# Differential simplicity in polynomial rings and algebraic independence of power series.

**Paulo Brumatti**[1] **Yves Lequain**[2]
**Daniel Levcovitz**[3]

**Abstract**

Let $k$ be a field of characteristic zero, $f(X,Y), g(X,Y) \in k[X,Y]$, $g(X,Y) \notin (X,Y)$ and $d := g(X,Y)\frac{\partial}{\partial X} + f(X,Y)\frac{\partial}{\partial Y}$. We establish a connection between the $d$-simplicity of the local ring $k[X,Y]_{(X,Y)}$ and the transcendency of the solution in $tk[[t]]$ of the algebraic differential equation $g\big(t, y(t)\big).\frac{\partial}{\partial t}y(t) = f\big(t, y(t)\big)$. We use this connection to obtain some interesting results in the theory of the formal power series and to construct new examples of differentially simple rings.

## Introduction

Let $k$ be a field of characteristic zero, $k[x_1, \cdots, x_n]$ a domain that is a finitely generated $k$-algebra, $\mathfrak{D}$ its module of $k$-derivations and $M$ a maximal ideal of $k[x_1, \cdots, x_n]$. A. Seidenberg has shown in [6] that the domain $k[x_1, \cdots, x_n]$ is regular if only if it is $\mathfrak{D}$-simple; later on, R. Hart has shown in [3] that, when $k[x_1, \cdots, x_n]$ is $\mathfrak{D}$-simple, there need not exist $d \in \mathfrak{D}$ such that $k[x_1, \cdots, x_n]$ be $d$-simple. For the local ring $R := k[x_1, \cdots, x_n]_M$, the situation is better: Hart has shown that if $R$ is regular, then there always exists a $k$-derivation $d$ of $R$ such that $R$ be $d$-simple. Since Seidenberg had already proved the converse, one has that the existence of a $k$-derivation $d$ of $R$ that makes $R$ $d$-simple is a characterization of the property of regularity for the local ring $R$.

In view of this geometric interpretation of the $d$-simplicity of a ring of the type $R := k[x_1, \cdots, x_n]_M$, it is natural to investigate the conditions under which a derivation $d$ of $R$ will make $R$ $d$-simple. In this paper, we will do that in the particular case of $R$ to be the localization of a polynomial ring in finitely many indeterminates. In this introduction, in order to avoid technicalities, we shall present our results in the special case of a polynomial ring in two variables.

Let $f(X,Y), g(X,Y)$ be two polynomials in $k[X,Y]$, $g(X,Y)$ not in the maximal ideal $(X,Y)$. On the local ring $k[X,Y]_{(X,Y)}$, consider the derivation $d := g(X,Y)\frac{\partial}{\partial X} + f(X,Y)\frac{\partial}{\partial Y}$.

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In section 1 of the paper, we show that $k[X,Y]_{(X,Y)}$ is $d$-simple if and only if the unique solution $y(t) \in tk[[t]]$ of the algebraic differential equation $(*)$: $g\big(t, y(t)\big).\frac{\partial}{\partial t} y(t) = f\big(t, y(t)\big)$ is transcendental over $k(t)$. Using this connection between $d$-simplicity of the local ring $k[X,Y]_{(X,Y)}$ and the transcendency of the solution $y(t) \in tk[[t]]$ of the equation $(*)$, we get some applications in the theory of formal power series: if neither $f(X,Y)$ nor $g(X,Y)$ belong to $(X,Y)$, we obtain in Theorem 1.5 that the solution $y(t) \in tk[[t]]$ of the equation

$$(*) \quad g\big(t, y(t)\big).\frac{\partial}{\partial t} y(t) = f\big(t, y(t)\big)$$

is transcendental over $k(t)$ if and only if the solution $x(t) \in tk[[t]]$ of the equation

$$(**) \quad f\big(x(t), t\big).\frac{\partial}{\partial t} x(t) = g\big(x(t), t\big)$$

is transcendental over $k(t)$. This is interesting: on one hand it may be difficult (or even impossible) to decide whether the solution of one of the equations is, or is not, transcedental over $k(t)$, using the classical methods; on the other hand, the same classical methods may easily give the solution for the other equation. In section 2 of the paper, we give three such examples.

In section 3, also using the connection established in section 1, we construct several families of differentially simple rings. For example, in Proposition 3.3, we show that $k[X,Y]_{(X,Y)}$ is $d$-simple if we take $d = \frac{\partial}{\partial X} + (Y^n + 1)\frac{\partial}{\partial Y}$ with $n \geq 1$. In Proposition 3.4 we show that $k[Y_1, \cdots, Y_r]_{(Y_1, \cdots, Y_r)}$ is $d$-simple if we take $d = \frac{\partial}{\partial Y_1} + \sum_{i=1}^{r} (1+Y_1)(1+Y_3)\cdots(1+Y_{i-1})\frac{\partial}{\partial Y_i}$.

In section 4, we study the ring $k[X,Y]$ endowed with the derivation $d := \frac{\partial}{\partial X} + (a(X)Y + b(X))\frac{\partial}{\partial Y}$ with $a(X), b(X) \in k[X]$. With the help of results on power series obtained in section 2, we complement and deepen the theorem of Shamsuddin in [7] which asserts that $k[X,Y]$ is $d$-simple if only if there does not exist any polynomial $h(X) \in k[X]$ such that $h'(X) = a(X)h(X) + b(X)$.

Let $R$ be a ring and $d$ a derivation of $R$. An ideal $I$ of $R$ is a $d$-ideal if $d(I) \subseteq I$. The ring $R$ is $d$-simple if $(0)$ and $R$ are the only $d$-ideals of $R$. The ring $R$ is differentially simple if for every ideal $I \neq (0), R$, there exists a derivation $d$ of $R$ such that $d(I) \not\subseteq I$.

# 1 Differential simplicity and algebraic independence

The objective of this section is to establish a connection between the local differential simplicity in a ring of polynomials and the algebraic independence of power series that are solutions of a certain associated system of algebraic differential equations:

**Theorem 1.1.** *Let $k$ be a field of characteristic zero and $r$ a positive integer.Let $(\alpha, \underline{\beta}) := (\alpha, \beta_1, \cdots, \beta_r) \in k^{r+1}$ and $t$, $X$, $\underline{Y} := \{Y_1, \cdots, Y_r\}$ be some indeterminates over $k$. Let $g(X,\underline{Y}), f_1(X,\underline{Y}), \cdots, f_r(X,\underline{Y}) \in k[X, Y_1, \cdots, Y_r]$, $g(X,\underline{Y}) \notin (X-\alpha, Y_1-\beta_1, \cdots Y_r-\beta_r)$.*

*Over the local ring $k[X, Y_1, \cdots, Y_r]_{(X-\alpha, Y_1-\beta_1, \cdots Y_r-\beta_r)}$, consider the derivation*

$$d := g(X,\underline{Y})\frac{\partial}{\partial X} + \sum_{i=1}^{r} f_i(X,\underline{Y})\frac{\partial}{\partial Y_i}$$

*and over $k[[t]]$, consider the system of differential equations*

$$(*.\alpha) \qquad \left\{ g\big(t+\alpha, y_1(t), \cdots, y_r(t)\big).\frac{\partial}{\partial t}(y_i(t)) = f_i\big(t+\alpha, y_1(t), \cdots, y_r(t)\big) \right\}_{i=1}^{r}.$$

*Then,*

**(a)** *The system of equations $(*.\alpha)$ has a unique solution $y_1(t,\underline{\beta}), \cdots, y_r(t,\underline{\beta}) \in k[[t]]$ such that $y_1(0,\underline{\beta}) = \beta_1, \cdots, y_r(0,\underline{\beta}) = \beta_r$,*

**(b)** *The height of the biggest $d-$ideal of $k[X, Y_1, \cdots, Y_r]_{(X-\alpha, Y_1-\beta_1, \cdots Y_r-\beta_r)}$ is equal to*

$$r - trdeg_{k(t)}\big(k\big(y_1(t,\underline{\beta}), \cdots, y_r(t,\underline{\beta})\big)\big),$$

**(c)** *The ring $k[X, Y_1, \cdots, Y_r]_{(X-\alpha, Y_1-\beta_1, \cdots Y_r-\beta_r)}$ is $d$- simple if and only if $y_1(t,\underline{\beta}), \cdots, y_r(t,\underline{\beta})$ are algebraically independent over $k(t)$.*

**Remark 1.2.** Let $k$ be a field of characteristic zero, $r$ a positive integer, $X, Y_1, \cdots, Y_r$ some indeterminates over $k$ and $d$ a derivation of $k[X, Y_1, \cdots, Y_r]$. Let $g(X, Y_1, \cdots, Y_r) := d(X)$ and $f_i(X, Y_1, \cdots, Y_r) := d(Y_i)$ for $i = 1, \cdots, r$. Let $(\alpha, \beta_1, \cdots, \beta_r) \in k^{r+1}$, $R$ the local ring $k[X, Y_1, \cdots, Y_r]_{(X-\alpha, Y_1-\beta_1, \cdots Y_r-\beta_r)}$ and $m$ its maximal ideal. Let the natural extension of $d$ to the ring $R$ be also denoted by $d$.

