SINGULAR-HYPERBOLIC SETS AND TOPOLOGICAL DIMENSION

C.A. Morales

November 2002

Abstract. A *singular-hyperbolic set* for flows is a partially hyperbolic set with singularities (hyperbolic ones) and volume expanding central direction [MPP1]. The class of transitive singular-hyperbolic sets includes the geometric Lorenz attractor and the singular horseshoe [GW, LP]. We prove that all compact, non-singular, invariant subsets of a transitive singular-hyperbolic set are one-dimensional. This generalizes the minimal set's results for Axiom A flows in [B2] to a class of flows studied in [MPP2].

1. Introduction

The *topological dimension* of a space is either -1 (if it is empty) or the last integer k for which every point has arbitrarily small neighborhoods whose boundaries have dimension less than k [HW]. Spaces with topological dimension k are said to be k*-dimensional* for short. There is a relation between topological dimension and hyperbolic dynamics. Indeed, Anosov diffeomorphisms on n-manifolds have no compact invariant $(n-1)$-dimensional sets [H]. Hyperbolic automorphisms on n-tori have compact invariant k-dimensional sets $\forall 0 \le k \le n-2$ ([P]). Minimal systems for Axiom A diffeomorphisms are always 0-dimensional, while minimal systems for Axiom A flows are 1-dimensional, or a single point (hence 0-dim) see [B1,B2]. Similar results hold for expansive homeomorphisms and certain expansive flows on compact metric spaces [KS,M].

It is natural to extend the above properties to non-expansive systems with some hyperbolicity. However we need some extra hypotheses to perform such an extension. In fact, multiplying the irrational toral flow by a strong contraction we obtain a non-expansive partially hyperbolic set containing two-dimensional non-singular minimal subsets. Another examples of non-expansive partially hyperbolic sets are the geometric Lorenz attractor and the singular horseshoe [GW,LP]. The obvious question is whether the non-singular minimal subsets of such sets are one-dimensional.

2000 MSC: 37D30 (Primary), 37C45 (Secondary).

Key words and phrases: Partially hyperbolic set, Transitive set, Topological dimension.

Partially supported by CNPq-Brazil, FAPERJ-Brazil and Pronex-Brazil.

In this paper we study the topological dimension for *singular-hyperbolic sets*, a class of non-expansive partially hyperbolic sets including the Lorenz attractor and the singular horseshoe [MPP1]. We prove that all compact, non-singular, invariant subsets of a transitive singular-hyperbolic set are one-dimensional. In particular, the non-singular minimal subsets of the geometric Lorenz attractor and the singular horseshoe are one-dimensional. This allow us to generalize the results on minimal sets for Axiom A flows [B2] to the class of singular-Axiom A flows [MPP2].

Let us state our result in a precise way. Hereafter M denotes a closed 3-manifold and $X = X_t$ denotes a C^1 flow on M. We always assume that X is orientable and still denote by X the corresponding vector field. If $q \in M$ we define $\omega_X(q)$, the *ω-limit set of* q, as the set of points $x \in M$ for which there is a real number sequence $t_n \to \infty$ such that $X_{t_n}(q) \to x$. A subset T of M is *invariant* if $X_t(T) = T$ for all $t \in \mathbf{R}$. T is: *transitive* if $T = \omega_X(p)$ for some $p \in T$ and *singular* if it has a singularity (i.e. an invariant singleton). All the invariant sets in this paper are not empty. A compact, singular, invariant set Λ of X is *singular-hyperbolic* if its singularities are hyperbolic and there is a continuous, invariant, tangent bundle decomposition $E^s \oplus E^c$ over Λ such that E^s dominates E^c, E^s is contracting and E^c is volume expanding [MPP1]. See Section 2 for precise definitions. Singular-hyperbolic sets are non-expansive partially hyperbolic sets but not conversely. We shall prove the following result.

Theorem. *All compact, non-singular, invariant subsets of a transitive singular-hyperbolic set of X are one-dimensional.*

The proof goes as follows. Let Λ be a transitive singular-hyperbolic set of X. In Lemma 1 we construct, for all $z \in \Lambda$ regular, a cross-section Σ of X containing z with some special properties. For all $x \in \Lambda \cap \Sigma$ we denote by $\mathcal{F}^s(x, \Sigma)$ the stable foliation of Λ restricted to Σ. If $T \subset \Sigma$ and $x \in T$, we denote by $(T)_x$ the connected component of T containing x. In Lemma 2 we show that if $H \subset \Lambda$ is a compact, non-singular, invariant set of X then $(H \cap \Sigma)_x$ either is $\{x\}$ or contains a non-trivial curve in $\mathcal{F}^s(x, \Sigma) \cup \partial\Sigma$. This lemma is used in Lemma 3 to show that for all $z \in H$ there is a cross-section Σ_z of X containing z in its interior such that $H \cap \Sigma_z$ is 0-dimensional. The proof uses the fact that H has unstable manifolds as it is hyperbolic [MPP2]. In Lemma 4 we prove that H can be covered by a finite collection of closed one-dimensional subsets. The proof uses the cross-section in Lemma 3 and well known properties of the topological dimension [HW]. The Theorem follows from Lemma 4.

