A REFINEMENT OF THE HOFER–ZEHNDER THEOREM ON THE EXISTENCE OF CLOSED TRAJECTORIES NEAR A HYPERSURFACE

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We consider a smooth symplectic manifold (M,ω). A *hypersurface* S in M is a smooth compact connected orientable codimension 1 submanifold of $M \setminus \partial M$ without boundary. A characteristic on S is an embedded circle in S all of whose tangent lines belong to the distinguished line bundle

$$\mathcal{L}_S \;=\; \{(x,\xi) \in TS \mid \omega(\xi,\eta) = 0 \text{ for all } \eta \in T_xS\}\,.$$

We denote by $\mathcal{P}(S)$ the set of closed characteristics on S. Examples show that $\mathcal{P}(S)$ can be empty, see [2, 3]. We therefore follow [4] and consider parametrized neighbourhoods of S. Since S is orientable, there exists an open neighbourhood I of 0 and a smooth diffeomorphism

$$\psi\colon S \times I \;\to\; U \subset M$$

such that $\psi(x,0) = x$ for $x \in S$. We call ψ a *thickening of* S, and we abbreviate $S_\epsilon = \psi\left(S \times \{\epsilon\}\right)$.

Given an open subset $U \subset M$ we consider the function space $\mathcal{H}(U)$ of smooth functions $H\colon U \to [0, \max H]$ such that

- $H|_V = 0$ for some nonempty open set $V \subset U$;
- $H|_{U\setminus K} = \max H$ for some compact set $K \subset U$.

We say that $H \in \mathcal{H}(U)$ is HZ-*admissible* if the flow φ_H^t has no non-constant T-periodic orbit with period $T \leq 1$, and we set

$$\mathcal{H}_{\mathrm{HZ}}(U) \;=\; \{H \in \mathcal{H}(U) \mid H \text{ is HZ-admissible}\}\,.$$

The Hofer–Zehnder capacity of U is defined as

$$c_{\mathrm{HZ}}(U) \;=\; \sup\left\{\max H \mid H \in \mathcal{H}_{\mathrm{HZ}}(U)\right\}.$$

It has been shown in [4, Sections 4.1 and 4.2] that for any thickening $\psi\colon S \times I \to U \subset M$ for which $c_{\mathrm{HZ}}(U) < \infty$, the set $\{\epsilon \in I \mid \mathcal{P}\left(S_\epsilon\right) \neq \emptyset\}$ is dense in I, and that

$$\mu\left\{\epsilon \in I \mid \mathcal{P}\left(S_\epsilon\right) \neq \emptyset\right\} \;=\; \mu(I)$$

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if S bounds a compact submanifold of M and $c_{\rm HZ}(M) < \infty$. Here, μ denotes Lebesgue measure. In this note we prove

Theorem 1. *For any thickening $\psi\colon S \times I \to U \subset M$ we have*

$$\mu \left\{\epsilon \in I \mid \mathcal{P}\left(S_\epsilon\right) \neq \emptyset\right\} \,=\, \mu(I)$$

provided that $c_{\rm HZ}(U) < \infty$.

Proof. Consider a thickening $\psi\colon S \times I \to U \subset M$. We can assume that $I = \,]-1,1[$. Let $G = \{\epsilon \in \,]-1,1[\mid \mathcal{P}\left(S_\epsilon\right) \neq \emptyset\}$ be the parameters of the good hypersurfaces.

Step 1. *The set G is measurable.*

Proof. We define the smooth function $K\colon U \to \,]-1,1[$ by

$$K(z) = \epsilon \quad \text{if } \; z \in S_\epsilon .$$

The set $\mathcal{P}_K(\epsilon)$ of closed orbits of the Hamiltonian flow φ_K^t of K on S_ϵ corresponds to $\mathcal{P}\left(S_\epsilon\right)$. For each closed orbit x of φ_K^t we denote by $T(x)$ its period defined as

$$T(x) \,=\, \min\left\{t > 0 \mid \varphi_K^t(p) = p\right\}$$

where p is any point on x. For each $n \in \mathbb{N}$ we set

$$G_n \,=\, \{\epsilon \in \,]-1,1[\mid \text{ there exists } x \in \mathcal{P}_K(\epsilon) \text{ such that } T(x) \leq n\}\,.$$

It follows from the Arzelà–Ascoli Theorem that G_n is a closed subset of $]-1,1[$, see [4, page 109, Proposition 1] The set $G = \bigcup_{n\geq 1} G_n$ is thus measurable.