**i)** If $R$ is $d-$simple, then it is clear that $d(X-\alpha) \notin m$ or $d(Y_i-\beta_i) \notin m$ for some $i$; without loss of generality, we may suppose that $d(X-\alpha) \notin m$, i.e., that $g(X, Y_1, \cdots, Y_r) \notin (X-\alpha, Y_1-\beta_1, \cdots Y_r-\beta_r)$. This is the hypothesis we make in Theorem 1.1.

**ii)** The results **(b)** and **(c)** of Theorem 1.1 can be reformulated in the following way:

**(b')** The height of the biggest $d-$ideal of $k[X, Y_1, \cdots, Y_r]$ contained in $(X-\alpha, Y_1-\beta_1, \cdots, Y_r-\beta_r)$ is equal to $r - trdeg_{k(t)}\big(k\big(y_1(t,\underline{\beta}), \cdots, y_r(t,\underline{\beta})\big)\big)$.

**(c')** The biggest $d-$ideal of $k[X, Y_1, \cdots, Y_r]$ contained in $(X-\alpha, Y_1-\beta_1, \cdots, Y_r-\beta_r)$ is equal to $(0)$ if and only if $y_1(t,\underline{\beta}), \cdots, y_r(t,\underline{\beta})$ are algebraically independent over $k(t)$.

Theorem 1.1 will be obtained as a consequence of several auxiliary results.

**Proposition 1.3.** *Let $k$, $r$, $(\alpha, \underline{\beta})$, $g(X, \underline{Y})$, $f_1(X, \underline{Y}), \cdots, f_r(X, \underline{Y})$ and $d$ be as in Theorem 1.1; let $D := \frac{1}{g(X,\underline{Y})} d$.*

*Let $e^{tD}_{\alpha,\underline{\beta}} : k[X, Y_1, \cdots, Y_r]_{(X-\alpha, Y_1-\beta_1, \cdots Y_r-\beta_r)} \longrightarrow k[[t]]$ be the map defined by*

$$e^{tD}_{\alpha,\underline{\beta}}(\xi) = \sum_{i\geq 0} \frac{D^i(\xi)}{i!}(\alpha, \underline{\beta})t^i.$$

*Let $\wp_{\alpha,\underline{\beta}}$ be the kernel of $e^{tD}_{\alpha,\underline{\beta}}$. Then,*

**(a)** *$e^{tD}_{\alpha,\underline{\beta}}$ is a homomorphism of $k$-algebras such that*

$$e^{tD}_{\alpha,\underline{\beta}}(X) = \alpha + t, \quad e^{tD}_{\alpha,\underline{\beta}}(Y_i) \in \beta_i + tk[[t]] \text{ for every } i = 1, \cdots, r,$$

**(b)** $\frac{\partial}{\partial t} \circ e^{tD}_{\alpha,\underline{\beta}} = e^{tD}_{\alpha,\underline{\beta}} \circ D$,

**(c)** *$\wp_{\alpha,\underline{\beta}}$ is the biggest $D-$ideal of $k[X, Y_1, \cdots, Y_r]_{(X-\alpha, Y_1-\beta_1, \cdots Y_r-\beta_r)}$,*

**(d)** *height of $\wp_{\alpha,\underline{\beta}} = r - trdeg_{k(t)} k\big(e^{tD}_{\alpha,\underline{\beta}}(Y_1), \cdots, e^{tD}_{\alpha,\underline{\beta}}(Y_r)\big)$,*

**(e)** *$k[X, Y_1, \cdots, Y_r]_{(X-\alpha, Y_1-\beta_1, \cdots Y_r-\beta_r)}$ is $D-$simple if and only if $e^{tD}_{\alpha,\underline{\beta}}$ is injective, if and only if $e^{tD}_{\alpha,\underline{\beta}}(Y_1), \cdots, e^{tD}_{\alpha,\underline{\beta}}(Y_r)$ are algebraically independent over $k(t)$.*

**Proof. (a)** It is clear that $e^{tD}_{\alpha,\underline{\beta}}$ leaves every element of $k$ fixed.
Let $\xi, \eta \in k[X, Y_1, \cdots, Y_r]_{(X-\alpha, Y_1-\beta_1, \cdots Y_r-\beta_r)}$. For every $n \geq 0$, we have

$$D^n(\xi\eta) = \sum_{i=0}^n C_n^i D^i(\xi) D^{n-i}(\eta), \quad \text{where } C_n^i = \frac{n!}{i!(n-i)!},$$

hence $\frac{D^n(\xi\eta)}{n!} = \sum_{i=0}^n \frac{C_n^i}{n!} D^i(\xi) D^{n-i}(\eta) = \sum_{i=0}^n \frac{D^i(\xi)}{i!} \frac{D^{n-i}(\eta)}{(n-i)!}$.

Then, $e^{tD}_{\alpha,\underline{\beta}}(\xi\eta) = \sum_{n\geq 0} \frac{D^n(\xi\eta)}{n!}(\alpha, \underline{\beta})t^n = \sum_{n\geq o}(\sum_{i=0}^n \frac{D^i(\xi)}{i!}(\alpha, \underline{\beta})t^i . \frac{D^{n-i}(\eta)}{(n-i)!}(\alpha, \underline{\beta})t^{n-i}) =$

$(\sum_{i\geq 0} \frac{D^i(\xi)}{i!}(\alpha, \underline{\beta})t^i).(\sum_{i\geq 0} \frac{D^i(\eta)}{i!}(\alpha, \underline{\beta})t^i) = e^{tD}_{\alpha,\underline{\beta}}(\xi).e^{tD}_{\alpha,\underline{\beta}}(\eta)$.

Finally it is clear that $e^{tD}_{\alpha,\underline{\beta}}(X) = \alpha + t$ and that $e^{tD}_{\alpha,\underline{\beta}}(Y_i) \in \beta_i + tk[[t]]$ for every $i = 1, \cdots, r$.

**(b)** Let $\xi \in k[X, Y_1, \cdots, Y_r]_{(X-\alpha, Y_1-\beta_1, \cdots Y_r-\beta_r)}$. We have

$$e^{tD}_{\alpha,\underline{\beta}} \circ D(\xi) = \sum_{i\geq 0} \frac{D^i(D(\xi))}{i!}(\alpha,\underline{\beta})t^i = \sum_{i\geq 0} \frac{D^{i+1}(\xi)}{i!}(\alpha,\underline{\beta})t^i =$$

$$\sum_{j\geq 1} \frac{D^j(\xi)}{(j-1)!}(\alpha,\underline{\beta})t^{(j-1)} = \frac{\partial}{\partial t}(\sum_{j\geq 0} \frac{D^j(\xi)}{j!}(\alpha,\underline{\beta})t^j) = \frac{\partial}{\partial t} \circ e^{tD}_{\alpha,\underline{\beta}}(\xi).$$

**(c)** By part **(b)** it is clear that $\wp_{\alpha,\underline{\beta}}$ is a $D-$ideal of $R := k[X, Y_1, \cdots, Y_r]_{(X-\alpha, Y_1-\beta_1, \cdots Y_r-\beta_r)}$. Now, let $J \subseteq m := (X-\alpha, Y_1-\beta_1, \cdots Y_r-\beta_r)R$ be a $D-$ideal of $R$. Then $e^{tD}_{\alpha,\underline{\beta}}(J) \subseteq e^{tD}_{\alpha,\underline{\beta}}(m) \subseteq tk[[t]]$ by part **(a)**; thus the ideal $< e^{tD}_{\alpha,\underline{\beta}}(J) >$ of $k[[t]]$ generated by $e^{tD}_{\alpha,\underline{\beta}}(J)$ is contained in $tk[[t]]$. On the other hand, since $J$ is a $D-$ideal, we have by part **(b)**,

$$\frac{\partial}{\partial t}(e^{tD}_{\alpha,\underline{\beta}}(J)) = e^{tD}_{\alpha,\underline{\beta}}(D(J)) \subseteq e^{tD}_{\alpha,\underline{\beta}}(J) \ \text{, hence } \frac{\partial}{\partial t}(< (e^{tD}_{\alpha,\underline{\beta}}(J)) >) \subseteq < (e^{tD}_{\alpha,\underline{\beta}}(J)) >.$$

Thus $< (e^{tD}_{\alpha,\underline{\beta}}(J)) >$ is a proper $\frac{\partial}{\partial t}-$ideal of $k[[t]]$. Since $k[[t]]$ is clearly $\frac{\partial}{\partial t}-$simple, this implies that $< (e^{tD}_{\alpha,\underline{\beta}}(J)) >= (0)$ and therefore that $J \subseteq ker(e^{tD}_{\alpha,\underline{\beta}}) = \wp_{\alpha,\underline{\beta}}$. Thus $\wp_{\alpha,\underline{\beta}}$ is the biggest $D-$ideal of $R$.