It follows from the Theorem that a positive orbit of a three-dimensional flow accumulates a singularity if: (1) it lies in a transitive singular-hyperbolic set, and (2) the dimension of its ω-limit set is > 1. This remark has been used elsewhere [MP, p. 370]. The Theorem is false without the non-singular hypothesis. The counterexample is the geometric Lorenz attractor considered as a compact, invariant subset of itself. This leads to the question whether all the singular-hyperbolic attractors of X are two-dimensional.

2. Proof

We start with the precise definition of hyperbolic and singular-hyperbolic sets. A compact, invariant set H of a flow X is *hyperbolic* [PT] if there are a continuous, tangent bundle, invariant, splitting $T\Lambda = E^s \oplus E^X \oplus E^u$ and positive constants C, λ such that $\forall x \in H$ one has:

1. E^X_x is the direction of $X(x)$ in T_xM.
2. E^s *is contracting*: $\| DX_t(x)/E^s_x \| \leq Ce^{-\lambda t}$, $\forall t \geq 0$.
3. E^u *is expanding*: $\| DX_t(x)/E^u_x \| \geq C^{-1}e^{\lambda t}$, $\forall t \geq 0$.

A periodic orbit (or singularity) of X is hyperbolic if it is hyperbolic as a compact, invariant set of X. A hyperbolic set is *saddle-type* if $E^s \neq 0$ and $E^u \neq 0$. The most representative example of a saddle-type hyperbolic set for three-dimensional flows is the suspension of the horseshoe map in S^2 [PT].

The Invariant Manifold Theory [HPS] says that through each point $x \in H$ pass smooth injectively immersed submanifolds $W^{ss}(x), W^{uu}(x)$ tangent to E^s_x, E^u_x at x. The manifold $W^{ss}(x)$, the strong stable manifold at x, is characterized by $y \in W^{ss}(x)$ if and only if $d(X_t(y), X_t(y))$ goes to 0 exponentially as $t \to \infty$. Similarly $W^{uu}(x)$, the strong unstable manifold at x, is characterized by $y \in W^{uu}(x)$ if and only if $d(X_t(y), X_t(x))$ goes to 0 exponentially as $t \to -\infty$. These manifolds are invariant, i.e. $X_t(W^{ss}(x)) = W^{ss}(X_t(x))$ and $X_t(W^{uu}(x)) = W^{uu}(X_t(x))$, $\forall t$. For all $x, x' \in H$ we have that $W^{ss}(x)$ and $W^{ss}(x')$ either coincides or are disjoint. The maps $x \in H \to W^{ss}(x)$ and $x \in H \to W^{uu}(x)$ are continuous (in compact parts). For all $x \in H$ we define $W^s(x) = \cup_{t\in\mathbf{R}} W^{ss}(X_t(x))$ and $W^u(x) = \cup_{t\in\mathbf{R}} W^{uu}(X_t(x))$. Then, $W^s(x)$ and $W^u(x)$ are submanifolds depending continuously on x. Note that $dim(W^s(x)) = dim(E^s)+1$ and $dim(W^u(x)) = dim(E^u)+1$. Henceforth, as M is three-dimensional, both $W^s(x)$ and $W^u(x)$ are two-dimensional if H is saddle-type. If $x, x' \in H$, then W^s_x and $W^s_{x'}$ either coincide or are disjoint (similarly for W^u). For more details see [PS, p. 152] or [HPSP, p. 132].

On the other hand a compact, singular, invariant set Λ of X is *singular-hyperbolic* if its singularities are hyperbolic and satisfies the following property. There are a continuous, tangent bundle invariant, splitting $T\Lambda = E^s \oplus E^c$ and positive constants C, λ such that $\forall x \in \Lambda$ one has:

1'. E^s *dominates* E^c: $\| DX_t(x)/E^s_x \| \cdot \| DX_{-t}(X_t(x))/E^c_{X_t(x)} \| \leq Ce^{-\lambda t}$, $\forall t \geq 0$.

2'. E^s *is contracting*: $\| DX_t(x)/E^s_x \| \leq Ce^{-\lambda t}$, $\forall t \geq 0$.