Step 2. *The set G has full measure.*

Proof. Applying the argument in the proof of Theorem 3 in Section 4.2 of [4] to Hamiltonian functions $H\colon U \to \mathbb{R}$ as in the figure below such that

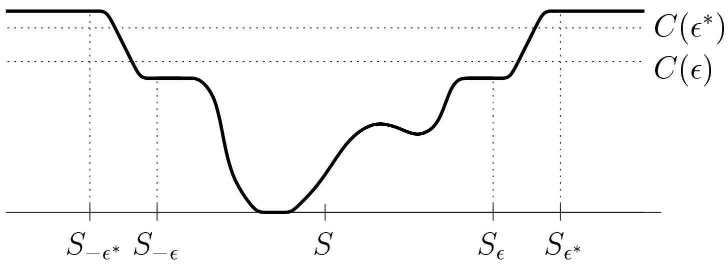
$$H(x) \,=\, f\left(|K(x)|\right) \quad \text{if } \; x \in S_{-t} \cup S_t, \;\epsilon \leq t \leq 1,$$

we see that $\mathcal{P}\left(S_{-\epsilon}\right) \neq \emptyset$ or $\mathcal{P}\left(S_\epsilon\right) \neq \emptyset$ whenever the function $C(t)\colon [0,1[\,\to \mathbb{R}$, $t \mapsto c_{\rm HZ}\left(\psi\left(S \times]-t,t[\right)\right)$ is Lipschitz continuous at ϵ. Since $C(t)$ is monotone increasing, it is differentiable almost everywhere, and thus Lipschitz continuous almost everywhere. Therefore,

$$\mu \left\{\epsilon \in [0,1[\mid \mathcal{P}\left(S_{-\epsilon}\right) \neq \emptyset \text{ or } \mathcal{P}\left(S_\epsilon\right) \neq \emptyset\right\} \,=\, 1. \tag{1}$$

Let $B = \,]-1,1[\setminus G$ be the parameters of the bad hypersurfaces. Arguing by contradiction we assume that $\mu(B) > 0$. Choose an interval $]a,b[\,\subset\,]-1,1[$ such that

$$\mu\left(]a,b[\cap B\right) \,\geq\, \tfrac{2}{3}(b-a). \tag{2}$$



Applying (1) to $]a,b[$ instead of $]-1,1[$ we see that

$$\mu\left(]a,b[\cap G\right) \;\geq\; \tfrac{1}{2}(b-a). \tag{3}$$

Combining (2) with (3) we find that

$$\begin{aligned} b-a &= \mu(]a,b[) \\ &= \mu\left(]a,b[\cap B\right) + \mu\left(]a,b[\cap G\right) \\ &\geq \tfrac{2}{3}(b-a) + \tfrac{1}{2}(b-a) \\ &> b-a. \end{aligned}$$

This contradiction shows that $\mu(B)=0$, and so $\mu(G) = \mu\left(]-1,1[\right)$. $\square$

It is often of interest to find closed characteristics in a restricted set of homotopy classes. We thus fix a subset Γ of the fundamental group $\pi_1(S)$, and given a thickening $\psi\colon S\times I \to U \subset M$, we denote by $\mathcal{P}^\Gamma(S_\epsilon)$ the set of closed characteristics on S_ϵ representing an element of $\Gamma \subset \pi_1(S_\epsilon) = \pi_1(S)$. We can assume that $I =]-1,1[$, and for each $\epsilon \in]0,1[$ we set $U_\epsilon = \psi\left(S\times]-\epsilon,\epsilon[\right)$. We say that $H \in \mathcal{H}(U_\epsilon)$ is HZ^Γ-*admissible* if the flow φ_H^t has no non-constant T-periodic orbit with period $T \leq 1$ which represents an element of $\Gamma \subset \pi_1(U_\epsilon) = \pi_1(S)$, and we set

$$\mathcal{H}^\Gamma_{\mathrm{HZ}}(U_\epsilon) \;=\; \left\{ H \in \mathcal{H}(U_\epsilon) \mid H \text{ is } \mathrm{HZ}^\Gamma\text{-admissible} \right\}.$$

The Γ-sensitive Hofer–Zehnder capacity of U_ϵ is defined as

$$c^\Gamma_{\mathrm{HZ}}(U_\epsilon) \;=\; \sup\left\{ \max H \mid H \in \mathcal{H}^\Gamma_{\mathrm{HZ}}(U_\epsilon) \right\}.$$

Notice that the map $\epsilon \mapsto c^\Gamma_{\mathrm{HZ}}(U_\epsilon)$ is monotone increasing. Repeating the above proof with c_{HZ} replaced by c^Γ_{HZ} we find the following refinement of Theorem 1.

Theorem 2. *For any thickening $\psi\colon S\times I \to U \subset M$ of a hypersurface S in (M,ω) we have*

$$\mu\left\{ \epsilon \in I \mid \mathcal{P}^\Gamma(S_\epsilon) \neq \emptyset \right\} \;=\; \mu(I)$$

provided that $c^\Gamma_{\mathrm{HZ}}(U) < \infty$.

While a version of Theorem 1 is applied in [1], Theorem 2 is used in [5] to show that the flow describing the dynamics of a charge in a rational symplectic magnetic field has a closed orbit on almost every small energy level.

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