**(d)** We have

$$\begin{aligned}
\text{height of } \wp_{\alpha,\underline{\beta}} &= \text{height of } (\wp_{\alpha,\underline{\beta}} \cap k[X, Y_1, \cdots, Y_r]) \\
&= r + 1 - dim \frac{k[X, Y_1, \cdots, Y_r]}{\wp_{\alpha,\underline{\beta}} \cap k[X, Y_1, \cdots, Y_r]} \\
&= r + 1 - dim\ k[\alpha + t, e^{tD}_{\alpha,\underline{\beta}}(Y_1), \cdots, e^{tD}_{\alpha,\underline{\beta}}(Y_r)] \\
&= r + 1 - trdeg_k\ k\big(t, e^{tD}_{\alpha,\underline{\beta}}(Y_1), \cdots, e^{tD}_{\alpha,\underline{\beta}}(Y_r)\big).
\end{aligned}$$

**(e)** Since $\wp_{\alpha,\underline{\beta}}$ is its biggest $D-$ideal, then $k[X, Y_1, \cdots, Y_r]_{(X-\alpha, Y_1-\beta_1, \cdots Y_r-\beta_r)}$ is $D-$simple if and only if $\wp_{\alpha,\underline{\beta}} = (0)$. By part **(d)**, this happens if and only if $e^{tD}_{\alpha,\underline{\beta}}(Y_1), \cdots, e^{tD}_{\alpha,\underline{\beta}}(Y_r)$ are algebraically independent over $k(t)$. But $\wp_{\alpha,\underline{\beta}}$ is also the kernel of $e^{tD}_{\alpha,\underline{\beta}}$. Thus $\wp_{\alpha,\underline{\beta}} = (0)$ happens also if and only if $e^{tD}_{\alpha,\underline{\beta}}$ is injective. □

**Proposition 1.4.** *Let the notations be the same as in Theorem 1.1 and Proposition 1.3. Then, the system of differential equations* $(*.\alpha)$ *has a unique solution* $y_1(t, \underline{\beta}), \cdots, y_r(t, \underline{\beta}) \in k[[t]]$ *such that* $y_1(0, \underline{\beta}) = \beta_1, \cdots, y_r(0, \underline{\beta}) = \beta_r$. *This solution is given by* $y_1(t, \underline{\beta}) = e^{tD}_{\alpha,\underline{\beta}}(Y_1), \cdots, y_r(t, \underline{\beta}) = e^{tD}_{\alpha,\underline{\beta}}(Y_r)$.

**Proof.** Let $i \in \{1, \cdots, r\}$. By Proposition 1.3(a), we have $e^{tD}_{\alpha,\underline{\beta}}(Y_i)(0) = \beta_i$. Now,

$$\begin{aligned}
\frac{\partial}{\partial t}\big(e^{tD}_{\alpha,\underline{\beta}}(Y_i)\big) &= e^{tD}_{\alpha,\underline{\beta}} \circ D(Y_i) \quad \text{by Proposition 1.3(b)} \\
&= e^{tD}_{\alpha,\underline{\beta}}\Big(\frac{f_i(X, Y_1, \cdots, Y_r)}{g(X, Y_1, \cdots, Y_r)}\Big) \quad \text{by the definition of } D \\
&= \frac{f_i\big(t+\alpha, e^{tD}_{\alpha,\underline{\beta}}(Y_1), \cdots, e^{tD}_{\alpha,\underline{\beta}}(Y_r)\big)}{g\big(t+\alpha, e^{tD}_{\alpha,\underline{\beta}}(Y_1), \cdots, e^{tD}_{\alpha,\underline{\beta}}(Y_r)\big)} \quad \text{by Proposition 1.3(a).}
\end{aligned}$$

Thus $e^{tD}_{\alpha,\underline{\beta}}(Y_1), \cdots, e^{tD}_{\alpha,\underline{\beta}}(Y_r))$ is a solution of the system $(*.\alpha)$ with the desired property.

Now, we check the unicity. For $i = 1, \cdots, r$ let $y_i(t) := \beta_i + \sum_{j\geq 1} b_{ij}t^j \in k[[t]]$ and suppose that $y_1(t), \cdots, y_r(t)$ is a solution of the system $(*.\alpha)$. Write $g(X, Y_1, \cdots, Y_r)$ as a polynomial in the indeterminates $Y_1 - \beta_1, \cdots, Y_r - \beta_r$, with coefficients in $k[X - \alpha]$:

$$(1) \qquad g(X, Y_1, \cdots, Y_r) = \sum_{(l_1,\cdots,l_r)} g_{(l_1,\cdots,l_r)}(X-\alpha).(Y_1-\beta_1)^{l_1}\cdots(Y_r-\beta_r)^{l_r}$$

We then have

$$(2) \qquad g\big(t+\alpha, y_i(t), \cdots, y_r(t)\big) = \sum_{(l_1,\cdots,l_r)} g_{(l_1,\cdots,l_r)}(t).\Big(\sum_{j\geq 1} b_{1j}t^j\Big)^{l_1}\cdots\Big(\sum_{j\geq 1} b_{rj}t^j\Big)^{l_r}.$$

Note that the constant term in (2) is equal to the constant term of $g_{(0,\cdots,0)}(t)$; by (1) this constant term is equal to $g(\alpha, \beta_1, \cdots, \beta_r)$, which is different from zero since, by hypothesis, $g(X, Y_1, \cdots, Y_r))$ does not belong to the ideal $(X - \alpha, Y_1 - \beta_1, \cdots, Y_r - \beta_r)$.

Write also $f_i(X, Y_1, \cdots, Y_r)$ as a polynomial in the variables $Y_1 - \beta_1, \cdots, Y_r - \beta_r$ with coefficients in $k[X - \alpha]$:

$$(3) \qquad f_i(X, Y_1, \cdots, Y_r) = \sum_{i,(l_1,\cdots,l_r)} f_{i,(l_1,\cdots,l_r)}(X-\alpha).(Y_1-\beta_1)^{l_1}\cdots(Y_r-\beta_r)^{l_r}.$$

We then have

$$(4) \qquad f_i\big(t+\alpha, y_i(t), \cdots, y_r(t)\big) = \sum_{(l_1,\cdots,l_r)} f_{i,(l_1,\cdots,l_r)}(t)\Big(\sum_{j\geq 1} b_{1j}t^j\Big)^{l_1}\cdots\Big(\sum_{j\geq 1} b_{rj}t^j\Big)^{l_r}.$$

Note that the constant term in (4) is equal to the constant term of $f_{i,(0,\cdots,0)}(t)$ which, by (3), is equal to $f_i(\alpha, \beta_1, \cdots, \beta_r)$. Looking at the constant terms in the following equality

$$(5) \qquad g\big(t+\alpha, y_1(t), \cdots, y_r(t)\big).y_i'(t) = f_i\big(t+\alpha, y_1(t), \cdots, y_r(t)\big),$$

we obtain $g(\alpha, \beta_1, \cdots, \beta_r).b_{i1} = f_i(\alpha, \beta_1, \cdots, \beta_r)$. Since $g(\alpha, \beta_1, \cdots, \beta_r)$ is not zero, then $b_{i1}$ is uniquely determined by $g(X, Y_1, \cdots, Y_r)$ and $f_i(X, Y_1, \cdots, Y_r)$.