3'. E^c *is volume expanding*: $\mid det(DX_t(x)/E^c_x) \mid \geq C^{-1}e^{\lambda t}$, $\forall t \geq 0$ ($det(\cdot)$ denotes the jacobian).

The singular-hyperbolic sets considered in this paper have regular orbits and satisfy $E^s \neq 0$. For three-dimensional flows every transitive singular-hyperbolic set satisfies $dim(E^s) = 1$. Examples of transitive singular-hyperbolic sets for three-dimensional flows are the geometric Lorenz attractor [GW] and the singular horseshoe [LP].

The Invariant Manifold Theory [HPS] also applies for transitive singular-hyperbolic sets Λ. Indeed, if E^s is the contracting direction of Λ, then through each $x \in \Lambda$ passes a smooth injectively immersed one-dimensional submanifold $\mathcal{F}^{ss}_x$ tangent to E^s_x at x. This manifold is invariant (i.e. $X_t(\mathcal{F}^{ss}_x) = \mathcal{F}^{ss}_{X_t(x)}$, $\forall t \in \mathbf{R}$) and characterized by $y \in \mathcal{F}^{ss}_x$ if and only if $d(X_t(x), X_t(y))$ goes exponentially to 0 as $t \to \infty$. For all $x, x' \in \Lambda$ we have that $\mathcal{F}^{ss}_x$ and $\mathcal{F}^{ss}_{x'}$ either coincides or are disjoint. The map $x \in \Lambda \to \mathcal{F}^{ss}_x$ is continuous. For all $x \in \Lambda$ we define $\mathcal{F}^s_x = \cup_{t \in \mathbf{R}} \mathcal{F}^{ss}_{X_t(x)}$. If $x \in \Lambda$ is regular then $\mathcal{F}^s_x$ is a two-dimensional submanifold. If $x, x' \in \Lambda$, then $\mathcal{F}^s_x$ and $\mathcal{F}^s_{x'}$ either coincide or are disjoint. The map $x \in \Lambda \to \mathcal{F}^s_x$ is continuous at the regular points $x \in \Lambda$.

A *cross-section* of X is a 2-disk D transverse to X. The interior and the boundary of D are denoted by $Int(D)$ and ∂D respectively. A *rectangle for X* is a cross-section D diffeomorphic to $[0,1] \times [0,1]$. In this case ∂D consists of two parts $\partial D = \partial^v D \cup \partial^h D$ in a way that $\partial^v D = D^l_v \cup D^r_v$ and $\partial^h D = D^t_h \cup D^b_h$, where the curves D^i_v ($i \in \{l, r\}$) and D^i_h ($i = t, b$) are as in Figure 1. We always assume that I_D, the image of $0 \times [0,1]$, is a curve tangent to the centre direction $TD \cap E^c$. Here TD denotes the tangent space of D.

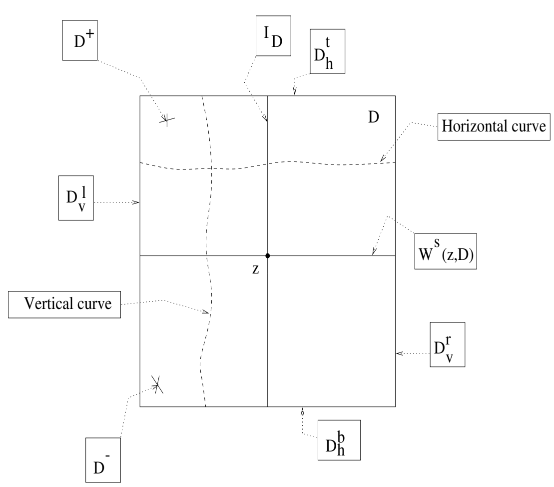

FIGURE 1

Let H be a hyperbolic set of X and D be is a cross-section intersecting H. If $x \in H \cap D$ we denote by $W^s(x, D)$ the connected component of $W^s(x) \cap D$ containing x. The continuity of $x \in H \to W^s(x)$ in compact parts implies that the map $x \in H \cap D \to W^s(x, D)$ is continuous. The family of curves $\{W^s(x, D) : x \in H \cap D\}$ is pairwise disjoint. Similarly we define the pairwise disjoint family of curves $\{W^u(x, D) : x \in H\}$ depending continuously on $x \in H$.

