PARAMETER EXCLUSIONS IN HÉNON-LIKE SYSTEMS

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1. Introduction

This survey is a presentation of the arguments in the proof that Hénon-like maps

$$\Phi_a(x,y) = (1 - ax^2, 0) + R(a,x,y), \quad \|R(a,x,y)\|_{C^3} \leq b$$

have a *strange attractor*, with positive Lebesgue probability in the parameter a, if the perturbation size b is small enough. We first sketch a *geometric model* of the strange attractor in this context, emphasising some of its key geometrical properties, and then focus on the construction and estimates required to show that this geometric model does indeed occur for *many parameter values*.

Our ambitious aim is to provide an exposition at one and the same time intuitive, synthetic, and rigorous. We think of this text as an introduction and study guide to the original papers [BenCar91] and [MorVia93] in which the results were first proved. We shall concentrate on describing in detail the overall structure of the argument and the way it breaks down into its (numerous) constituent sub-arguments, while referring the reader to the original sources for detailed technical arguments. Let us begin with some technical and historical remarks aimed at motivating the problem and placing it in its appropriate mathematical context.

1.1. **Uniform and non-uniform hyperbolicity.** The arguments which we shall discuss lie at the heart of a certain branch of dynamics. To formulate its aims and scope we recall first of all two notions of hyperbolicity: *uniform hyperbolicity* where hyperbolic estimates are assumed to hold uniformly at every point of some set, and *non-uniform hyperbolicity* which is formulated in terms of asymptotic hyperbolicity estimates (non-zero Lyapunov exponents) holding only almost everywhere with respect to some invariant probability measure. The notion of uniform hyperbolicity was introduced by Smale (see [Sma67] and references therein) and was central to a large part of the development the field of Dynamics experienced through the sixties and the seventies, including the fundamental work of Anosov [Ano67] on ergodicity of geodesic flows; the notion of non-uniform hyperbolicity was formulated by the work of Pesin [Pes77] and was subsequently much developed by him and several other mathematicians.

In both cases, one may distinguish two related but distinct aspects. On the one hand there is the general theory which *assumes* hyperbolicity and addresses the question of its geometrical and dynamical *implications* such as the existence of

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stable and unstable manifolds, questions of ergodicity, entropy formulas, statistical properties etc. This aspect of the theory is well developed in both cases although results are naturally stronger in the uniformly hyperbolic case. See the comprehensive texts [Man87, Shu87, Pol93, KatHas94, AnoSol95, Yoc95a, Via97, Bal00, BarPes01] for details and extensive bibliographies.

On the other hand there is the question of constructing and finding examples and, more generally, of *verifying* hyperbolicity in specific classes of systems. In this respect, the difference between uniform and non-uniform hyperbolicity is striking. Uniformly hyperbolic systems can be constructed relatively easily and in principle, and often also in practice, it is possible to verify the uniform hyperbolicity conditions by considering only a finite number of iterations of the map. A main technique for verifying uniform hyperbolicity is the method of *conefields* which involves checking some open set of relations on the partial derivatives of the map.

Verifying non-uniform hyperbolicity is generally much more problematic, partly because this notion is asymptotic in nature, that is, it depends on the behavior of iterates as time goes to infinity. Also, non-uniformly hyperbolic systems may contain tangencies between stable and unstable leaves in which case they cannot admit complementary stable and unstable continuous invariant conefields. In fact, invariant objects for this type of systems tend to live in the measurable category rather than the topological category. Moreover, there is an a priori *impasse* related to the fact that the very definition of non-uniform hyperbolicity requires an invariant measure. Such a measure is not usually given to begin with and one needs to take advantage of hyperbolicity features of the system to even prove that it exists. All in all, we still lack a good understanding of what makes a dynamical system non-uniformly hyperbolic and it seems more examples of such systems need yet to be found for such an understanding to be achieved.

The research which we describe in this paper is at the heart of ongoing work towards developing a toolbox of concrete conditions which can play a similar role to that of the conefield conditions in the uniformly hyperbolic case, implying both the existence of an invariant measure and the property of non-uniform hyperbolicity essentially at the same time. The difficulties we mentioned before are even more significant in the context of Hénon-like systems because non-uniformly hyperbolicity cannot be expected to be persistent in parameter space and thus cannot be checked using an open set of conditions which only take into account a finite number of iterations. We shall try to describe here how these difficulties have been resolved, in a series of spectacular developments over the last quarter of a century or so.

1.2. **The Hénon family.** The Hénon family $H_{a,b}(x, y) = (1-ax^2+by, x)$ was introduced in the mid-seventies [Hen76] as a simplified model of the dynamics associated to the Poincaré first return map of the Lorenz system of ordinary differential equations [Lor63] and as the simplest model of a two-dimensional dynamical system exhibiting chaotic behavior. The numerical experiments carried out by Hénon suggested the presence of a non-periodic attractor for parameter values $a \approx 1.4$ and $b \approx 0.3$. However, numerics cannot tell a truly strange attractor from a periodic one having large period, and rigorous proofs that a strange aperiodic attractor does exist have proved to be extremely challenging. Hénon's original assertion remains unproved to-date for the parameter range he considered even though remarkable progress has been made in this direction.

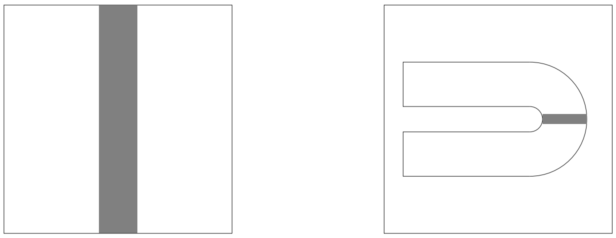

FIGURE 1. Folding behavior

The distinctive feature of these maps, which makes them a model for much more general systems, is the occurrence of "folds" as described in Figure 1: in the shaded region horizontal/expanding and vertical/contracting directions are, roughly, interchanged. This may give rise to tangencies between stable and unstable manifolds where expanding and contracting behavior gets mixed up and implies that if some hyperbolicity is present it will have to be strictly non-uniform and the dynamics will be structurally unstable.

1.2.1. *The case* $b = 0$. In the strongly dissipative limit $b = 0$, the Hénon family reduces to a family of quadratic one-dimensional maps and the fold reduces to a *critical point*. It is in this context that the first results appeared. Abundance of aperiodic and non-uniform hyperbolic behaviour, was first proved by Jakobson [Jak81] less than a quarter of a century ago in a paper which pioneered the parameter exclusion techniques for proving the existence of dynamical phenomena which occur for nowhere dense positive measure sets.

The starting point was the formulation of some geometrical condition which implies non-uniform hyperbolicity; in [Jak81] this was defined as the existence of an induced map with certain expansion and distortion properties. A conceptual breakthrough was the realization that since this condition requires information about all iterates of the map, it was not reasonable to try to prove it for a particular given map. Instead one should start with a family of maps for which some finite number of steps in the construction of the required induced map can be carried out. One then tries to take the construction further one step at a time, and at each step excludes from further consideration those parameters for which this cannot be done. The problem then reduces to showing that not all parameters are excluded in the limit and this is resolved by a probabilistic argument which shows that the proportion of excluded parameters decreases exponentially fast with n implying that the total measure of the exclusions is relatively small and a positive measure set of parameters remains for which the constructions can be carried out for all iterations. All corresponding maps are therefore non-uniformly hyperbolic.

There have been many generalizations of Jakobson's Theorem, using different geometric conditions to define the notion of a "good" parameter, and considering more general families of one-dimensional smooth maps [BenCar85, Ryc88, MelStr88, Tsu93, Tsu93a, ThiTreYou94, Cos98, Luz00, HomYou02] as well as maps with critical points and singularities with unbounded derivative [PacRovVia98, LuzTuc99, LuzVia00]. Many of these papers use an intermediate geometric condition formulated in terms of the properties of the orbits of the critical points which is sometimes easier to work with than the full induced map. It is then possible to show by independent arguments that the appropriate conditions on the orbits of the critical points imply the existence of an induced map and thus the existence of an invariant measure and non-uniform hyperbolicity.

1.2.2. *Hénon-like systems.* Extension of these results to the two-dimensional case requires a significant amount of new arguments and new ideas. Several issues will be discussed below when we make a more detailed comparison with the one-dimensional case. For the moment we just mention the "conceptual" problem mentioned above of what a good parameter looks like. It turns out that it is possible to generalize the one-dimensional approach mentioned above, formulated in terms of recurrence properties of the orbits of critical points. However, even the precise formulation of such a generalization is highly non-trivial and occupies a central part of the theory. One outstanding contribution of Benedicks and Carleson [BenCar91] was to invent a geometrical structure encompassing tangencies between stable and unstable leaves, which play the role of critical points, together with non-uniformly hyperbolic dynamics. They were then able to generalize the parameter exclusion argument to conclude that this structure does occur in the Hénon family $H_{a,b}$, for a positive Lebesgue measure set of parameters (a, b) with $b \approx 0$.

Shortly afterwards, [MorVia93] extended Benedicks and Carleson's approach to general Hénon-like families, thus freeing the arguments from any dependence on the explicit expression of the Hénon maps, and also established the connection between these systems and general bifurcation mechanisms such as homoclinic tangencies. Moreover, [Via93] extended the conclusions of [MorVia93] to arbitrary dimension. The ergodic theory of Hénon-like systems was then developed, including the existence of a Sinai-Ruelle-Bowen measure [BenYou93] (in particular proving that the attractors of [BenCar91, MorVia93] are indeed non-uniformly hyperbolic in the standard sense), exponential mixing [BenYou00, You98], and the basin property [BenVia01]. Moreover, [DiaRocVia96] extended [MorVia93] to the, more global, strange attractors arising from saddle-node cycles and, in doing so, observed that the original approach applies to perturbations of very general families of uni- or multimodal maps in one dimension, besides the quadratic family. Recently, [WanYou01] showed that the whole theory extends to such a generality, and also isolated a small set of conditions under which it works. Similar ideas have also been applied in related contexts in [Cos98, PumRod01, PalYoc01, WanYou02].

1.3. **General remarks and overview of the paper.** One key point in the construction in [BenCar91] is the notion of *dynamically defined critical point*, a highly non-trivial generalization of the notion of critical point in the one-dimensional context, and the associated notion of dynamically defined *finite time approximation* to a critical point. The definition of a good parameter is formulated in terms of the existence of a suitable set of such critical points satisfying certain hyperbolicity conditions along their forward orbits. However the very existence of the critical points is tied to their satisfying such hyperbolicity properties and thus to the parameter being a good parameter, and we are faced with another *impasse* analogous to the one discussed above.

The solution lies in the observation that a set of finite time *approximations* to these critical points can be defined for all parameter values in some sufficiently small parameter interval. One can then set up an inductive argument where a certain condition satisfied by the critical approximations implies that the approximations can be refined to a better approximation. Parameters for which the condition is not satisfied are excluded from further consideration. Then, as in the one dimensional case, one has to estimate the size of the exclusions at each step to conclude that there is a substantial set of parameters for which all critical approximations always satisfy the required condition and in particular converge to a "true" critical set which also satisfies these conditions.

The overall argument is set up as an induction which is quite involved, and the exposition in the original papers is occasionally terse, especially when describing the parameter exclusions procedure. More explanations on some important points have been provided subsequently, for instance in [PacRovVia98], where the handling of infinitely many critical points was formalized in detail, in a one-dimensional set-up. However, it has been suggested that it would be useful to have in a single text a conceptual reader-friendly survey of the whole procedure with particular emphasis on parameter exclusions. The present text is a response to that suggestion.

In Section 2 we briefly outline the main geometrical properties of the attractor for the Benedicks-Carleson "good" parameter values, including the definition of dynamically defined critical points. This corresponds to formulating precisely the conditions which determine the parameters which will be excluded at each step of the parameter exclusion argument. In the remainder of the paper we discuss the second stage: proving that the set of good parameters has positive Lebesgue measure.

Another text, with a similar goal, has been written at about the same time by Benedicks, Carleson [BenCar02], and another presentation of parameter exclusions is contained in [WanYou01] in a related more general setting. Our presentation is based on the original papers [BenCar91] and [MorVia93], although we present here a new (previously unpublished) formalization of the arguments by introducing the notion of an *extended parameter space* to keep track of the combinatorics of each individual critical point approximation at each stage and to make more explicit the effect of exclusions due to multiple critical points. This formalism was developed as part of ongoing joint work on Lorenz-like attractors [LuzVia] and was first announced in [Luz98].

2. Geometrical structure in dynamical space

In this section we review the basic geometric properties of a "good" parameter value, and introduce the notation and definitions required to set up the parameter exclusion argument. When this is not a source of confusion we will consider the parameter a to be fixed and will not mention it explicitly.

2.1. **The one-dimensional case.**

The argument in the one-dimensional case $\phi_a : x \mapsto 1 - ax^2$ breaks down into three basic steps.

2.1.1. *Uniform expansion outside a critical neighbourhood.* The first step is a manifestation of the general principle in one dimensional dynamics, proved by Mañé [Man85], according to which orbits behave in a uniformly hyperbolic fashion as long as they remain outside a neighborhood of the critical points and the periodic attractors. More specifically, we use

Proposition 1. *There exists a constant $\lambda > 1$ such that for every $\delta > 0$ there exists $a_0(\delta) < 2$ such that for every $a_0 \leq a \leq 2$, the dynamics of ϕ_a outside a δ-neighbourhood of the critical point $c = 0$ is uniformly expanding with expansion rate λ.*

Thus by choosing a parameter interval Ω sufficiently close to $a = 2$ we can work with maps which satisfy uniform expansion estimates, uniformly also in the parameter, outside any *arbitrarily small* neighbourhood of the critical point with an expansion coefficient λ which does not depend on the size of the neighbourhood. This fact is crucial to the whole sequel of the arguments.

2.1.2. *Bounded recurrence and non-uniform expansivity.* Once a constant $\delta > 0$, the corresponding critical neighbourhood Δ, and a suitable parameter interval Ω have been fixed, we define a good parameter a by the recurrence condition

$$(*) \qquad \sum_{\substack{1\leq j\leq n \\ c_j\in\Delta}} \log|c_j - c|^{-1} \leq \alpha n \quad \text{for all } n \geq 1,$$

where $c_j = \phi_a(c)$ are the iterates of the critical point, and $\alpha > 0$ is some small constant. We remark that this is different, but essentially equivalent to, the *basic assumption* and the *free period assumption* taken together, which are the conditions originally formulated in [BenCar91, MorVia93]. It is similar to the condition of [Tsu93b] and has proved particularly useful in [LuzTuc99, LuzVia00, LuzVia] where the presence of a discontinuity set requires an additional bounded recurrence condition which remarkably takes exactly the same form. Moreover, a straightforward calculation using the expansion estimates of Proposition 1, see [Luz00], shows that under this condition the critical orbit exhibits exponential growth of the derivative:

$$(\mathrm{EG}) \qquad |D\phi^n(\phi(c))| \geq e^{\kappa n} \quad \text{for all } n \geq 1,$$

for some constant $\kappa > 0$. By [ColEck83, NowStr88] condition (EG) implies that the corresponding map is non-uniformly hyperbolic.