Now suppose that for every $i = 1, \cdots, r$ and every $j \leq n$, $b_{ij}$ is uniquely determined by $g(X, Y_1, \cdots, Y_r), f_1(X, Y_1, \cdots, Y_r), \cdots, f_r(X, Y_1, \cdots, Y_r)$.We want to show that $b_{i(n+1)}$ is also uniquely determined for every $i$. First note that our supposition implies that for every $j \leq n$, the coefficient $a_j$ of $t^j$ in (2) and the coefficient $c_j$ of $t^j$ in (4) are also uniquely determined by $g(X, Y_1, \cdots, Y_r), f_1(X, Y_1, \cdots, Y_r), \cdots, f_r(X, Y_1, \cdots, Y_r)$. Now, looking at the terms in $t^n$ in the equality (5) we have

$$(n+1)a_0 b_{i(n+1)} + n a_1 b_{in} + \cdots + a_n b_{i1} = c_n.$$

Since $a_0, a_1, \cdots, a_n, b_{i1}, \cdots, b_{in}, c_n$ are all uniquely determined by $g(X, Y_1, \cdots, Y_r)$, $f_1(X, Y_1, \cdots, Y_r), \cdots, f_r(X, Y_1, \cdots, Y_r)$ and since $a_0 = g(\alpha, \beta_1, \cdots, \beta_r) \neq 0$, then $b_{i(n+1)}$ is indeed also uniquely determined by $g(X, Y_1, \cdots, Y_r), f_1(X, Y_1, \cdots, Y_r), \cdots, f_r(X, Y_1, \cdots, Y_r)$. □

**Proof of Theorem 1.1**:

**(a)** This is given by Proposition 1.4.

**(b)** Let $\wp_{\alpha,\underline{\beta}}$ be the biggest $D-$ideal of $R := k[X, Y_1, \cdots, Y_r]_{(X-\alpha, Y_1-\beta_1, \cdots Y_r-\beta_r)}$; since $g(X, \underline{Y})$ is a unit in $R$, then $\wp_{\alpha,\underline{\beta}}$ is also the biggest $d$-ideal of $R$. By Proposition 1.3(d), height of $\wp_{\alpha,\underline{\beta}} = r - trdeg_{k(t)} k\big(e^{tD}_{\alpha,\underline{\beta}}(Y_1), \cdots, e^{tD}_{\alpha,\underline{\beta}}(Y_r)\big)$. By Proposition 1.4, the set $\{e^{tD}_{\alpha,\underline{\beta}}(Y_1), \cdots, e^{tD}_{\alpha,\underline{\beta}}(Y_r)\}$ is equal to the set $\{y_1(t, \underline{\beta}), \cdots, y_r(t, \underline{\beta})\}$.

**(c)** This is consequence of part **(b)**, or of Proposition 1.3(e) and Proposition 1.4. □

As a consequence of Theorem 1.1, we shall obtain an important result on the algebraic independence of some sets of power series. For convenience, we first state the result in a very particular case.

**Theorem 1.5.** *Let $k$ be a field of characteristic zero, $(\alpha, \beta) \in k^2$ and $t, X, Y$ some indeterminates over $k$. Let $f(X, Y), g(X, Y) \in k[X, Y], f(\alpha, \beta) \neq 0$ and $g(\alpha, \beta) \neq 0$. Over the local ring $k[X, Y]_{(X-\alpha, Y-\beta)}$, consider the derivation $d := g(X, Y)\frac{\partial}{\partial X} + f(X, Y)\frac{\partial}{\partial Y}$. Over $k[[t]]$, consider the differential equations*

$$(*.\alpha) \qquad g\big(t+\alpha, y(t)\big).y'(t) = f\big(t+\alpha, y(t)\big)$$

*and*

$$(**.\beta) \qquad f\big(x(t), t+\beta\big).x'(t) = g\big(x(t), t+\beta\big) \quad .$$

*Let $y(t) \in k[[t]]$ be the unique solution of $(*.\alpha)$ such that $y(0) = \beta$ and $x(t) \in k[[t]]$ be the unique solution of $(**.\beta)$ such that $x(0) = \alpha$. Then, the following statements are equivalent:*

- **(i)** *$y(t)$ is transcendental over $k(t)$.*

**(ii)** *$R$ is $d-$simple.*

**(iii)** *$x(t)$ is transcendental over $k(t)$.*

**Proof.** By Theorem 1.1 **c)**, $R$ is $d-$simple if and only if $y(t)$ is transcendental over $k(t)$ and also if and only if $x(t)$ is transcendental over $k(t)$. □

As a generalization of Theorem 1.5 we have:

**Theorem 1.6.** *Let $k$ be a field of characteristic zero and $r, s$ positive integers. Let $\underline{\alpha} = (\alpha_1, \cdots, \alpha_{r+s}) \in k^{r+s}$ and $t, \underline{Y} := \{Y_1, \cdots, Y_{r+s}\}$ some indeterminates over $k$. Let $f_1(\underline{Y}), \cdots, f_{r+s}(\underline{Y}) \in k[\underline{Y}]$ such that $f_i(\underline{\alpha}) \neq 0$ for all $i \in \{1, \cdots, r\}$. Over the local ring $R := k[\underline{Y}]_{(Y_1-\alpha_1, \cdots, Y_{r+s}-\alpha_{r+s})}$ consider the derivation*

$$D := \sum_{i=1}^{s+r} f_i(\underline{Y}) \frac{\partial}{\partial Y_i}.$$

*For each $i \in \{1, \cdots, r\}$ consider, over $k[[t]]$, the system of differential equations in $(r+s-1)$ unknowns*

$$(*_i.\alpha_i) \qquad \begin{aligned} &\{f_i(y_1(t), \cdots, y_{i-1}(t), t+\alpha_i, y_{i+1}(t), \cdots, y_{r+s}(t)).\tfrac{\partial}{\partial t}(y_l(t)) \\ &= f_l(y_1(t), \cdots, y_{i-1}(t), t+\alpha_i, y_{i+1}(t) \cdots, y_{s+r}(t))\}_{l \in \{1, \cdots, r+s\} \setminus \{i\}} \end{aligned}$$

*and let $y_{i1}(t), \cdots, y_{i(i-1)}(t), y_{i(i+1)}(t), \cdots, y_{i(r+s)}(t) \in k[[t]]$ be the solution of $(*_i.\alpha_i)$ such that $y_{il}(0) = \alpha_l$ for every $l \in \{1, \cdots, r+s\} \setminus \{i\}$. Then the following statements are equivalent:*

**(i)** *There exists $i \in \{1, \cdots, r\}$ such that the elements $y_{il}(t)$ ,with $l \in \{1, \cdots, r+s\} \setminus \{i\}$, are algebraically independent over $k(t)$.*

**(ii)** *The ring $R$ is d-simple.*

**(iii)** *For every $i \in \{1, \cdots, r\}$, the elements $y_{il}(t)$ ,with $l \in \{1, \cdots, r+s\} \setminus \{i\}$, are algebraically independent over $k(t)$.*

□

## 2 Transcendental Power Series

In Theorem 1.5, we have obtained that the solution in $\beta+tk[[t]]$ of the equations $(*.\alpha)$ is transcendental over $k(t)$ if and only if the solution in $\alpha + tk[[t]]$ of the equation $(**.\beta)$ is transcendental over $k(t)$. This is interesting: in one hand it may be difficult (or even impossible) to decide whether the solution of one of the equations is, or is not, transcendental over $k(t)$, using the classical methods; on the other hand, the same classical methods may easily give the solution for the other equation. We shall give two such examples. We shall also give an example to illustrate the more complex context of theorem 1.6.

**Proposition 2.1.** *Let $k$ be a field of characteristic zero, $t$ an indeterminate over $k$, $n$ a positive integer and $q \in \mathbb{Q} \setminus \{0\}$. Then:*

**(a)** *The solution in $tk[[t]]$ of the equation*

$$(**.0) \qquad (qt^n + 1)x'(t) = 1$$

*is transcendental over $k(t)$.*

**(b)** *The solution in $tk[[t]]$ of the equation*

$$(*.0) \qquad y'(t) = qy^n(t) + 1$$

*is transcendental over $k(t)$.*

*Proof.* **(a)** We have

$$\frac{1}{qt^n + 1} = 1 - qt^n + q^2t^{2n} - q^3t^{3n} + \cdots ,$$

hence

$$x(t) = \int \frac{1}{qt^n + 1} = t - \frac{qt^{n+1}}{n+1} + \frac{q^2t^{2n+1}}{2n+1} - \frac{q^3t^{3n+1}}{3n+1} + \cdots .$$

By Dirichlet's Theorem on arithmetic progressions, the set $S := \{1, n+1, 2n+1, \cdots\}$ of the denominators of the series $x(t)$ contains infinitely many prime integers. Then, by a theorem of Eisenstein [4, 32., page 44], $x(t)$ is transcendental over $k(t)$.

**(b)** Is a consequence of **(a)** and Theorem 1.5. □

**Proposition 2.2.** *Let $k$ be a field of characteristic zero and $t, X, Y$ some indeterminates over $k$. Let $f(X,Y) := a(X)Y + b(X)$ with $a(X), b(X) \in k[X]$, deg $a(X) >$deg $b(X)$, $b(X) \neq 0$. Then,*

**(a)** *For every $\alpha \in k$, all the solutions in $k[[t]]$ of the equation*

$$(*.\alpha) \qquad y'(t) = a(t+\alpha)y(t) + b(t+\alpha)$$

*are transcendental over $k(t)$.*

**(b)** *For every $\alpha, \beta \in k$ such that $f(\alpha, \beta) \neq 0$, the solution $x(t) \in \alpha + tk[[t]]$ of the equation*

$$(**.\beta) \qquad \big(a\big(x(t)\big).(t+\beta) + b\big(x(t)\big)\big).x'(t) = 1$$

*is transcendental over $k(t)$.*

We shall prove part **(a)** of Proposition 2.2 and obtain part **(b)** as a consequence of **(a)** and Theorem 1.5. But before that, we need an auxiliary result that has its own interest.

