K (resp. to $\widetilde{K}$).

Now we have the following

Lemma 5. *: Suppose that K and $\widetilde{K}$ are as before, $HD(K) + HD(\widetilde{K}) = 1$ and $\frac{\log \lambda_1}{\log \lambda_2} \notin \mathbb{Q}$. Then $\widehat{d}(K + \lambda\widetilde{K}, a + \lambda\widetilde{a}) = 0$ for every $\lambda > 0$.*

Proof. : We know that $m(K + \lambda\widetilde{K}) = 0$ for almost every λ. Given $m \in \mathbb{N}$ and $\lambda > 0$ there is a unique $n(m) \in \mathbb{Z}$ such that $\lambda\lambda_1^m\lambda_2^{-n(m)} \in [1,\lambda_2)$ (indeed $n(m) = \left\lfloor \frac{\log \lambda + m \log \lambda_1}{\log \lambda_2} \right\rfloor$), and $\frac{\log(\lambda\lambda_1^m\lambda_2^{-n(m)})}{\log \lambda_2} = \left(\frac{\log \lambda}{\log \lambda_2} + m\frac{\log \lambda_1}{\log \lambda_2}\right) - \left\lfloor \frac{\log \lambda}{\log \lambda_2} + m\frac{\log \lambda_1}{\log \lambda_2} \right\rfloor$ is uniformly distributed in [0,1) when m varies.Given $\epsilon > 0$ and $M_0 \gg 1$, there is an open set $L \subset [1, \lambda_2)$ with $m([1, \lambda_2)\backslash L) < \epsilon$ and $M > M_0$ such that $\frac{m(\{t \in [0,M]\,;\, a+\lambda\widetilde{a}+e^{-t} \in K+\lambda\widetilde{K}\})}{M} < \epsilon$, for every $\lambda \in L$. This implies that, for every $\lambda > 0, \widehat{d}(K + \lambda\widetilde{K}, a + \lambda\widetilde{a}) < 2\epsilon$ (since for most values of $m \in \mathbb{N}$, $\lambda\lambda_1^m\lambda_2^{-n(m)} \in L$), and this clearly implies the Lemma. $\square$

Now let (K_μ, K'_μ) be a family of pairs of regular Cantor sets as in the statement of Theorem 1. We can apply the above Lemma to the pairs of Cantor sets $(K_{0,p}, K'_{0,p'})$, where $(p, p') \in P(K_0) \times P'(K'_0)$, and the previous claim and remarks in order to show that, if $a + b, a \in K_\mu, b \in K'_\mu$, then taking open and closed neighbourhoods $\widetilde{K}_\mu$ and $\widetilde{K}'_\mu$ of a and b in K_μ and K'_μ respectively, diffeomorphic to some sets $K_{\mu,p}$ and $K'_{\mu,p'}$, where $(p, p') \in P(K_\mu) \times P'(K'_\mu)$, we have $\widehat{d}(\widetilde{K}_\mu + \widetilde{K}'_\mu, a + b) < \frac{1}{2}$ for μ small enough.

Since each extreme x of a gap of $K_\mu + K'_\mu$ can be written in at most two ways as $a + b$, where $a \in K_\mu, b \in K'_\mu$, say $x = a_1 + b_1 = a_2 + b_2$, there is $\delta > 0$ such that $(K_\mu + K'_\mu) \cap (x - \delta, x + \delta) =$

$$\left((\widetilde{K}_\mu^{(1)} + \widetilde{K}'^{(1)}_\mu) \cap (a_1 + b_1 - \delta, a_1 + b_1 + \delta)\right) \cup$$

$$\left((\overset{\sim}{K_\mu}^{(2)} + \overset{\sim}{K'_\mu}^{(2)}) \cap (a_2 + b_2 - \delta, a_2 + b_2 + \delta) \right),$$

