EQUILIBRIUM STATES FOR RANDOM NON-UNIFORMLY EXPANDING MAPS

ALEXANDER ARBIETO, CARLOS MATHEUS AND KRERLEY OLIVEIRA

Abstract. We show that, for a robust (C^2-open) class of random non-uniformly expanding maps, there exists equilibrium states for a large class of potentials.In particular, these sytems have measures of maximal entropy. These results also give a partial answer to a question posed by Liu-Zhao. The proof of the main result uses an extension of techniques in recent works by Alves-Araújo, Alves-Bonatti-Viana and Oliveira.

1. Introduction

Particles systems, as they appear in kinetic theory of gases, have been an important model motivating much development in the field of Dynamical Sytems and Ergodic Theory. While these are deterministic systems, ruled by Hamiltonian dynamics, the evolution law is too complicated, given the huge number of particles involved. Instead, one uses a stochastic approach to such systems.

More generally, ideais from statistical mechanics have been brought to the setting of dynamical systems, both discrete-time and continuous-time, by Sinai, Ruelle, Bowen, leading to a beautiful and very complete theory of equilibrium states for uniformly hyperbolic diffeomorphisms and flows. In a few words, equilibrium states are invariant probabilities in the phase space which maximaze a certain variational principle (corresponding to the Gibbs free energy in the statistical mechanics context). The theory of Sinai-Ruelle-Bowen gives that for uniformly hyperbolic systems equilibrium states exist, and they are unique if the associiated potential is Hölder continuous.

Several authors have worked on extending this theory beyond the uniformly hyperbolic case. See e.g. [5], [13], among other important authors. Our present work is more directly motivated by the results of Oliveira [12] where he constructed equilibrium states associated to potentials with not-too-large variation, for a robust (C^1-open) class of non-uniformly expanding maps introduced by Alves-Bonatti-Viana [3].

On the other hand, corresponding problems have been studied also in the context of the theory of random maps, which was much developed by Kifer [6] and Arnold [1], among other mathematicians. Indeed, Kifer [6]

Date: March 12, 2003.

A.A. was partially supported by CNPq-Brazil, C.M. was partially supported by Faperj-Brazil and K.O. was partially supported by CNPq-Brazil.

proved the existence of equilibrium states for random uniformly exapnding systems, and Liu [8] extended this to uniformly hyperbolic systems.

In the present work, we combine these two approaches to give a construction of equilibrium states for non-uniformly hyperbolic maps. In fact, some attempts to show the existence of equilibrium states were made by Khanin-Kifer [7]. However, our point of view here is quite different from their work. Before stating the main result, we recall that a random map is a continuous map $f:\Omega\to C^r(M,M)$ where M is a compact manifold Ω is a Polish space, and $T:\Omega\to\Omega$ a measurably invertible continuous map with an invariant ergodic measure $\mathbb{P}$. The main result is the following :

"For a C^2-open set $\mathcal{F}$ of non-uniformly expanding local diffeomorphisms, potentials ϕ with low variation *and $f:\Omega\to\mathcal{F}$, there are equilibrium states for the random system associated to f and T. In particular, f admits measures with maximal entropy."*

A potential has low variation if it is not far from being constant. See the precise definition in section 3. In particular, constant functions have low variation; their equilibrium states are measures of maximal entropy.

The proof, which we present in the next sections extends ideias from Alves-Araújo [2], Alves-Bonatti-Viana [3] and Oliveira [12].

It is very natural to ask whether these equilibrium states we construct are unique and whether they are (weak) Gibbs states, Another very interesting question is whether existence (and uniqueness) of equilibrium states extends to (random r deterministic) non-uniformly hyperbolic maps with singularities, such as the Viana maps [2]. Although our present methods do not solve these questions, we believe the answers are affirmative.

2. Definitions

Random Transformations and Invariant Measures

Let M^l be a compact l-dimensional Riemannian manifold and $\mathcal{D}$ the space of C^2 local diffeomorphisms of M. Let $(\Omega,T,\mathbb{P})$ a measure preserving system, where $T:\Omega\to\Omega$ is $\mathbb{P}$-invariant ($\mathbb{P}$ is a Borel measure) and Ω is a Polish space, i.e., Ω is a complete separable metric space. By a random transformation we understand a continuous map $f:\Omega\to\mathcal{D}$. Then we define:

$$f^n(w)=f(T^{n-1}(w))\circ\cdots\circ f(w),\ f^{-n}(w)=(f^n(w))^{-1}. \tag{1}$$

We also define the skew-product generating by f:

$$F:\Omega\times M\to\Omega\times M,\ F(w,x)=(Tw,f(w)x).$$

We denote $\mathcal{P}(\Omega\times M)$ the space of probability measures μ on $\Omega\times M$ such that the marginal of μ on Ω is $\mathbb{P}$. Let $\mathcal{M}(\Omega\times M)\subset\mathcal{P}(\Omega\times M)$ be the measures μ which are F-invariant.

Because M is compact, invariant measures always exists and the property of $\mathbb{P}$ be the marginal on Ω of a invariant measures can be characterized by its disintegration:

$$d\mu(w,x) = d\mu_w(x)d\mathbb{P}(w).$$

μ_w are called *samples measures* of μ (see [9], [10]).

An invariant measure is called *ergodic* if (F,μ) is ergodic, the set of all ergodic measures is denoted by $\mathcal{M}_e(\Omega \times M)$. Furthermore, each invariant measure can be decomposed into its ergodic components by integration when the σ-algebra on Ω is countably generated and $\mathbb{P}$ is ergodic.

In what follows, as usual, **we always assume** $(\Omega, \mathcal{A}, \mathbb{P})$ **is a Lebesgue space,** $(T,\mathbb{P})$ **is ergodic and** T **is measurably invertible and continuous**. Observe that these assumptions are satified in the canonical case of left-shift operators τ, Ω being $C^r(M,M)^{\mathbb{N}}$ or $C^r(M,M)^{\mathbb{Z}}$.