**Proposition 2.3.** *Let $l \supseteq k \supseteq \mathbb{Z}$ be domains of characteristic zero, where $k$ is a field. Let $t$ be an indeterminate over $l$, $a(t), b(t) \in k[t]$, $a(t) \neq 0$.Over $l[[t]]$, consider the differential equation*

$$(*) \qquad y'(t) = a(t)y(t) + b(t).$$

*Then:*

**(a)** *With the possible exception of one of them, all the solutions of $(*)$ in $l[[t]]$ are transcendental over $k[t]$.*

**(b)** *If there exists a solution of $(*)$ in $l[[t]]$ that is not transcendental over $k[t]$, then it belongs to $k[t]$.*

This proposition will be obtained as an immediate consequence of the next proposition in witch we take $A := k[t]$, $R := l[[t]]$ and $d := \frac{\partial}{\partial t}$.

We recall that if $R$ is a ring that contains the rational numbers and if $d$ is a derivation of $R$, then $Nil(d)$ is defined to be $\{\xi \in R; \exists s \in \mathbb{N} \text{ such that } d^s(\xi) = 0\}$. Following [2] we define the function $deg_d$ on $Nil(d)$ by $deg_d(\xi) = max\{s \in \mathbb{N}; d^s(\xi) = 0\}$. Using Leibnitz's formula, it is easy to check that, if $R$ is a domain, then one has $deg_d(\xi\eta) = deg_d(\xi)deg_d(\eta)$.

**Proposition 2.4.** *Let $R \supseteq A \supseteq k \supseteq \mathbb{Z}$ be domains of characteristic zero. Let $d$ be a $k$-derivation of $R$ such that $d(A) \subseteq A$ and $A \subseteq Nil(d)$.*

*Let $a, b \in A$, $a \neq 0$, and over $R$ consider the differential equation*

$$(*.b) \qquad d(y) = ay + b$$

**(a)**
- *If $y_0 \in R$ is a solution of $(*.b)$, then*

$$\{solutions\ of\ (*.b)\ in\ R\} = \{y_0 + z; z \in R,\ z\ solution\ of\ d(y) = ay\}.$$

- *If $z_0 \in R$ is a non-zero solution of $d(y) = ay$, then*

$$\{solutions\ of\ d(y) = ay\ in\ R\} = \{cz_0;\ c \in qf(R),\ d(c) = 0,\ cz_0 \in R\}.$$

**(b)** *With the possible exception of one of them, all the solutions of $(*.b)$ in $R$ are transcendental over $A$.*

**(c)** *Suppose that $A$ is a U.F.D and that $ker(d) \cap (A \setminus \{0\}) = \{units\ of\ A\}$. If there exists a solution of $(*.b)$ in $R$ that is not transcendental over $A$, then it belongs to $A$.*

**Remark 2.5.** *Part* **(b)** *of Proposition 2.4 was proved in the particular case of $A \subseteq \ker(d)$ by Nousiainen and Sweedler in [5, Lemma 3.5]*

**Proof of Proposition 2.4**. **(a)** These are routine verifications.

**(b)** First we prove the following:

**Claim 1.** Let the natural extension of $d$ to the quotient field $qf(R)$ of $R$ be also denoted by $d$. If $z \in qf(R)$ is a non-zero solution of the differential equation

$$(*.0) \qquad d(y) = ay,$$

then $z$ is transcendental over $A$.

Let $z$ be any non-zero solution of (*.0) in $qf(R)$ and suppose that $z$ is algebraic over $A$. Let

$$\alpha_n z^n + \cdots + \alpha_1 z + \alpha_0 = 0$$

be an algebraic relation, with $\alpha_n, \cdots, \alpha_1, \alpha_0 \in A$, $n$ minimal. Applying $d$ to this relation, we obtain

$$(6) \qquad \big(d(\alpha_n) + n\alpha_n a\big) z^n + \cdots + \big(d(\alpha_1) + \alpha_1 a\big) z + d(\alpha_0) = 0.$$

We assert that $d(\alpha_n) + n\alpha_n a$ is not equal to zero. Indeed, if it were equal to zero, then we would have $d(\alpha_n) = -na\alpha_n$, hence $deg_d(d(\alpha_n)) = deg_d(a) + deg_d(\alpha_n) \geq deg_d(\alpha_n)$ which is absurd since $deg_d(d(\alpha_n)) = deg_d(\alpha_n) - 1$. Thus the left side of (6) has a non-zero term in $z^n$. Then, by the minimality of $n$, $d(\alpha_0)$ must be different from zero. Applying $d$ to the relation (6), we shall obtain a new algebraic relation with a non-zero term in $z^n$ and with $d^2(\alpha_0) \neq 0$ as a constant. Going on with this process, we see that $d^s(\alpha_0) \neq 0$ for every $s \geq 0$ which is absurd since $\alpha_0 \in A \subseteq Nil(d)$. This terminates the proof of our claim.

Now, if there exists a solution $y_0$ of $(*.b)$ in $R$ that is algebraic over $A$, then all the other solutions of $(*.b)$ in $R$ will be transcendental over $A$ since they will be of the type $y_0 + z$ with $z$ a non-zero solution of (*.0) in $R$.

**(c)** Let $y \in R$ be a solution of $(*.b)$ that is not transcendental over $A$. Let

$$P(X) := X^n + \beta_{n-1}X^{n-1} + \cdots + \beta_0$$

be the irreducible polynomial of $y$ over the quotient field $qf(A)$ of $A$. Applying $d$ to the equality $y^n + \beta_{n-1}y^{n-1} + \cdots + \beta_0 = 0$, we obtain

$$nay^n + [nb + (n-1)\beta_{n-1}a + d(\beta_{n-1})]y^{n-1} + \cdots = 0.$$

Clearly, the polynomial

$$g(X) = naX^n + [nb + (n-1)\beta_{n-1}a + d(\beta_{n-1})]X^{n-1} + \cdots$$

has degree $n$, has its coefficients in $qf(A)$ and admits $z$ as a root; thus $g(X)$ is a multiple of $P(X)$. Looking at the terms in $X^n$, we see that $g(X) = naP(X)$. Then, $nb + (n-1)\beta_{n-1}a + d(\beta_{n-1}) = na\beta_{n-1}$, hence

$$d(\beta_{n-1}) = a\beta_{n-1} - nb. \tag{7}$$

Let $w := ny + \beta_{n-1}$. We have

$$\begin{aligned} d(w) &= nd(y) + d(\beta_{n-1}) \\ &= n(ay+b) + a\beta_{n-1} - nb \qquad \text{by } (*.b) \text{ and } (7) \\ &= a(ny + \beta_{n-1}) \\ &= aw \end{aligned}$$

Thus $w$ belongs to $qf(R)$ and is a solution of (*.0) which is algebraic over $A$ since $y$ is algebraic over $A$ and since $\beta_{n-1} \in qf(A)$. Then , by the previous claim, we have $w = 0$ and hence $y = -\frac{\beta_{n-1}}{n} \in qf(A)$ .

Now since $A$ is a U.F.D we can write $y = \frac{\gamma}{\delta}$ with $\gamma, \delta \in A$, $\delta \neq 0$, $mdc(\gamma, \delta) = 1$. We have $ay + b = \frac{a\gamma + b\delta}{\delta}$ and $d(y) = \frac{d(\gamma)\delta - \gamma d(\delta)}{\delta^2}$, hence $\frac{a\gamma+b\delta}{\delta} = \frac{d(\gamma)\delta - \gamma d(\delta)}{\delta^2}$ since $y$ is a solution of $(*.b)$. From this we get $\delta[d(\gamma) - a\gamma - b\delta] = \gamma d(\delta)$. Since $mdc(\gamma, \delta) = 1$, this implies that $\delta$ divides $d(\delta)$ in $A$, say $d(\delta) = u\delta$ with $u, \delta, d(\delta) \in A$. Looking at the $d$-degree, we see that necessarily $d(\delta) = 0$, i.e., that $\delta \in Ker(d) \cap (A \setminus \{0\}) \subseteq \{\text{units of } A\}$. Thus $y = \frac{\gamma}{\delta} \in A$.

**Proof of Proposition 2.2 (a)**: For reasons of degree, it is clear that the equation $(*.\alpha)$ does not have any solution in $k[t]$. Then, by Proposition 2.3(b), all the solutions of $(*.\alpha)$ in $k[[t]]$ are transcendental over $k(t)$.