Let Λ be a transitive singular-hyperbolic set of X and D be a cross-section intersecting Λ. Every $x \in D$ is regular, and so, $\mathcal{F}^s_x$ is two-dimensional $\forall x \in \Lambda \cap D$. If $x \in \Lambda \cap D$ we denote by $\mathcal{F}^s(x, D)$ the connected component of $\mathcal{F}^s_x \cap D$ containing x. The continuity of $\mathcal{F}^s_x$ at the regular points implies that the map $x \in \Lambda \cap D \to \mathcal{F}^s(x, D)$ is continuous. The family $\{\mathcal{F}^s(x, D) : x \in \Lambda \cap D\}$ is pairwise disjoint. If $H \subset \Lambda$ is hyperbolic, then $W^s(x, D) = \mathcal{F}^s(x, D)$ by the Property (1') above.

For all $z \in \Lambda$ regular it is easy to find a rectangle D, $z \in Int(D)$, such that $\mathcal{F}^s(z, D)$ is a curve crossing D as in Figure 1. In this case we call a curve in D *horizontal* if it is the graph of a map $\mathcal{F}^s(z, D) \to I_D$ transverse to $\partial^u D$. Similarly, we call a curve in D *vertical* if it is the graph of a map $I_D \to \mathcal{F}^s(z, D)$ transverse to $\partial^h D$.

Lemma 1. *Let Λ be a transitive singular-hyperbolic set of a three-dimensional flow X. Then, for every regular point $z \in \Lambda$ there is a rectangle Σ of X such that:*

(1) $z \in Int(\Sigma)$.

(2) $\mathcal{F}^s(x, \Sigma)$ is a horizontal curve for all $x \in \Lambda \cap \Sigma$.

(3) If $\Lambda \cap \Sigma^t_h \neq \emptyset$, then $\Sigma^t_h = \mathcal{F}^s(y, \Sigma)$ for some $y \in \Lambda \cap \Sigma$.

(4) If $\Lambda \cap \Sigma^b_h \neq \emptyset$, then $\Sigma^b_h = \mathcal{F}^s(y, \Sigma)$ for some $y \in \Lambda \cap \Sigma$.

Proof. Let z be a regular point of Λ. As remarked before there is a cross-section D, $z \in Int(D)$, such that $\mathcal{F}^s(z, D)$ is a horizontal curve as in Figure 1. In particular $\mathcal{F}^s(z, D)$ divides D in two subrectangles D^+, D^-. Shrinking D if necessary we can assume that $\partial^v D$ is tangent to the centre direction $TD \cap E^c$. The proof breaks into the following cases.

Case I: z is isolated $\Lambda \cap D$. In this case $\Sigma = D$ and z satisfy the desired properties.

Case II: $\Lambda \cap D \subset \mathcal{F}^s(z, D) \cup \partial D$. Shrinking D if necessary we can assume that $\Lambda \cap D \subset \mathcal{F}^s(z, D)$. In particular $\Lambda \cap \partial^h D = \emptyset$. We define $\Sigma = D$. Clearly $z \in Int(\Sigma)$ and so the Property (1) of Lemma 1 holds. If $x \in \Lambda \cap \Sigma$ then $\mathcal{F}^s(x, D) = \mathcal{F}^s(z, D)$ since $\Lambda \cap D \subset \mathcal{F}^s(z, D)$, and so, the Property (2) of the lemma holds. The remainer properties follow from $\Lambda \cap \partial^h \Sigma = \emptyset$.

Case III: $\Lambda \cap Int(D^+) = \emptyset$ *or* $\Lambda \cap Int(D^-) = \emptyset$. By Case I we can assume that z is not isolated in $\Lambda \cap D$. We construct Σ only if $\Lambda \cap Int(D^-) = \emptyset$ since the remainer construction is similar. As z is not isolated and $\Lambda \cap Int(D^-) = \emptyset$ we can choose $w \in Int(D^+) \cap \Lambda$ arbitrarily close to z. As $x \in \Lambda \cap D \to \mathcal{F}^s(x, D)$ is continuous it follows that $\mathcal{F}^s(w, D)$ is a well defined horizontal curve close to $\mathcal{F}^s(z, D)$. In particular $\mathcal{F}^s(z, D), \mathcal{F}^s(w, D)$ form the horizontal boundary of a non-trivial rectangle R in D, i.e. $\partial^h R = \{\mathcal{F}^s(z,D), \mathcal{F}^s(w,D)\}$. We define $\Sigma = R \cup D^-$. Then, $\partial^h \Sigma = \{D_h^b, \mathcal{F}^s(w,D)\}$. As $z \in Int(R)$ by construction we have that the Property (1) of the lemma holds. Shrinking D^-, by the continuity of $x \in \Lambda \cap D \to \mathcal{F}^s(x,D)$, we have that if $x \in \Lambda \cap \Sigma$, then $\mathcal{F}^s(x,\Sigma)$ is close to $\mathcal{F}^s(z,\Sigma) = \mathcal{F}^s(z,D)$. This implies the Property (2) of the Lemma. The properties (3)-(4) of the lemma follows from $\partial^h \Sigma = \{D_h^b, \mathcal{F}^s(w,D)\}$ and $D_h^b \cap \Lambda = \emptyset$ since $\Lambda \cap Int(D^-) = \emptyset$.