Entropy

We follow Liu [9] on the definition of the Kolmogorov-Sinai entropy for random transformations:

Let μ an F-invariant measure like above. Let ξ a finite Borel partition of M. We set:

$$h_\mu(f,\xi) = \lim_{n\to+\infty} \frac{1}{n} \int H_{\mu_w}(\vee_{k=0}^{n-1} f^k(w)^{-1}\xi)d\mathbb{P}(w), \tag{2}$$

where $H_\nu(\eta) := -\sum_{C\in\eta} \nu(C)\log\nu(C)$ (and $0\log 0 = 0$), for a finite partition η and ν a probability on M (and μ_w are the sample measures of μ).

Definition 2.1. *The entropy of* (f,μ) *is:*

$$h_\mu(f) := \sup_\xi h_\mu(f,\xi)$$

with the supremum taken over all finite Borel partitions of M*.*

Definition 2.2. *The* topological entropy *of* f *is* $h_{top}(f) = \sup_\mu h_\mu(f)$

Theorem 2.3 ("Random" Kolmogorov-Sinai theorem)**.** *If* $\mathcal{B}$ *is the Borel* σ*-algebra of* M *and* ξ *is a generating partition of* M*, i.e.,*

$$\bigvee_{k=0}^{+\infty} f^{-k}(w)\ \xi = \mathcal{B} \ \ for\ \mathbb{P}-a.e.\ w,$$

then

$$h_\mu(f) = h_\mu(f,\xi).$$

For a proof of this theorem see [10] or [4].

Equilibrium States

Let $L^1(\Omega, C(M))$ the set of all families $\{\phi = \{\phi_w \in C^0(M)\}\}$ such that the map $(w,x) \to \phi_w(x)$ is a measurable map and $\|\phi\|_1 := \int_\Omega |\phi_w|_\infty d\mathbb{P}(w) < +\infty$.

For a $\phi \in L^1(\Omega, C(M))$, $\varepsilon > 0$ and $n \geq 1$, we define:

$$\pi_f(\phi)(w,n,\varepsilon) = \sup\{\sum_{x\in K} e^{S_f(\phi)(w,n,x)}; K \text{ is a } (n,\varepsilon) - \text{separated set}\},$$

where $S_f(\phi)(w,n,x) := \sum_{k=0}^{n-1} \phi_{T^k(w)}(f^k(w)x)$.

Definition 2.4. *The map $\pi_f : L^1(\Omega, C(M)) \to \mathbb{R}\cup\{\infty\}$ given by:*

$$\pi_f(\phi) = \lim_{\varepsilon\to 0}\limsup_{n\to+\infty}\frac{1}{n}\int_\Omega \log \pi_f(\phi)(w,n,\varepsilon)d\mathbb{P}(w).$$

is called the pressure map.

It is well know that the variational principle occurs (see [9]):

Theorem 2.5. *If Ω is a Lebesgue space, then for any $\phi \in L^1(\Omega, C(M))$ we have:*

$$\pi_f(\phi) = \sup_{\mu\in\mathcal{M}(\Omega\times M)}\{h_\mu(f) + \int \phi d\mu\} \tag{3}$$

Remark 1. If $\mathbb{P}$ is ergodic then we can take the supremum over the set of ergodic measures.

Definition 2.6. *A measure $\mu \in \mathcal{M}(\Omega\times M)$ is an equilibrium state for f, if μ attains the supremum of (3).*

Physical Measures

As in the deterministic case, we follow [2] on the definition of *physical measure* in the context of random transformations :

Definition 2.7. *A measure μ is a physical measure if for positive Lebesgue measure set of points $x \in M$ (called the* basin $B(\mu)$ *of μ),*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n-1}\phi(f^j(w)(x)) = \int \phi d\mu \text{ for all continuous } \phi : M \to \mathbb{R}. \tag{4}$$

for $\mathbb{P}$-ae w.

3. Statement of the results

Before starting abstract definitions, we comment that in next section, it is showed that there are examples of random transformations satisfying our hypothesis below.

We say that a local diffeomorphism f of M is in $\widetilde{\mathcal{F}}$ if f is in $\mathcal{D}$ and satisfies, for positive constants δ_0, β, δ_1, σ_1, and p, $q \in \mathbb{N}$, the following properties :

(H1) There exists a covering $B_1, \ldots, B_p, \ldots, B_{p+q}$ of M such that every $f|B_i$ is injective and

- f is uniformly expanding at every $x \in B_1 \cup \cdots \cup B_p$:
$$\|Df(x)^{-1}\| \leq (1+\delta_1)^{-1}.$$
- f is never too contracting: $\|Df(x)^{-1}\| \leq (1+\delta_0)$ for every $x \in M$.

(H2) f is everywhere volume-expanding: $|\det Df(x)| \geq \sigma_1$ with $\sigma_1 > q$.
Define
$$V = \{x \in M; \|Df(x)^{-1}\| > (1+\delta_1)^{-1}\}.$$

(H3) There exists a set $W \subset B_{p+1} \cup \cdots \cup B_{p+q}$ containing V such that
$$M_1 > m_2 \quad \text{and} \quad m_2 - m_1 < \beta$$
where m_1 and m_2 are the infimum and the supremum of $|\det Df|$ on V, respectively, and M_1 and M_2 are the infimum and the supremum of $|\det Df|$ on W^c, respectively.

This kind of transformations was considered by [3], [12], [2], where they construct C^1-open sets of such maps.

We will consider a subset $\mathcal{F} \subset \widetilde{\mathcal{F}}$ such that :

(C1) There is a uniform constant A_0 s.t. $|\log \|f\|_{C^2}| \leq A_0$ for any $f \in \mathcal{F}$ and the constants m_1, m_2, M_1, M_2 are uniform on $\mathcal{F}$;

From now on, our random transformations will be given by a map $f : \Omega \to \mathcal{F}$, and f satisfy the following condition :

(C2) f admits an ergodic absolutely continuous *physical measure* $\mu_{\mathbb{P}}$ (see section 2).

