ROBUST ENTROPY EXPANSIVENESS IMPLIES GENERIC DOMINATION

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ABSTRACT. Let $f : M \to M$ be a C^r-diffeomorphism, $r \geq 1$, defined on a compact boundaryless d-dimensional manifold M, $d \geq 2$, and let $H(p)$ be the homoclinic class associated to the hyperbolic periodic point p. We prove that if there exists a C^1 neighborhood $\mathcal{U}$ of f such that for every $g \in \mathcal{U}$ the continuation $H(p_g)$ of $H(p)$ is entropy-expansive then there is a Df-invariant dominated splitting for $H(p)$ of the form $E \oplus F_1 \oplus \cdots \oplus F_c \oplus G$ where E is contracting, G is expanding and all F_j are one dimensional and not hyperbolic.

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1. INTRODUCTION

In this paper we study what are the consequences at the dynamical behavior of the tangent map Df of a diffeomorphism $f : M \to M$, assuming that f is robustly entropy expansive. In this direction we obtain that the tangent bundle has a Df-invariant dominated splitting of the form $E \oplus F_1 \oplus \cdots \oplus F_c \oplus G$ where E is contracting, G is expanding and all F_j are one dimensional and not hyperbolic.

Let M be a compact connected boundary-less Riemannian d-dimensional manifold, $d \geq 2$, and $f : M \to M$ a homeomorphism. Let K be a compact invariant subset of M and $\text{dist} : M \times M \to \mathbb{R}^+$ a distance in M compatible with its Riemannian structure. For $E, F \subset K$, $n \in \mathbb{N}$ and $\delta > 0$ we say that E (n,δ)-spans F with respect to f if for each $y \in F$ there is $x \in E$ such that $\text{dist}(f^j(x), f^j(y)) \leq \delta$ for all $j = 0,\ldots,n-1$. Let $r_n(\delta,F)$ denote the minimum cardinality of a set that (n,δ)-spans F. Since K is compact $r_n(\delta,F) < \infty$. We define

$$h(f,F,\delta) \equiv \limsup_{n\to\infty} \frac{1}{n} \log(r_n(\delta,F))$$

and the topological entropy of f restricted to F as

$$h(f,F) \equiv \lim_{\delta\to 0} h(f,F,\delta)\,.$$

The last limit exists since $h(f,F,\delta)$ increases as δ decreases to zero.

Definition 1.1. *For $x \in K$ let us denote*

$$\Gamma_\varepsilon(x,f) \equiv \{y \in M \,/\, d(f^n(x), f^n(y)) \leq \varepsilon,\, n \in \mathbb{Z}\}\,.$$

We will simply write $\Gamma_\varepsilon(x)$ instead of $\Gamma_\varepsilon(x,f)$ when it is understood which f we refer to.

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Following Bowen (see [Bo]*) we say that* f/K *is* entropy-expansive *or* h-expansive *for short, if and only if there exists* $\varepsilon > 0$ *such that*

$$h^*_f(\varepsilon) \equiv \sup_{x\in K} h(f,\Gamma_\varepsilon(x)) = 0\,.$$

Theorem 1.1. [Bo, Theorem 2.4] *For all homeomorphism* f *defined on a compact invariant set* K *it holds*

$$h(f,K) \le h(f,K,\varepsilon) + h^*_f(\varepsilon) \text{ in particular } h(f,K) = h(f,K,\varepsilon) \text{ if } h^*_f(\varepsilon) = 0\,.$$

A similar notion to h-expansiveness, albeit weaker, is the notion of *asymptotically h-expansiveness* introduced by Misiurewicz [Mi]: let K be a compact metric space and $f : K \to K$ an homeomorphism. We say that f is asymptotically h-expansive if and only if

$$\lim_{\varepsilon\to 0} h^*_f(\varepsilon) = 0\,.$$

Thus, we do not require that for a certain $\varepsilon > 0$, $h^*_f(\varepsilon) = 0$ but that $h^*_f(\varepsilon) \to 0$ when $\varepsilon \to 0$. It has been proved by Buzzi, [Bu], that any C^∞ diffeomorphism defined on a compact manifold is asymptotically h-expansive. The interessed reader can found examples of diffeomorphisms that are not entropy expansive neither asymptotically entropy expansive in [Mi, PaVi].

Next we recall the notion of dominated splitting.

Definition 1.2. *We say that a compact* f*-invariant set* $\Lambda \subset M$ *admits a dominated splitting if the tangent bundle* $T_\Lambda M$ *has a continuous* Df*-invariant splitting* $E \oplus F$ *and there exist* $C > 0$, $0 < \lambda < 1$, *such that*

$$\|Df^n|E(x)\| \cdot \|Df^{-n}|F(f^n(x))\| \le C\lambda^n \;\forall x \in \Lambda,\, n \ge 0. \tag{1}$$

Observe that if the topological entropy of a map $f : M \to M$ vanishes, $h(f) = 0$, then automatically f is h-expansive. For instance Morse-Smale diffeomorphisms $\varphi : M \to M$ are h-expansive. We remark that Morse-Smale diffeomorphisms are C^1-stable under perturbations and so they constitute a class which is robustly h-expansive.

Here we are interested in diffeomorphisms that exhibit a chaotic behavior, i.e.: their topological entropy is positive. Moreover, we restrict our study to homoclinic classes $H(p)$ associated to saddle-type hyperbolic periodic points. Recall that the homoclinic class $H(p)$ of a saddle-type hyperbolic periodic point p of $f \in \mathrm{Diff}^1(M)$ is the closure of the intersections between the unstable manifold $W^u(p)$ of p and the stable manifold $W^s(p)$ of p. These classes persist under perturbations and we wish to establish the property of those classes under the assumption that h-expansiveness is robust.