Case IV: $\Lambda \cap Int(D^+) \neq \emptyset$ *and* $\Lambda \cap Int(D^-) \neq \emptyset$. We still assume that z is not isolated in $\Lambda \cap D$. If z is not accumutated by points in $\Lambda \cap D^-$, then we fall in Case III by shrinking D if necessary. Similarly if z is not accumulated by points in $\Lambda \cap D^+$. So, without loss of generality, we can assume that there are points $w^+ \in \Lambda \cap D^+$ and $w^- \in \Lambda \cap D^-$ arbitrarily close to z. Again the continuity of $x \in \Lambda \cap D \to \mathcal{F}^s(x,D)$ implies that $\mathcal{F}^s(w^+,D)$ and $\mathcal{F}^s(w^-,D)$ are horizontal curves close to $\mathcal{F}^s(z,D)$. It follows that $\mathcal{F}^s(w^+,D), \mathcal{F}^s(w^-,D)$ form the horizontal boundary of a non-trivial rectangle R in D, i.e $\partial^h R = \{\mathcal{F}^s(w^+,D), \mathcal{F}^s(w^-,D)\}$. We define $\Sigma = R$. The Property (1) of the lemma follows since $z \in Int(R)$ by construction. If $x \in \Lambda \cap \Sigma$, then $\mathcal{F}^s(x,\Sigma)$ is close to $\mathcal{F}^s(z,\Sigma)$ by continuity. This proves the Property (2) of the lemma. The properties (3)-(4) of the lemma follows as in the previous case using that $\partial^h R = \{\mathcal{F}^s(w^+,D), \mathcal{F}^s(w^-,D)\}$. $\square$

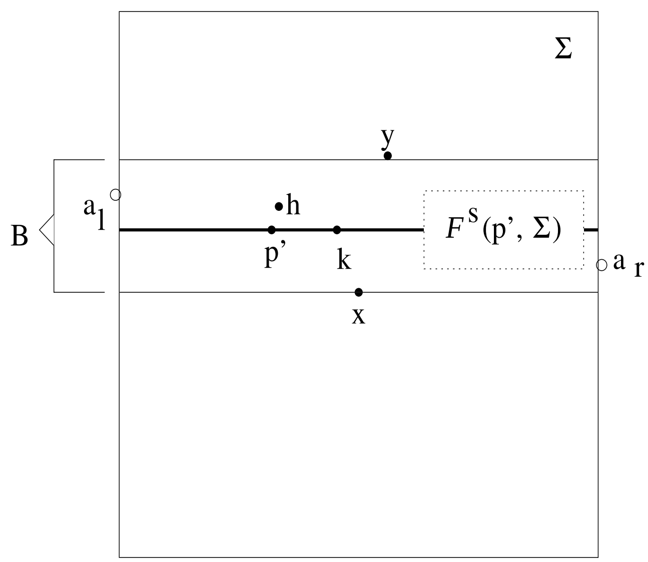

FIGURE 2

Given a cross section Σ, $T \subset \Sigma$ and $x \in T$ we denote by $(T)_x$ the connected component of T containing x.

Lemma 2. *Let* Λ *be a transitive singular hyperbolic set of a three-dimensional flow* X*. Let* $H \subset \Lambda$ *be a compact, non-singular, invariant set of* X*. For* $z \in H$ *we let* Σ *be the* *rectangle of X as in Lemma 1. If $x \in H \cap \Sigma$ and $(H \cap \Sigma)_x \neq \{x\}$, then $(H \cap \Sigma)_x$ contains a non-trivial curve in $\mathcal{F}^s(x, \Sigma) \cup \partial\Sigma$.*

Proof. Fix $z \in H$ and let Σ be the rectangle in Lemma 1 for z (it is well defined since z is regular). Let us prove that Σ satisfies the conclusion of the lemma. By contradiction assume that it is not so, i.e. $\exists x \in H \cap \Sigma$ such that:

(i) $(H \cap \Sigma)_x \neq \{x\}$ and

(ii) $(H \cap \Sigma)_x$ does not contain non-trivial curves in $\mathcal{F}^s(x, \Sigma) \cup \partial\Sigma$.