**(b)** This is a consequence of **(a)** and of Theorem 1.5. □

We shall apply Theorem 1.6 on a set of polynomials that is related to the example constructed by Hart in [3]

**Proposition 2.6.** *Let $k$ be a field of characteristic zero and $t, Y_1, \cdots, Y_r$ some indeterminates over $k$. Let $f_1(\underline{Y}) := 1$ and for $i = 2, \cdots, r$, let $f_i(\underline{Y}) := 1(1+Y_1)(1+Y_2)\cdots(1+Y_{i-1})$. For each $i \in \{1, \cdots, r\}$, consider over $k[[t]]$ the system of differential equations in $(r-1)$ unknowns*

$$(*_i.0) \qquad \begin{aligned} &\{f_i\big(y_1(t), \cdots, y_{i-1}(t), t, y_{i+1}(t), \cdots, y_r(t)\big).\tfrac{\partial}{\partial t}(y_l(t)) \\ &\quad = f_l\big(y_1(t), \cdots, y_{i-1}(t), t, y_{i+1}(t) \cdots, y_r(t)\big)\}_{l \in \{1, \cdots, r\} \setminus \{i\}} \end{aligned}$$

*and let $y_{i1}(t), \cdots, y_{i(i-1)}(t), y_{i(i+1)}(t), \cdots, y_{ir}(t)$ be the solution of $(*_i.0)$ in $tk[[t]]$. Then, for every $i \in \{1, \cdots, r\}$, the elements $y_{il}(t)$ with $l \in \{1, \cdots, r\} \backslash \{i\}$ are algebraically independent over $k(t)$.*

*Proof.* By Theorem 1.6, it suffices to show that the series $y_{12}(t), y_{13}(t), \cdots, y_{1r}(t)$ are algebraically independent over $k(t)$. The system of equations $(*_1.0)$ is particularly easy to handle, and it is routine to check that its solution in $tk[[t]]$ is given by

$$\begin{cases} y_{12}(t) & = e^t - 1 \\ y_{13}(t) & = e^{y_{12}(t)} - 1 \\ & \vdots \\ y_{1r}(t) & = e^{y_{1(r-1)}(t)} - 1 \end{cases}$$

Now it is well known that $y_{12}(t), y_{13}(t), \cdots, y_{1r}(t)$ are algebraically independent over $k(t)$.(For example, see [1, Corollary 1, p.253].) □

**Remark 2.7.** *(a) In the previous proposition, instead of working with the system of equations $(*_1.0)$, we could have worked with the system $(*_r.0)$. It would have been routine to check that the solution of this system in $tk[[t]]$ is given by:*

$$\begin{cases} y_{r(r-1)}(t) & = log(1+t) \\ y_{r(r-2)}(t) & = log(1 + y_{r(r-1)}(t)) \\ & \vdots \\ y_{r1}(t) & = log(1 + y_{r2}(t)) \end{cases}$$

*Then, we would have used the fact that $y_{r(r-1)}(t), \cdots, y_{r1}(t)$ are algebraically independent over $k(t)$.( See [3, Lemma, p.292].)*

*(b) For $i \neq 1, r$, it is not clear at all what is the form of the solution $y_{i1}(t), \cdots, y_{i(i-1)}(t)$, $y_{i(i+1)}(t), \cdots, y_{ir}(t)$ of $(*_i.0)$ in $tk[[t]]$, and it is even less clear how one could see directly that the elements $y_{il}(t)$, with $l \in \{1, \cdots, r\} \setminus \{i\}$, are algebraically independent over $k(t)$.*

## 3 Differentially Simple Rings

In this section, we apply the results of section 1 to obtain interesting families of differential simple rings.

**Theorem 3.1.** *Let $k \subseteq k'$ be fields of characteristic zero and $Y_1, \cdots, Y_r$ some indeterminates over $k'$. Let $M$ be a maximal ideal of $k[Y_1, \cdots, Y_r]$ and $M'$ a maximal ideal of $k'[Y_1, \cdots, Y_r]$ that lies over $M$. Let $d$ be a $k$-derivation of $k[Y_1, \cdots, Y_r]$ and $d'$ the extension of $d$ to a $k'$-derivation of $k'[Y_1, \cdots, Y_r]$; let $d$ and $d'$ also denote the extensions to*

*$k[Y_1, \cdots, Y_r]_M$ and $k'[Y_1, \cdots, Y_r]_M$ respectively. Then, the following statements are equivalent:*

- **(i)** *The local ring $k[Y_1, \cdots, Y_r]_M$ is d-simple.*
- **(ii)** *The local ring $k'[Y_1, \cdots, Y_r]_{M'}$ is $d'$-simple.*

*Proof.* Let $\overline{k}$ be the algebraic closure of $k$ and $\overline{k'}$ the algebraic closure of $k'$. Since the extension $k'[Y_1, \cdots, Y_r] \subseteq \overline{k'}[Y_1, \cdots, Y_r]$ is integral, there exists a maximal ideal $\overline{M'}$ of $\overline{k'}[Y_1, \cdots, Y_r]$ lying over $M'$. Let $\overline{M} := \overline{M'} \cap \overline{k}[Y_1, \cdots, Y_r]$. The local ring $k'[Y_1, \cdots, Y_r]_{M'}$ is $d'$-simple if and only if $\overline{k'}[Y_1, \cdots, Y_r]_{\overline{M'}}$ is $d'$-simple. Indeed, if $I$ is a non-zero $d'$-ideal contained in $M'$, then $I\overline{k'}[Y_1, \cdots, Y_r]$ is a non-zero $d'$-ideal in $\overline{M'}$; conversely, if $J$ is a non-zero $d'$-ideal contained in $\overline{M'}$, then $J \cap k'[Y_1, \cdots, Y_r]$ is a non-zero $d'$-ideal contained in $M'$. Similarly, $k[Y_1, \cdots, Y_r]_M$ is $d$-simple if and only if $\overline{k}[Y_1, \cdots, Y_r]_{\overline{M}}$ is $d$-simple. Thus, without loss of generality, we may suppose that $k$ and $k'$ are algebraically closed.

**(ii) $\Rightarrow$ (i)** The local ring $k'[Y_1, \cdots, Y_r]_{M'}$ is $d'$-simple if an only if for every $\xi \in M'$, there exists $s \geq 1$ such that $d'^s(\xi) \notin M'$; when this occurs then evidently, for every $\eta \in M$, there exists $s$ such that $d^s(\eta) \notin M' \cap k[Y_1, \cdots, Y_r] = M$, hence $k[Y_1, \cdots, Y_r]$ is $d$-simple.

**(i) $\Rightarrow$ (ii)** Let $\beta_1, \cdots, \beta_r \in k$ such that $M = (Y_1 - \beta_1, \cdots, Y_r - \beta_r)k[Y_1, \cdots, Y_r]$. Then, necessarily, the ideal $M'$ that lies above $M$ is $M' = (Y_1 - \beta_1, \cdots, Y_r - \beta_r)k'[Y_1, \cdots, Y_r]$. For every $i = 1, \cdots, r$, let $f_i(Y_1, \cdots, Y_r) := d(Y_i)$. Since $k[Y_1, \cdots, Y_r]_M$ is $d$-simple,there exists $i \in \{1, \cdots, r\}$, say $i = 1$, such that $d(Y_1) = d(Y_1 - \beta_1) \notin M$. Let $y_{12}(t), \cdots, y_{1r}(t) \in k[[t]]$ be the solution of the system of differential equations

$$(*_1.\beta_1) \quad \{f_1\big(t + \beta_1, y_2(t), \cdots, y_r(t)\big).\frac{\partial}{\partial t}(y_i(t)) = f_i\big(t + \beta_1, y_2(t), \cdots, y_r(t)\big)\}_{i=2}^{r}$$

such that $y_{12}(0) = \beta_2, \cdots, y_{1r}(0) = \beta_r$. By Theorem 1.1(c), $k[Y_1, \cdots, Y_r]_M$ is $d$-simple if and only if $y_{12}(t), \cdots, y_{1r}(t)$ are algebraically independent over $k(t)$. Also by Theorem 1.1(c), $k'[Y_1, \cdots, Y_r]_{M'}$ is $d'$-simple if and only if $y_{12}(t), \cdots, y_{1r}(t)$ are algebraically independent over $k'(t)$. Then, we can conclude by the following easy to prove lemma.

□

**Lemma 3.2.** *Let $k \subseteq k'$ be fields of characteristic zero, $t$ an indeterminate over $k'$ and $y_1(t), \cdots, y_r(t) \in k[[t]]$. Then $y_1(t), \cdots, y_r(t)$ are algebraically independent over $k(t)$ if and only if they are algebraically independent over $k'(t)$.*

Most published examples of derivations $D$ that make the local ring $k[X, Y]_{(X,Y)}$ $D$-simple are of the type $D = \frac{\partial}{\partial X} + f(X, Y)\frac{\partial}{\partial Y}$ with $f(X, Y) \in k[X, Y]$, $deg_Y f(X, Y) = 1$, or variations of that type; in particular, there does not seem to be any example with $deg_Y f(X, Y)$ an arbitrary positive integer. Our next proposition gives such an example.