Remark 2. We will show in the appendix that $(H1), (H2)$ implies the following property:

(F1) There exists some $\gamma_0 = \gamma_0(\delta_1, \sigma_1, p, q) < 1$ such that the random orbits of Lebesgue almost every point spends at most a fraction of time $\gamma_0 < 1$ inside $B_{p+1} \cup \cdots \cup B_{p+q}$, depending only on σ_1, p, q. I.e., for $\mathbb{P}$-a.e. w and Lebesgue almost every x
$$\lim_{n\to\infty} \frac{\#\{0 \leq j \leq n-1 : f^j(w)(x) \in B_{p+1} \cup \cdots \cup B_{p+q}\}}{n} \leq \gamma_0.$$

Then we analyse the existence of an equilibrium state for low-variation potentials:

Definition 3.1. *A potential $\phi \in L^1(\Omega, C(M))$ has ρ_0-low variation if*

$$\|\phi\|_1 < \pi_f(\phi) - \rho_0 h_{top}(f). \tag{5}$$

Remark 3. We call ϕ above a ρ_0-low variation potential because in the deterministic case (i.e., $\phi(w,x) = \phi(x)$), if $\max \phi - \min \phi < (1-\rho_0) h_{top}(f)$ then ϕ satisfies (5).

The main result is :

Theorem A. *Assume hypotheses (H1), (H2), (H3) hold, with δ_0 and β sufficiently small and assume also conditions (C1), (C2). Then, there exists*

ρ_0 such that if ϕ is a continuous potential with ρ_0-low variation then ϕ has some equilibrium state. Moreover, these equilibrium states are hyperbolic measures, with all Lyapunov exponents bigger than some $c = c(\delta_1, \sigma_1, p, q) > 0$.

4. Examples

In this section we exhibit a C^1-open class of C^2-diffeomorphism which are contained in $\widetilde{\mathcal{F}}$. To start the construction, we now follow [12] *ipsis-literis* and construct examples of 'deterministic' non-uniformly expanding maps. After this, we construct the desired random non-uniformly expanding maps in $\mathcal{F}$ a C^2-neighborhood of a fixed diffeomorphism of $\widetilde{\mathcal{F}}$.

We observe that the class $\mathcal{F}$ contains an open set of non-uniformly exapanding which *are not uniformly exapnding.*

We start by considering any Riemannian manifold that supports an expanding map $g : M \to M$. For simplicity, choose $M = \mathbb{T}^n$ the n-dimensional torus, and g an endomorphism induced from a linear map with eigenvalues $\lambda_n > \cdots > \lambda_1 > 1$. Denote by $E_i(x)$ the eigenspace associated to the eigenvalue λ_i in T_xM.

Since g is an expanding map, g admits a transitive Markov partition $R_1, \ldots, R_d$ with arbitrary small diameter. We may suppose that $g|R_i$ is injective for every $i = 1, \ldots, d$. Replacing by a iterate if necessary, we may suppose that there exists a fixed point p_0 of g and, renumbering if necessary, this point is contained in the interior of the rectangle R_d of the Markov partition.

Considering a small neighborhood $W \subset R_d$ of p_0 we deform g inside W along the direction E_1. This deformation consists essentially in rescaling the expansion along the invariant manifold associated to E_1 by a real function α. Let us be more precise:

Considering W small, we may identify W with a neighborhood of 0 in $\mathcal{R}^n$ and p_0 with 0. Without loss of generality, suppose that $W = (-2\epsilon, 2\epsilon) \times B_{3r}(0)$, where $B_{3r}(0)$ is the ball or radius $3r$ and center 0 in $\mathcal{R}^{n-1}$. Consider a function $\alpha : (-2\epsilon, 2\epsilon) \to \mathcal{R}$ such $\alpha(x) = \lambda_1 x$ for every $|x| \geq \epsilon$ and for small constants γ_1, γ_2:

1. $(1 + \gamma_1)^{-1} < \alpha'(x) < \lambda_1 + \gamma_2$
2. $\alpha'(x) < 1$ for every $x \in (-\frac{\epsilon}{2}, \frac{\epsilon}{2})$;
3. α is C^0-close to λ_1: $\sup\limits_{x \in (-\epsilon, \epsilon)} |\alpha(x) - \lambda_1 x| < \gamma_2$,

Also, we consider a bump function $\theta : B_{3r}(0) \to \mathcal{R}$ such $\theta(x) = 0$ for every $2r \leq |x| \leq 3r$ and $\theta(x) = 1$ for every $0 \leq |x| \leq r$. Suppose that $\|\theta'(x)\| \leq C$ for every $x \in B_{3r}(0)$. Considering coordinates $(x_1, \ldots, x_n)$ such that $\partial_{x_i} \in E_i$, define f_0 by:

$$f_0(x_1, \ldots, x_n) = (\lambda_1 x_1 + \theta(x_2, \ldots, x_n)(\alpha(x_1) - \lambda_1 x_1), \lambda_2 x_2, \ldots, \lambda_n x_n)$$

Observe that by the definition of θ and α we can extend f_0 smoothly to $\mathbb{T}^n$ as $f_0 = g$ outside W. Now, is not difficult to prove that f_0 satisfies the conditions (H1), (H2), (H3) above.