Definition 1.3. *Let* M *be a compact boundaryless* C^∞ *manifold and* $f : M \to M$ *be a* C^r *diffeomorphism,* $r \ge 1$. *Let* $H(p)$ *be a* f*-homoclinic class associated to the* f*-hyperbolic periodic point* p. *Assume that there is a* C^r *neighborhood* $\mathcal{U}$ *of* f, *such that for any* $g \in \mathcal{U}$ *it holds that the continuation* $H(p_g)$ *of* $H(p)$ *is h-expansive. Then we say that* $f/H(p)$ *is* C^r*-robustly h-expansive.*

In [PaVi, Theorem B] we obtain that if $H(p,f)$ is isolated and the finest dominated splitting on $H(p,f)$ is

$$T_{H(p,f)}M = E \oplus F_1 \oplus \cdots \oplus F_k \oplus G$$

with E contracting, G expanding and all F_j, $j = 1,\dots,k$, one dimensional and not hyperbolic, then $f/H(p, f)$ is h-expansive. Moreover, since the dominated splitting is preserved under C^1-perturbations this result holds for a C^1-neighborhood $\mathcal{U}(f) \subset \mathrm{Diff}^1(M)$, i.e.: h-expansiveness is C^1-robust.

Roughly speaking, [PaVi, Theorem B] says that the domination property implies that small neighbourhoods in $H(p)$ have an 'ordered dynamics' and there cannot appear 'arbitrarily small horseshoes', i.e:, horseshoes generated by homoclinic points in $W^s_\xi(x) \cap W^u_\xi(x)$ for $\xi > 0$ arbitrarily small and $x \in H(p)$ periodic, as in the example given in [PaVi][Section 2] for a surface diffeomorphism. The presence of these arbitrarily small horseshoes would imply that $\sup_{x\in H(p)} h(f,\Gamma_\varepsilon(x)) > 0$ for any $\varepsilon > 0$.

This paper is intended to continue [PaVi] in the reverse direction: we analyze the consequences of h-expansiveness to hold in a C^1-neighbourhood $\mathcal{U}(f) \subset \mathrm{Diff}^1(M)$ of f. Our main results are the following:

Theorem A. *Let M, $f : M \to M$ and $H(p)$ be as in Definition 1.3 for $r = 1$. Then $H(p)$ has a dominated splitting $E \oplus F$.*

In fact [PaVi, Example 2] shows that in dimension greater or equal to three the existence of a dominated splitting for $H(p)$ is not enough tho guarantee h-expansiveness, so it is natural to search for a stronger property.

Let us recall the concept of *finest dominated splitting* introduced in [BDP].

Definition 1.4. *Let $\Lambda \subset M$ be a compact f-invariant subset such that $TM/\Lambda = E_1 \oplus E_2 \oplus \cdots \oplus E_k$ with E_j Df invariant, $j = 1,\dots,k$. We say that $E_1 \oplus E_2 \oplus \cdots \oplus E_k$ is dominated if for all $1 \le j \le k-1$*

$$(E_1 \oplus \cdots E_j) \oplus (E_{j+1} \oplus \cdots \oplus E_k)$$

has a dominated splitting. We say that $E_1 \oplus E_2 \oplus \cdots \oplus E_k$ is the finest dominated splitting when for all $j = 1,\dots,k$ there is no possible decomposition of E_j as two invariant sub-bundles having domination.

An improvement of Theorem A is the following.

Theorem B. *Let M, $f : M \to M$ and $H(p)$ be as in Definition 1.3 for $r = 1$. Then the finest dominated splitting in $H(p)$ has the form $E \oplus F_1 \oplus \cdots \oplus F_c \oplus G$ where all F_j are one dimensional and not hyperbolic.*

If $H(p)$ *is isolated* then we may refine the previous result. Before we announce precisely this result, let us recall the definitions of: chain recurrent set, isolated homoclinic class and heterodimensional cycles..

Definition 1.5. *The chain recurrent set of a diffeomorphism f, denoted by $R(f)$, is the set of points x such that, for every $\varepsilon > 0$, there is a closed ε-pseudo orbit joining x to itself: there is a finite sequence $x = x_0, x_1, \dots, x_n = x$ such that* $\mathrm{dist}(f(x_i), x_{i+1}) < \varepsilon$.

Definition 1.6. *We say that $H(p)$ is isolated if there are neighborhoods $\mathcal{U}$ of f in* $\mathrm{Diff}^1(M)$ *and U of the homoclinic class class $H(p)$ in M such that, for every $g \in \mathcal{U}$, the continuation $H(p_g)$ of*

$H(p)$ coincides with the intersection of the chain recurrence set of g, $R(g)$ with the neighborhood U.

Remark 1.2. *Generically a recurrence class which contains a periodic point p_g coincides with $H(p_g)$*, [BC].

Definition 1.7. *We say that Γ is a cycle if $\Gamma = \{p_i, 0 \leq i \leq n, p_0 = p_n\}$, where p_i are hyperbolic periodic points of f and $W^u(p_i) \cap W^s(p_{i+1}) \neq \emptyset$, for all $0 \leq i \leq n-1$. Γ is called a heterodimensional cycle if, for some $i \neq j$, $\dim(W^u(p_i)) \neq \dim(W^u(p_j))$.*

Recall that the *index* of a hyperbolic periodic point p is the dimension of its unstable manifold $W^u(p)$.

Theorem C. *Let M, $f : M \to M$ and $H(p)$ be as in Definition 1.3 for $r = 1$. Assume moreover that $f/H(p)$ is isolated. Then for g in $\mathcal{U}(f)$, $H(p_g)$ has a dominated splitting of the form $E \oplus F_1 \oplus \cdots \oplus F_k \oplus G$ where E is contracting, G is expanding and all F_j are not hyperbolic and $\dim(F_j) = 1$. Moreover, in case that the index of periodic points in $H(p_g)$ are in a C^1 robust way equal to index(p) then for an open dense subset $\mathcal{V} \subset \mathcal{U}(f)$, $H(p_g)$ is hyperbolic, i.e.: $k = 0$.*

On the other hand, if there are g arbitrarily C^1-close to f such that in $H(p_g)$ there are periodic points of different index then $H(p)$ is approximated by robust heterodimensional cycles, [BDi].