Firstly we prove $(H \cap \Sigma)_x \cap (Int(\Sigma) \setminus \mathcal{F}^s(x, \Sigma)) \neq \emptyset$. By (i) we can choose $y \in (H \cap \Sigma)_x \setminus \{x\}$. Suppose by contradiction that $(H \cap \Sigma)_x \cap (Int(\Sigma) \setminus \mathcal{F}^s(x, \Sigma)) = \emptyset$. Then, $(H \cap \Sigma)_x \subset \mathcal{F}^s(x, \Sigma) \cup \partial\Sigma$. Since the later set is a one-dimensional manifold and $(H \cap \Sigma)_x$ is connected, it would exist a curve in $\mathcal{F}^s(x, \Sigma) \cup \partial\Sigma$ joining x to y. Such a curve is non-trivial as $x \neq y$. This is a contradiction by (ii) and the result follows.

By the previous observation there is $y \in (H \cap \Sigma)_x \cap (Int(\Sigma) \setminus \mathcal{F}^s(x, \Sigma))$. Clearly $y \in \Lambda \cap \Sigma$ and so, by (2) in Lemma 1, $\mathcal{F}^s(y, \Sigma)$ is a horizontal curve in Σ. Because $y \notin \mathcal{F}^s(x, \Sigma)$ we have that $\mathcal{F}^s(y, \Sigma)$ and $\mathcal{F}^s(x, \Sigma)$ are disjoint. Then, there is a rectangle $B \subset \Sigma$ of X such that $\partial^h B = \mathcal{F}^s(y, \Sigma) \cup \mathcal{F}^s(x, \Sigma)$. See Figure 2.

Now we prove $H \cap Int(B) \neq \emptyset$. By (ii) the interior of the two vertical curves forming $\partial^v B$ are *not* contained in $(H \cap \Sigma)_x$. This allows us to choose $a_l, a_r \notin (H \cap \Sigma)_x$ in each vertical curve (see Figure 2). Clearly $Int(B) \cup \{a_l, a_r\}$ separates Σ in two connected componens one of which containing x and the other one containing y. It would follow that x, y belong to different connected components of $H \cap \Sigma$, a contradiction. This proves the result.

Finally we get the contradiction as follows. By the previous observation we can choose $h \in H \cap Int(B)$. Let $p \in \Lambda$ be such that $\Lambda = \omega_X(p)$. Because $h \in \Lambda$ we have that $h \in \omega_X(p)$. By the Tubular Flow-Box Theorem (recall that Σ is transverse) there is $p' \in \Sigma$ in the positive orbit of p arbitrarily close to h. In particular, $p' \in Int(B)$ as in Figure 2. Clearly $p' \in \Lambda \cap \Sigma$ and so, by (2) in Lemma 1, $\mathcal{F}^s(p', \Sigma)$ is a horizontal curve in $Int(B)$. Note that $\mathcal{F}^s(p', \Sigma)$ separates Σ in two connected components containing y and x respectively. See Figure 2. Since x, y are in the same connected component of $H \cap \Sigma$ we conclude that there is $k \in \mathcal{F}^s(p', \Sigma) \cap H$. By the definition of $\mathcal{F}^s(p', \Sigma)$ it follows that k and p' have the same ω-limit set. Since p' is in the orbit of p we conclude that $\omega_X(k) = \omega_X(p)$. But $k \in H$, H is compact and $\omega_X(p) = \Lambda$. It follows that $H = \Lambda$ and so H contains a singularity of X since Λ does. Because H has no singularities we get the contradiction. This contradiction proves that Σ satisfies the conclusion of the lemma. □

Lemma 3. *Let Λ be a transitive singular-hyperbolic set of a three-dimensional flow X. Let $H \subset \Lambda$ be a compact, non-singular, invariant set of X. Then, for every $z \in H$ there* *is a cross-section Σ_z of X such that $z \in Int(\Sigma_z)$ and $H \cap \Sigma_z$ is 0-dimensional.*

Proof. Fix $z \in H$ and let $\Sigma = \Sigma_z$ be the rectangle in Lemma 2. Let us prove that Σ satisfies the conclusion of Lemma 3. As Σ comes from Lemma 1 we have that $z \in Int(\Sigma)$. It remains to prove that $H \cap \Sigma$ is 0-dimensional. Since $H \cap \Sigma$ is compact it suffices to show that $H \cap \Sigma$ is totally disconnected, namely $(H \cap \Sigma)_x = \{x\}$ for all $x \in \Sigma \cap H$ (see [HW, (D) p. 22]). Assume by contradiction that it is not so. Then, there is $x \in H \cap \Sigma$ such that $(H \cap \Sigma)_x \neq \{x\}$. By Lemma 2 we have that $(H \cap \Sigma)_x$ contains a non-trivial curve $J \subset \mathcal{F}^s(x, \Sigma) \cup \partial\Sigma$. We can assume either $J \subset \mathcal{F}^s(x, \Sigma)$ or $J \subset \partial\Sigma$ without loos of generality. Then the proof breaks into the following cases.