First, we have that $\|Df_0(x)^{-1}\|^{-1} \geq \min_{i=1,\ldots,n} \|\partial_{x_i} f_0\|$. Observe that:

$$\partial_{x_1} f_0(x_1,\ldots,x_n) = (\alpha'(x_1)\theta(x_2,\ldots,x_n) + (1-\theta(x_2,\ldots,x_n))\lambda_1, 0,\ldots,0)$$
$$\partial_{x_i} f_0(x_1,\ldots,x_n) = ((\alpha(x_1)-\lambda_1)\partial_{x_i}\theta(x_2,\ldots,x_n), 0,\ldots,\lambda_i,0,\ldots,0), \text{ for } i \geq 2.$$

Then, since $\|\partial_{x_i}\theta(x)\| \leq C$ for every $x \in B_{3r}(0)$, and $\alpha(x_1) - \lambda_1 x_1 \leq \gamma_2$ we have that $\|\partial_{x_i} f_0\| > (\lambda_i - \gamma_2 C)$ for every $i = 2,\ldots,n$. Moreover, by condition 1, $\|\partial_{x_1} f_0\| \leq \max\{\alpha'(x_1), \lambda_1\} \leq \lambda_1 + \gamma_2$, if we choose γ_2 small in such way that $\lambda_2 - \gamma_2 C > \lambda_1 + \gamma_2$ then:

$$\|\partial_{x_i} f_0\| > \|\partial_{x_1} f_0\|, \text{ for every } i \geq 2.$$

Notice also that $\|\partial_{x_1} f_0\| \geq \min\{\alpha'(x_1), \lambda_1\} \geq (1+\gamma_1)^{-1}$. This prove that:

$$\|Df_0(x)^{-1}\|^{-1} \geq \min_{i=1,\ldots,n} \|\partial_{x_i} f_0\| \; (1+\gamma_1)^{-1}.$$

Since f coincides with g outside W, we have $\|Df_0(x)^{-1}\| \leq \lambda_1^{-1}$ for every $x \in W^c$. Together with the above inequality, this proves condition (H1), with $\delta_0 = \gamma_1$.

Choosing γ_1 small and $p = d-1$, $q = 1$, $B_i = R_i$ for every $i = 1,\ldots,d$, condition (H2) is immediate. Indeed, observe that the Jacobian of f_0 is given by the formula:

$$\det Df_0(x) = (\alpha'(x_1)\theta(x_2,\ldots,x_n) + (1-\theta(x_2,\ldots,x_n))\lambda_1) \prod_{i=2}^{n} \lambda_i.$$

Then, if we choose $\gamma_1 < \prod_{i=2}^{n} \lambda_i - 1$:

$$\det Df_0(x) > (1+\gamma_1)^{-1} \prod_{i=2}^{n} \lambda_i > 1.$$

Therefore, we may take $\sigma_1 = (1+\gamma_1)^{-1} \prod_{i=2}^{n} \lambda_i > 1$.

To verify property (H3) for f_0, observe that if we denote by

$$V = \{x \in M; \|Df_0(x)^{-1}\| > (1+\delta_1)^{-1}\},$$

with $\delta_1 < \lambda_1 - 1$, then $V \subset W$. Indeed, since $\alpha(x_1)$ is constant equal to $\lambda_1 x_1$ outside W we have that $\|Df_0(x)^{-1}\| \leq \lambda_1^{-1} < (1+\delta_1)^{-1}$, for every $x \in W^c$. Given γ_3 close to 0, we may choose δ_1 close to 0 and α satisfying the conditions above in such way that,

$$\sup_{x,y \in V} \alpha'(x_1) - \alpha'(y_1) < \gamma_3.$$

If m_1 and m_2 are the infimum and the supremum of $|\det Df_0|$ on V, respectively,

$$m_2 - m_1 \leq C(\sup_{x,y\in V} \alpha'(x_1) - \alpha'(y_1)) < \gamma_3 C,$$

where $C = \prod_{i=2}^{n} \lambda_i$. Then, we may take $\beta = \gamma_3 C$ in (H3). If M_1 is the infimum of $|\det Df_0|$ on W^c, $M_1 > m_2$, since $\lambda_1 > (1+\delta_1) \geq \sup_{x\in V} \alpha'(x)$.

The arguments above show that the hypotheses $(H1),(H2),(H3)$ are satisfied by f_0. Moreover, if we one takes $\alpha(0) = 0$, then p_0 is fixed point for f_0, which is not a repeller, since $\alpha'(0) < 1$. Therefore, f_0 is not a uniformly expanding map.

It is not difficult to see that this construction may be carried out in such way that f_0 does not satisfy the expansiveness property: there is a fixed hyperbolic saddle point p_0 such that the stable manifold of p_0 is contained in the unstable manifold of two other fixed points.

Now, if $\mathcal{F}$ denotes a small C^2-neighborhood of f_0 in $\widetilde{\mathcal{F}}$, and $h : \Omega \to \widetilde{\mathcal{F}}$ is a continuous map, Alves-Araújo [2] shows that if $w^* \in \Omega$ is such that $f(w^*) = f_0$ and θ_ε is a sequence of measures, $\mathrm{supp}(\theta_\varepsilon) \to \{w_0\}$ then for small $\varepsilon > 0$ there are physical measures for the RDS $f : \Omega^{\mathbb{Z}} \to \mathcal{F}$, $f(\dots, w_{-k}, \dots, w_0, \dots, w_k, \dots) = H(w_0)$. This concludes the construction of examples satisfying $(H1),(H2),(H3),(C1),(C2)$.

5. Proof of the theorem A

We now precise the conditions on δ_0 and β. We consider γ_0 given in condition $(F1)$. By condition $(C1)$, there exists $\epsilon_0 > 0$ s.t. for any $\eta \in B_{\epsilon_0}(\xi)$ and $\mathbb{P}$-a.e. w holds :

$$\frac{||Df(w)^{-1}(\xi)||}{||Df(w)^{-1}(\eta)||} \leq e^{\frac{c}{2}},$$

where c is such that for some $\alpha > \gamma_0$, we have $(1+\delta_0)^{\alpha}(1+\delta_1)^{-(1-\alpha)} < e^{-2c} < 1$ and $\alpha m_2 + (1-\alpha)M_2 < \gamma_0 m_1 + (1-\gamma_0)M_1 - l\log(1+\delta_0)$ $(l := \dim(M))$, if δ_0 and β are sufficiently small. Now, the constants fixed above allows us to prove good properties for the objects defined below, which are of fundamental interest in the proof of theorem A.