where $\overset{\sim}{K_\mu}^{(1)}, \overset{\sim}{K_\mu}^{(2)}, \overset{\sim}{K'_\mu}^{(1)}, \overset{\sim}{K'_\mu}^{(2)}$ are the above mentioned open and closed neighbourhoods of a_1, b_1, a_2 and b_2 in K_μ or K'_μ. Now, since for every sets X, Y and $a \in \mathbb{R}$ we have $\widehat{d}(X \cup Y, a) \leq \widehat{d}(X, a) + \widehat{d}(Y, a)$, we have $\widehat{d}(K_\mu + K'_\mu, x) < \frac{1}{2} + \frac{1}{2} = 1$, so x is accumulated by gaps of $K_\mu + K'_\mu$, and this is enough to conclude the proof of Theorem 1.□

Remark 6. *The above proof of Theorem 1 gives the existence of an open set* $\widetilde{\mathcal{U}}$ *of pairs of regular Cantor sets such that* $\widetilde{\mathcal{U}} \cap \mathcal{H}$ *is (open and) dense such that if* $(K, \widetilde{K}) \in \widetilde{\mathcal{U}}$ *and* $K, \widetilde{K}$ *are linearized, i.e., self-similar near their left extreme points* $a, \widetilde{a}$ *and* $g, \widetilde{g}$ *are orientation-preserving diffeomorphisms then* $\widehat{d}(g(K) + \widetilde{g}(\widetilde{K}), g(a) + \widetilde{g}(\widetilde{a})) < \frac{1}{2}$. *We may also change* $\frac{1}{2}$ *by any fixed* $\epsilon > 0$ *and get a similar result.*

Remark 7. *We can adapt this proof in order to show an analogous version of Theorem 1 for families with* k *parameters for any* $k \in \mathbb{N}$*: for a residual set of families of pairs of regular Cantor sets* $(K_\mu, K'_\mu), \mu \in (-1,1)^k$*, if* $d(\mu) = HD(K_\mu) + HD(K'_\mu)$*, there is a neighbourhood* V *of* $\{\mu \in (-1,1)^k : d(\mu) = 1\}$ *such that* $\mu \in V$*,* $d(\mu) > 1$ *implies that* $K_\mu + K'_\mu$ *is an M-Cantorval. Indeed, it is enough to take* $\epsilon = \frac{1}{k+1}$ *in the previous remark and to use the fact that for generic* k*-parameter families of regular Cantor sets (contained in the complement of a countable union of codimension* $k+1$ *submanifolds) we never have distinct extremes of gaps* $a_1, a_2, .., a_{k+2}$ *of* K_μ *and* $b_1, b_2, .., b_{k+2}$ *of* K'_μ *with* $a_i + b_i = a_j + b_j$*,* $\forall\, i, j \leq k+2$*, for any* $\mu \in (-1,1)^k$.

3 Homoclinic Bifurcations

We will prove that in homoclinic bifurcations in surfaces associated to horseshoes with Hausdorff dimension close to 1, one has positive density of hyperbolicity at the first parameter of bifurcation:

Consider a first homoclinic bifurcation associated to a hyperbolic set Λ, i.e., a family of diffeomorphisms $\varphi_\mu \colon M^2 \to M^2$, $\mu \in (-1,1)$ such that:

• φ_μ is persistently hyperbolic for $\mu < 0$;

• φ_0 presents a homoclinic tangency associated to the periodic saddle point p_0 belonging to the hyperbolic set Λ, which unfolds into transversal homoclinic orbits for $\mu > 0$.