If we do not assume that $H(p)$ is isolated but we know that f cannot be approximated by g exhibiting a heterodimensional cycle we have the following result:

Theorem D. *Let $\mathcal{C}(M) = \{f \in \mathrm{Diff}^1(M); f \text{ has no cycles}\}$, and $H(p)$ be as in Definition 1.3 for $r = 1$. Assume that $f \in \mathrm{Diff}^1(M) \setminus \overline{\mathcal{C}(M)}$. Then for g in a residual subset $\mathcal{R} \subset \mathcal{U}(f)$, $H(p_g)$ has a dominated splitting of the form $E^s \oplus E^c \oplus E^u$ where E^c is not hyperbolic and $\dim(E^c) \leq 2$, E^s is contracting and E^u is expanding. Moreover, if $\dim(E^c) = 2$ then $E^c = E^c_1 \oplus E^c_2$ dominated.*

1.1. **Idea of the proofs.** The proofs of Theorems A and B go by contradiction: under the hypothesis that there is not a dominated splitting in $T_{H(p)}M$, we profit from some ideas of [PV] and [Ro] to create a flat tangency between $W^s(p)$ and $W^u(p)$. We remark that in [PV, Ro] for the case that $\dim(M) > 2$ it was proved that if $r \geq 2$ and g has a homoclinic tangency then there are diffeomorphisms arbitrarily C^r-close to g exhibiting persistent homoclinic tangencies (thus generalizing results of [Nh1], see also [Nh2]). In our case, since we can perform the perturbations in the C^1 topology, our arguments are simplier than theirs to obtain a C^2 diffeomorphism g exhibiting a flat tangency, and afterward create an arc of tangencies between $W^s(p)$ and $W^u(p)$.

Next we follow [DN], to perform another C^1-perturbation with support in a small neighborhood of the arc of tangencies leading to the appearance of arbitrarily small horseshoes with positive entropy contradicting h-expansiveness. Therefore $Df/T_{H(p,f)}M$ admits a dominated spliting.

Moreover, either the finest dominated splitting (see Definition 1.4) has the form $E \oplus F_1 \oplus \cdots \oplus F_c \oplus G$ where all F_j are one dimensional and not hyperbolic or again we contradict robustness of h-expansiveness using [Go, Theorem 6.6.8].

For the proof of Theorem C we assume some specific generic properties described in Section 3 and that $H(p)$ is isolated. These allow to prove that the extremal sub-bundles E and G are respectively contracting and expanding. Moreover if the index of periodic points of $H(p_g)$ is robustly the index of p then for an open dense subset of $\mathcal{U}(f)$ the dominated splitting defined on $T_{H(p)}M$ is hyperbolic. This proof is done in two steps: (1) First we prove in Lemma 3.2 that the extremal sub-bundles are hyperbolic using the fact that $H(p)$ is isolated, [BDPR]. (2) Second we show in Lemma 3.3 that if in a C^1-robust way the index of periodic points in $H(p_g)$ are the same for $g \in \mathcal{U}(f)$ then for an open and dense subset $\mathcal{U}_1$ of $\mathcal{U}(f)$ we have that $H(p_g)$ is hyperbolic.

Finally in Theorem D, where we do not assume that $H(p)$ is isolated, we see, under the generic assumptions described at Section 3, that for a residual subset $\mathcal{R} \subset \mathcal{U}(f)$ we have a dominated splitting $E^s \oplus E^c \oplus E^u$ defined on $T_{H(p)}M$ such that E^s is contracting, E^u is expanding and E^c is dominated and at most two dimensional. For this we assume further that $f \in \mathrm{Diff}^1(M)\backslash\overline{\mathcal{C}}(M)$ which allows to use [Cr, MainTheorem].

2. ENTROPY EXPANSIVENESS IMPLIES DOMINATION.

In this section we prove Theorem B assuming that $f/H(p)$ is robustly h-expansive.

Let $H(p)$ be a f-homoclinic class associated to the hyperbolic periodic point p. Assume that there is a C^1 neighborhood $\mathcal{U}$ of f such that for any $g \in \mathcal{U}$ it holds that there is a continuation $H(p_g)$ of $H(p)$ such that $H(p_g)$ is h-expansive.

We may assume that p is a *hyperbolic fixed point* since $f/H(p)$ is h-expansive if and only if $f^m/H(p)$ is h-expansive. This follows from the fact that for any compact f-invariant set Λ we have that $h(f^m,\Lambda) = m \cdot h(f,\Lambda)$ which implies that $h(f^m,\Lambda) = 0 \iff h(f,\Lambda) = 0$.

Let $x \in W^s(p) \cap W^u(p)$ be a transverse homoclinic point associated to the periodic point p. We define $E(x) \equiv T_xW^s(p)$ and $F(x) \equiv T_xW^u(p)$. Since p is hyperbolic we have that $E(x) \oplus F(x) = T_xM$. Moreover, $E(x)$ and $F(x)$ are Df-invariant, i.e.: $Df(E(x)) = E(f(x))$ and $Df(F(x)) = F(f(x))$. Denote by $H_t(p)$ the set of the transverse homoclinic points associated to p. Then, it can be proved that $H(p) \equiv \overline{H_t(p)}$. Here $\overline{A}$ stands for the closure in M of the subset $A \subset M$. So if we prove that there is a dominated splitting for $H_t(p)$ we are done since we can extend by continuity the splitting to the closure $H(p)$. Moreover, since C^2-diffeomorphisms are dense in the C^1-neighbourhood $\mathcal{U}$ we may assume that f is of class C^2 taking into account that we are assuming that h-expansiveness is C^1-robust.

We will use the following result proved in [Fr]:

Lemma 2.1. [Fr, Lemma 1.1] *Let M be a closed n-manifold, $f : M \to M$ a C^1 diffeomorphism, and $\mathcal{U}(f)$ a given neighbourhood of f. Then, there exist $\mathcal{U}_0(f) \subset \mathcal{U}(f)$ and $\delta > 0$ such that if $g \in \mathcal{U}_0(f)$, $S = \{p_1, p_2, \dots p_m\} \subset M$ is a finite set, and L_i, $i = 1, \dots, m$ are linear maps, $L_i : TM_{p_i} \to TM_{f(p_i)}$, satisfying $\|L_i - D_{p_i}g\| \le \delta$, $i = 1, \dots, m$ then there is $\tilde{g} \in \mathcal{U}(f)$ satisfying $\tilde{g}(p_i) = g(p_i)$ and $D_{p_i}\tilde{g} = L_i$. Moreover, if U is any neighborhood of S then we may chose $\tilde{g}$ so that $\tilde{g}(x) = g(x)$ for all $x \in \{p_1, p_2 \dots p_m\} \cup (M\backslash U)$.*

Remark 2.2. *The statement given there is slightly different from that above, but the proof of our statement is contained in* [Fr].