Case 1: $J \subset \mathcal{F}^s(x, \Sigma)$. As H has no singularities we have that H is a saddle-type hyperbolic set [MPP2]. So, $W^u(x, \Sigma)$ is well defined $\forall x \in H \cap \Sigma$ and $x \in H \to W^u(x, \Sigma)$ is continuous. Because of this continuity we can fix an open interval $K \subset J$ with the following property : The union $\cup_{y \in J} W^u(y, \Sigma)$ covers a vertical band D as in Figure 3.

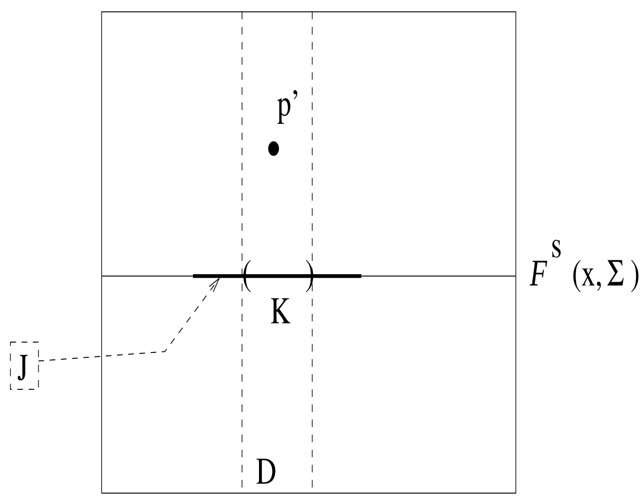

FIGURE 3

Because Λ is transitive there is $p \in \Lambda$ such that $\omega_X(p) = \Lambda$. As $K \subset J \subset H \subset \Lambda$, there is a point $p' \in Int(D)$ in the positive orbit of p as in Figure 3. It follows that $p' \in W^u(y, \Sigma)$ for some $y \in J$ with the distance $d(p', y)$ being small. Then, by taking the negative orbit of $y \in H$, it follows that p is arbitrarily close to H. Because H is closed we would have that $p \in H$, and so, $\Lambda = H$ a contradiction since H has no singularities. This yields the desired contradiction in Case 1.

Case 2: $J \subset \partial\Sigma$. In this case we can assume that either $J \subset \partial^h\Sigma$ or $J \subset \partial^v\Sigma$ holds. If $J \subset \partial^h\Sigma$, then we fix $y \in J$ and a rectangle Σ' around y' and close enough to y' such that

Σ' is foliated by $\{\mathcal{F}^s(y,\Sigma') : y \in J\}$. The existence of y', Σ' satisfying these hypothesis can be seen by enlarging the rectangle Σ a bit and choosing y' in the interior of the resulting rectangle. Afterward we proceed as in the previous case replacing y and Σ by y' and Σ' respectively. Now assume that $J \subset \partial^v\Sigma$. Fix $y' \in J$ and let Σ' be a small rectangle around y' and close to y' enough such that Σ' is foliated by $\{\mathcal{F}^s(y,\Sigma') : y \in J\}$. As before, the existence of y', Σ' can be seem by enlarging Σ a bit and choosing y' in the interior of the resulting rectangle. Because Λ is transitive there is $p \in \Lambda$ such that $\Lambda = \omega_X(p)$. Clearly $y' \in \Lambda$, and so, there is $p' \in \Sigma'$ in the positive orbit of p close to y'. Since Σ' is foliated by stable manifolds we have that $p' \in \mathcal{F}^s(y,\Sigma')$ for some $y \in J$. As $J \subset H$ we conclude as before that $\Lambda = H$. As H has no singularities we obtain the contradiction in Case 2. $\square$

Lemma 4. *Let Λ be a transitive singular-hyperbolic set of a three-dimensional flow X. If H is a compact, non-singular, invariant set of X, then H can be covered by a finite collection of closed one-dimensional subsets.*

Proof. If $z \in H$ we let Σ_z be the cross-section in Lemma 3. It follows that $z \in Int(\Sigma_z)$ and $H \cap \Sigma_z$ is 0-dimensional $\forall z \in H$. For $\delta = \delta_z > 0$ sufficiently small the map $(y,t) \to \Sigma_z \times [-\delta,\delta] \to X_t(y)$ is a diffeomorphism onto a compact neighborhood U_z of z. Since H is invariant we have that this diffeomorphism carries $(H \cap \Sigma_z) \times [-\delta,\delta]$ precisely into $H \cap U_z$. In particular, $(H \cap \Sigma_z) \times [-\delta,\delta]$ and $H \cap U_z$ are homeomorphic.