Let $(\widetilde{K}_1)_0$ and $(\widetilde{K}_2)_0$ be the linearizations of the stable and the unstable Cantor sets of Λ. Suppose that $((\widetilde{K}_1)_0, (\widetilde{K}_2)_0)$ belongs to the open set $\widetilde{\mathcal{U}}$ described in Remark 6 above. Recall that $\widetilde{\mathcal{U}} \cap \mathcal{H}$ is dense in $\mathcal{H}$. Then we have the following

Theorem 8. *Under the above conditions, (φ_μ) presents positive density of hyperbolicity at $\mu = 0$. More precisely, he have*

$$(*) \qquad \liminf_{\delta \to 0} \frac{m(\{\mu \in [0,\delta] : \varphi_\mu \text{ is persistently hyperbolic}\})}{\delta} > 0.$$

If $((\widetilde{K}_1)_0, (\widetilde{K}_2)_0)$ belongs to $\widetilde{\mathcal{U}} \cap \mathcal{U}$, where $\mathcal{U} \subset \Omega$ was defined in Section 2 (so $\widetilde{\mathcal{U}} \cap \mathcal{U}$ is a generic set in the intersection of Ω with a neighbourhood of $\mathcal{H}$) then both $(*)$ and

$$\liminf_{\delta \to 0} \frac{m(\{\mu \in [0,\delta] : \varphi_\mu \text{ presents persistent homoclinic tangencies}\})}{\delta} > 0.$$

hold.

Proof. We will use the Theorem V.1 of **[Mo1]**, which we describe next.

At the first moment of bifurcation (at the parameter $\mu = 0$) there is a line of tangencies which intersects the stable and unstable foliations of Λ at the Cantor sets $(K_1)_\mu$ and $(K_2)_\mu$, whose extreme points (p and p', respectively) coincide (in the sense that their support intervals intersect at one point). The sets $(\widetilde{K}_1)_0$ and $(\widetilde{K}_2)_0$ are the linearizizations of $(K_1)_0$ and $(K_2)_0$ relative to p and p', respectively. The mentioned Theorem says that, if there exists $t > 0$ for which $(\widetilde{K}_1)_0 + t$ is left-linked with $(\widetilde{K}_2)_0$ and $\left((\widetilde{K}_1)_0 + t\right) \cap (\widetilde{K}_2)_0 = \emptyset$, then

$$(*) \qquad \liminf_{\delta\to 0} \frac{m(\{\mu\in[0,\delta] : \varphi_\mu \text{ is persistently hyperbolic}\})}{\delta} > 0.$$

In particular, if $((\widetilde{K}_1)_0, (\widetilde{K}_2)_0)$ (or, equivalently, $((K_1)_0,(K_2)_0)$) belongs to the open set $\widetilde{\mathcal{U}}$ described at Remark 6 of the last section, the condition above about $(\widetilde{K}_1)_0$ and $(\widetilde{K}_2)_0$, and therefore, the $(*)$ condition are both satisfied, persistently.

If $((\widetilde{K}_1)_0,(\widetilde{K}_2)_0) \in \mathcal{U}\cap\widetilde{\mathcal{U}}$, then the $(*)$ condition and

$$(**) \qquad \liminf_{\delta\to 0} \frac{m(\{\mu\in[0,\delta] : (K_1)_\mu\cap(K_2)_\mu\neq\emptyset, \text{ persistently}\})}{\delta} > 0,$$

both hold by Theorem I.2 of **[Mo1]**. This concludes the proof. $\square$

Remark 9. *The same techniques can be used to show that for generic families $\varphi_{\mu,s}$, $(\mu,s) \in (-1,1)^2$ of diffeomorphisms such that, for each s, $\varphi_{0,s}$ presents a first homoclinic tangency and $d(s) = HD((K_1)_{0,s}) + HD((K_2)_{0,s})$ satisfying $d(0) = 1, d'(0) > 0$, we have that $(*)$ and $(**)$ hold for fixed small positive values of s (where φ_μ is replaced by $\varphi_{\mu,s}$), and moreover*

$$\lim_{s\to 0}\liminf_{\delta\to 0} \frac{m(\{\mu\in[0,\delta] : \varphi_{\mu,s} \text{ is persistently hyperbolic}\})}{\delta} = 1.$$

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