2.1. Existence of dominated splitting: proof of Theorem A.

Under the hypothesis of Theorem A, let us assume that f is of class C^r, $r \ge 2$ and prove that there is a dominated splitting for $H_t(p)$.

The proof goes by contradiction and it is done in several steps: (1) at Lemma 2.3 we perform a pertubation g of f exhibting a homoclinic point $x_g \in H(p_g)$ with small angle between $W^s_{loc}(x_g, g)$ and $W^u_{loc}(x_g, g)$, (2) at Proposition 2.5 we perform another perturbation (that we still denote by g) of f to create a tangency between $E^s(x, g)$ and $E^u(x, g)$, $x \in H(p_g)$, (3) at Proposition 2.1 through another pertubation of f we create an arc of flat tangencies $\beta \subset H(p_g)$, (4) finally in Subsection 2.1.1 we perform a sequence of perturbations of f leading to G near f presenting a sequence of two by two disjoint small horseshoes $H_{\varepsilon_n} \subset H(p_G)$, $\varepsilon_n \to 0$ as $n \to \infty$. Moreover, we can select the sequence ε_n in such a way that none of then are a constant of h-expansiveness of G. Since the entropy of each of these small horseshoes is positive, we arrive to a contradiction to h-expansiveness of f.

To start, let us assume, by contradiction, that $H_t(p)$ has no dominated splitting. Then, by [MPP, § 3.6 Proof of Theorem F] it holds

(AD) for all $m \in \mathbb{Z}^+$ there exists x_m such that for all $0 \leq n \leq m$,

$$\|Df^n|E(x_m)\| \cdot \|Df^{-n}|F(f^n(x_m))\| > 1/2\ ,$$

Lemma 2.3. *Assume that (AD) holds. Then, given $\gamma > 0$ and $\varepsilon > 0$ there is $m > 0$ and g an ε-C^1-perturbation of f with a homoclinic point x_g associated to p_g such that the angle at x_g between $W^s_{loc}(x_g, g)$ and $W^u_{loc}(x_g, g)$ is less than γ.*

Proof. Arguing by contradiction let us assume that there is $\gamma_0 > 0$ such that for all g in $\mathcal{U}_0$ the angle at x_g between $W^s_{loc}(x_g, g)$ and $W^u_{loc}(x_g, g)$ is greater or equal than γ_0.

By hypothesis there exist vectors $v_m \in F(x_m)$ and $w_m \in E(x_m)$ with $\|v_m\| = \|w_m\| = 1$ such that

$$\frac{\|Df^j(w_m)\|}{\|Df^j(v_m)\|} > \frac{1}{2}, \quad \forall\, j,\, 1 \leq j \leq m.$$

Take $\varepsilon > 0$ small such that any C^1-ε-perturbation of f gives a diffeomorphism $g \in \mathcal{U}_0$ where $\mathcal{U}_0$ is the C^1-neighborhood of f where we have h-expansiveness. Let $\varepsilon' > 0$ be such that any perturbation of the derivatives along a finite orbit of f can be realized via Lemma 2.1 by a C^1-ε-perturbation of f.

Let us define $T_j : T_{f^j(x_m)}M \to T_{f^j(x_m)}M$ a linear map such that $T_j|_{E(f^j(x_m))} = (1+\varepsilon')id$ and $T_j|_{F(f^j(x_m))} = id$, $j = 0, \ldots, m$. Note that T_j stretches $E = T_{x_m}W^s_\varepsilon(x_m, f)$ and left $F = T_{x_m}W^u_\varepsilon(x_m, f)$ unchanged. Let $P : T_{x_m}M \to T_{x_m}M$ be a linear map satisfying $P = id$ in $E(x_m)$ and $P = id + L$ in $F(x_m)$ where $L : F(x_m) \to E(x_m)$ is a linear map such that $L(v_m) = \varepsilon' w_m$ and $\|L\| = \varepsilon'$. Finally define $G_0 = T_1 \cdot Df_{x_m}, \cdot P$, and $G_j = T_{j+1} \cdot Df_{f^j(x_m)}$ for $j = 1, \ldots, m-1$. By Lemma 2.1 there exists a diffeomorphism $g : M \to M$ such that g is ε-near f, keeps the orbit of x_m unchanged for $j = 0, 1, \ldots, m$, and such that $Dg_{f^j(x_m)} = G_j$. We may assume (and do) that the support of the perturbation does not cut a small neighborhood of p. It follows that x_m continues to be a homoclinic point of g. Moreover, we do not change $E(f^j(x_m))$, $j \in \mathbb{Z}$, and $F(f^j(x_m))$ is changed only for $j \geq 0$. Thus such bundles are the stable and unstable directions of a homoclinic point of a diffeomorphism $g \in \mathcal{U}_0$. We obtain that $v_m \mapsto v_m + \varepsilon' w_m = u$ and after m iterates we have $u_m = Dg^m(u) = Dg^m(v_m + \varepsilon' w_m) = Df^m(v_m) + (1+\varepsilon')^m Df^m(\varepsilon' w_m)$.

Given $\varepsilon' > 0$ we may find $m > 0$ such that $\varepsilon'(1+\varepsilon')^m \geq 4 + 2/\gamma_0$ where $\gamma_0 > 0$ is, by hypothesis of absurd, such that $\angle(E(x), F(x)) > \gamma_0$ for all $x \in H_t(p_g)$, $g \in \mathcal{U}_0$, where $\angle(E(x), F(x))$ stands for the angle between $E(x)$ and $F(x)$. With this choice of m, by [Ma2, Lemma II.10] we have

$$\|Df^m(v_m)\| = \|u_m - (1+\varepsilon')^m Df^m(\varepsilon' w_m)\| \geq$$

$$\geq \frac{\gamma_0}{1+\gamma_0}\|u_m\| \geq \frac{\gamma_0}{1+\gamma_0}\left| \|\varepsilon'(1+\varepsilon')^m Df^m(w_m)\| - \|Df^m(v_m)\| \right|.$$

Dividing the inequality $\|Df^m(v_m)\| \geq \frac{\gamma_0}{1+\gamma_0}\left| \|\varepsilon'(1+\varepsilon')^m Df^m(w_m)\| - \|Df^m(v_m)\| \right|$ by $\frac{\gamma_0}{1+\gamma_0}\|Df^m(v_m)\|$ and taking into account that by hypothesis

$$\frac{\|Df^m(w_m)\|}{\|Df^m(v_m)\|} > \frac{1}{2} \text{ and } \varepsilon'(1+\varepsilon')^m \geq 4 + 2/\gamma_0$$

we find

$$\frac{1+\gamma_0}{\gamma_0} > \frac{\varepsilon'(1+\varepsilon')^m}{2} - 1 > 1 + 1/\gamma_0 = \frac{1+\gamma_0}{\gamma_0},$$

arriving to a contradiction. Hence $\angle(Dg^m(u), w_m) < \gamma$, proving Lemma 2.3.