2.1.3. *Parameter exclusions.* Thus the problem has been reduced to showing that many parameters in Ω satisfy the bounded recurrence condition $(*)$. This is essentially a consequence of Proposition 1 and the observation that the uniform expansion estimates given there transfer to expansion estimates for the derivatives with respect to the parameter. Thus the images of the critical orbit for different parameter values tend to be more and more "randomly" distributed and the probability of them falling very close to the critical point gets smaller and smaller. Thus the probability of satisfying the bounded recurrence conditions is positive even over all iterates. In section 3.1 we sketch the combinatorial construction and the estimates required to formalize this strategy. This is also a special case of the strategy applied to the two-dimensional case which will be discussed in some detail.

2.2. **The two-dimensional case.** In the two dimensional situation we can also break down the overall argument into three steps as above, although each one is significantly more involved. In particular, the very formulation of the recurrence condition $(*)$ requires substantial work and we concentrate on this issue here, leaving the issues related to the exclusion of parameters to the later sections.

2.2.1. *Uniform hyperbolicity outside a critical neighbourhood.* In two dimensions we define the critical neighbourhood Δ as a small vertical strip of width 2δ.

Proposition 2. *There exists a constant $\lambda > 1$ such that for every $\delta > 0$ there exists $b_0(\delta) > 0$ and $a_0(\delta) < 2$ such that for every $0 \leq b \leq b_0$ and $a_0 \leq a \leq 2$, the dynamics of $\Phi_{a,b}$ outside a vertical strip Δ around $x = 0$ is uniformly hyperbolic with expansion rate λ and contraction rate b.*

The proof of this proposition relies on the fact that b is small and thus $\Phi_{a,b}$ is close to the one dimensional family of maps for which the estimates of Proposition 1 hold. One other place in which the strong dissipativeness assumption $b \approx 0$ is used is for estimating the cardinality of the critical set at each step n, as we shall see.

2.2.2. *Bounded recurrence and non-uniform hyperbolicity.* We now suppose that the constants δ, a_0, b_0 are fixed and that for some $0 < b \leq b_0$ we have chosen an interval $\Omega \subset (a_0, 2)$ of a-parameters. We want to formulate some condition with which to characterize the good parameters in Ω. Our aim is to remain as close as possible to the one-dimensional formulation, and to identify a *critical set* $\mathcal{C} \subset \Delta$ containing an infinite number of *critical points* such that each point satisfies the bounded recurrence condition

$$(*) \qquad \sum_{\substack{1 \leq j \leq n \\ c_j \in \Delta}} \log |c_j - \mathcal{C}|^{-1} \leq \alpha n \quad \text{for all } n \geq 1,$$

for some $\alpha > 0$ sufficiently small. Here the distance $|c_j - \mathcal{C}|$ does not refer exactly to the standard Hausdorff distance between the point c_j and the set $\mathcal{C}$ but to the distance between c_j and some particular point of $\mathcal{C}$ which is chosen by a procedure to be discussed below.

A not-so-straightforward calculation (which is in fact a large part of the proof of Theorem 1 below) shows that this bounded recurrence condition implies the two dimensional analogue of the exponential growth condition:

$$\text{(EG)} \qquad |D\phi^n_{\phi(c)} w| \ge e^{\kappa n} \quad \text{for all } n \ge 1,$$

for some constant $\kappa > 0$ and for a horizontal or "almost horizontal" vector w and for any critical point $c \in \mathcal{C}$. We shall not discuss here why $(*)$ (together with the uniform hyperbolicity conditions outside Δ) is also sufficient to guarantee the global non-uniform hyperbolicity of the corresponding map, and refer to the papers [BenYou93, BenYou00, BenVia01, HolLuz] for the construction of the Sinai-Ruelle-Bowen measure under these, or other essentially equivalent, conditions. We also postpone the discussion of the verification that this condition is satisfied by many parameters in Ω to the following sections. Instead, in the remaining parts of this section we focus on the problem of the definition of the critical set $\mathcal{C}$.

2.2.3. *Dynamically defined critical points.* Since Φ is a diffeomorphism its Jacobian never vanishes and thus there are no a priori given critical points as in the one-dimensional case. However something "bad" does happen in the critical region because the uniformly hyperbolic estimates outside Δ cannot be extended to Δ. Geometrically this is due to the folds described above, which are reflected at the level of the differential by the fact that (roughly) horizontal vectors get mapped to (roughly) vertical vectors. Dynamically this is problematic because the (roughly) horizontal direction is expanding while the (roughly) vertical direction is strongly contracting. Thus any expansion gained over several iterates may be lost during the iterates following a return to Δ. It is necessary to have a finer control over the way in which vectors rotate in order to show that after some bounded time and some bounded contraction, they return to a (roughly) horizontal direction and start expanding again.

One can hope to characterize geometrically as *critical points* those points on which the fold has the most dramatic effect, i.e. those for which the almost horizontal vector which is *precisely* in the expanding direction, i.e. that vector which is tangent to an unstable manifold, gets mapped to the almost vertical vector which is precisely in a contracting direction, i.e. tangent to a stable manifold. This turns out indeed to be the case and the set of critical points $\mathcal{C}$ is formed by a set of points of tangential intersection between some stable and some unstable manifolds. However, as mentioned above, such manifolds cannot be assumed to exist for all parameter values, and thus the construction requires an inductive approximation argument by *finite time critical points* which are also tangencies between pieces of unstable manifold and some *finite time stable leaves* to be described below. In the following sections we shall explain in more detail the local geometry associated to critical points and their approximations. For the moment we clarify the formal structure of the induction.

2.2.4. *The induction.* We start by defining a critical set $\mathcal{C}^{(0)}$ and then suppose inductively that a set $\mathcal{C}^{(k)}$ of critical points of order k is defined for $1 \le k \le n-1$, such that each critical point satisfies certain hyperbolicity conditions $(\text{EG})_{n-1}$

which are finite time versions of condition (EG) given above, together with a condition $(\mathrm{BD})_{n-1}$ of bounded distortion in a neighbourhood to be stated below. The existence of the set $\mathcal{C}^{(n-1)}$ allows us to *state* a condition $(*)_{n-1}$ on the recurrence of points of $\mathcal{C}^{(n-1)}$ to the set itself. This is a finite time version of condition $(*)$ given above, with $\mathcal{C}^{(n-1)}$ replacing $\mathcal{C}$.

The main inductive step then consists of showing that if all points of $\mathcal{C}^{(n-1)}$ satisfy this recurrence condition, then conditions $(\mathrm{EG})_n$ and $(\mathrm{BD})_n$ hold in a neighborhood of $\mathcal{C}^{(n-1)}$. Now the fact that these conditions hold is enough to allow us to *define* a new critical set $\mathcal{C}^{(n)}$ close enough in the Hausdorff metric to $\mathcal{C}^{(n-1)}$ so that its points also automatically satisfy $(*)_{n-1}$ and $(\mathrm{EG})_n$ and $(\mathrm{BD})_n$. This completes the inductive step. If all points of $\mathcal{C}^{(n)}$ satisfy $(*)_n$ the argument can be repeated to obtain a critical set $\mathcal{C}^{(n+1)}$ and so on. The sets $\mathcal{C}^{(n)}$ eventually converge to a critical set $\mathcal{C}$ which consists of tangencies between stable and unstable leaves. We summarize this reasoning in the following

Theorem 1. *Suppose that for some $a \in \Omega$ a finite critical set $\mathcal{C}^{(n-1)} \subset \Delta$ has been defined.*

1. *If $\mathcal{C}^{(n-1)}$ satisfies $(*)_{n-1}$ then it satisfies $(EG)_n$ and $(BD)_n$;*
2. *If $\mathcal{C}^{(n-1)}$ satisfies $(*)_{n-1}$, $(EG)_n$ and $(BD)_n$, then a finite set $\mathcal{C}^{(n)}$ can be defined whose elements are critical points of order n and satisfy $(*)_{n-1}$, $(EG)_n$ and $(BD)_n$. Moreover $\mathcal{C}^{(n)}$ and $\mathcal{C}^{(n-1)}$ are exponentially close in n in the Hausdorff sense.*

In particular, if $(*)_n$ continues to hold for increasing values of n the set of critical approximations converges to a set $\mathcal{C}$ of true *critical points satisfying $(*)_n$, $(EG)_n$ and $(BD)_n$ for all n.*

This result tells us that the bounded recurrence conditions $(*)_n$ are exactly the conditions we need to define a good parameter, and allows us to focus the parameter exclusion argument on the recurrence of the critical approximations at each stage n. The framework is henceforth similar to the one-dimensional case apart from the additional complications coming from the requirement to prove the inductive step and the fact that the exclusions need to be made with respect to each critical point.

2.2.5. *Why do we need a critical* set *?* The reason one needs a whole critical set, and not just a single critical point, is the way iterates hitting the critical region Δ are compensated for in order to recover exponential growth. Whenever a point $z \in \mathcal{C}^{(n)}$ returns to Δ at time n one looks for some point $\zeta \in \mathcal{C}^{(n)}$ close to $\Phi^n(z)$, and transmits information about hyperbolicity on the first iterates of ζ, inductively, to the stretch of orbit z that follows the return. This works out well if the two points ζ and $\Phi^n(z)$ are *in tangential position*, that is, contained in the same almost horizontal curve. For this, in general, ζ must be different from z. This step forces the critical sets $\mathcal{C}^{(n)}$ to be fairly large, indeed, their cardinality has to go to infinity as $n \to \infty$. Fortunately, as we are going to see, one can do with a sequence of critical sets whose cardinality grows slowly enough, as long as one supposes that b is small. See Section 2.8. In the sequel we define more formally the notion of critical point and sketch the argument in the proof of Theorem 1.

2.2.6. *Constants and notation.* First we introduce some notation which will be used extensively below. Given a point ξ_0 and a vector w_0 we denote $\xi_j = \Phi^j(\xi_0)$ and $w_j = D\Phi^j(\xi_0)w_0$ for all $j \geq 0$. The vector w_0 will be assumed to have slope $\leq 1/10$ unless we explicitly mention otherwise. We fix $\log 2 > \kappa \gg \alpha \gg \delta > 0$. These constants have the following meaning:

- κ is a lower bound for the hyperbolicity of the two-dimensional map in condition (EG)$_n$;
- α is used in formulating the recurrence condition $(*)_n$, as well as in defining the notion of binding;
- δ defines the width of the critical neighborhood Δ.

The parameter interval Ω is chosen close enough to $a = 2$ depending on κ, α, δ. The perturbation size b is taken to be small, depending on all the previous choices. A few ancillary constants appear in the course of the arguments, related to the previous ones. $K = 5$ is an upper bound for the C^3 norm of our maps. A small $\rho > 0$ e.g. $\rho = (10K^2)^{-2}$ is used to describe the radius of an admissible segment of unstable manifold around every critical point. We use $\tau > 0$ in the treatment of the recurrence condition. It is chosen in Section 4.3.4, much smaller than α and independent of δ. Constants $\kappa_1, \ldots, \kappa_4 > 0$ depending only on κ describe expansion during binding periods. And $\theta = C/|\log b|$ is used when bounding the number of critical points, where $C > 0$ is some large constant e.g. $C = 10 \log |\rho|$. Notice that $\theta \to 0$ as $b \to 0$.

2.3. **Hyperbolic coordinates.** The definition of finite time critical point is based on the notion of hyperbolic coordinates which we discuss in this section.

2.3.1. *Non-conformal linear maps.* Suppose that the derivative map $D\Phi^k_{\xi_0}$ at some point ξ_0 is non-conformal (a very mild kind of hyperbolicity). Then there are well defined *orthogonal* subspaces $E^{(k)}(\xi_0)$ and $F^{(k)}(\xi_0)$ of the tangent space, for which vectors are *most contracted* and *most expanded* respectively by $D\Phi^k_{\xi_0}$. This follows by the elementary observation from linear algebra that a linear map L which sends the unit circle $\mathcal{S}^1$ to an ellipse $L(\mathcal{S}^1) \neq \mathcal{S}^1$ defines two *orthogonal* vectors e and f whose images map to the minor and major axis of the ellipse respectively. The directions $E^{(k)}$ and $F^{(k)}$ can in principle be obtained explicitly as solutions to the differential equation $d|D\Phi^k_{\xi_0} \cdot (\sin\theta, \cos\theta)|/d\theta = 0$ which gives

$$\tan 2\theta = \frac{2(\partial_x \Phi_1^k \partial_y \Phi_1^k + \partial_x \Phi_2^k \partial_y \Phi_2^k)}{(\partial_x \Phi_1^k)^2 + (\partial_x \Phi_2^k)^2 - (\partial_y \Phi_1^k)^2 - (\partial_y \Phi_2^k)^2}. \tag{1}$$

This shows that the direction fields given by $E^{(k)}$ and $F^{(k)}$ depend smoothly on the base point and extend to some neighbourhood of ξ_0 on which the derivative continues to satisfy the required non-conformality. Therefore they can be integrated to give two smooth *orthogonal* foliations $\mathcal{E}^{(k)}$ and $\mathcal{F}^{(k)}$. The individual leaves of these foliations are the natural *finite time version* of classical local stable and unstable manifolds. Indeed they are canonically defined precisely by the property that they are the most contracted and most expanded respectively for a certain finite number of iterations. The estimates to be developed below will show that in certain situations the stable leaves $\mathcal{E}^{(k)}(\xi_0)$ converge as $k \to \infty$ to the classical local stable manifold $W^s_\varepsilon(\xi_0)$.

The notions of most contracted directions and most contracted integral curves play a central role in the original papers [BenCar91, MorVia93], although they are not exploited as systematically as in here. Our formalism was developed in the context of Lorenz-like systems [LuzVia, HolLuz] and leads to a significant simplification of several steps of the argument, in particular it plays an important role in allowing us to formulate the induction of Theorem 1 in such a straightforward way. The idea of approximating the classical local stable manifold by finite time local stable manifolds has been further refined in [HolLuz03] where it forms the basis of a new approach to the local stable manifold theorem in more classical contexts.