Expansive Measures and Hyperbolic Times

Definition 5.1. *We say that a measure* $\nu \in \mathcal{M}(\Omega \times M)$ *is* f-expanding with exponent c *if for* ν*-almost every* $(w,x) \in \Omega \times M$ *we have:*

$$\lambda(w,x) = \limsup_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Df(T^j(w))(f^j(w)(x))^{-1}\| \leq -2c < 0.$$

Definition 5.2. *We say that* n *is a hyperbolic time for* (w,x) *with exponent* c*, if for every* $1 \leq k \leq n$*:*

$$\prod_{j=n-k}^{n-1} \|Df(T^{j+1}(w))(f^j(w)(x))^{-1}\| \leq e^{-ck}.$$

As in lemma 3.1 of [3], lemma 4.8 of [12] and lemma 2.2 of [2], we have infinity many hyperbolic times for expanding measures. For this we need a lemma due to Pliss (see [3]).

Lemma 5.3. *Let $A \geq c_2 > c_1 > 0$ and $\zeta = \frac{c_2 - c_1}{A - c_1}$. Given real numbers $a_1, \cdots, a_N$ satisfying:*

$$\sum_{j=1}^{N} a_j \geq c_2 N \text{ and } a_j \leq H \text{ for all } 1 \leq j \leq,$$

there are $l > \zeta N$ and $1 < n_1 < \cdots < n_l \leq N$ such that:

$$\sum_{j=n+1}^{n_i} a_j \geq c_1(n_1 - n) \text{ for each } 0 \leq n < n_i,\ i = 1, \cdots, l.$$

Lemma 5.4. *For every invariant measure ν which is c-expanding, there exists a full ν-measure set $H \subset \Omega \times M$ such that every $(w,x) \in H$ has infinitely many hyperbolic times $n_i = n_i(w,x)$ with exponent c and, in fact, the density of hyperbolic times at infinity is larger than some $d_0 = d_0(c) > 0$:*

1. $\displaystyle\prod_{j=n-k}^{n-1} \|Df(T^{j+1}(w))(f^j(w)(x))^{-1}\| \leq e^{-cj}$ *for every* $1 \leq k \leq n_i$
2. $\displaystyle\liminf_{n\to\infty} \frac{\sharp\{0 \leq n_i \leq n\}}{n} \geq d_0 > 0.$

Proof. Let $H \subset \Omega \times M$ with full ν-measure. For any $(w,x) \in H$ and n large enough, we have:

$$\sum_{j=0}^{n-1} \log \|Df(T^j(w))(f^j(w)(x))^{-1}\| \leq -\frac{3c}{2} n$$

Now, by $(C1)$ we can apply lemma 5.3 with $A = \sup\limits_{(w,x)} (-\log ||Df(w)^{-1}(x)||)$, $c_1 = c$, $c_2 = \frac{3c}{2}$ and $a_i = -\log \|Df(T^j(w))(f^j(w)(x))^{-1}\|$ and the statement follows. ∎

Lemma 5.5. *$\exists\ \epsilon_0 > 0$ such that for $\mathbb{P}$-a.e. w, if n_i is a hyperbolic time of (w,x) and $f^{n_i}(w)(z) \in B_{\epsilon_0}(f^{n_i}(w)(x))$ then $d(f^{n_i-j}(w)(z), f^{n_i-j}(w)(x)) \leq e^{\frac{-cj}{2}} d(f^{n_i}(w)(z), f^{n_i}(w)(x))$, $\forall 1 \leq j \leq n_i$.*

Proof. By $(C1)$ we know that there exists $\varepsilon_0 > 0$ such that for any $\eta \in B_{\varepsilon_0}(\xi)$ we have:

$$\frac{||Df(w)^{-1}(\xi)||}{||Df(w)^{-1}(\eta)||} \leq e^{\frac{c}{2}} \text{ for } \mathbb{P}\text{-ae } w.$$

In fact, this hold in the T-orbit of w $\mathbb{P}$-ae. Indeed, let $C = \{w; \frac{||Df(w)^{-1}(\xi)||}{||Df(w)^{-1}(\eta)||} \leq e^{\frac{c}{2}}\}$ for any ξ and $\eta \in B_{\varepsilon_0}(\xi)$, then $\bigcap T^j(C)$ has full measure and the estimate follows. Because $f^{n_i}(w)(z) \in B_{\epsilon_0}(f^{n_i}(w)(x))$, by the estimative above, we have that w $\mathbb{P}$-ae if we take the inverse branch of $f^{n_i}(w)$ which sends

$f^{n_i}(w)(x)$ to $f^{n_i-1}(w)(x)$ (restricted to $B_{\varepsilon_0}(f^{n_i}(w)(x))$) and has derivative with norm less than $e^{-\frac{c}{2}}$, then we have $d(f^{n_i-1}(w)(z), f^{n_i-1}(w)(x)) \leq \varepsilon_0$. Using the estimate along the orbit (and induction), we have:

$$\prod_{j=n-k}^{n-1} \|Df(T^{j+1}(w))(f^j(w)(z))^{-1}\| \leq e^{-\frac{ck}{2}} \text{ for all } 0 \leq k \leq n_i.$$

The statement follows. □

Now we define a set of measures where the "bad set" V has small measure.

Definition 5.6. *We define the convex set K_α by*

$$K_\alpha = \{\mu : \mu(\Omega \times V) \leq \alpha\}$$

Lemma 5.7. *$K_\alpha \neq \emptyset$ is a compact set.*

Proof. Let $\{\mu_n\} \subset K_\alpha$. By compacity, we can assume that $\mu_n \to \mu$. Since V is open then $\mu(\Omega \times V) \leq \liminf(\mu_n)(\Omega \times V) \leq \alpha$. This implies compacity. The physical measure given by condition (C2) (see equation (4)) is in K_α, because *Leb*-a.e. random orbit stay at most $\gamma_0 < \alpha$ inside V (by $(F1)$). By definition of physical measure (limit of average of Dirac measures supported on random orbits) and the absolute continuity with respect to the Lebesgue measure, $\mu_w(V) \leq \alpha$ for w $\mathbb{P}-$ a.e. holds. In particular, $\mu(\Omega \times V) \leq \alpha$. □

We recall that the ergodic decomposition theorem holds for RDS. With this in mind, we distinguish a set $\mathcal{K} \subset K_\alpha$:

Definition 5.8. $\mathcal{K} = \{\mu : \mu_{(w,x)} \in K_\alpha$ *for* $\mu -$ *a.e.*$(w, x)\}$ *($\mu_{(w,x)}$ is the ergodic decomposition of μ).*

Lemma 5.9. *Every measure $\mu \in \mathcal{K}$ is f-expanding with exponent c :*

$$\limsup_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Df(T^j(w))(f^j(w)(x))^{-1}\| \leq -2c$$

for μ-a.e. $(w, x) \in M$.