□

Let us recall the following result which may be found in [BDP, Lemma 4.16], see also [BDPR, Lemma 3.8].

Theorem 2.4. *Let p be a hyperbolic periodic point and $H(p)$ its homoclinic class. Assume that $H(p)$ is not trivial. Then there exists and arbitrarily small C^1-perturbation g of f and a hyperbolic periodic point q of $H(p_g)$ with period $\pi(q)$ and homoclinically related with p_g such that $Df_q^{\pi(q)}$ has only positive real eigenvalues of multiplicity one.*

Observe that in the previous result, since $q_g \in H(p_g)$, we have $H(p_g) = H(q_g)$. So, to simplify notation, we may assume directly that $p = q$ and moreover that $g = f$, and that p is a fixed point. We order the eigenvalues of Df_p labeling them as $0 < \lambda_k < \cdots < \lambda_1 < 1 < \mu_1 < \cdots < \mu_{d-k}$ so that the less contracting and the less expanding ones are respectively λ_1 and μ_1.
By a small C^1-preturbations we may also assume that locally, in a neighborhood V of p, we have linearizing coordinates so that

$$f(x) = \sum_{j=1}^{k} \lambda_j a_j u_j + \sum_{j=1}^{d-k} \mu_j a_{k+j} u_{k+j}$$

where we write $x = \sum_{j=1}^{d} a_j u_j$ for $x \in V$.

The lines in $W^s_{loc}(p)/V$ corresponding to the eigenvalues λ_j may be extended to all of $W^s(p)$ by backward iteration by f giving us a foliation by lines of dimension k. Similarly for $W^u(p)$ we have a $(d-k)$-foliation by lines obtained by forward iteration by f.

Now, let us assume that g is near f, $f = g$ in a small neighborhood of p and that there is a small angle between $T_xW^s(p,g)$ and $T_xW^u(p,g)$ where x is a g-homoclinic point associated to p. That is: there is γ small such that

$$\angle(T_xW^s(p,g), T_xW^u(p,g)) < \gamma.$$

By Theorem 2.4, we may assume that all the eigenvalues of $Df_p^{\pi_p}$ are positive with multiplicity one and that we have linearizing coordinates in a small neighborhood of p.

The next proposition stablishes that if the angle between $T_xW^s(p,g)$ and $T_xW^u(p,g))$ is small than we can create a tangency between $T_xW^s(p,\tilde{g})$ and $T_xW^u(p,\tilde{g}))$, for some $\tilde{g}$ near g.

Proposition 2.5. *There is $\gamma > 0$ and $\mathcal{U}_0(g) \subset \mathcal{U}(f)$ so that for some $\tilde{g} \in \mathcal{U}_0(g)$ there is a tangency between $E^s(x,\tilde{g})$ and $E^u(x,\tilde{g})$ if $\angle(E^s(x,g),E^u(x,g)) < \gamma$. Moreover x is a homoclinic point of $\tilde{g}$, $E^s(x,\tilde{g}) \oplus E^u(x,\tilde{g})$ has dimension $d-1$ and there is $N > 0$ so that if $< u >$ is the subspace common to $E^s(x,\tilde{g})$ and $E^u(x,\tilde{g})$ then $(D\tilde{g})^N(< u >)$ is tangent to the line corresponding to the less contracting eigenvalue and $(D\tilde{g})^{-N}(< u >)$ is tangent to the line corresponding to the less expanding eigenvalue of $D_p\tilde{g}$.*

Proof. Let $\mathcal{U}(f)$, $\mathcal{U}_0(f)$ and δ be as in Lemma 2.1. Shrinking $\mathcal{U}_0$ if it were necessary we may assume that $\operatorname{clos}\mathcal{U}_0(f) \subset \mathcal{U}(f)$. Hence we may assume without loss of generality that there is some $C > 0$ such that $\sup\{\|D_xg\| : g \in \mathcal{U}_0(f)\} \leq C$.

By hypothesis there is $g \in \mathcal{U}_0(f)$, $x \in W^s(p_g,g) \overline{\pitchfork} W^u(p_g,g)$ and $\gamma > 0$ small so that

$$\angle(E^s(x,g),E^u(x,g)) < \gamma.$$

Taking $\gamma < \delta/C$, since $\angle(E^s(x,g),E^u(x,g)) < \gamma$, there exist $v \in E^{s\perp}$ and $u \in E^s$ such that $v+u \in E^u, \|u\| = 1, \|v\| < \gamma$. Let $T : T_xM \to T_xM$ be such that $T_{|E^{s\perp}} = 0,\ T(u) = -v$ and $\|T\| < \delta/C$. Let $L : T_{g^{-1}(x)}M \to T_xM$ be defined by $L = (Id+T)\circ D_{g^{-1}(x)}g$. Then we have

$$\|L - D_{g^{-1}(x)}g\| < \delta, \quad \text{and} \quad u \in L(E^u(g^{-1}(x)).$$

Take a neighborhood U of $g^{-1}(x)$ such that $O_g(x) \cap U = \{g^{-1}(x)\}$. Using Lemma 2.1 we find $\tilde{g} \in \mathcal{U}(f)$ such that $g^j(x) = \tilde{g}^j(x)$ for all j, $\tilde{g} = g$ outside U, and $D_{g^{-1}(x)}\tilde{g} = L$. Hence $x \in W^s(p_{\tilde{g}},\tilde{g}) \cap W^u(p_{\tilde{g}},\tilde{g})$ since its forward and backward orbits continue to converge to $p_{\tilde{g}}$. Moreover $u \in E^s(x,\tilde{g}) \cap E^u(x,\tilde{g})$ and so the intersection of $W^s(p_{\tilde{g}})$ and $W^u(p_{\tilde{g}})$ is not transverse at the point x.