On the other hand, $z \in Int(U_z)$ since $z \in Int(\Sigma_z)$. It follows that the family $\{Int(U_z) : z \in H\}$ is an open covering of H. By compacity there are $z_1, \cdots, z_r \in H$ such that $H \subset \cup_{i=1}^r Int(U_{z_i})$, and so, $H = \cup_{i=1}^r (H \cap U_{z_i})$. For all i we have that $H \cap \Sigma_{z_i}$ is 0-dimensional. It follows that $H \cap U_z$ is at most one-dimensional since it is homeomorphic to $(H \cap \Sigma_{z_i}) \times [-\delta,\delta]$ ([HW, Theorem III 4 p. 33]). This proves that $\{H \cap U_{z_i} : i = 1, \cdots, r\}$ is a finite collection of closed one-dimensional subsets covering H. The result is proved. $\square$

Proof of the Theorem. Let X be a smooth flow on a closed 3-manifold M. Let Λ be a transitive singular-hyperbolic set of X. Let $H \subset \Lambda$ be a compact, non-singular, invariant set of X. By Lemma 4 we have that H can be covered by a finite collection of closed one-dimensional subsets. It follows from [HW, Theorem III 2 p. 30] that H is at most one-dimensional. Because H is formed by regular orbits we have that H is at least one-dimensional. It follows that H is one-dimensional and the result follows. $\square$

References

[B1] Bowen, R., *Markov partitions and minimal sets for Axiom A diffeomorphisms*, Amer. J. Math. **92** (1970), 907-918.

[B2] Bowen, R., *Symbolic dynamics for hyperbolic flows*, Amer. J. Math. **95** (1973), 429-460.

[GW] Guckenheimer, J., Williams, R. F., *Structural stability of Lorenz attractors*, Inst. Hautes Études Sci. Publ. Math. **50** (1979), 59-72.

[H] Hirsch, M., W., *On invariant subsets of hyperbolic sets*, Essays on Topology and Related Topics, Mémoires dédiés á Georges de Rham (1970), 126-135.

[HPS] Hirsch, M., W., Pugh, C., C., Shub, M., *Invariant manifolds*, Lecture Notes in Mathematics **583** (1977), Springer-Verlag, Berlin-New York.

[HPSP] Hirsch, M., Palis, J., Pugh, C., Shub, M., *Neighborhoods of hyperbolic sets.*, Invent. Math. **9** (1969/1970), 121-134.

[KS] Keynes, H., B., Sears, M., *Real-expansive flows and topological dimension*, Ergodic Theory Dynamical Systems **1** (1981), 179-195.

[LP] Labarca, R., Pacifico, M., J., *Stability of singular horseshoes*, Topology **25** (1986), 337-352.

[M] Mañé, R., *Expansive homeomorphisms and topological dimension*, Trans. Amer. Math. Soc. **252** (1979), 313-319.

[MP] Morales, C., A., Pacifico, M., J., *Mixing attractors for 3-flows*, Nonlinearity **14** (2001), 359-378.

[MPP1] Morales, C., A., Pacifico, M., J., Pujals, E., R., *On* C^1 *robust singular transitive sets for three-dimensional flows*, C. R. Acad. Sci. Paris Sér. I Math. **326** (1998), 81-86.

[MPP2] Morales, C., A., Pacifico, M., J., Pujals, E., R., *Singular-hyperbolic systems*, Proc. Amer. Math. Soc. **127** (1999), 3393-3401.

[PT] Palis, J., Takens, F., *Hyperbolicity and sensitive chaotic dynamics at homoclinic bifurcations. Fractal dimensions and infinitely many attractors*, Cambridge Studies in Advanced Mathematics **35** (1993) Cambridge University Press, Cambridge.

[PS] Pugh, C., Shub, M., *The* Ω*-stability theorem for flows*, Invent. Math. **11** (1970), 150-158.

[P] Przytycki, F., *Construction of invariant sets for Anosov diffeomorphisms and hyperbolic attractors*, Studia Math. **68** (1980), 199-213.

[HW] Hurewicz, W., Wallman, H., *Dimension Theory*, Princeton University Press, Princeton, NJ, (1984).

C.A.M.

Instituto de Matematica

Universidade Federal do Rio de Janeiro

C. P. 68.530, CEP 21.945-970, Rio de Janeiro, Brazil

e-mail: *morales@impa.br*