2.3.2. *Notation.* Before explaining how these foliations are used to define the notion of critical point, we introduce some more notation. We let $H^{(k)}(\xi_0) = \{F^{(k)}(\xi_0), E^{(k)}(\xi_0)\}$ denote the coordinate system in the tangent space at ξ_0 determined by the directions $F^{(k)}$ and $E^{(k)}$ and by $\mathcal{H}^{(k)} = \{\mathcal{F}^{(k)}, \mathcal{E}^{(k)}\}$ the family of such coordinate systems in the neighbourhood in which they are defined. We also let $f^{(k)}(\xi_0)$ and $e^{(k)}(\xi_0)$ denote unit vectors in the directions $F^{(k)}(\xi_0)$ and $E^{(k)}(\xi_0)$ respectively. For $k = 0$ we let $F^{(0)}$ and $E^{(0)}$ denote the horizontal and vertical direction respectively and thus $\mathcal{F}^{(0)}$ and $\mathcal{E}^{(0)}$ are horizontal and vertical foliations respectively. Notice that $\mathcal{H}^{(k)}$ can be thought of as living in the tangent bundle as a family of coordinate systems, or in the phase space as a foliation; we will not distinguish formally between these two interpretations. For all the objects defined above we use a subscript j to denote their images under the map Φ^j, or the differential map $D\Phi^j$ as appropriate. In particular we let $e^{(k)}_j = D\Phi^j(e^{(k)})$, $f^{(k)}_j = D\Phi^j(f^{(k)})$ and $\mathcal{H}^{(k)}_j = \Phi^j \mathcal{H}^{(k)}$. Notice moreover, that $\mathcal{H}^{(k)}_k$ is also an orthogonal system of coordinates, whereas $\mathcal{H}^{(k)}_j$ is *not* orthogonal in general for $j \neq k$. Notice that the differential map $D\Phi^k_{\xi_0}$, expressed as a matrix with respect to the hyperbolic coordinates $\mathcal{H}^{(k)}$ and $\mathcal{H}^{(k)}_k$, has the diagonal form

$$D\Phi^k_{\xi_0} = \begin{pmatrix} \|f^{(k)}_k(\xi_0)\| & 0 \\ 0 & \|e^{(k)}_k(\xi_0)\| \end{pmatrix}.$$

Finally, for $j, k \geq 0$, we consider the angle between the leaves of $\mathcal{H}^{(k)}$ and $\mathcal{H}^{(k+1)}$ at some point ξ_0 at which both foliations are defined, and the corresponding angle between the images:

$$\theta^k = \theta^{(k)}(\xi_0) = \sphericalangle(e^{(k)}(\xi_0), e^{(k+1)}(\xi_0)) \quad \text{and} \quad \theta^{(k)}_j = D\Phi^j_{\xi_0}\theta^{(k)}(\xi_0);$$

as well as the derivatives of these angles with respect to the base point ξ_0:

$$D\theta^{(k)} = D_{\xi_0}\theta^{(k)}(\xi_0) \quad \text{and} \quad D\theta^{(k)}_j = D_{\xi_0}\theta^{(k)}_j(\xi_0)$$

2.3.3. *Convergence of hyperbolic coordinates.* For $k=1$ and $\xi_0 \notin \Delta$, relation (1) implies

$$|\theta^{(0)}| = \mathcal{O}(b) \tag{2}$$

Thus by taking b small we can guarantee that the stable and unstable foliations $\mathcal{E}^{(1)}, \mathcal{F}^{(1)}$ are arbitrarily close to the vertical and horizontal foliations respectively. It turns out that the angle between successive contractive directions is related to the hyperbolicity along the orbit in question, and we get a quite general estimate which says that as long as the inductive assumption (EG)$_k$ continues to be satisfied,we have

$$|\theta^{(k)}| = \mathcal{O}(b^k). \tag{3}$$

In particular, in the limit they converge to a well defined direction which is contracted by all forward iterates. This convergence is in the C^1 norm (even C^r for any fixed r) if b is sufficiently small.

2.4. **Critical points.** We are now ready to define the notion of *critical point of order* k, generally denoted by $z^{(k)}$. The definition will be given inductively. The set of such points will be denoted $\mathcal{C}^{(k)}$ and is contained in the critical neighborhood Δ defined by

$$\Delta = \{(x,y) : |x| \le \delta\}.$$

All critical points are on the global unstable manifold $W^u(P)$ of the hyperbolic fixed point $P \approx (1/2, 0)$; notice that $W^u(P)$ has many folds and $W^u(P) \cap \Delta$ has infinitely many connected components. Fix a compact admissible neighborhood W of P inside $W^u(P)$ with length $\lesssim 2$ and extending to the left of P across the critical region Δ. By definition, the intersection of W with the vertical line $\{x=0\}$ is the unique critical point $z^{(0)}$ of order zero.

2.4.1. *First step of the induction.* Now consider the curve $W_0 = \Phi(W \cap \Delta)$. As described above, the map Φ gives rise to a *fold* precisely in Δ and therefore W_0 is *folded* horizontally. The quadratic nature of Φ guarantees that it is in fact a quadratic parabola (positive curvature) laying on its side. Notice moreover that $W_0 \cap \Delta = \emptyset$ and therefore, contractive directions $e^{(1)}_{\xi_0}$ of order 1 are defined at each point $\xi_0 \in W_0$. The smoothness of these directions, the fact that they are essentially vertical, see (2), and the fact that W_0 is quadratic, guarantee that there must be a point $z_0^{(1)} \in W_0$ which is tangent to a contracting leaf of the foliation $\mathcal{E}^{(1)}$. A bit more work shows that the leaves of the contracting foliation $\mathcal{E}^{(1)}$ have small curvature which further implies that there can be at most one point of tangency and that this tangency is quadratic.

We define $z^{(1)} = \Phi^{-1}(z_0^{(1)})$ as the unique critical point of order 1, i.e. the unique element of the set $\mathcal{C}^{(1)}$, and $z_0^{(1)}$ as the corresponding critical value. Notice that taking b small implies that the "tip" of the parabola is close to $z_0^{(0)}$ and that the stable leaves are almost vertical. Therefore the point of tangency $z_0^{(1)}$ is close to $z_0^{(0)}$ and the distance between the critical points $z^{(0)}$ and $z^{(1)}$ is $\mathcal{O}(b)$. This constitutes the first step in the inductive definition of the critical set.

The characteristic feature of a critical point $z^{(k)}$ of order k will be that the unstable manifold is tangent to the stable foliation of order k at the critical value $z_0^{(k)} = \Phi(z^{(k)})$. More formally, we assume that for each $k = 1, \dots, n-1$, the critical set $\mathcal{C}^{(k)}$ contains points $z^{(k)}$ with the following properties.

2.4.2. *Generation of critical points.* We introduce the notion of the *generation* of a critical point which is quite different from the notion of the *order* of the critical point. We say that the critical point k is of generation $g \geq 1$ if it belongs to $\Phi^g(W) \setminus \Phi^{g-1}(W)$, where W is the component of $W^u(P)$ defined above, of length $\lesssim 2$ containing the fixed point P and crossing Δ completely. By convention we say that a critical point is of generation 0 if it belongs to W. The critical points $z^{(0)}$ and $z^{(1)}$ defined above, are critical points of generation 0 and so are all their refinements $z^{(k)}$, $k \geq 1$ to be defined below. As part of the construction we impose the condition that critical points of order k must be of generation $\leq \theta k$, where $\theta \approx 1/\log b^{-1}$. In particular the only admissible critical points of order $\lesssim \log b^{-1}$ are those of generation 0.

2.4.3. *Admissible segments.* The neighborhood ω of radius $\rho^{\theta k}$ around $z^{(k)}$ inside the unstable manifold $W^u(P)$ is an admissible curve contained in $\Phi^{\theta k}(W) \cap \Delta$; we say that a curve is almost horizontal, or *admissible*, if it is a graph $\{x, y(x)\}$ with $|y'| \leq 1/10$, $|y''| \leq 1/10$. Moreover, $z^{(k)}$ is the unique element of $\mathcal{C}^{(k)}$ in ω (we really mean the iterate Φ^ℓ with $\ell =$ integer part of θk, but do not want to overload the notations). Notice that this condition is satisfied for $k = 0, 1$ since W can be chosen to be admissible for b sufficiently small.

The fact that each critical point has some minimum space on either side, on which no other critical points lie, and that the critical points up to order k must lie on a piece of $W^{(u)}(p)$ of finite length (since they are of generation $\leq \theta k$) implies a bound on the possible number of critical points of order k. This bound will be made explicit below and will play an important part in the estimates.

2.4.4. *Bound neighbourhoods.* For all $n - 1 \geq k \geq j \geq 0$ and $z^{(k)} \in \mathcal{C}^{(k)}$ we let $z_j^{(k)} = \Phi^{j+1}(z^{(k)})$ and

$$B^{(j)}(z_0^{(k)}) = \{\xi_0 : |\xi_i - z_i^{(k)}| \leq e^{-2\alpha i} + 10^{-k} \text{ for all } i \in [0, j]\}.$$

This is a way of formalising the idea that there is a set of points which *shadow*, or remain *bound* to, the orbit of $z^{(k)}$ up to time j. The sequence of iterates $0, \dots, k$ is divided into *free iterates* and *bound iterates*: j is bound if it belongs to the binding period associated to a return $\nu < j \leq k$, i.e. all the points bound to $z^{(k)}$ up to time k are also bound to another critical point ζ between the iterates $\nu + 1$ and j. This is explained precisely in Section 2.6.1 below. If j is not a bound iterate, it is called a free iterate. By convention 0 is a free iterate.

2.4.5. *Hyperbolicity and distortion.* We assume that the differential map $D\Phi^k$ satisfies uniformly hyperbolic estimates on the bound neighborhood of every $z^{(k)}$:

$$\text{(EG)}_k \qquad \|w_j(\xi_0)\| := \|D\Phi^j_{\xi_0}(w_0)\| \geq e^{\kappa j}.$$

for all $j \in [0,k]$, any $\xi_0 \in B^{(k)}(z_0^{(k)})$, and any tangent vector w_0 with slope $\leq 1/10$. In particular the stable and unstable foliations $\mathcal{E}^{(j)}, \mathcal{F}^{(j)}$ are defined in the whole of $B^{(j)}(z_0^{(k)})$ and, as part of the inductive assumptions, the leaves of $\mathcal{F}^{(k)}$ are admissible curves. Let $f^{(j)}$ be a norm 1 vector field tangent to the leaves of $\mathcal{F}^{(j)}$ respectively, and $f_i^{(j)}$ its image under the differential map $D\Phi^i$, for $i \geq 0$. For every free iterate $j < k$ of the critical point $z^{(k)} \in \mathcal{C}^{(k)}$, the hyperbolic coordinates $\mathcal{H}^{(j)}$ as well as their images $\mathcal{H}_j^{(j)}$ are C^2 close to the standard coordinate system $\mathcal{H}^{(0)}$ (i.e. the unstable leaves are admissible) and satisfy uniform distortion bounds: there exists a constant $D_0 > 0$ such that for all points $\xi_0, \eta_0 \in B^{(k)}(z_0^{(k)})$ we have

$$\text{(BD)}_k \qquad \begin{gathered} \log \frac{\|f_j^{(j)}(\xi_0)\|}{\|f_j^{(j)}(\eta_0)\|} \leq D_0 \sum_{i<j} \frac{|\xi_i - \eta_i|}{e^{-\alpha i}} \\ \text{and} \\ \sphericalangle(f_j^{(j)}(\xi_0), f_j^{(j)}(\eta_0)) \leq D_0 \frac{|\xi_j - \eta_j|}{e^{-\alpha j}}. \end{gathered}$$

The bounded distortion property says that the orbits of all these points are in a sense indistinguishable from an analytic point of view.

2.4.6. *Quadratic tangencies.* As mentioned above, the exponential growth condition guarantees that hyperbolic coordinates $\mathcal{H}^{(k)}$ are defined in the whole of the bound neighbourhood $B^{(k)}(z_0^{(k)})$. The critical point $z^{(k)}$ is then characterized by the property that the corresponding critical value $z_0^{(k)}$ is a point of tangency between the image γ_0 of the admissible curve γ containing $z^{(k)}$ and a the leaf $\mathcal{E}^{(k)}(z_0^{(k)})$ of the stable foliation of order k. Again it is possible to show that the curvature of γ_0 is much larger than the curvature of the stable leaves of $\mathcal{E}^{(k)}$ and thus this tangency is unique and quadratic.

2.4.7. *Nested neighborhoods and ancestors.* There exists a sequence

$$z^{(k-1)}, \ldots, z^{(0)}$$

of *ancestors* of $z^{(k)}$ such that for $i = 0, \ldots, k-1$ we have $z^{(i)} \in \mathcal{C}^{(i)}$ and

$$B^{(i+1)}(z_0^{(i+1)}) \subset B^{(i)}(z_0^{(i)}).$$

Notice that the term 10^{-k} in the definition of the bound neighbourhoods is much smaller than $e^{-2\alpha i}$, for any $i \leq k$, and so it is negligible from a geometrical point of view. It is introduced for formal reasons only, to ensure this nested property, see Section 2.7.1.

2.4.8. *True critical points.* The set of critical points $\mathcal{C}$ is obtained as the set of limit points of any sequence $\{z^{(k)}\}$ with $z^{(k)} \in \mathcal{C}^{(k)}$ and such that $z^{(k-1)}, \ldots, z^{(0)}$ are ancestors of $z^{(k)}$ for each k.

2.5. **Bounded recurrence.** We assume that the critical sets $\mathcal{C}^{(k)}$ are defined and satisfy the conditions stated above for all $k \le n-1$ and explain how to formulate a bounded recurrence condition $(*)_{n-1}$ on the set $\mathcal{C}^{(n-1)}$.

2.5.1. *The recurrence condition.* Let $0 \le \nu \le k$ be a free iterate and $z^{(\nu)}$ be the corresponding ancestor of $z^{(k)}$. If the ν'th image of $B^{(\nu)}(z_0^{(\nu)})$ intersects Δ we say that ν is a *free return* for $z^{(k)}$. Then there is an algorithm, the *capture argument*, which associates to $z_\nu^{(k)}$ a particular critical point $\zeta^{(\nu)} \in \mathcal{C}^{(\nu)}$ in *tangential position* to it. We just give a snapshot of this algorithm at time ν, referring the reader to [BenCar91, § 6] or [MorVia93, § 9] for the detailed construction.

As part of the argument, one constructs a whole sequence of candidates $\zeta_{(g_j)}$ which are critical points sitting on admissible segments of radii ρ^{g_j} inside $\Phi^{g_j}(W)$ such that the *vertical* distance between $z_\nu^{(k)}$ and $\zeta_{(g_j)}$ is $\le b^{cg_j}$ as shown in Figure 2. These points are defined for an increasing sequence of g_i which is not too sparse:

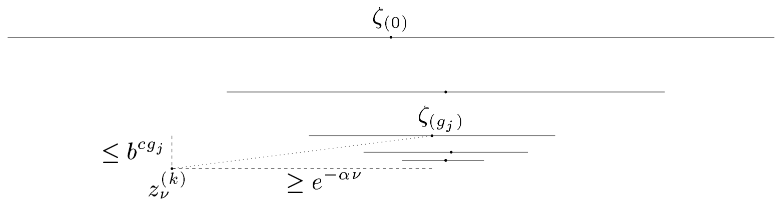

FIGURE 2. Looking for a binding point

$g_{i+1} \le 3g_i$. Then one chooses as the binding point $\zeta^{(\nu)} = \zeta_{(g_j)}$ where g_j is largest such that $\zeta_{(g_j)}$ is defined and (the constant $\theta = C/|\log b|$ was introduced in Section 2.2.6)

$$g_j \le \theta\nu\,. \tag{4}$$

This condition will be explained in Section 2.5.2.

Then we define the "distance" of $z_\nu^{(k)}$ from the critical set $\mathcal{C}^{(\nu)}$ as the minimum distance between ξ_ν and $\zeta^{(\nu)}$ over all points ξ_ν where $\xi_0 \in B^{(k)}(z_0^{(k)})$:

$$d(z_\nu^{(k)}) := |z_\nu^{(k)} - \mathcal{C}^{(\nu)}| := \min_{\xi_0 \in B^{(k)}(z_0^{(k)})} |\xi_\nu - \zeta^{(\nu)}|$$

With the notion of distance to the critical set defined above, we can formulate precisely the bounded recurrence condition

$$(*)_k \qquad \sum_{\substack{\text{free returns}\\ \nu \le k}} \log d(z_\nu^{(k)})^{-1} \le \alpha k.$$

Notice that this implies in particular $d(z_\nu^{(k)}) \ge e^{-\alpha\nu}$ and even $d(\xi_\nu) \ge e^{-\alpha\nu}$ for all $\xi_0 \in B^{(k)}(z_0^{(k)})$ and for all $\nu \le k$. We assume that all critical sets $\mathcal{C}^{(k)}$ satisfy condition $(*)_k$ (as well as (EG)$_k$, (BD)$_k$ and the other conditions given above, for all $k \le n-1$ and prove that this implies that conditions (EG)$_n$ and (BD)$_n$ hold.