Proof. We assume first that μ is ergodic. By definition of K_α, we have $\mu(\Omega \times V) \leq \alpha$. But Birkhoff's Ergodic Theorem applied to (F, μ) says that in the random orbit of (w, x) $\mu-$a.e. we have:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \chi_V(f^i(w)(x)) \leq \alpha.$$

Now, we use hypothesis (H1): $\|Df(w, y)^{-1}\| \leq (1 + \delta_0)$ for any $y \in V$ and $\|Df(w, y)^{-1}\| \leq (1 + \delta_1)^{-1}$ for any $y \in V^c$, obtaining:

$$\frac{1}{n} \sum_{j=0}^{n-1} \log \|Df(T^j(w))(f^j(w)(x))^{-1}\| \leq \log[(1 + \delta_0)^\alpha (1 + \delta_1)^{1-\alpha}] \leq -2c < 0$$

$(w, x) - \mu-$a.e.

In the general case we use the ergodic decomposition theorem (see [12] and [10]). ☐

Entropy lemmas

Definition 5.10. *Given $\varepsilon > 0$, we define :*

$$A_\varepsilon(w,x) = \{y : d(f^n(w)(x), f^n(w)(y)) \leq \varepsilon \text{ for every } n \geq 0\}.$$

Lemma 5.11. *Suppose that $\mu \in \mathcal{K}$ is ergodic and let ε_0 given by lemma 5.5. Then, for $\mathbb{P}$-almost every w and any $\varepsilon < \varepsilon_0$,*

$$A_\varepsilon(w,x) = x.$$

Proof. By lemma 5.4 we have infinity hyperbolic times $n_i = n_i(w,x)$ for $(w,x) \in H$ (where $\mu(H) = 1$). For each w set $H_w = \{x; (w,x) \in H\}$, then $\mathbb{P}$-a.e. w we have $\mu_w(H_w) = 1$ and infinity hyperbolic times for μ_w-a.e. x. Now, by lemma 5.5, if $z \in A_\varepsilon(w,x)$ with $\varepsilon < \varepsilon_0$ we have:

$$d(x,z) \leq e^{-\frac{cn_i}{2}} d(f^{n_i}(w)(x), f^{n_i}(w)(z)) \leq e^{-\frac{cn_i}{2}} \varepsilon.$$

The lemma follows. ☐

Let $\mathcal{P}$ be a partition of M in measurable sets with diameter less than ε_0. From the above lemma, we get :

Lemma 5.12. *$\mathcal{P}$ is a generating partition for every $\mu \in \mathcal{K}$.*

Proof. As usual we will write:

$$\mathcal{P}_w^n = \{\mathcal{C}_w^n = (\mathcal{P}_w)_{i_0} \cap \cdots \cap f^{-(n-1)}(w)(\mathcal{P}_w)_{i_{n-1}})\} \text{ for each } n \geq 1,$$

where $(\mathcal{P}_w)_{i_k}$ is an element of the partition $\mathcal{P}$. By the previous lemma, we know that for $\mathbb{P}$-a.e. w, we have $A_\varepsilon(w,x) = x$ for x μ_w-a.e. Let A a measurable set of M and $\delta > 0$. Given $K_1 \subset A$ and $K_2 \subset A^c$ two compact sets such that $\mu_w(K_1 \triangle A) \leq \delta$ and $\mu_w(K_2 \triangle A^c) \leq \delta$. Now if $r = d(K_1, K_2)$, the previous lemma says that if n is big enough then $diam \mathcal{P}_w^n(x) \leq \frac{r}{2}$ for x in a set of μ_w-measure bigger than $1 - \delta$. The sets $(\mathcal{C}_w^n)_1, \cdots, (\mathcal{C}_w^n)_k$ that intersects K_1 satisfy:

$$\begin{aligned} \mu(\bigcup (C_w^n)_i \Delta A) &= \mu(\bigcup (C_w^n)_i - A) + \mu(A - \bigcup (C_w^n)_i) \\ &\leq \mu(A - K_1) + \mu(A^c - K_2) + \delta \leq 3\delta. \end{aligned}$$

This end the proof. ☐

Corollary 5.13. *For every $\mu \in \mathcal{K}$, $h_\mu(f) = h_\mu(f, \mathcal{P})$*

Proof. The result follows from lemma 5.12 and the theorem 2.3. ☐

We have that the map $\mu \to h_\mu(f, \mathcal{P})$ is upper semi-continuous at μ_0 measure s.t. $(\mu_0)_w(\partial P) = 0$ for $\mathbb{P}$-a.e. w, $P \in \mathcal{P}$. In fact, we have :

$$h_\mu(f, \mathcal{P}) = \lim_{n\to\infty} \frac{1}{n} \int H_{\mu_w}(\mathcal{P}_w^n) d\mathbb{P} = \inf_n \frac{1}{n} \int H_{\mu_w}(\mathcal{P}_w^n) d\mathbb{P}(w).$$

But, if $(\mu_0)_w(\partial P) = 0$ for any $P \in \mathcal{P}$ and $\mathbb{P}$-a.e. w, then the function $H(\mu, n)$ given by $\mu \to \int H_{\mu_w}(\mathcal{P}_w^n) d\mathbb{P}$ is upper semi-continuous at μ_0. Indeed, since we are assuming that T is continuous, the same argument in the proof of theorem 1.1 of [11] shows this result. In particular, because the infimum of a sequence of upper semi-continuous functions is itself upper semi-continuous, this proves the claim.