Since the eigenvalues of Df_p are all real positive and of multiplicity one and $f = g$ in a small neighborhood of p, by N forward iterations we have a vector $D^N\tilde{g}(u)$ almost tangent to the straight line $< v_1 >$ corresponding to the less contracting eigenvalue at p. Again by Lemma 2.1 we can perturb $\tilde{g}$ outside a small neighborhood of p to let the direction of $(D\tilde{g})^N(u)$ coincide with $< v_1 >$. Similarly we obtain $(D\tilde{g})^{-N}(u)$ tangent to the line corresponding to the less expanding eigenvector of $D\tilde{g}_p$. □

From Proposition 2.5 we may assume for f itself that there is a homoclinic point of tangency $x \in W^s(p) \cap W^u(p)$ with properties analogous to those of $\tilde{g}$. The next lemma asserts that under these hypothesis, we can obtain an arc β of non-tranversal homoclinic points in $W^s(p) \cap W^u(p)$.

Proposition 2.1. *Let p be a hyperbolic fixed point for f of index k and $x \in W^s(p) \cap W^u(p)$ such that the intersection at x is not transversal. Then by an arbitrarily small C^1-perturbation we may obtain a diffeomorphism g with $x \in W^s(p_g,g) \cap W^u(p_g,g)$ such that the intersection at x is flat, there exists a small arc β contained in the intersection of the stable and unstable manifolds of p. Moreover, there is $N > 0$ such that $g^N(\beta) \subset W^s_{loc}(p,g)$ is tangent to the eigenvector corresponding to the less contracting eigenvalue and analogously $g^{-N}(\beta) \subset W^u_{loc}(p,g)$ is tangent to the eigenvector corresponding to the less expanding eigenvalue.*

Proof. Since p is a hyperbolic saddle, $W^s(p)$ is an Euclidean k-dimensional hyperplane and $W^u(p)$ an Euclidean $(d-k)$-dimensional hyperplane both immersed in M. If the intersection at x of $W^s(p)$ and $W^u(p)$ is not transversal we should have a vector $u \neq 0$ in $T_xW^u(p) \cap T_xW^s(p)$, i.e.: we have a tangency between $W^s(p)$ and $W^u(p)$ at the homoclinic point x. Using Lemma 2.1 we may assume that the subspace generated by u is the unique in common between $T_xW^u(p)$ and $T_xW^s(p)$, that is $T_xW^u(p) + T_xW^s(p)$ has dimension $d-1$. Moreover, we also may assume that $k \geq d-k$ (otherwise we may take f^{-1} instead of f) and, again by Lemma 2.1, that the tangent space $T_xW^u_\varepsilon(x)$ intersects trivially $(T_xW^s_\varepsilon(x))^\perp$ the orthogonal complement of $T_xW^s_\varepsilon(x)$. Under these assumptions the orthogonal projection of $W^u_\varepsilon(x)$ into $W^s_\varepsilon(x)$ is locally a diffeomorphism in a suitable neighborhood of x. Let us choose $D_x \subset W^s_\varepsilon(x)$ a small disk and $N > 0$ such that $f^N(D_x) \subset W^s_\varepsilon(p)$, and let L_x be a small disk in $W^u_\varepsilon(x)$ such that $f^{-N}(L_x) \subset W^u_\varepsilon(p)$. L_x projects onto $L'_x \subset D_x$ diffeomorphicaly. Via a local coordinate map we may identify D_x with

$$\{y \in \mathbb{R}^d \,/\, y_{k+1} = \cdots = y_d = 0; y_1^2 + \cdots + y_k^2 = 1\},$$

with x identified with the origin 0 and u having the direction of Oy_1 which is tangent at 0 to L'_x too. L_x may be viewed as the graph of a map $\Gamma : L'_x \to (T_xW^s_\varepsilon(x))^\perp$ with $\frac{\partial \Gamma}{\partial y_1}|_0 = 0$. To simplify notation we write $(y_1, \ldots, y_k) = Y_1$ and $(y_{k+1}, \ldots, y_d) = Y_2$. Hence if $(Y_1, Y_2) \in L_x$ then $Y_2 = \Gamma(Y_1(Z))$, where, given L'_x, $Y_1(Z)$ is a local coordinate map from a neighborhood of 0 in $\mathbb{R}^{d-k}$ to D_x.

Claim 2.2. *There exists a C^1 perturbation of f that produces a diffeomorphism $g \in \mathcal{U}(f)$ with a flat intersection at $x \in D_x \cap L_x$, with $D_x \subset W^s_\varepsilon(x)$ and $L_x \subset W^u_\varepsilon(x)$. This flat intersection contains a small arc β.*

Proof. Define $h : M \to M$ by

$$h(Y_1, Y_2) = (Y_1, Y_2 - G(Y_1, Y_2)\Gamma(y_1, 0 \ldots, 0)).$$

Here G is a C^∞-bump function , $0 \leq G(Y_1, Y_2) \leq 1$, that vanishes in the boundary of the ball $B(0, \varepsilon')$, is equal to 1 in $B(0, \varepsilon'/4)$, and such that $\|\nabla G\| < \frac{2}{\varepsilon'}$, where ∇ means the gradient. Let us see that h is a diffeomorphism ε'-C^1-close to the identity.