2.5.2. *Tangential position.* A key consequence of the bounded recurrence condition and the capture argument outlined in Section 2.5.1 is that the binding point $\zeta^{(k)}$ and ξ_k are in *tangential position* for all $\xi_0 \in B^{(k)}(z_0^{(k)})$: there exists an admissible curve γ which is tangent to the vector $w_k(\xi_0)$ at ξ_k and tangent to the unstable manifold at $\zeta^{(k)}$. In particular, the critical point $\zeta^{(k)}$ chosen via the capture argument has essentially the same vertical coordinate as any of these ξ_k, including the critical iterate $z_k^{(k)}$.

Indeed, the bounded recurrence condition $(*)_k$ implies that the horizontal distance from z_k to the binding point is $\geq e^{-\alpha k}$. Hence, to ensure tangential position we have the choice of any $\zeta_{(g_j)}$ with

$$e^{-\alpha k} \gg b^{c g_j} \quad \text{or equivalently} \quad g_j \geq \frac{\text{const}}{|\log b|}\, k.$$

This shows, in other words, that it is sufficient to consider critical points of generations $g \leq \text{const}\ k/|\log b| = \theta k$ to guarantee the existence of one in tangential position. Indeed, this is how the expression $\theta = \theta(b)$ and condition (4) come about. As we shall see below, this also guarantees that the number of critical points of a given order are not too many to destroy the parameter exclusion estimates.

The reason being in tangential position is so crucial is that it allows for estimates at returns which are very much the same as in the one-dimensional situation. In particular, the "loss of expansion" is roughly proportional to the distance to the binding critical point. See [BenCar91, § 7], [MorVia93, § 9] and Section 2.6.2 below.

Remark 1. In [BenCar91, MorVia93] the critical set is constructed in such a way that the tangential position property at free returns is satisfied for the critical points themselves. One main contribution in [BenYou93] was to show that the capture argument works for essentially any other point in $W^{(u)}(p)$ as well and this implies the existence of a hyperbolic Sinai-Ruelle-Bowen measure. Further results such as exponential decay of correlations [BenYou00] and other hyperbolicity and topological properties [WanYou01] ultimately rely on this fact. Moreover, [BenVia01] went one step further and proved that for Lebesgue almost all points *in the basin of attraction* returns are *eventually* tangential. This is crucial in their proof that the basin has "no holes": the time average of Lebesgue almost every point (not just a positive measure subset) coincides with the Sinai-Ruelle-Bowen measure.

2.6. **Hyperbolicity and distortion at time n.** We outline the proof of the first part of Theorem 1 where the bounded recurrence condition on the critical set $\mathcal{C}^{(n-1)}$ is shown to imply some hyperbolicity and distortion estimates in a neighbourhood of each point of $\mathcal{C}^{(n-1)}$ up to time n. The situation we have to worry about is when the critical point has a return at time $n-1$, otherwise the calculations are relatively straightforward. In the case of a return however, as we mentioned above, vectors get rotated and end up in almost vertical directions which are then violently contracted for many iterations, giving rise to a possibly unbounded loss of expansion accumulated up to time n. The idea therefore is to use condition $(*)_{n-1}$ to control the effect of these returns. We assume that $n-1$ is a free return as in the previous section and let $\zeta = \zeta^{(n-1)}$ denote the corresponding associated critical point.

2.6.1. *Binding periods.* Our inductive assumptions imply that hyperbolic coordinates $\mathcal{H}^{(k)}$ are defined in the neighbourhoods $B^{(k)}(\zeta_0)$ for all $k \leq n-1$. Notice that these bound neighbourhoods shrink as k increases, but start off relatively large for small values of k. Therefore there must be some values of k for which the image $\gamma_0 = \Phi(\gamma) \in B^{(k)}(\zeta_0)$. We denote by p the *largest* such k. In principle we do not know that $p < n-1$ but it is not difficult to prove that in fact that $p \sim |\log d(z_{n-1}^{(n-1)})|$. In particular, under condition $(*)_{n-1}$ we get $p \sim \alpha n \ll n$ (fixing α small). Thus the point ξ_n will *shadow* ζ_0 for exactly p iterations. We say that p is the length of the *binding period* associated to the return of z_0 to Δ at time $n-1$.

2.6.2. *Local geometry.* We now want to analyse carefully the geometry of γ_0 and $w_n(\xi_0)$ with respect to the hyperbolic coordinates $\mathcal{H}^{(p)}$. The information we have is that γ_0 is tangent at ζ_0 to a stable leaf $\mathcal{E}^{(n-1)}(\zeta_0)$ and that the curve γ_0 is quadratic with respect to the coordinate system $\mathcal{H}^{(n-1)}$. Now suppose for the moment that $p = n-1$. Then the quadratic nature of γ_0 with respect to $\mathcal{H}^{(n-1)}$ implies that the *slope* of $w_n(\xi_0)$ *in these coordinates* is related to the distance between ξ_{n-1} and ζ and more specifically the "horizontal" component of $w_n(\xi_0)$, that is, the component in the direction of $f^{(n-1)}(\xi_n)$ is proportional to $d(z_{n-1}^{(n-1)})$.

These estimates do not apply immediately to hyperbolic coordinates for arbitrary $p \leq n-1$, for example they may not apply to the standard coordinates $\mathcal{H}^{(0)}$ as the $w_n(\xi_0)$ may actually be completely vertical in these coordinates and therefore have no horizontal component. Nevertheless it follows from (3) that the angle between leaves associated to $\mathcal{H}^{(n-1)}$ and $\mathcal{H}^{(p)}$ for $p \leq n-1$ is of order b^p. Moreover $p \sim \log d(z_{n-1}^{(n-1)})^{-1} \ll \log d(z_{n-1}^{(n-1)})/\log b$ and therefore $b^p \ll d(z_{n-1}^{(n-1)})$ and therefore the length of the horizontal component in the coordinates $\mathcal{H}^{(p)}$ is essentially the same in $\mathcal{H}^{(n-1)}$.

2.6.3. *Recovering hyperbolicity.* The fact that the horizontal component of $w_n(\xi_0)$ (in hyperbolic coordinates) is proportional to $d(z_{n-1}^{(n-1)})$ is a two-dimensional analogue of the simple fact that in the one-dimensional case, the loss of derivative incurred after a return to Δ is proportional to the distance to the critical point. Thus, even though the vector $w_n(\xi_0)$ may be very close to vertical (in fact it may be vertical in the standard coordinates) and therefore suffer strong contraction for arbitrarily many iterates, we do not need to worry about the contraction because we know that it has a component of strictly positive length proportional to $d(z_{n-1}^{(n-1)})$ and thus of the order of $e^{-\alpha n}$ in the "horizontal" direction and this component is being expanded, providing us with a lower bound for the real size of the vector. Using the inductive assumptions we can show that an average exponential rate of growth is recovered by the end of the binding period:

$$\|D\Phi^{p+1}(\xi_{n-1})w_{n-1}\| \geq d(\xi_\nu)^{-\kappa_1} \geq e^{\kappa_2 p} \gg 1, \tag{5}$$

for all $\xi \in B^{(n+p)}(z_0^{(k)})$, where the constants $\kappa_1, \kappa_2 > 0$ depend only on κ. In fact the strong contraction is useful at this point because it implies that the "vertical" component, i.e. the component in the direction of $e^{(n-1)}(\xi_n)$ is shrinking very fast and this implies that the slope of the vector is decreasing very fast and that it returns to an almost horizontal position very quickly.

2.6.4. *Bounded distortion.* Using the geometrical structure and estimates above one also proves that the bounded distortion property $(\mathrm{BD})_n$ holds. This is a technical calculation and we refer to [BenCar91, MorVia93] or [LuzVia] for the proof in much the same formal setting as that given here.

2.7. **New critical points.** We give two algorithms for generating the new critical set $\mathcal{C}^{(n)}$. Both of them depend on the fact that since condition $(\mathrm{EG})_n$ is satisfied by all points of $\mathcal{C}^{(n-1)}$ it follows in particular that the hyperbolic coordinates $\mathcal{H}^{(n)}$ are also defined in neighbourhoods of these points.

2.7.1. *Refining the set of critical points of order* $n-1$. Since $\mathcal{H}^{(n)}$ does not generally coincide with $\mathcal{H}^{(n-1)}$, the critical points $z^{(n-1)}$ are no longer tangent to the new stable foliations $\mathcal{E}^{(n)}$. Instead, these foliations define new points of tangencies with the new stable leaves $\mathcal{E}^{(n)}$ close to the old ones. By definition these belong to the new set $\mathcal{C}^{(n)}$ of critical points of order n. By the estimates on the convergence of hyperbolic coordinates, see e.g. (3), the "distance" between the leaves of $\mathcal{E}^{(n-1)}$ and the leaves of $\mathcal{E}^{(n)}$ is of the order b^{n-1} and therefore the distance between the new points of tangencies, i.e. the new critical points, and the old ones will also be of the order of b^{n-1}, which is extremely small. It is then easy to see that the distance between the iterates $z_j^{(n-1)}$ and $z_j^{(n)}$ will continue to be essentially negligible for all $j \leq n$. In particular the nested property of bound neighbourhoods is satisfied, and the new point $z^{(n)}$ inherits all the properties of its ancestor $z^{(n-1)}$ as far as bounded recurrence, exponential growth, and bounded distortion are concerned.

2.7.2. *Adding really new critical points.* Notice that there may be other admissible pieces of the unstable manifold $W^u(P)$ which are too small or not on the right section of $W^u(P)$ to admit critical points of order $n-1$, recall property 2.4.3, but can in principle admit critical points of order n. We add these points to the new critical set $\mathcal{C}^{(n)}$ as long as they are close enough to $\mathcal{C}^{(n-1)}$ so that in particular the nested property of bound neighbourhoods is satisfied. This completes the definition of $\mathcal{C}^{(n)}$ and the sketch of the proof of Theorem 1.

2.8. **The cardinality of the critical set.** Before going on to discuss the parameter dependence of the objects defined above, we make a couple of important remarks regarding the definition of the set $\mathcal{C}^{(n)}$.

2.8.1. *Why we need many critical points.* We recall that the overall objective of our discussion is to prove the existence of many parameters for which some (non-uniform) hyperbolicity conditions are satisfied. As a first step in this direction, it is useful to start with the relatively modest objective of showing that the unstable manifold $W^{(u)}(p)$ is not contained in the basin of attraction of an attracting periodic orbit, a necessary, though not sufficient, condition for the hyperbolicity conditions to hold. To prove this it is enough to show that almost all points $z \in W^u(P)$ satisfy the exponential growth condition $(\mathrm{EG})_n$ for all $n \geq 1$. The proof of this fact requires controlling returns to Δ and the argument presented here relies on achieving this control by identifying a set of *critical points* as explained above, with the crucial property that a critical point in tangential position can always be found at every free return as long as the bounded recurrence condition is satisfied. This critical point can then be used to implement the shadowing (binding) argument to show that the exponential growth condition can be maintained through the passage in Δ. Since returns can occur at various "heights", tangential position can only be guaranteed if there are *sufficiently many* critical points.

2.8.2. *Why we need not-too-many critical points.* A choice of critical set which contains many points becomes problematic in view of our strategy of defining a good parameter in terms of some recurrence conditions on such points. The more critical points there are the greater the likelihood that at least one of them will fail to satisfy such condition and will lead to having to exclude a particular parameter value. Therefore, it is crucial to ensure that there are *relatively few* critical points such that by imposing the recurrence condition on their orbits one controls the whole dynamics, in the sense that one si able to prove hyperbolicity. Ultimately, at least at the present stage of the theory, this requires a strong (smallness) restriction on the perturbation size b.

2.8.3. *A reasonable compromise.* The main restriction on the number of critical points of a given order k comes from the requirement that they are of generation $g \leq \theta k$ and that they have some space around them where there is no other critical point, see Section 2.4.3. These properties immediately imply the following crucial bound on the total number of critical points of order k:

$$\#\mathcal{C}^{(k)} \leq \frac{|\Phi^{\theta k}(W)|}{2\rho^{\theta k}} \leq (5/\rho)^{\theta k}. \tag{6}$$

The constant 5 is an upper bound for the norm of the derivative.

We shall see in the parameter exclusion argument that this bound is good enough to ensure that not too many parameters get excluded. On the other hand, the reason we can afford to use only critical points with the above properties is related to the features of the constructions in Sections 2.5.1 and 2.5.2: as we have seen, a binding critical in tangential position can always be found among the critical points of generation $g \leq \theta k$ and lying on admissible unstable segments of radius $\rho^g \geq \rho^{\theta k}$.

3. POSITIVE MEASURE IN PARAMETER SPACE

Next we explain why the set of parameter values a for which the previous construction works has positive Lebesgue measure. It is assumed that b is sufficiently small and that a varies in an interval Ω close to $a = 2$ and not too small.

Theorem 2. *There exists a set $\Gamma^* \subset \Omega$ such that:*

1. *the Lebesgue measure* $|\Gamma^*| > 0$*;*
2. *for all* $a \in \Gamma^*$ *a critical set* $\mathcal{C}$ *is defined and satisfies* $(*)_n$ *for all* $n \geq 0$.

The precise condition for the choice of the interval Ω is in terms of the limiting one-dimensional map $\phi_a(x) = 1 - ax^2$. Firstly, the iterates c_n of the critical point remain outside the critical region Δ for the first N iterates, for some large N. Secondly, $c_N(a) = \phi_a^{N+1}(c)$ describes an interval of length $> \delta/10$ in a monotone fashion when a varies in Ω. This last requirement ensures that N is an *escape situation* (this notion will be recalled in a while). By simple perturbation, these properties extend to the two-dimensional Hénon-like map Φ_a if b is sufficiently small.

The proof relies on the construction of a nested sequence of sets $\Gamma^{(n)}$ such that each parameter value in $\Gamma^{(n)}$ has a critical set $\mathcal{C}^{(n)}$ satisfying $(*)_{n-1}$. The set Γ^* is then just the intersection of all $\Gamma^{(n)}$. The main estimate concerns the probability of exclusions at each time n, that is, the Lebesgue measure of $\Gamma^{(n-1)} \setminus \Gamma^{(n)}$. We begin here with a sketch of the construction of the sets $\Gamma^{(n)}$ in the one-dimensional case and discuss the main issues with the generalizations of the construction to the two-dimensional setting.

3.1. **The one-dimensional case.** Given the critical point $z = c = 0$ and an integer $k \geq 0$ we define the map

$$z_k : \Omega \to Q, \qquad z_k(a) = \phi_a^{k+1}(z(a)), \tag{7}$$

from parameter space to phase space associating the k:th iterate of the critical value $z_0(a) = \phi_a(z)$ to each parameter value $a \in \Omega$. Whenever $z_k(\Omega)$ intersects the critical neighborhood Δ we subdivide it into subintervals by pulling back a certain partition $\mathcal{I}$ of Δ. Roughly, the partition consists of the intervals bounded by the sequence $\pm e^{-r}$ for $r \geq |\log \delta|$ (for distortion reasons these intervals must be subdivided a bit further). Then we exclude those parameter subintervals for which condition $(*)$ does not hold at time k.

We obtain in this way a sequence of good parameter sets $\Gamma^{(n-1)} \subset \cdots \subset \Gamma^{(0)} = \Omega$ and corresponding partitions $\mathcal{P}^{(n-1)}, \ldots, \mathcal{P}^{(0)}$ such that all parameters in any given $\gamma \in \mathcal{P}^{(k)}$ have essentially indistinguishable itineraries (in particular as far as the critical recurrence is concerned) and essentially equivalent derivative estimates up to time $k+1$ (in particular z_k restricted to elements of $\mathcal{P}^{(k)}$ is a diffeomorphism onto its image).