**Proposition 3.3.** *Let $k$ be a field of characteristic zero, $n$ a positive integer and $q \in \mathbb{Q}\backslash\{0\}$. On the local ring $k[X,Y]_{(X,Y)}$ consider the derivation $D := \frac{\partial}{\partial X} + (qY^n + 1)\frac{\partial}{\partial Y}$. Then, $k[X,Y]_{(X,Y)}$ is $D$-simple.*

*Proof.* As seen in Proposition2.1, the solution in $tk[[t]]$ of the equation $(qt^n + 1).x'(t) = 1$ is transcendental over $k(t)$. Then, by Theorem1.1(c), the ring $k[X,Y]_{(X,Y)}$ is $D$-simple □

We can recover the main result of [3] witch asserts that there exists a derivation $d$ that makes $k[Y_1, \cdots, Y_r]_{(Y_1,\cdots,Y_r)}$ $d$-simple:

**Proposition 3.4.** *Let $k$ be a field of characteristc zero and $\underline{Y} := \{Y_1, \cdots, Y_r\}$ some indeterminates over $k$. Let $f_1(\underline{Y}) := 1$ and for $i = 2, \cdots, r$, let $f_i(\underline{Y}) := 1(1 + Y_1)(1 + Y_2)\cdots(1 + Y_{i-1})$. On the local ring $R := k[Y_1, \cdots, Y_r]_{(Y_1,\cdots,Y_r)}$, consider the derivation $d := \sum_{i=1}^{r} f_i(\underline{Y})\frac{\partial}{\partial Y_i}$. Then, $R$ is $d$-simple.*

*Proof.* Consider the system of differential equations

$$(*_1.0) \qquad \{f_1\big(t, y_2(t), \cdots, y_r(t)\big).\frac{\partial}{\partial t}y_l(t) = f_l\big(t, y_2(t), \cdots, y_r(t)\big)\}_{l=2}^{r}$$

i.e.,

$$\{\frac{\partial}{\partial t}y_l(t) = (1+t)\big(1 + y_2(t)\big)\cdots\big(1 + y_{l-1}(t)\big)\}_{l=2}^{r}.$$

As it was seen in the proof of Proposition 2.6, the solution of this system in $tk[[t]]$ is given by $y_{12}(t) = e^t - 1$, $y_{13}(t) = e^{y_{12}(t)} - 1$, $\cdots$, $y_{1r} = e^{y_{1(r-1)}(t)} - 1$.

Since these power series are algebraically independent over $k(t)$, then the ring $R$ is $d$-simple by Theorem 1.1(c).

□

Finally using a theorem of Ax on the transcendency of certain formal power series, we obtain examples of derivations $d$ that makes the ring $k[X, Y_1, \cdots, Y_r]_{(X,Y_1,\cdots,Y_r)}$ $d$-simple:

**Proposition 3.5.** *Let $k$ be a field of characteristic zero. Let $y_1(t) := \frac{h_1(t)}{l(t)}, \cdots, y_r(t) := \frac{h_r(t)}{l(t)} \in tk[t]_{(t)}$ be $\mathbb{Q}$-linearly independent. Let $d$ be the $k$-derivation of $k[X, Y_1, \cdots, Y_r]_{(X,Y_1,\cdots,Y_r)}$ defined by $d(X) := l^2(X)$, $d(Y_i) := \big(h_i'(X)l(X) - h_i(X)l'(X)\big)Y_i + \big(h_i'(X)l(X) - h_i(X)l'(X)\big)$. Then:*

- **(a)** *$e^{y_1(t)} - 1, \cdots, e^{y_r(t)} - 1$ are algebraically independent over $k(t)$.*
- **(b)** *$k[X, Y_1, \cdots, Y_r]_{(X,Y_1,\cdots,Y_r)}$ is $d - simple$.*

*Proof.* **(a)** Note that for every $i$, $y_i(t) \in k[[t]] \cap k(t)$. Then, applying [1, Corollary 1, p.253] we obtain that $e^{y_1(t)} - 1, \cdots, e^{y_r(t)} - 1$ are algebraically independent over $k(t)$.

**(b)** Follows from Theorem 1.1 since, for every $i = 1, \cdots, r$, $z_i(t) := e^{y_i(t)} - 1$, satisfies the differential equation

$$l^2(t).z_i'(t) = \big(h_i'(t)l(t) - h_i(t)l'(t)\big)z_i(t) + \big(h_i'(t)l(t) - h_i(t)l'(t)\big).$$

□

# 4 The Ring k[X,Y] with the derivation $d$ defined by $d(X) = 1$, $d(Y) = a(X)Y + b(X)$ with $a(X), b(X) \in k[X]$.

In this section, we consider the domain $k[X,Y]$ (and not a localization of it), endowed with the derivation $d$ defined by $d(X) = 1$, $d(Y) = a(X)Y + b(X)$ with $a(X), b(X) \in k[X]$. By a Theorem of Shamsuddin [7], it is known that $k[X,Y]$ is $d$-simple if and only if there does not exist any polynomial $h(X) \in k[X]$ such that $h'(X) = a(X)h(X) + b(X)$. Using some of our results on power series obtained in section 2, we shall complement and deepen that result of Shamsuddin.

**Theorem 4.1.** *Let $k$ be a field of characteristic zero and $X, Y$ two indeterminates over $k$. Let $a(X), b(X) \in k[X]$ and $d := \frac{\partial}{\partial X} + \big(a(X)Y + b(X)\big)\frac{\partial}{\partial Y}$.*

**a)** *The following statements are equivalent:*

- **(i)** *$k[X,Y]$ is $d$-simple.*
- **(ii)** *There does not exist any polynomial $h(X) \in k[X]$ such that $h'(X) = a(X)h(X) + b(X)$.*
- **(iii)** *There exists an irreducible polynomial $f(X) \in k[X]$ such that, for every maximal ideal $< f(X), - >$ of $k[X,Y]$ that contains $f(X)$, the ring $k[X,Y]_{<f(X),->}$ is $d$-simple.*

**b)** *If $k[X,Y]$ is not $d$-simple, then:*

- *There exists a unique polynomial $h(X) \in k[X]$ such that $h'(X) = a(X)h(X) + b(X)$.*
- *The ring $k[X,Y]$ has a unique non-zero prime $d$-ideal $\wp$. It is equal to $(Y - h(X))$. In particular $\frac{k[X,Y]}{\wp} \simeq k[X]$.*
- *Given any irreducible polynomial $f(X) \in k[X]$ there exists a unique maximal ideal $< f(X), - >$ of $k[X,Y]$ that contains $f(X)$ such that $k[X,Y]_{<f(X),->}$ is not $d$-simple. It is equal to $\big(f(X), Y - h(X)\big)$.*

**Proof (a)** Let $\overline{k}$ be the algebraic closure of $k$. We first make three observations:

1) $k[X,Y]$ is $d$-simple if and only if $\overline{k}[X,Y]$ is $d$-simple. This is routine to check.

2) If the equation $h'(X) = a(X)h(X) + b(X)$ does not have a solution in $\overline{k}[X]$, then clearly it does not have a solution in $k[X]$ either. Conversely, if it has a solution in $\overline{k}[X]$, then by Proposition 2.3(b) (with $l = \overline{k}$), this solution belongs to $k[X]$.

3) If $f(X)$ is an irreducible polynomial of $k[X]$ and if $\alpha \in \overline{k}$ is a root of $f(X)$, then it is routine to check that $k[X,Y]_{<f(X),->}$ is $d$-simple for every maximal ideal $< f(X), - >$ of $k[X,Y]$ that contains $f(X)$ if and only if $\overline{k}[X,Y]_{<X-\alpha,->}$ is $d$- simple for every maximal ideal $< X - \alpha, - >$ of $\overline{k}[X,Y]$ that contains $X - \alpha$.

Thus, in order to prove the equivalence of (i)-(iii) in part **(a)** of the Theorem , we may suppose that $k$ is algebraically closed.

If $a(X) = 0$, then clearly, the equation $h'(X) = b(X)$ has a solution $\int b(X)$ in $k[X]$ and $k[X,Y]$ is not $d$-simple since $(Y - \int b(X))$ is a non-zero proper $d$-ideal of $k[X,Y]$. Thus, in this case , both statements **(i)** and **(ii)** are false. That statement **(iii)** is also false is a clear consequence of the, soon to be proved, part **(b)** of the theorem.