(a) h is injective: Indeed, $h(Y_1, Y_2) = h(Y'_1, Y'_2)$ implies that $Y_1 = Y'_1$. Hence

$$Y_2 - G(Y_1, Y_2)\Gamma(y_1, 0 \ldots, 0) = Y'_2 - G(Y_1, Y'_2)\Gamma(y_1, 0 \ldots, 0).$$

Therefore

$$\|Y_2 - Y'_2\| = \|(G(Y_1, Y_2) - G(Y_1, Y'_2))\Gamma(y_1, 0, \ldots, 0)\| \leq \|\Gamma(y_1, 0, \ldots, 0)\|,$$

where we have used that $0 \leq G(Z_1, Z_2) \leq 1$ for all (Z_1, Z_2). Taking into account that

$$\Gamma(0,0) = 0, \ \frac{\partial \Gamma}{\partial y_1}\Big|_0 = 0$$

we obtain that $\Gamma(y_1, 0 \ldots, 0) = o(\varepsilon')$. Therefore

$$|(G(Y_1, Y_2) - G(Y_1, Y'_2))| =< \nabla G(Y_1, \Theta_2), Y_2 - Y'_2 > \leq \|\nabla G\| \|\Gamma(y_1, 0 \ldots, 0)\| < \frac{2}{\varepsilon'} o(\varepsilon').$$

Here (Y_1,Θ_2) is a point in the segment joining (Y_1,Y_2) with (Y_1,Y_2'). Let us choose $\varepsilon'>0$ so small that $\frac{2}{\varepsilon'}\cdot o(\varepsilon')<\frac{1}{2}$. It follows that

$$\|Y_2-Y_2'\|=\|(G(Y_1,Y_2)-G(Y_1,Y_2'))\Gamma(y_1,0,\ldots,0)\|\leq\frac{1}{2}\|\Gamma(y_1,0,\ldots,0)\|.$$

By induction we have that for all $n\in I\!N$

$$\|Y_2-Y_2'\|=\|(G(Y_1,Y_2)-G(Y_1,Y_2'))\Gamma(y_1,0,\ldots,0)\|\leq\frac{1}{2^n}\|\Gamma(y_1,0,\ldots,0)\|.$$

Therefore $Y_2=Y_2'$ and h is injective.

(b) h is a diffeomorphism: Indeed, we have

$$Dh=\begin{pmatrix} Id & \vdots & 0\\ \ldots & \vdots & \ldots\\ -G\frac{\partial\Gamma}{\partial y_1}^t-\Gamma^t\frac{\partial G}{\partial Y_1} & \vdots & Id-\Gamma^t\frac{\partial G}{\partial Y_2}\end{pmatrix}$$

Here $\Gamma=\Gamma(y_1,0\ldots,0)$, analogously $\frac{\partial\Gamma}{\partial y_1}$ only depends on y_1, and Γ^t is the transpose of Γ. As $\frac{\partial\Gamma}{\partial y_1}|_0=0$ we have that $-G\frac{\partial\Gamma}{\partial y_1}^t$ is small if ε' is sufficiently small and the same is true with respect to $\Gamma^t\frac{\partial G}{\partial Y_1}$ and $\Gamma^t\frac{\partial G}{\partial Y_2}$, taking into account that $\Gamma(y_1,0,\ldots,0)=o(\varepsilon')$ and $\|\nabla G\|<\frac{2}{\varepsilon'}$. Thus Dh is invertible.

Items (a) and (b) above prove that h is a diffeomorphism as C^1-close to the identity map as we wish and $h=id$ off a small ball $B(x,\varepsilon')$. Now consider $g=h\circ f$. Then g is a small pertubation of f.

Claim 2.3. *x is a flat g-homoclinic point and there is an arc $\beta\subset W^s(p,g)\cap W^u(p,g)$ with $x\in\beta$.*

Indeed, since $x\in W^s(p,f)\cap W^u(p,f)$ we have that $\lim_{n\to+\infty}f^n(x)=\lim_{n\to-\infty}f^n(x)=p$ and so x is neither forward recurrent nor backward recurrent. This implies that we may choose the support, $B(x,\varepsilon')$, of the perturbation in such a way that for $n\neq 0$, $g^n(B(x,\varepsilon'))\cap B(x,\varepsilon')=\emptyset$. Hence if $y\in W^s_\varepsilon(x,f)$ then for $\varepsilon>0$ small we obtain that $y\in W^s_\varepsilon(x,g)$. But h sends and arc β passing through x in $W^u_\varepsilon(x,f)$ onto an arc γ included in $W^s_\varepsilon(x,f)=W^s_\varepsilon(x,g)$ and passing through x too. Therefore $g^{-1}=f^{-1}\circ h^{-1}$ sends the arc γ into β which iterated sucessively by f^{-1} converges to p. Hence β is an arc contained in *both* the local stable and unstable manifold of x which is contained in $W^s(p,g)\cap W^u(p,g)$. Thus β is an arc of flat intersection between $W^s(p,g)$ and $W^u(p,g)$. This finishes both the proofs of Claim 2.3 and Claim 2.2. □

It is not difficult to see that this perturbation g may be done in such a way that for $N>0$ great enough $g^N(\beta)\subset W^s_{loc}(p,g)$ is tangent to the eigenvector corresponding to the less contracting eigenvalue and analogously $g^{-N}(\beta)\subset W^u_{loc}(p,g)$ is tangent to the eigenvector corresponding to the less expanding eigenvalue.

All together finishes the proof Proposition 2.1.

□

2.1.1. *Creating small horseshoes.* The previous result gives a diffeomorphism g, C^1-near f, such that the intersection between $W^u(p,g)$ and $W^s(p,g)$, in a local chart around x such that $T_xW^s_\varepsilon(x)\cap T_xW^u_\varepsilon(x) = <u>$, contains a segment $\beta=\{su : -\delta\le s\le\delta\}$. Moreover, Dg^Nu is tangent to the line corresponding to the less contracting eigenvector of Dg_p and $Dg^{-N}u$ is tangent to the line corresponding to the less expanding eigenvector of Dg_p.

Next we shall do a perturbation of g, which will give a diffeomorphism G such that G coincides with g outside a small neighborhood of β, similar to those of [DN, Lemma 5.1, Lemma 6.3] in order to create a sequence of small horseshoes $H_n\subset H(p,G)$ associated to $W^s_{loc}(x,G)$ and $W^u_{loc}(x,G)$. These horseshoes will have positive topological entropy and will be built in such a way that neither $\varepsilon>0$, nor $\varepsilon/2,\varepsilon/4,\dots,\varepsilon/2^n,\dots$ will be constants of h-expansiveness for $H(p,G)$. Therefore the diffeomorphism G is not h-expansive, contradicting our hypothesis.