At each step we refine $\mathcal{P}^{(k)}$ to a partition $\widehat{\mathcal{P}}^{(k)}$ of $\Gamma^{(k)}$ by pulling back the intersection of elements of $\mathcal{P}^{(k)}$ under the map z_{k+1} with $\mathcal{I}$. We then exclude those elements of $\widehat{\mathcal{P}}^{(k)}$ for which the recurrence condition fails, and define $\Gamma^{(k+1)}$ as the union of the remaining elements and $\mathcal{P}^{(k+1)}$ as the restriction of $\widehat{\mathcal{P}}^{(k)}$ to $\Gamma^{(k+1)}$. A large deviations type of argument shows that the measure of the excluded set decreases exponentially fast with k:

$$|\Gamma^{(k)} \setminus \Gamma^{(k+1)}| \leq e^{-\tau_0 k}|\Omega| \quad \text{for all } k \geq N,$$

where $\tau_0 > 0$ is independent of N. Taking N large enough (no exclusions are needed inside Ω before time N), this gives that a positive measure set remains after all exclusions. We do not give the details here as this is a special case of the argument in the two-dimensional context, which will be discussed in some detail below.

3.2. **Two-dimensional issues.** We mention here the key differences between the one-dimensional and two-dimensional situations and the main difficulties in generalizing the scheme sketched above to the two-dimensional case.

3.2.1. *Many critical points.* The most obvious difference is that in two dimensions there is a large number of critical points of order n at each stage n and *all* these critical points must satisfy $(*)_n$. Thus many more parameter exclusions are necessary. However we have seen in (6) that the cardinality of $\mathcal{C}^{(n)}$ grows at most exponentially fast with n, with exponential growth rate which can be made arbitrarily small by reducing b. This is crucial to guarantee that the total proportion of parameters excluded at time n continues to be exponentially small in n: the measure of parameters excluded by imposing the recurrence condition on each individual critical point decreases exponentially fast with n, with decay rate τ_0 which is essentially the same as in dimension one, and so is independent of b. Section 3.3.1 makes these explanations more quantitative.

3.2.2. *Interaction between different critical points.* A second important issue is that the one dimensional argument relies on keeping track of combinatorial and analytic data related to the history of the critical orbit for various parameter values. Here we can do the basically the same, but each one of the critical orbits has its own associated data, since the dynamical history and pattern of recurrence to the critical neighbourhood vary from one critical orbit to the other. For this reason, it will be convenient to introduce an extended parameter space, with separate combinatorial structures (partitions, itineraries) relating to each critical point. While we try to think of these structures as being essentially independent, this is not entirely accurate because different critical orbits do interact with each other. Namely, a critical point z may require a different one ζ as the binding point associated to some free return. Then, if a parameter is deleted because ζ fails to satisfy condition $(*)_n$ for that parameter value, this deletion must be somehow registered in the combinatorial structure of the other critical points z. Section 3.3.3 explains how this is handled.

Remark 2. Neither of these two points is really related to dimensionality: multiplicity of critical points and the difficulties connected to interactions between their orbits occur already for *multi*-modal maps in dimension 1. In fact, [PacRovVia98] treated those difficulties in the extreme case of *infinite*-modal maps of the interval, that is with infinitely many critical points, using this strategy of defining different but not-quite-independent combinatorics and exclusion rules for each critical point that we just outlined and will be detailing a bit more in a while.

3.2.3. *Continuation of critical points.* Another fundamental difficulty, this time intrinsically two-dimensional, is the problem of talking about a given critical point for different parameter values, as was implicitly assumed in the discussion of the previous two points. It is not immediately obvious how to do this because critical points are defined dynamically: the definition requires certain hyperbolicity properties to be satisfied and the precise location of the point depends on the geometrical and dynamical features of the map for a specific parameter value, which are very unstable under parameter changes. We shall use the fact that critical points of finite order do admit a *critical continuation* to a neighborhood in parameter space: the condition of quadratic tangency that defines such points has a unique smooth solution on that neighborhood. As a matter of fact, we make it here an additional requirement on a tangency of order k, for it to be in the critical set $\mathcal{C}_a^{(k)}$, that it should have a suitable continuation in parameter space.

To appreciate the situation better, suppose for example that a critical point of order k admits a critical continuation to a parameter interval ω. Suppose however that there exists a subinterval $\tilde{\omega} \subset \omega$ such that $\omega \setminus \tilde{\omega}$ has two connected components and such that the required bounded recurrence condition $(*)_k$ fails to be satisfied by the critical point for $a \in \tilde{\omega}$. Then the critical point cannot be refined to an approximation of order $k+1$ for critical points in $\tilde{\omega}$ although it can in the two components of $a \in \omega \setminus \tilde{\omega}$. We need to address the questions of whether these refinements can still be thought of as continuations of each other, i.e. whether we can still talk about a single critical point with a critical continuation on the (disconnected) set $\omega \setminus \tilde{\omega}$ or whether we should think of having two independent critical points defined in the two distinct parameter intervals. See Sections 3.3.1 and 4.2 for the details of how these issues are resolved and how we manage to relate critical points existing for different parameter values.

3.3. **Overview of the argument.** All of these issues will be dealt with formally by defining an *extended parameter space* where each critical point (of finite order) comes with its own interval of parameters on which it admits a continuation as a critical point, and with its own combinatorial and analytical data. In the remaining part of this section we describe the structure of this extended parameter space at each iterate k, and outline the main calculation that proves that $\Gamma^* = \cap_k \Gamma^{(k)}$ has positive Lebesgue measure.

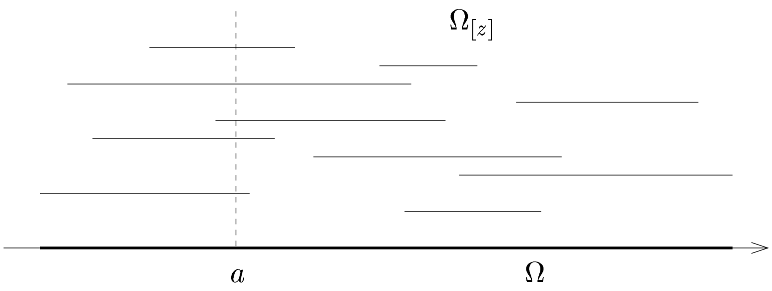

FIGURE 3. Extended parameter space

3.3.1. *The extended parameter space.* The parameter space at time k consists of a disjoint union of copies $\Omega_{[z]}$ of (not necessarily disjoint) subintervals of Ω, as described in Figure 3. Each of them comes with a critical point $z^{(k)}(a)$ of order k defined on some subset of $\Gamma_{[z]}^{(k)}$ of $\Omega_{[z]}$. The symbol $[z]$ parametrizes the set of these segments $\Omega_{[z]}$ and may be thought of as an "equivalence class" of critical points in the sense that there exists some $\nu \leq k$ and a critical point $z^{(\nu)}(a)$ which admits a continuation as a tangency of order ν over the entire $\Omega_{[z]}$ and which is ancestor to $z^{(k)}(a)$ whenever the latter is defined. For this reason, it makes sense to think of $z^{(k)}(a)$ as "the same critical point" for different parameter values in its domain. In addition, $z^{(\nu)}$ has an *escape situation* at time ν : the image of

$$\Omega_{[z]} \ni a \mapsto z_{\nu}^{(\nu)}(a)$$

is an admissible curve of length $\geq \delta/10$. This exactly corresponds to the requirement, in the one-dimensional setting, that the initial parameter interval should not be too small.

The set $\Gamma_{[z]}^{(k)}$ is a finite union of subintervals of $\Omega_{[z]}$ and is a subset of parameters in $\Omega_{[z]}$ for which the corresponding critical point $z^{(k)}(a)$ satisfies the recurrence conditions up to time k. It also comes with a combinatorial structure in the form of a finite partition $\mathcal{P}_{[z]}^{(k)}$ into subintervals defined in such a way that all critical points $z^{(k)}(a)$ with a belonging to any one element of this partition have the same history, that is, essentially the same analytic, hyperbolicity, distortion, and recurrence estimates, up to time $k-1$.

3.3.2. *The parameter exclusion argument.* For each $[z]$ we exclude a set of parameters $E_{[z]}^{(k)} \subset \Omega_{[z]}$ to enforce condition $(*)_k$. These individual exclusions are estimated in much the same way as in dimension 1, the details will be given in the following sections. For the moment we just mention that we begin by defining a refined partition $\widehat{\mathcal{P}}_{[z]}^{(k)} \succ \mathcal{P}_{[z]}^{(k)}$ of the parameter set $\Gamma_{[z]}^{(k)}$, depending on the position of the critical points $z_k^{(k)}(a)$ for each parameter $a \in \Gamma_{[z]}^{(k)}$. We then decide which parameters to exclude on the basis of this additional combinatorial information. We always exclude whole elements of this refined partition, and not just individual parameters, even if this may mean excluding somewhat more parameters than is actually necessary. This is important because it ensures that the remaining set

$$\Gamma_{[z]}^{(k+1)} = \Gamma_{[z]}^{(k)} \setminus E_{[z]}^{(k)}$$

of parameters which are good for $[z]$ up to time k inherits a combinatorial structure, the family of atoms of the refined partition which have not been excluded, and these are relatively large intervals. Indeed, our exclusion estimates depend crucially on lower bounds on the size of parameter intervals (small intervals might even be completely deleted at one given return!), and removing individual parameter values could lead to the formation of such small connected components in parameter space. In Section 5.5 we get

$$|E^{(k)}_{[z]}| \le e^{-\tau_0 k}|\Omega_{[z]}| \tag{8}$$

with $\tau_0 > 0$ independent of b and N. By definition, the new set of good parameters is

$$\Gamma^{(k+1)} = \Gamma^{(k)} \setminus \bigcup_{[z]} E^{(k)}_{[z]}. \tag{9}$$

This means *we only consider a parameter value good at any given time* $k+1$ *if it is good for all critical points up to that time*. Notice also that there are no partitions associated to Γ^k or $\Gamma^{(k+1)}$, these are just "raw" sets of parameter values.

To estimate the total size of exclusions, we remark that if a parameter a belongs to intervals $\Omega_{[z]}$ and $\Omega_{[w]}$ then, by definition of critical points, the corresponding ancestor points $z^{(\nu)}(a)$ and $w^{(\mu)}(a)$ are $\ge 2\rho^{\theta k}$ away from each other in the intrinsic metric of the unstable manifold $W^{(u)}(p)$. Together with the fact that they must be contained in a compact part of $W^{(u)}(p)$ of length $\le 2 \times 5^{\theta k}$ (since we started with a leaf of length ≤ 2 and iterated this for at most θk iterates with a maximum expansion of a factor 5 at each iteration), the same calculation as for (6) gives us the following bound on the size of this family of intervals (see Section 4.3.5):

$$\#\{[z] \in \mathcal{C}^{(k)} : a \in \Omega_{[z]}\} \le (5/\rho)^{\theta k} \quad \text{for any } a \in \Omega. \tag{10}$$

So the total exclusions at this iterate are

$$|\bigcup_{[z]} E^{(k)}_{[z]}| \le \sum_{[z]} |E^{(k)}_{[z]}| \le e^{-\tau_0 k} \sum_{[z]} |\Omega_{[z]}| \le e^{-\tau_0 k}(5/\rho)^{\theta k}|\Omega|, \tag{11}$$

by (8) and (10). Assuming b is small, the term on the right is $\le e^{-(\tau_0/2)k}|\Omega|$.

$$|\Omega \setminus \Gamma^*| = |\bigcup_{k=N}^{\infty} \bigcup_{[z]} E^{(k)}_{[z]}| \le \sum_{k=N}^{\infty} e^{-(\tau_0/2)k}|\Omega| < |\Omega|, \tag{12}$$

where Γ^* is the intersection of all $\Gamma^{(k)}$. The last inequality assumes N was chosen large enough, and implies that $|\Gamma^*| > 0$.

3.3.3. *Interaction between different critical orbits.* Observe that each individual parameter interval $\Gamma^{(k+1)}_{[z]}$ typically contains some parameter values which are not in $\Gamma^{(k+1)}$: at each stage there may exist (globally) bad parameters which, nevertheless, are good for *some* of the critical points, at least up to that stage. This is inevitable, given that we always exclude entire partition intervals, as explained before, and that different critical points have different partitions. However, a little bit of thought shows that this is also most natural to happen.

To explain why, let us consider any parameter value $\bar{a}$ for which there is a homoclinic point z associated to the fixed point P (these parameters form a zero measure set, we mention this situation because it sheds light into the general case). The forward orbit of z converges to P and, thus, never goes to the critical region. The recurrence condition is automatically satisfied, and hyperbolicity features on the homoclinic orbit follow simply from Proposition 2: there is no need for the binding argument, etc. The point z is a true critical point (point of tangency between true stable and unstable manifolds) and from its point of view the parameter $\bar{a}$ is perfectly good, *notwithstanding the fact that $\bar{a}$ may be a bad parameter for some other critical point (in which case it is excluded from Γ^*) and the map may even exhibit periodic attractors:* this one critical point never becomes aware of it!

Having said this, different critical orbits do interact with each other in general. In terms of our inductive construction this interaction materializes when a critical point $[w] = w^{(k)}$ is used as the binding point associated to some free return k of a different critical point $[z] = z^{(k)}$ (this does not occur in the special situation discussed before): parameters that have been excluded because $[w]$ failed to satisfy condition $(*)_j$ at some iterate $j \leq k$ must be excluded from the parameter space of $[z]$ as well. We do indeed exclude an additional set of partition elements γ of $\widehat{\mathcal{P}}^{(k)}_{[z]}$, but only those which have already been *completely eliminated* due to parameter exclusions associated to other critical points, i.e. such that $\gamma \cap \Gamma^{(k)} = \emptyset$. This means we are really excluding a somewhat larger set

$$\hat{E}^{(k)}_{[z]} \supset E^{(k)}_{[z]}$$

from the parameter interval of $[z]$ at time k and defining

$$\Gamma^{(k+1)}_{[z]} = \Gamma^{(k)}_{[z]} \setminus \hat{E}^{(k)}_{[z]} \subset \Gamma^{(k)}_{[z]} \setminus E^{(k)}_{[z]}.$$

An easy, yet important observation is that these exclusions *do not* affect the calculation made before: by definition, any parameter in the difference belongs to $E^{(j)}_{[w]}$ for some $[w]$ and some $j \leq k$, hence

$$\Gamma^{(k)} \setminus \bigcup_{[z]} \hat{E}^{(k)}_{[z]} = \Gamma^{(k)} \setminus \bigcup_{[z]} E^{(k)}_{[z]}. \tag{13}$$

In other words, (9) and (12) are not changed at all!

The success of this strategy is based also on the important observation that *we do not need to exclude an element $\gamma \in \mathcal{P}^{(k)}_{[z]}$ as long as at least one parameter $a \in \gamma$ belongs to $\Gamma^{(k)}$*. This is explained in more detail in Section 4.2.5 and is essentially due to the fact that as long as there is even a single parameter $\tilde{a} \in \gamma \cap \Gamma^{(k)}$ then the capture argument works and there is a binding critical point $\zeta(\tilde{a})$ in tangential position if k is a free return for γ.

4. THE COMBINATORIAL STRUCTURE

We are now going to detail the construction outlined in the previous section. We begin by giving explicit definitions of the extended parameter space and the set of good parameters for small values of k.