So, for part **(a)** of the theorem, we may suppose that $a(X) \neq 0$.

(**i**) $\Longrightarrow$ (**ii**) Suppose that there exists a polynomial $h(X) \in k[X]$ such that $h'(X) = a(X)h(X) + b(X)$. Then, $d(Y - h(X)) = a(X)(Y - h(X))$ and therefore, $(Y - h(X))$ is a non-zero proper $d$-ideal of $k[X,Y]$, which is a contradiction with the hypothesis.

(**ii**) $\Longrightarrow$ (**iii**) Since the equation $h'(X) = a(X)h(X)+b(X)$ does not have any solution in $k[X]$, then by Proposition 2.3, all the solutions of this equation in $k[[X]]$ are transcendental over $k(X)$. By Theorem 1.1(c), this implies that $k[X,Y]_{(X,Y-\beta)}$ is $d$-simple for every $\beta \in k$.

(**iii**) $\Longrightarrow$ (**i**) By hypothesis, there exists $\alpha_0 \in k$ such that $k[X,Y]_{(X-\alpha_0,Y-\beta)}$ is $d$-simple for every $\beta \in k$. Then, by Theorem 1.1 (c), we already have:

**Fact 1.** All the solutions in $k[[X]]$ of the equation

$$(*.\alpha_0) \qquad y'(X) = a(X + \alpha_0)y(X) + b(X + \alpha_0)$$

are transcendental over $k(X)$.

Now suppose that $k[X,Y]$ is not $d$-simple. Then there exists $\alpha_1, \beta_1 \in k$ such that $k[X,Y]_{(X-\alpha_1,Y-\beta_1)}$ is not $d$-simple. By Theorem 1.1 (c), this implies that the element $h_1(X) \in \beta_1 + Xk[[X]]$ satisfying

$$(8) \qquad h_1'(X) = a(X + \alpha_1)h_1(X) + b(X + \alpha_1)$$

is algebraic over $k(X)$; even more, by Proposition 2.3, $h_1(X) \in k[X]$.

Let $h_0(X) := h_1(X - \alpha_1 + \alpha_0)$. We have $h_0(X) \in k[X]$ and

$$\begin{aligned} h_0'(X) &= (h_1(X - \alpha_1 + \alpha_0))' \\ &= a(X + \alpha_0)h_1(X - \alpha_1 + \alpha_0) + b(X + \alpha_0) \\ &= a(X + \alpha_0)h_0(X) + b(X + \alpha_0) \end{aligned}$$

Thus $h_0(X) \in k[X]$ is a solution of $(*.\alpha_0)$; this contradicts **Fact 1**. Thus $k[X, Y]$ is $d$-simple.

**(b)** Suppose that $k[X, Y]$ is not $d$-simple. Once again, using the fact that $\overline{k}[X, Y]$ is an integral extension of $k[X, Y]$ and using Proposition 2.3, it easy to see that, without loss of generality, we may suppose that $k$ is algebraically closed.

As seen before, there exists $\alpha_1 \in k$ and $h_1(X) \in k[X]$ satisfying (8). Let $h(X) := h_1(X - \alpha_1)$. We have $h(X) \in k[X]$ and

$$\begin{aligned} h'(X) &= (h_1(X - \alpha_1))' \\ &= a(X).h_1(X - \alpha_1) + b(X) \quad \text{by} \quad (8) \\ &= a(X)h(X) + b(X). \end{aligned}$$

Then $d(Y - h(X)) = a(X)\big(Y - h(X)\big) \in \big(Y - h(X)\big)$ and $\wp := \big(Y - h(X)\big)$ is a non-zero prime $d$-ideal of $k[X, Y]$.

**Claim 2.** Given an arbitrary $\alpha \in k$, then:

1) $\wp$ is the biggest $d$-ideal contained in $(X - \alpha, Y - h(\alpha))$.

2) $k[X, Y]_{(X-\alpha,Y-\gamma))}$ is $d$-simple for every $\gamma \in k$, $\gamma \neq h(\alpha)$.

Indeed, with $\beta := h(\alpha)$, we have $\wp = (Y - h(X)) \subseteq (X - \alpha, Y - \beta)$. Since $\wp$ is a prime $d$-ideal of height one and since $(X - \alpha, Y - \beta)$ is not a $d$-ideal then $\wp$ is the biggest $d$-ideal contained in $(X - \alpha, Y - \beta))$.

Since $k[X, Y]_{(X-\alpha,Y-\beta))}$ is not $d$-simple then by Theorem 1.1 (c), the solution in $\beta + Xk[[X]]$ of the equation

$$(*.\alpha) \;\; y'(X) = a(X + \alpha)y(X) + b(X + \alpha)$$

is algebraic over $k(X)$, and by Proposition 2.3 (a), all the other solutions in $k[[X]]$ are transcendental over $k(X)$. Thus, by Theorem 1.1 (c), $k[X, Y]_{(X-\alpha,Y-\gamma))}$ is $d$-simple for every $\gamma \neq \beta$, $\gamma \in k$. This terminates the proof of our claim .

Now, let $\wp'$ be any non-zero prime $d$-ideal of $k[X, Y]$. Let $\alpha_2, \beta_2 \in k$ such that $\wp' \subseteq (X - \alpha_2, Y - \beta_2)$. Clearly $(X - \alpha_2, Y - \beta_2)$ is not a $d$-ideal, hence $\wp'$ is the biggest $d$-ideal contained in $(X - \alpha_2, Y - \beta_2)$. Since $\wp' \neq (0)$, the ring $k[X, Y]_{(X-\alpha_2,Y-\beta_2))}$ is not $d$-simple. Thus, by Claim 2 .2), we necessarily have $\beta_2 = h(\alpha_2)$, and by Claim 2.1), $\wp \subseteq (X - \alpha_2, Y - \beta_2)$. Since both $\wp$ and $\wp'$ are the biggest $d$-ideal of $k[X, Y]$ contained in

$(X-\alpha_2, Y-\beta_2)$, we obtain that $\wp = \wp'$ and therefore that $\wp$ is the unique non-zero prime $d$-ideal of $k[X,Y]$.

The prime ideal $\wp$ has been defined as $\wp := \big(Y - h(X)\big)$ where $h(X)$ was an element of $k[X]$ such that $h'(X) = a(X)h(X) + b(X)$. The uniqueness of such an element $h(X)$ is given by Proposition 2.3.

Finally, given an arbitrary irreducible polynomial of $k[X]$,i.e., given an arbitrary polynomial of the type $X - \alpha$ with $\alpha \in k$, then by Claim 2, $(X-\alpha, Y-h(\alpha))$ is the unique maximal ideal $M$ of $k[X,Y]$ that contains $X-\alpha$ such that $k[X,Y]_M$ is not $d$-simple; since $\big(X-\alpha, Y-h(\alpha)\big) = \big(X-\alpha, Y-h(X)\big)$, we are through. □

Using Theorem4.1 and Theorem1.1, we can prove the following:

**Proposition 4.2.** *Let $k$ be a field of characteristic zero. Let $t$ be an indeterminate over $k$, $a(t), b(t) \in k[t]$, $a(t) \neq 0$. Then, the following statements are equivalent:*

**(i)** *The equation*

$$y'(t) = a(t)y(t) + b(t)$$

*does not have any solutions in $k[t]$.*

**(ii)** *For every $\alpha \in k$, all the solutions in $k[[t]]$ of the equation*

$$y'(t) = a(t+\alpha)y(t) + b(t+\alpha)$$

*are transcendental over $k[t]$.*

**Proof** (**ii**) $\Longrightarrow$ (**i**) Clear

(**i**) $\Longrightarrow$ (**ii**) By Theorem4.1 (a), $k[X,Y]$ is $d$-simple and, if $\overline{k}$ denotes the algebraic closure of $k$, $\overline{k}[X,Y]$ is $d$-simple. Then $\overline{k}[X,Y]_{(X-\alpha,Y-\beta)}$ is $d$-simple for every $\alpha, \beta \in \overline{k}$ and, by Theorem 1.1 (c), for every $\alpha \in \overline{k}$, all the solutions in $\overline{k}[[t]]$ of $y'(t) = a(t+\alpha)y(t) + b(t+\alpha)$ are transcendental over $\overline{k}(t)$, hence in particular over $k[t]$. □

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IMECC, Universidade Estadual de Campinas, 13801-970 Campinas, SP, Brazil (brumatti@ime.unicamp.br)

Instituto de Matemática Pura e Aplicada, Estrada Dona Castorina 110, J. Botânico, 22460-320 , Rio de Janeiro, RJ, Brazil (ylequain@impa.br)

Instituto de Ciências Matemáticas e de Computação, USP-SC, Av. Dr. Carlos Botelho, 1465, 13560-970 São Carlos, SP, Brazil (lev@icmsc.sc.usp.br)