Lemma 5.14. *All ergodic measures η outside $\mathcal{K}$ have small entropy : there exists $\rho_0 < 1$ such that*

$$h_\eta(f) \leq \rho_0 h_{top}(f).$$

Proof. By the random versions of Oseledet's theorem and Ruelle's inequality (see [9]), we have:

$$h_\eta(f) \leq \int \sum_{i=1}^{s} \lambda^{(i)}(w,x) m^{(i)}(w,x) d\eta.$$

where $\lambda^{(i)}(w,x)$ and $m^{(i)}(w,x)$ are the Lyapunov exponents of f at (w,x) and its multiplicity respectively (and $\lambda^{(1)}(w,x), \cdots, \lambda^{(s)}(w,x)$ are the positive Lyapunov exponents). Furthermore, by hypothesis the measure is ergodic, then these objects are constant a.e. then $h_\eta(f) \leq \sum_{i=1}^{s} \lambda^{(i)}$ and $\int \log \| \det Df(w)(x) \| d\eta = \sum_i \lambda^{(i)}$. Since $\|Df(w)(x)^{-1}\| \leq (1+\delta_0)$ we have $\lambda_l > -\log(1+\delta_0)$. By the definitions of m_2, M_2 and the above estimates, we have by (C1):

$$\begin{aligned} h_\eta(f) &\leq \int \log \|Df(w)(x)\| d\eta - \sum_{i=s+1}^{l} \lambda^i \\ &\leq \eta(\Omega \times V) m_2 + (1 - \eta(\Omega \times V)) M_2 + (l-s)(1+\delta_0) \\ &\leq \alpha m_2 + (1-\alpha) M_2 + l \log(1+\delta_0) \end{aligned}$$

Now the physical measure $\mu_{\mathbb{P}}$ given by condition (C2) satisfy $\mu_{\mathbb{P}}(W) < \gamma_0$ (by $(F1)$). The Random Pesin's formulae gives:

$$h_{\mu_{\mathbb{P}}}(f) = \int \log \|detDf\| d\mu_{\mathbb{P}} \geq \mu_{\mathbb{P}}(W) m_1 + (1 - \mu_{\mathbb{P}}(W)) M_1.$$

But $m_1 < M_1$ then $\gamma_0 m_1 + (1-\gamma_0) M_1 \leq h_{\mu_{\mathbb{P}}}(f)$. Using that $\eta \notin K$, $m_2 < M_2$ and (C1) we have:

$$\alpha m_2 + (1-\alpha) M_2 < \gamma_0 m_1 + (1-\gamma_0) M_1 - l \log(1+\delta_0).$$

Then, we can choose $\rho_0 < 1$ such that

$$\alpha m_2 + (1-\alpha) M_2 + l \log(1+\delta_0) < \rho_0(\gamma_0 m_1 + (1-\gamma_0) M_1) < \rho_0 h_{\mu_{\mathbb{P}}}(f)$$

This gives: $h_\eta(f) \leq \rho_0 h_{top}(f)$. ☐

Corollary 5.15. $\pi_F(\phi) = \sup_{\mu \in \mathcal{K}} \{h_\mu(f) + \int \phi d\mu\}$.

Proof. By remark 1, we need to show that:

$$\sup_{\mu\in\mathcal{K}}\{h_\mu(f)+\int\phi d\mu\}=\sup_{\mu\in\mathcal{M}_e(\Omega\times M)}\{h_\mu(f)+\int\phi d\mu\}$$

By the previous lemma, if $\eta\notin\mathcal{K}$ then:

$$h_\eta(f)+\int\phi d\eta\leq\rho_0 h_{top}(f)+\|\phi\|_1<\pi_f(\phi)$$

□

Proof of theorem A. We will use the following notation: $\Psi(\mu)=h_\mu(f)+\int\phi d\mu$. Let $\{\mu_k\}\subset\mathcal{K}$ such that $\Psi(\mu_k)\to\pi_f(\phi)$, by compacity we can suppose that μ_k converge to μ weakly.

Fix $\mathcal{P}$ a partition with diameter less than ε_0, and for w-a.e., $\mu_w(\partial P)=0$, for any $P\in\mathcal{P}$. By corollary 5.13 we have $h_{\mu_k}(f)=h_{\mu_k}(f,\mathcal{P})$. Then $\pi_f(\phi)=\sup_{\eta\in\mathcal{K}}\Psi(\eta)=\limsup\Psi(\mu_k)$. By the comments after corollary 5.13 we know that $\eta\to h_\eta(f,\mathcal{P})$ is upper semicontinuous in η *over* $\mathcal{K}$, then:

$$\limsup\Psi(\mu_k)\leq h_\mu(f,\mathcal{P})+\int\phi d\mu\leq\Psi(\mu).$$

But, $\Psi(\mu)\leq\pi_f(\phi)$. This implies that μ is an equilibrium state.

In the other hand, if η is a measure which attain the supremum in (3) then let $\eta_{(w,x)}$ the ergodic decomposition of η. Then the entropy of η is equal to the integral of entropies of its ergodic components (see [9], page 1289 and references there in), of course the same occurs with the $\Psi(\eta)$ (*). If $(x,w)\notin\{(x,w);\eta_{(x,w)}\in\mathcal{K}_\alpha\}$ then by lemma 5.14:

$$\Psi(\eta_x)=h_{\eta_{(x,w)}}(f)+\int\phi d\eta_{(x,w)}\leq\rho_0 h_{top}(f)+\|\phi\|_1<\pi_f(\phi).$$

Then if $\eta(\{(x,w);\eta_{(x,w)}\in\mathcal{K}_\alpha\}^c)>0$, (*) says that $\Psi(\eta)<\pi_f(\eta)$ a contradiction, so every equilibrium state is in $\mathcal{K}$. The proof of the theorem is now complete. □

Remark 4. Liu-Zhao [11] show the semi-continuity of the entropy under the hypothesis that $T:\Omega\to\Omega$ is *continuous* and f is expansive *at very point of* M. From this result, a natural question is : "What about the semi-continuity without topological assumptions (e.g., continuity) ? And the case of weak expansiveness assumptions ?". We point out that the proof of theorem A shows the semicontinuity of the entropy map *in the set* $\mathcal{K}$. This partially answer the question since, although we need to assume continuity, only a weak expansion at Lebesgue a.e. point of M is required (*this assumption is the sole reason of the restriction to the set of measures* $\mathcal{K}$). Indeed, non-uniform expansion on Lebesgue a.e. point obligates us to restrict the proof of our lemmas on semicontinuity to the set $\mathcal{K}$.