4.1. **First step of the induction.** The hyperbolic fixed point P has a continuation $P(a)$ for all $a \in \Omega$ and we can also consider a continuation $W(a)$ of the compact interval $W \subset W^u(P)$ introduced in Section 2.4. Note that $P(a)$ and $W(a)$ depend smoothly on the parameter. Given $1 \leq k \leq N$, we have contracting directions $e^{(k)}$

of order k defined at each point of $W_0(a) = \Phi_a(W(a) \cap \Delta)$ and there exists a unique point $z^{(k)}(a) \in W(a)$ such that the k'th contracting direction is tangent to $W_0(a)$ at the critical value $z_0^{(k)}(a) = \Phi_a(z^{(k)}(a))$. For $1 \leq k \leq N$ we let $\Gamma^{(k)} = \Omega$ and for each $a \in \Gamma^{(k)}$ we let the critical set $\mathcal{C}_a^{(k)}$ consist exactly of this critical point $z^{(k)}(a)$. This defines the critical set $\mathcal{C}_a^{(k)}$ for all $a \in \Omega$. The extended parameter space reduces to the single interval $\Omega_{[z]} = \Omega$.

4.2. **Properties of parametrized critical points.** We now fix $n \geq N$ and suppose inductively that for each $1 \leq k \leq n$ we have already constructed

- a family of intervals $\{\Omega_{[z]} : [z]\}$ each one with associated critical point $[z] = z^{(k)}$ defined on a set of good parameters $\Gamma_{[z]}^{(k)} \subset \Omega_{[z]}$: these critical points satisfy $(*)_{k-1}$ and (EG)$_k$ for all $a \in \Gamma_{[z]}^{(k)}$;
- and a set $\Gamma^{(k)}$ of parameters good for all critical points: $\Gamma^{(k)} \cap \Omega_{[z]}$ is contained in $\Gamma_{[z]}^{(k)}$ for all $[z]$.

From now on $\mathcal{C}^{(k)}$ will represent the set of $[z]$ parametrizing the family of intervals above, which we think of as the set of all critical points or order k. To avoid confusion with the notation below, notice that points $\mathcal{C}^{(k)}$ are well defined in virtue of their satisfying condition (EG)$_k$ but are only assumed to satisfy $(*)_{k-1}$ (not $(*)_k$). The extended parameter space is the disjoint union of intervals:

$$\Omega_*^{(k)} = \coprod_{[z] \in \mathcal{C}^{(k)}} \Omega_{[z]}.$$

These objects have the following additional properties:

4.2.1. *Globally defined ancestor.* There exists a critical point $z^{(\nu)}(a)$ of some order $\nu \leq k$ defined on the whole $\Omega_{[z]}$ which is an ancestor to $z^{(k)}(a)$ whenever the latter is defined. In addition, ν is an escape situation for $z^{(\nu)}$ so that the image of $\Omega_{[z]}$ under $a \mapsto z_\nu^{(\nu)}(a)$ is an admissible curve with length $\geq \delta/10$.

4.2.2. *Location and uniqueness.* Every $z^{(\nu)}(a)$ is the midpoint of an admissible curve ω_a of radius $\rho^{\theta\nu}$ inside $\Phi_a^{\theta\nu}(W(a))$, for $a \in \Omega_{[z]}$. The critical value $z_0^{(\nu)}(a) = \Phi_a(z^{(\nu)}(a))$ is a point of quadratic tangency between $\Phi_a(\omega_a)$ and the stable foliation of order ν in the bound neighborhood of $z_0^{(\nu)}(a)$. Moreover, $z^{(\nu)}(a)$ is the unique element of $\mathcal{C}_a^{(\nu)}$ in ω_a.

4.2.3. *Itinerary information.* Each element $[z] \in \mathcal{C}^{(k)}$ has successive sets of good parameters $\Gamma_{[z]}^{(k)} \subset \cdots \subset \Gamma_{[z]}^{(\nu)} = \Omega_{[z]}$ and corresponding partitions $\mathcal{P}_{[z]}^{(k)}, \ldots, \mathcal{P}_{[z]}^{(\nu)}$. They are defined in essentially the same way as in dimension one, as we shall explain in a while. We let $\mathcal{P}_*^{(k)}$ denote the corresponding induced partition of $\Omega_*^{(k)}$: to each $\gamma \in \mathcal{P}_*^{(k)}$ is implicitly associated a critical point $[z] \in \mathcal{C}^{(k)}$ with $\gamma \in \mathcal{P}_{[z]}^{(k)}$. We always assume that γ intersects $\Gamma^{(k)}$ in at least one point. Otherwise we just delete γ: obviously, this has no effect whatsoever on the measure estimates. Each

$\gamma \in \mathcal{P}_*^{(k)}$ has associated combinatorial information which we call the *itinerary* of γ. This consists of:

- A sequence of *escape times*
$$\nu = \eta_0 < \eta_1 < \cdots < \eta_s < k \qquad s = s(\gamma) \geq 0.$$
- Between any two escape times η_{i-1} and η_i (and between η_s and k) there is a sequence of *essential returns*
$$\eta_{i-1} < \nu_1 < \cdots < \nu_t < \eta_i \qquad t = t(\gamma, i) \geq 0.$$
- Between any two essential returns ν_{j-1} and ν_j (and between ν_t and η_i) there is a sequence of *inessential returns*
$$\nu_{j-1} < \mu_1 < \cdots < \mu_u < \nu_j \qquad u = u(\gamma, i, j) \geq 0.$$

Any of these sequences may be empty, except for the first one because the construction ensures that ν is always an escape time. Any iterate j after an escape time and before the subsequent return, including the escape time itself, is called an *escape situation*. The corresponding image curve $\gamma_j = \{z_j^{(k)}(a) : a \in \gamma\}$ is admissible and long. Escape times and the essential and inessential return times are *free returns*. Any returns to Δ occurring during binding periods associated to a previous return are called *bound returns*. Binding periods for all the returns may be chosen constant on the interval γ.

Associated to each free return ν is a positive integer $|r|$ that we call the *return depth*. This corresponds to the position of γ_ν relative to the partition $\mathcal{I}^\zeta = \{I_{r,m}^\zeta\}$ as we shall see in the completion of the inductive step in Section 4.3. By convention the return depth is zero at escape times. We let

$$\mathcal{R}_{[z]}^{(k-1)} : \Omega_{[z]} \to \mathbb{N} \qquad \text{and} \qquad \mathcal{E}_{[z]}^{(k-1)} : \Omega_{[z]} \to \mathbb{N}.$$

be the functions which associate to each $a \in \gamma$, respectively, the sum of all free (essential and inessential) return depths and the sum of the essential return depths, both for returns $\nu \leq k-1$. These functions are constant on partition elements, so they naturally induce functions

$$\mathcal{R}^{(k-1)} : \mathcal{P}_*^{(k)} \to \mathbb{N} \qquad \text{and} \qquad \mathcal{E}^{(k-1)} : \mathcal{P}_*^{(k)} \to \mathbb{N}.$$

4.2.4. *Phase and parameter derivatives.* For each $\gamma \in \mathcal{P}_*^{(k)}$ and associated critical point the velocity $D_a z_k^{(k)}(a)$ of the curve $\gamma \ni a \mapsto z_k^{(k)}(a)$ is uniformly comparable, in argument and magnitude, to the image of the most expanded vector $f^{(k)}(z_0^{(k)}(a))$ under the differential $D\Phi_a^k(z_0^{(k)}(a))$. Using also bounded distortion in phase space (BD)$_k$, we get a uniform constant $D > 0$ such that for every free iterate k the curve

$$\gamma_k = \{z_k^{(k)}(a) : a \in \gamma\} = \{\Phi_a^{k+1}(z^{(k)}(a)) : a \in \gamma\}$$

is admissible and satisfies

$$\frac{1}{D}\frac{|\tilde{\gamma}_k|}{|\gamma_k|} \leq \frac{|\tilde{\gamma}|}{|\gamma|} \leq D\frac{|\tilde{\gamma}_k|}{|\gamma_k|} \tag{14}$$

for any subinterval $\tilde{\gamma} \subset \gamma$. See [BenCar91, Lemmas 8.1, 8.4] and [MorVia93, Lemmas 11.3, 11.5, 11.6] for proofs of these properties. An important ingredient (cf. [MorVia93, Lemma 11.2] or [Via93, Lemma 9.2]) is to prove that critical points vary slowly with the parameter a:

$$\|D_a z^{(k)}(a)\| \leq b^{1/20} \ll 1.$$

In particular, since γ is connected, this gives

$$|z^{(k)}(a) - z^{(k)}(\tilde{a})| \leq b^{1/20}|a - \tilde{a}| \tag{15}$$

for all $a, \tilde{a}$ belonging to the same element $\gamma \in \mathcal{P}^{(k)}_{[z]}$.

4.2.5. *Existence of binding points.* For each $\gamma \in \mathcal{P}^{(k)}_*$ such that k is a free return there is $\tilde{a} \in \gamma \cap \Gamma^{(k)}$ and $\zeta^{(k)} \in \mathcal{C}^{(k)}_{\tilde{a}}$ a suitable binding point for all $z^{(k)}_k(a), a \in \gamma$ satisfying $(*)_k$. By suitable we mean that $z^{(k)}_k(a)$ and $\zeta^{(k)}(\tilde{a})$ are in tangential position for all $a \in \gamma$ satisfying the recurrence condition at time k. This corresponds to the condition in Section 2.5.

Note that a critical point for some *fixed* parameter $\tilde{a}$ in the intersection $\gamma \cap \Gamma^{(k)}$ is used as the binding critical point for all $a \in \gamma$, we do not need the continuation $\zeta(a)$ of $\zeta(\tilde{a})$ to be good for all the parameters in γ. This is useful when dealing with the exclusions in Section 3.3.3: we only need to remove γ if *all* its parameters have anyhow already been excluded from the set of good parameters. The reason this is possible is that the interval γ is quite small, $|\gamma| \leq e^{-\kappa_0 k}$ for some constant κ_0 related to κ, and critical points vary slowly with the parameter, see (15), while $(*)_k$ implies $|\zeta^{(k)}(\tilde{a}) - z^{(k)}_k(a)| \geq e^{-\alpha k} \gg e^{-\kappa_0 k}$. See [MorVia93, pp 65-66] or the second Remark in [Via93, § 9] for explicit estimates.

Moreover, if p represents the binding period associated to the return ν, then we have

$$\frac{|\gamma_{\nu+p+1}|}{|\gamma_\nu|} \geq e^{\kappa_3 r} \geq e^{\kappa_4 p} \gg 1, \tag{16}$$

where r is the return depth, and the constants κ_3, κ_4 depend only on κ. Indeed, this follows from the corresponding statement in phase space (5) and the property 4.2.4 that phase and parameter derivatives are uniformly comparable.

4.3. **The parameter space at time n.** We now explain how the parameter exclusions are determined and how the parameter space and the combinatorial structure are "updated". Part of this description involves explaining the way that this structure is updated to take into account the "new" critical points, recall Section 2.7.2, as well as the refinements of "old" critical points, recall Section 2.7.1. We start with the latter.

By definition every interval $\Omega_{[z]}$, with $[z] \in \mathcal{C}^{(n)}$, also belongs to the extended parameter space at the next iterate $n + 1$. We define a refinement $\widehat{\mathcal{P}}^{(n)}_{[z]}$ of $\mathcal{P}^{(n)}_{[z]}$ based on the dynamics up to time n and we update the itinerary information to time n. Based on this information we decide to exclude some elements of $\widehat{\mathcal{P}}^{(n)}_{[z]}$ and then define

- the set $\Gamma^{(n+1)}_{[z]}$ to be the union of the remaining elements, and
- the partition $\mathcal{P}^{(n+1)}_{[z]}$ to be simply $\widehat{\mathcal{P}}^{(n)}_{[z]}$ restricted to $\Gamma^{(n+1)}_{[z]}$.

The corresponding critical point function $z^{(n)}(a)$ is replaced by an improvement $z^{(n+1)}(a)$ of order $n+1$, defined on $\Gamma^{(n+1)}_{[z]}$. We proceed to explain these steps in detail.

4.3.1. *Critical neighborhood and partitions.* We begin with defining some partition in the dynamical space. It is no restriction to let $r_\delta = |\log\delta|$ be an integer. For every integer $r \geq r_\delta$ let

$$I_{\pm r} = \{z = (x,y) \in Q : \pm x \in (e^{-r-1}, e^{-r}]\}.$$

Now let each I_r be further subdivided into r^2 vertical strips $I_{r,m}$ of equal width. This defines a partition

$$\mathcal{I} = \{I_{r,m} : |r| \geq r_\delta \text{ and } m \in [1, r^2]\}$$

of Δ (disregarding $\{x = 0\}$). Given $I_{r,m}$ we denote $\hat{I}_{r,m} = I^{\ell}_{r,m} \cup I_{r,m} \cup I^{\rho}_{r,m}$ where $I^{\ell}_{r,m}$ and $I^{\rho}_{r,m}$ are the left and right elements adjacent to $I_{r,m}$. If $I_{r,m}$ happens to be an extreme element of $\mathcal{I}$ we just let $I^{\ell}_{r,m}$ or $I^{\rho}_{r,m}$ denote an adjacent interval of length $\delta/10$. Given a point ζ we define an analogous partition $\mathcal{I}^{\zeta}$ centered at ζ simply by translating horizontally $\mathcal{I}$. Let I^{ζ}_{r}, $I^{\zeta}_{r,m}$, $\hat{I}^{\zeta}_{r,m}$, be the corresponding partition elements.

4.3.2. *Partitioning.* Let $[z] \in \mathcal{C}^{(n)}$ be fixed. For each $\gamma \in \mathcal{P}^{(n)}_{[z]}$ we distinguish two cases. We call n a non-chopping time in either of the following situations:

(a) $\gamma_n \cap \Delta$ is empty or contained in an outermost partition element of $\mathcal{I}^{\zeta}$;
(b) n belongs to the binding period associated to some return time $\nu < n$ of γ.

In both situations we simply let $\gamma \in \widehat{\mathcal{P}}^{(n)}_{[z]}$.

We say that n is a chopping time in the remaining cases, that is, if n is a free iterate and γ_n intersects Δ significantly. In this case we may need to divide γ_n into subintervals to guarantee that the distortion bounds and other properties continue to hold. This depends on the position of γ_n relative to the partition $\mathcal{I}^{\zeta}$ of Δ centered at the binding point ζ (recall that γ_n is admissible, and thus "transverse" to the partitions). Indeed, there are two different situations.

If γ_n intersects at most two partition elements then we let $\gamma \in \widehat{\mathcal{P}}^{(n)}_{[z]}$, that is, we do not subdivide it. In this case we say that n is an inessential return and add this to the itinerary information already associated to γ. If γ_n does intersect at least three partition elements then we partition it

$$\gamma = \gamma^{\ell} \cup \bigcup_{(r,m)} \gamma^{(r,m)} \cup \gamma^{\rho}, \tag{17}$$

where (using notations from Section 4.3.1)

- each $\gamma^{(r,m)}_n = \{z^{(n)}_n(a) : a \in \gamma^{(r,m)}\}$ crosses $I^{\zeta}_{r,m}$ and is contained in $\hat{I}^{\zeta}_{r,m}$;
- γ^{ℓ}_n and γ^{ρ}_n are either empty or components of $\gamma_n \setminus \Delta$ with length $\geq \delta/10$.

If the connected components of $\gamma_n \setminus \Delta$ have length $< \delta/10$ we just glue them to the adjacent interval of the form $\gamma_n^{(r,m)}$.

By definition the resulting subintervals of γ are elements of $\widehat{\mathcal{P}}_z^{(n)}$. The intervals γ^ℓ , γ^ρ are called *escape components* and are said to have an escape at time n. All the other intervals are said to have an essential return at time n and the corresponding values of $|r|$ are the associated essential return depths. By convention, escaping components have return depth zero.

4.3.3. *Parameter exclusions.* We now have itinerary information up to time n and in particular return depths are defined and so are the functions $\mathcal{R}_{[z]}^{(n)}$ and $\mathcal{E}_{[z]}^{(n)}$ on $\widehat{\mathcal{P}}_{[z]}^{(n)}$. We fix a small constant $\tau > 0$ and then let

$$E_{[z]}^{(n)} = \bigcup \{\gamma \in \widehat{\mathcal{P}}_{[z]}^{(n)} : \mathcal{E}_{[z]}^{(n)}(\gamma) > \tau n\}. \tag{18}$$

We claim that if τ is small with respect to α then all $\gamma \in \widehat{\mathcal{P}}_{[z]}^{(n)}$ that are *not* in $E_{[z]}^{(n)}$ satisfy the recurrence condition $(*)_n$. The proof is postponed to Section 4.3.4. We define

$$\Gamma_{[z]}^{(n+1)} = \Gamma_{[z]}^{(n)} \setminus \left(E_{[z]}^{(n)} \cup \{\gamma \in \widehat{\mathcal{P}}_{[z]}^{(n)} : \gamma \cap \Gamma^{(n)} = \emptyset\} \right). \tag{19}$$

Moreover, we define $\mathcal{P}_{[z]}^{(n+1)}$ as the restriction of $\widehat{\mathcal{P}}^{(n)}$ to $\Gamma_{[z]}^{(n+1)}$.

Observe that we also remove from the parameter space of $[z]$ intervals that have already been completely deleted because of other critical points, even if they may look like good parameters for the point $[z]$. This deals with the interaction between distinct critical orbits discussed in Section 3.3.3. Let us stress once more that these additional exclusions have no effect at all on the new set of parameters good for all critical points:

$$\Gamma^{(n+1)} = \Gamma^{(n)} \setminus \bigcup_{[z] \in \mathcal{C}^{(n)}} E_{[z]}^{(n)}. \tag{20}$$

For parameters $a \in \Gamma_{[z]}^{(n+1)}$ we replace $z^{(n)}(a)$ by its refinement $z^{(n+1)}(a)$ as discussed in Sections 2.7 and 4.3.5. This completes the construction in this case.

4.3.4. *Verifying the recurrence condition.* We explain why all $[z] = z^{(n)} \in \mathcal{C}^{(n)}$ satisfy the condition $(*)_n$ for all $a \in \Gamma_{[z]}^{(n+1)}$. We begin with the remark that if ν is a return for $[z]$ then, by construction, the return depth $|r_\nu| \approx |\log d(z_\nu)|$. In particular the sum in $(*)_n$ is $\leq 2\mathcal{R}^{(n)}([z])$. We claim moreover that the sum of all free return depths is bounded by a multiple of the sum of essential return depths: there exists a uniform constant $C > 0$ such that

$$\mathcal{R}^{(n)}([z]) \leq C \mathcal{E}^{(n)}([z]). \tag{21}$$

Assuming this statement, and keeping (18) and (20) in mind, we get that for all the parameters $a \in \Gamma_{[z]}^{(n+1)}$

$$\sum_{\substack{\text{free}\\ \text{returns}\\ \nu \le n}} \log \frac{1}{d(z_\nu^{(n)})} \le 2\mathcal{R}^{(n)}([z]) \le 2C\mathcal{E}^{(n-1)}([z]) \le 2C\tau.$$

The conclusion follows choosing $\tau < \alpha/(2C)$.

We are left to prove (21). Let $\mu_1 < \cdots < \mu_s$ be the inessential returns in between consecutive essential returns $\nu_i < \nu_{i+1}$. Also let r_i be the return depth associated to ν_i and ρ_j be the return depth associated to each μ_j, $1 \le j \le s$. Property (16) says that the iterates $|\gamma_j|$ are expanded over the complete binding period associated to any free return. Due to the hyperbolic behavior of our maps outside the critical region, we know that these curves are not contracted during free periods. This gives that

$$|\gamma_{\nu_{i+1}}| \ge e^{\kappa_3(r_i+\rho_1+\cdots+\rho_s)}|\gamma_{\nu_i}|.$$

Clearly, $|\gamma_{\nu_{i+1}}| \le 2$. On the other hand,

$$|\gamma_{\nu_i}| \ge \text{const } e^{-r_i} r_i^{-2} \ge 2e^{-2r_i}$$

because ν_i is an essential return. Putting these two estimates together we find

$$e^{\kappa_3(r_i+\rho_1+\cdots+\rho_s)-2r_i} \le 1 \quad \Rightarrow \quad \rho_1+\cdots+\rho_s \le (2/\kappa_3)r_i\,.$$

Adding these inequalities for every essential return ν_i, we get

$$\mathcal{R}^{(n)}([z]) \le C\mathcal{E}^{(n)}([z])$$

with $C = 2/\kappa_3$.

4.3.5. *New critical points.* Finally, we must include in the construction new critical points of order $n+1$. First of all we "upgrade" the old critical points $[z] = z^{(n)}$ as described in Section 2.7.1. These are easily seen to satisfy the inductive assumptions stated in Sections 4.2.1 and 4.2.2. Then we add "really new" critical points $[\zeta] = \zeta^{(n+1)}$ as mentioned in Section 2.7.2. To ensure that the inductive assumptions continue to hold for these points we proceed as follows. For every critical point $[z] = z^{(n)}$ of order n and every γ in the corresponding partition $\mathcal{P}_{[z]}^{(n+1)}$ such that n is an escape situation, as defined in the previous section, we introduce the points $\zeta^{(n+1)}$ such that

- $\zeta^{(n+1)}$ is defined as a critical continuation over the whole γ;
- $\zeta^{(n+1)}(a)$ is contained in $\Phi_a^{\theta(n+1)}(W(a)) \setminus \Phi_a^{\theta n}(W(a))$ for all $a \in \gamma$;
- $z^{(n)}(a)$ is ancestor to $\zeta^{(n+1)}(a)$ for all $a \in \gamma$.

Essentially, these $\zeta^{(n+1)}$ are the additional elements of the critical set at time $n+1$. However, there is the possibility that two or more critical points $z^{i,(n)}$, defined on intervals γ_i, generate by this procedure critical functions $\zeta^{i,(n+1)}$ which turn out to coincide at some parameters:

(22) $\qquad \zeta^{i,(n+1)}(a) = \zeta^{j,(n+1)}(a) \quad$ for some (and hence all) $a \in \gamma_i \cap \gamma_j$.

If we were to consider all these $\zeta^{i,(n+1)}(a)$ as different critical points, the counting argument to prove (10) that we give in the next paragraph would not be valid. Instead, we begin by (almost) eliminating redundancy as follows. From any family of critical functions $\zeta^{i,(n+1)}$ as in (22) we extract a minimal subfamily $\zeta^{i_j,(n+1)}$ such that the union of their domains γ_{i_j} coincides with the union of all γ_i. We retain these $\zeta^{i_j,(n+1)}$ but eliminate all the other $\zeta^{k,(n+1)}$ as they are clearly redundant. The key, if quite easy observation is that by minimality a parameter a belongs to not more than two of these domains γ_{i_j}.

The critical functions $\zeta^{(n+1)}$ obtained in this way, after the redundancy elimination we just described, are the remaining elements $[\zeta]$ of $\mathcal{C}^{(n+1)}$. For each one of them we set

$$\Omega_{[\zeta]} = \Gamma^{(n+1)}_{[\zeta]} = \gamma \quad \text{and} \quad \mathcal{P}^{(n+1)}_{[\zeta]} = \{\Omega_{[\zeta]}\}.$$

We think of these critical points as being "born" at time n. Thus the iterate $\nu = n$ is the first escape time for each $[\zeta]$; apart from this the combinatorics of $[\zeta]$ is blank and there are no exclusions corresponding to these points at this time. This procedure *defines* the new critical set $\mathcal{C}^{(n+1)}$ and, in particular, makes precise the meaning of the symbol $[z]$ at the next stage of the construction. By the condition in Section 4.2.2, there are at most

$$\frac{2|\Phi_a^{\theta(n+1)}(W(a)) \setminus \Phi_a^{\theta(n)}(W(a))|}{2\rho^{\theta(n+1)}}$$

of these new critical points whose domains $\Omega_{[\zeta]}$ contain a given $a \in \Omega$ (here we have $\nu = n$). The factor 2 in the numerator accounts for the fact that a given point may represent two "different" critical points, but not more, as discussed in the previous paragraph. In other words,

$$\begin{aligned} \#\{[w] \in \mathcal{C}^{(n+1)} : a \in \Omega_{[w]}\} &\leq \\ &\leq \#\{[z] \in \mathcal{C}^{(n)} : a \in \Omega_{[z]}\} + \frac{|\Phi_a^{\theta(n+1)}(W(a)) \setminus \Phi_a^{\theta(n)}(W(a))|}{\rho^{\theta(n+1)}}. \end{aligned}$$

Now a simple induction argument yields the bound in (10)

$$\#\{[w] \in \mathcal{C}^{(n+1)} : a \in \Omega_{[w]}\} \leq \frac{|\Phi_a^{\theta(n+1)}(W(a))|}{\rho^{\theta(n+1)}} \leq (5/\rho)^{\theta(n+1)}.$$

The n'th step of the construction is complete.

5. THE PROBABILISTIC ARGUMENT

It remains to show that the set $\Gamma^* = \cap_n \Gamma^{(n)}$ has positive Lebesgue measure. For each critical point we use a large deviations argument similar to the one-dimensional proof to get the estimate as in (8). As discussed in Section 3.3.2, we then sum the exclusions associated to each critical point, using the bound on the number of critical points in (10). Thus, most of this section deals with the exclusions associated to a single critical point $[z] \in \mathcal{C}^{(n)}$ for some $n \geq N$. The issue of the multiplicity of critical point is taken care of by the formalism. For simplicity we omit the subscript $[z]$ where this does not give rise to confusion.

We split the argument into 4 sections. The first step is a useful re-organization of the combinatorial structure on each $\Omega_{[z]}$. The reason this is necessary is that our combinatorial data keeps track of the critical orbits itineraries in between escape returns (escape situations which coincide with chopping times) but not beyond. At escape returns the dynamics starts "afresh", in the sense that subsequent itineraries are very much independent of the previous behavior. This is a key feature of escape returns (the system "escapes its past") and, together with the fact that such returns are fairly frequent (large waiting time exponentially improbable), a crucial ingredient in the whole exclusion argument. On the other hand it means that, due to the possibility of many intermediate escapes, the same combinatorial data may correspond to several different partition intervals, even with unbounded multiplicity. The purpose of the re-organization we carry out in Section 5.1 is to decompose the whole collection of elements of all the partitions into a number of "blocks" on each of which we do have bounded multiplicity of the combinatorics. The strategy is to restrict our attention to the subintervals of an escaping component only up to the following escape time associated to each subinterval.

Focussing on each one of these blocks, we show in Section 5.2 that intervals are exponentially small in terms of the total sum of their return depths. Then in Section 5.3 we develop a counting argument to estimate the cardinality of the set of intervals whose return depths sum up to some given value. We show that this bound is exponentially increasing in the sum of the return depths, but with an exponential rate slower than that used to estimate the size of the intervals. Combining these two estimates immediately implies a bound on the *average* recurrence for points in a single block. In Corollary 6 we then show how to "sum" the contributions of each block to get an estimate of the overall average recurrence over all points of $\Gamma^{(n)}$. Finally, a large deviation argument implies the required estimate for the proportion of excluded parameters.

5.1. **Combinatorics renormalization.** By construction, for each $\gamma \in \widehat{\mathcal{P}}^{(n)}_{[z]}$ we have a sequence $\nu = \eta_0 < \eta_1 < \cdots < \eta_s \leq n$, $s \geq 0$ of escape times and for each $0 \leq i \leq s$ there exists an ancestor $z^{(\eta_i)} \in \mathcal{C}^{(\eta_i)}$ and an interval $\gamma^{(\eta_i)}$ with $\gamma \subset \gamma^{(\eta_i)} \subset \Omega_{[z]}$ and which is an escape component for $z^{(\eta_i)}$. In particular, $z^{(\eta_i)}$ admits a continuation to the whole $\gamma^{(\eta_i)}$. Because s may depend on γ, it is convenient to extend the definition of $\gamma^{(\eta_i)}$ to all $0 \leq i \leq n$ and we do this by letting $\gamma^{(\eta_i)} = \gamma$ for $s+1 \leq i \leq n$. Then we consider the disjoint union

$$\mathcal{Q}^{(i)} = \mathcal{Q}^{(i)}_{[z]} = \coprod \{\gamma^{(\eta_i)} : \gamma \in \widehat{\mathcal{P}}^{(n)}_{[z]}\}$$

Clearly, $\mathcal{Q}^{(0)} = \{\Omega_{[z]}\}$ and $\mathcal{Q}^{(n)} = \widehat{\mathcal{P}}^{(n)}_{[z]}$, which is a partition of $\Gamma^{(n)}_{[z]}$. Given $\gamma^{(\eta_i)} \in \mathcal{Q}^{(i)}$ and $\gamma^{(\eta_j)} \in \mathcal{Q}^{(j)}$ with $i < j$, we say that $\gamma^{(\eta_j)}$ is a *descendant* of $\gamma^{(\eta_i)}$ if $\gamma^{(\eta_j)} \subset \gamma^{(\eta_i)}$ and $z^{(\eta_i)}$ is ancestor to $z^{(\eta_j)}$. For $0 \leq i \leq n-1$ and $\gamma \in \mathcal{Q}^{(i)}$ we let

$$\mathcal{Q}^{(i+1)}(\gamma) = \{\gamma' \in \mathcal{Q}^{(i+1)} : \gamma' \text{ is a descendant of } \gamma\}.$$

The itineraries of all intervals in $\mathcal{Q}^{(i+1)}(\gamma)$ clearly coincide up to time η_i. Then we may define functions $\Delta\mathcal{E}_\gamma^{(i)} : \mathcal{Q}^{(i+1)}(\gamma) \to \mathbb{N}$ where

$$\Delta\mathcal{E}_\gamma^{(i)}(\gamma') = \mathcal{E}^{(\eta_{i+1})}(\gamma') - \mathcal{E}^{(\eta_i)}(\gamma)$$

is the sum of all essential return depths associated to the itinerary $\gamma' \in \mathcal{Q}^{(i+1)}(\gamma)$ between the escape times η_i and η_{i+1}. Finally we let

$$\mathcal{Q}^{(i+1)}(\gamma, R) = \{\gamma' \in \mathcal{Q}^{(i+1)}(\gamma) : \Delta\mathcal{E}_\gamma^{(i)}(\gamma') = R\}.$$

5.2. **Metric bounds.** Let $\bar\kappa = \kappa_3/5$, where κ_3 is the constant in (16). Recall that κ_3 is independent of δ and b.