Remark 5. Our theorem A holds in the context of *RDS bundles* (see [9] or [11]) with the extra assumption that T *and* the skew-product F are continuous.

6. Appendix

We now prove that $(F1)$ follows from (H1) and (H2), in fact, this is a well know argument (see for example [2]), but for sake of completeness we give the proof.

Fix (w, x), if $i = (i_0, \cdots i_{n-1}) \in \{1, \cdots, p+q\}^n$ let $[i] = B_{i_0} \cap f^{-1}(w)(B_{i_1}) \cap \cdots \cap f^{-n+1}(w)(B_{i_{n-1}})$ and $g(i) = \#\{0 \leq j < n; I_j \leq p\}$.

If $\gamma > 0$ then $\#\{i; g(i) < \gamma n\} \leq \sum\limits_{k \leq \gamma n} \binom{n}{k} p^{\gamma n} q^n$. By Stirling's formula this is bounded by $(e^{\xi} p^{\gamma} q)^n$ (here ξ depends of γ) and $\xi(\gamma) \to 0$ if $\gamma \to 0$.

Now (H1) and (H2) says that $m([i]) \leq \sigma_1^{-n} \sigma_1^{\gamma n}$. If we set $I(n, w) = \bigcup\{[i]; g([i]) < \gamma n\}$ then $m(I(n, w)) \leq \sigma_1^{-(1-\gamma)n} (e^{\xi} p^{\gamma} q)^n$ and since $\sigma_1 > q$ there is a γ_0 (small) such that $(e^{\xi} p^{\gamma} q)^n < \sigma_1^{(1-\gamma_0)}$. Then there is a $\tau = \tau(\gamma_0) < 1$ and $N = N(\gamma_0)$ such that if $n \geq N$ then $m(I(n, w)) \leq \tau^n$

Let $I_n = \bigcup\limits_w (\{w\} \times I(n, w))$ and by Fubini's theorem $\mathbb{P} \times Leb(I_n) \leq \tau^n$ if $n \geq N$. But $\sum\limits_n \mathbb{P} \times Leb(I_n) < \infty$ then Borel-Cantelli's lemma implies:

$$\mathbb{P} \times Leb\Big(\bigcap_{n \geq 1} \bigcup_{n \geq k} I_k\Big) = 0$$

Using Birkhoff's theorem we have that the set:

$$\{(w, x); \exists\ n \geq 1, \forall k \geq n, \lim \frac{\#\{0 \leq j < n; f^j(w)(x) \in B_1 \cup \cdots \cup B_p\}}{n}\}.$$

has $\mathbb{P} \times Leb$-measure at least γ_0. Now by Fubini's theorem again, we have (F1).

Acknowledgements. The authors are indebted to Professor Marcelo Viana for useful conversations, suggestions and advices. Also to Professor Pei-Dong Liu for give us helpful references (including the preprint [11]), for communicated to us some entropy results and for some corrections on old versions of the paper, and Professor Yuri Kifer for another valuable references. Finally, we are grateful to IMPA and his staff for the remarkable research ambient.

References

[1] L. Arnold Random Dynamical Systems. Springer, 1998.

[2] J. Alves and V. Araujo. Random perturbations of non-uniformly expanding maps. *preprint*, 2000.

[3] J. Alves, C. Bonnati, M. Viana. SRB measures for partially hyperbolic systems whose central direction is mostly expanding. *Inventiones Mathematicae*, 140: 298–351, 2000.

[4] T. Bogenschutz. Equilibrium states for random dynamical syatems. *Ph.D. thesis-Universitat Bremen*, Institut fur Dynamische Systeme, 1993.

[5] J. Buzzi. Thermodynamical formalism for piecewise invertible maps: absolutely continuous invariant measures as equilibrium states. *Proc. Sympos. Pure Math.*, 69, 749–783, 2001.

[6] Y. Kifer. Equilibrium states for random expanding transformations. *Random Comput. Dynam.*, 1, 1–31, 1992.
[7] K. Khanin, Y. Kifer. Thermodynamic Formalism for Random Transformations and Statistical Mechanics. *Amer. Math. Soc. transl.*, 171, 107–140, 1996.
[8] P.D. Liu Random Perturbations of Axiom A Basic Sets. *J. Stat. Phys.*, 90, 467–490, 1998.
[9] P.D. Liu. Dynamics of random transformations smooth ergodic-theory. *Ergodic Theory and Dynamical Systems*, 21: 1279–1319, 2001.
[10] P.D. Liu and M. Quian. Smooth Ergodic of Random Dynamical Systems. *Lecture Notes in Mathematics*,no. 1606, Springer-Verlag, 1995.
[11] P.D. Liu and Y. Zhao. Large Deviations in Random Perturbations of Axiom A Basic Sets, *to appear in J. of the London Math. Soc.*.
[12] K. Oliveira. Equilibrium States for non-uniformly expanding maps. *Ph.D. thesis-IMPA, preprint*, www.preprint.impa.br/Shadows/SERIE_C/2002/12.html, 2002.
[13] O. Sarig Thermodynamic formalism for countable Markov shifts. *Ergodic Theory Dynam. Systems*, 19, no. 6, 1565–1593, 1999.

Alexander Arbieto (`alexande@impa.br`)

Carlos Matheus (`cmateus@impa.br`)

Krerley Oliveira (`krerley@mat.ufal.br`)

IMPA, Estrada D. Castorina 110, Jardim Botânico, 22460-320 Rio de Janeiro, Brazil