Lemma 3. *For all* $[z] \in \mathcal{C}^{(n)}$, $0 \le i \le n-1$, $\gamma \in \mathcal{Q}^{(i)}$, $R \ge 0$, *and* $\gamma' \in \mathcal{Q}^{(i)}(\gamma, R)$ *we have*

$$|\gamma'| \le e^{-3\bar\kappa R}|\gamma|.$$

Proof. By construction there are nested intervals $\gamma' \subset \gamma^{(\nu_t)} \subset \cdots \subset \gamma^{(\nu_1)} \subset \gamma^{(\nu_0)} = \gamma$ such that ν_0 is an escape time for $\gamma = \gamma^{(\nu_0)}$ and for each $j = 1, \ldots, t$ the interval $\gamma^{(\nu_{j-1})}$ has an essential return at time ν_j which is when the interval $\gamma^{(\nu_j)}$ is created as a consequence of chopping. Write

$$\frac{|\gamma'|}{|\gamma|} = \frac{|\gamma^{(\nu_1)}|}{|\gamma^{(\nu_0)}|}\frac{|\gamma^{(\nu_2)}|}{|\gamma^{(\nu_1)}|}\cdots\frac{|\gamma^{(\nu_t)}|}{|\gamma^{(\nu_{t-1})}|}\frac{|\gamma'|}{|\gamma^{(\nu_t)}|}. \tag{23}$$

The last factor has the trivial bound $|\gamma'|/|\gamma^{(\nu_t)}| \le 1$. For the middle factors we use

Lemma 4. *For all* $j = 1, \ldots, t-1$ *we have*

$$\frac{|\gamma^{(\nu_{j+1})}|}{|\gamma^{(\nu_j)}|} \le e^{-r_{j+1}+(1-3\bar\kappa)r_j}.$$

Proof. By the bounded distortion property (14),

$$\frac{|\gamma^{(\nu_{j+1})}|}{|\gamma^{(\nu_j)}|} \le D\frac{|\gamma^{(\nu_{j+1})}_{\nu_j+p_j+1}|}{|\gamma^{(\nu_j)}_{\nu_j+p_j+1}|}. \tag{24}$$

For each of these essential returns (16) gives,

$$|\gamma^{(\nu_j)}_{\nu_j+p_j+1}| \ge e^{5\bar\kappa r_j}|\gamma^{(\nu_j)}_{\nu_j}| \ge \text{const } r_j^{-2}e^{(5\bar\kappa-1)r_j} \ge e^{(4\bar\kappa-1)r_j}.$$

We used here that r^2 is much smaller than $e^{\bar\kappa r}$ for $r \ge r_\delta \gg 1$. To get an upper bound for the numerator in (24) we use that $\gamma_k^{(\nu_{j+1})}$ remains outside the critical region and is an admissible curve between time $\nu_j + p_j$ and time ν_{j+1}. So, during this period its length can not decrease:

$$|\gamma^{(\nu_{j+1})}_{\nu_j+p_j+1}| \le |\gamma^{(\nu_{j+1})}_{\nu_{j+1}}| \le e^{-r_{j+1}}.$$

Replacing these two bounds in (24), we find (using that r_δ is large)

$$\frac{|\gamma^{(\nu_{j+1})}|}{|\gamma^{(\nu_j)}|} \le De^{-r_{j+1}+(1-4\bar\kappa)r_j} \le e^{-r_{j+1}+(1-3\bar\kappa)r_j}$$

as claimed in Lemma 4. □

A similar argument applies to the first factor ($j = 0$) of (23). The length of $\gamma_{\nu_1}^{(\nu_1)}$ is bounded by e^{-r_1}, by construction. Moreover, the escaping component $\gamma_{\nu_0}^{(\nu_0)}$ has length $\geq \delta/10$. Since this component is adjacent to Δ and all the iterates from time ν_0 to time ν_1 take place in the hyperbolic region Δ^c, a simple hyperbolicity argument gives $|\gamma_{\nu_1}^{(\nu_0)}| \geq \delta^{9/10}$. It is no restriction to suppose $5\bar{\kappa} = \kappa_3 < 1/10$. Thus, we get

$$
\frac{|\gamma^{(\nu_1)}|}{|\gamma^{(\nu_0)}|} \leq D \, \frac{|\gamma_{\nu_1}^{(\nu_1)}|}{|\gamma_{\nu_1}^{(\nu_0)}|} \leq D \, e^{-r_1} \delta^{-1+5\bar{\kappa}} \leq e^{-r_1} \delta^{-1+3\bar{\kappa}}. \tag{25}
$$

Replacing these bounds in (23) we find

$$
\begin{aligned}
\frac{|\gamma'|}{|\gamma|} &\leq \exp\big(- r_1 - (1-3\bar{\kappa}) \log \delta + \sum_{j=1}^{t-1} -r_{j+1} + \sum_{j=1}^{t-1} (1-3\bar{\kappa}) r_j \big) \\
&= \exp\big(- (1-3\bar{\kappa}) \log \delta - r_t - 3\bar{\kappa} \sum_{j=1}^{t-1} r_j \big) \leq \exp\big(- 3\bar{\kappa} \sum_{j=1}^{t} r_j \big).
\end{aligned}
$$

In the second inequality we have used $r_t \geq |\log \delta|$. The term on the right is $e^{-3\bar{\kappa} R}$, so this completes the proof of Lemma 3. ☐

5.3. **Combinatorial bounds.**

Lemma 5. *For all $[z] \in \mathcal{C}^{(n)}$, $0 \leq i \leq n-1$, $\gamma \in \mathcal{Q}^{(i)}$ and $R \geq 0$ we have*

$$
\# \mathcal{Q}^{(i+1)}(\gamma, R) \leq e^{\bar{\kappa} R}
$$

Proof. Given $\gamma' \in \mathcal{Q}^{(i+1)}(\gamma, R)$ let $\eta_i = \nu_0 < \nu_1 < \cdots < \eta_t < \eta_{i+1}$ be the corresponding sequence of essential returns between the consecutive escape situations. To each ν_i the chopping procedure in Section 4.3.2 assigns a pair of integers (r_i, m_i) with $|r_i| \geq r_\delta$ and $1 \leq m_i \leq r_i^2$. By definition $|r_1| + \cdots + |r_t| = R$. The sequence (r_i, m_i) determines γ' completely, except that the next escape situation η_{i+1} may be generating two escape components, which thus share the same sequence. So, apart from the harmless factor 2, the cardinality of $\mathcal{Q}^{(i+1)}(\gamma, R)$ is bounded by the number of sequences (r_i, m_i), $t \geq 1$, with $|r_1| + \cdots + |r_t| = R$ and $|r_i| \geq r_\delta$ and $1 \leq m_i \leq r_i^2$.

We begin by estimating the number of integer solutions to

$$
r_1 + \cdots + r_t = R \quad \text{with} \quad r_i \geq r_\delta. \tag{26}
$$

This corresponds to the number of ways of partitioning R objects into t disjoint subsets, which is well known to be bounded above by $(R+t)!/R!t!$. Using Stirling's approximation formula we have

$$
\frac{(R+t)!}{R!t!} \leq \text{const} \, \frac{(R+t)^{R+t}}{R^R t^t} = \text{const} \left[\left(1 + \frac{t}{R}\right)^{1+\frac{t}{R}} \left(\frac{R}{t}\right)^{\frac{t}{R}} \right]^R .
$$

Recalling the fact that $R \geq t \, r_\delta$ and $r_\delta \to \infty$ when $\delta \to 0$, we see that both factors in the last term tend to 1 when δ tends to zero. Therefore, choosing δ small enough we ensure that the number of solutions of (26) is less than $e^{\bar{\kappa} R/4}$.

Now to complete the proof we need to sum over all values of t and we also need to take into account the sign of each r_i and the variation of m_i from 1 to r_i^2. The latter means that each fixed sequence r_i corresponds to at most $\prod_{i=1}^t r_i^2$ partition elements. For fixed R the product is biggest when the r_i are approximately equal. So this multiplicity is bounded by $(R/t)^t$. In the range we are interested in, the function $t \mapsto (R/t)^t$ is monotone increasing on t. So we may bound it by r_δ^{R/r_δ}, which is $\le e^{\bar\kappa R/4}$ if r_δ is large. In this way we get

$$\#\mathcal{Q}^{(i+1)}(\gamma, R) \le 2 \sum_{t \le R/r_\delta} 2^t\, e^{\bar\kappa R/2} \le 4\, 2^{R/r_\delta}\, e^{\bar\kappa R/2} \le e^{\bar\kappa R}$$

if r_δ is large enough. □

5.4. **Average recurrence.** From Lemmas 3 and 5 we immediately get

$$\sum_{\gamma' \in \mathcal{Q}^{(i+1)}(\gamma, R)} |\gamma'| \le e^{-2\bar\kappa R} |\gamma|. \tag{27}$$

With the help of this we are going to give an estimate for the distribution of the recurrence function when the entire itinerary up to time n is taken into account.

Corollary 6. *For every $[z] \in \mathcal{C}^{(n)}$ we have*

$$\int_{\Gamma_{[z]}^{(n)}} e^{\bar\kappa \mathcal{E}^{(n)}} \le e^{n/r_\delta} |\Omega_{[z]}|.$$

Proof. Recall first of all that $\mathcal{Q}^{(n)} = \widehat{\mathcal{P}}_{[z]}^{(n)}$ is a partition of $\Gamma_{[z]}^{(n)}$ and that $\mathcal{E}^{(n)}$ is constant on elements of $\mathcal{Q}^{(n)}$. Thus

$$\int_{\Gamma_{[z]}^{(n)}} e^{\bar\kappa \mathcal{E}^{(n)}} = \sum_{\gamma^{(n)} \in \mathcal{Q}^{(n)}} e^{\bar\kappa \mathcal{E}^{(n)}} |\gamma^{(n)}|\,.$$

Moreover $\mathcal{E}^{(n)} = \mathcal{E}^{(n-1)} + \Delta\mathcal{E}^{(n-1)}$ where $\mathcal{E}^{(n-1)}$ depends only on the element $\gamma^{(n-1)}$ of $\mathcal{Q}^{(n-1)}$ containing $\gamma^{(n)}$. Therefore

$$\sum_{\gamma^{(n)} \in \mathcal{Q}^{(n)}} e^{\bar\kappa \mathcal{E}^{(n)}} |\gamma^{(n)}| \le \sum_{\gamma^{(n-1)} \in \mathcal{Q}^{(n-1)}} e^{\bar\kappa \mathcal{E}^{(n-1)}} \sum_{\gamma^{(n)} \in \mathcal{Q}^{(n)}} e^{\bar\kappa \Delta\mathcal{E}^{(n)}} |\gamma^{(n)}|$$

and iterating the argument

$$\begin{aligned} \sum_{\gamma^{(n)} \in \mathcal{Q}^{(n)}} e^{\bar\kappa \mathcal{E}^{(n)}} |\gamma^{(n)}| \le & \sum_{\gamma^{(1)} \in \mathcal{Q}^{(1)}(\gamma^{(0)})} e^{\bar\kappa \Delta\mathcal{E}^{(0)}} \cdots \sum_{\gamma^{(i+1)} \in \mathcal{Q}^{(i+1)}(\gamma^{(i)})} e^{\bar\kappa \Delta\mathcal{E}^{(i)}} \cdots \\ & \cdots \sum_{\gamma^{(n-1)} \in \mathcal{Q}^{(n-1)}(\gamma^{(n-2)})} e^{\bar\kappa \Delta\mathcal{E}^{(n-2)}} \sum_{\gamma^{(n)} \in \mathcal{Q}^{(n)}(\gamma^{(n-1)})} e^{\bar\kappa \Delta\mathcal{E}^{(n-1)}} |\gamma^{(n)}| \end{aligned} \tag{28}$$

For each $0 \le i \le n-1$ and $\gamma^{(i)} \in \mathcal{Q}^{(i)}$ we can write

$$\sum_{\gamma^{(i+1)}} e^{\bar\kappa \Delta\mathcal{E}^{(i)}(\gamma^{(i+1)})} |\gamma^{(i)}| = |Q^{i+1}(\gamma^{(i)}, 0)| + \sum_{R \ge r_\delta} e^{\bar\kappa R} |Q^{(i+1)}(\gamma^{(i)}, R)|$$

where the sum on the left is over all $\gamma^{(i+1)} \in \mathcal{Q}^{(i+1)}(\gamma^{(i)})$. The relation (27) gives

$$\sum_{R \geq r_\delta} e^{\bar{\kappa} R} |Q^{(i+1)}(\gamma^{(i)}, R)| \leq \sum_{R \geq r_\delta} e^{-\bar{\kappa} R} |\gamma^{(i)}| \leq 2e^{-\bar{\kappa} r_\delta} |\gamma^{(i)}|$$

and therefore

$$\sum_{\gamma^{(i+1)}} e^{\bar{\kappa} \Delta \mathcal{E}^{(i)}(\gamma^{(i+1)})} |\gamma^{(i)}| \leq (1 + 2e^{-\bar{\kappa} r_\delta}) |\gamma^{(i)}| \leq e^{1/r_\delta} |\gamma^{(i)}| \tag{29}$$

(the last inequality uses that $\log(1+2e^{-\bar{\kappa} r_\delta}) \leq 4e^{-\bar{\kappa} r_\delta} \leq 1/r_\delta$ assuming r_δ is large enough). Replacing (29) in (28), successively for all values of i, and recalling that $\gamma^{(0)} = \Gamma^{(0)}_{[z]} = \Omega_{[z]}$, we find

$$\sum_{\gamma^{(n)} \in \mathcal{Q}^{(n)}} e^{\bar{\kappa} \mathcal{E}^{(n)}} |\gamma^{(n)}| \leq e^{n/r_\delta} |\Omega_{[z]}|$$

as claimed. □

5.5. **Conclusion.** Using the Chebyshev inequality and the definition

$$E^{(n)}_{[z]} = \{\gamma \in \widehat{\mathcal{P}}^{(n)}_{[z]} : \mathcal{E}^{(n)} \geq \tau(n+1)\}$$

we get (recall that $\widehat{\mathcal{P}}^{(n)}_{[z]}$ is a partition of $\Gamma^{(n)}_{[z]}$)

$$|E^{(n)}_{[z]}| \leq e^{-\tau\bar{\kappa}(n+1)} \int_{\Gamma^{(n)}_{[z]}} e^{\bar{\kappa} \mathcal{E}^{(n)}} \leq e^{\left(\frac{1}{r_\delta} - \bar{\kappa}\tau\right)n} |\Omega_{[z]}| \leq e^{-\bar{\kappa}\tau n/2} |\Omega_{[z]}|.$$

Then, using (10),

$$|\bigcup_{[z] \in \mathcal{C}^{(n)}} E^{(n)}_{[z]}| \leq e^{-\bar{\kappa}\tau n/2} \sum_{[z]} |\Omega_{[z]}| \leq e^{-\bar{\kappa}\tau n/2} (5/\rho)^{\theta n} |\Omega| \,.$$

While $\bar{\kappa}$ and τ are independent of the perturbation size b, the constant θ can be made arbitrarily small by reducing b. So, we may suppose that the last expression is $\leq e^{-\bar{\kappa}\tau n/4} |\Omega|$.

This means that $|\Gamma^{(n)} \setminus \Gamma^{(n+1)}| \leq e^{-\bar{\kappa}\tau n/4} |\Omega|$ for all n, which implies

$$|\Gamma^{(n+1)}| \geq \Big(1 - \sum_{i=N}^{n} e^{-\bar{\kappa}\tau i/4}\Big) |\Omega|$$

and

$$|\Gamma^*| \geq \Big(1 - \sum_{n=N}^{\infty} e^{-\bar{\kappa}\tau n/4}\Big) |\Omega| > 0.$$

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