To do so we proceed as follows: first, since we are working in a C^1-neighborhood of f and C^r, $r\ge 2$, diffeomorphisms are dense in $\mathrm{Diff}^1(M)$ we may assume that g, the diffeomorphism obtained at Proposition 2.1, is of class C^r, $r\ge 2$.

Let us assume first that p is of index $d-1$, i.e.: $\dim(W^u(p,f))=1$. This will simplify the techniques involved. We may assume, taking a large positive iterate by g and possibly reducing δ, that β, the segment of tangency, is contained in the local stable manifold of p in a local chart which is a linearizing neighborhood $U(p)$ of p.

Let $\psi:[0,\delta]\to\mathbb{R}$ be a C^∞ bump function satisfying:

(1) $\psi(s)=1/5$, for $s\in[0,\delta/16]$. This implies that $\psi^{(k)}(0)=\psi^{(k)}(\delta/16)=0$ for all $k\ge 1$.
(2) $\psi'(s)<0$ for $s\in(\delta/16,\delta/8)$.
(3) $\psi(s)=0$ for all $s\in[\delta/8,\delta/4]$, this implies that $\psi^{(k)}(\delta/8)=\psi^{(k)}(\delta/4)=0$ for all $k\ge 1$.
(4) $\psi'(s)>0$ for $s\in(\delta/4,3\delta/8)$.
(5) $\psi(s)=1$ for all $s\in[3\delta/8,\delta]$, this implies that $\psi^{(k)}(3\delta/8)=\psi^{(k)}(\delta)=0$ for all $k\ge 1$.

Next, consider $b:(-\delta,5\delta/4]\to\mathbb{R}$ such that

$$b(s)=\psi(s) \text{ for all } s\in[0,\delta]\,,$$

$$b(s)=\frac{1}{5}\psi(2(s+\delta/2)) \text{ for all } s\in[-\delta/2,0]\,,$$

$$b(s)=\frac{1}{5^2}\psi(2^2(s+3\delta/4)) \text{ for all } s\in[-3\delta/4,-\delta/2]\,,$$

and in general

$$b(s)=\frac{1}{5^n}\psi(2^n(s+\delta(1-1/2^n)) \text{ for all } s\in[-\delta(1-1/2^n),-\delta(1-1/2^{n-1})]\,.$$

Put also

$$b(s)=5\psi(\frac{s-\delta}{2}) \text{ for } s\in[\delta,5\delta/4]\,.$$

It is easy to see that $b(s)$ is C^∞ at $(-\delta,5\delta/4]$. We may assume that for $s\in[0,\delta]$, $|b'(s)|\le 24/\delta$ and $|b''(s)|\le K/\delta^2$, for some $K>0$.

Hence for $s\in[-\delta(1-1/2^n),-\delta(1-1/2^{n-1}]$ we have

$$|b'(s)|=\frac{1}{5^n}2^n\left|\psi'(2^n(s+\frac{2^n-1}{2^n}\delta))\right|\le\frac{24\cdot 2^n}{5^n\delta}$$

and

$$|b''(s)| = \frac{4^n}{5^n}\left|\psi''(2^n(s+\frac{2^n-1}{2^n}\delta))\right| \leq \frac{4^n K}{5^n\delta^2}.$$

Therefore $|b'(s)| \to 0$ and $|b''(s)| \to 0$ when $s \to -\delta$. Setting $b(-\delta)=0$ we have that $b'(-\delta) = b''(-\delta) = 0$ and b is of class C^2 on $[-\delta, 5\delta/4]$.

Let w be the unit vector in T_xM tangent to the expanding eigenvector of Dg_p. Recall we are assuming that $\dim(W^u(p,G) = 1$. Then w is not contained in $T_xW^s(x,g) + T_xW^u(x,g)$ since $T_xW^u(x,g)$ is tangent to $T_xW^s(x,g)$. Recall that $(0,s,0)$ are the coordinates of β in a local chart and that the interval $(0,[-\delta,5\delta/4],0)$ is totally contained in β. In the plane given by the origin 0 (identified with x) and the vectors u and w we consider the graph of the function $\hat{l} : [\delta/4, 5\delta/4] \to \mathbb{R}$ given by

$$\hat{l}(s) = \varepsilon_1 \cdot (s-\delta/2)(\delta - s), \quad s \in [\delta/4, 5\delta/4].$$

Observe that for $s \in [\delta/4, 5\delta/4]$, $\hat{l}(s)$ vanishes at $s = \delta/2$ and $s = \delta$ and it has a maximum value equals to $\delta^2\varepsilon_1/16$ at $s = 3\delta/4$. Now we extend $\hat{l}$ to $[-\delta, 5\delta/4]$ in the following way:

$$\hat{l}(s) = \varepsilon_2 \cdot (s+\delta/4)(-s), \quad s \in [-3\delta/8, \delta/8],$$

$$\hat{l}(s) = \varepsilon_3 \cdot (s+5\delta/8)(-\delta/2-s), \quad s \in [-11\delta/16, -7\delta/16],$$

and in general for $n \geq 1$:

$$\hat{l}(s) = \varepsilon_{n+1} \cdot (s+\delta(1-3/2^{n+1}))(-\delta(1-1/2^{n-1})-s), \quad s \in [-\delta(1-5/2^{n+2}), -\delta(1-9/2^{n+2})].$$

For $s \in [-\delta(1-5/2^{n+2}), -\delta(1-9/2^{n+2})]$, $\hat{l}$ vanishes only at $s_{n_1} = -\delta(1-3/2^{n+1})$ and $s_{n_2} = -\delta(1-1/2^{n-1})$ and it has a maximum value $\delta^2\varepsilon_{n+1}/(5^n \cdot 2^{2n+4})$ at $(s_{n_1}+s_{n_2})/2$. We complete the definition of $\hat{l}$ in $[-\delta, 5\delta/4]$ setting $\hat{l}(s) = 0$ elsewhere.

Finally, let $l(s) = \hat{l}(s)b(s)$ for all $s \in [-\delta, 5\delta/4]$. Then $l(s)$ is C^∞ in $(-\delta, 5\delta/4]$ and C^2 in $[-\delta, 5\delta/4]$.

Put coordinates in the local chart $Y = (S,s,t)$ and denote by B_s a small $(d-1)$-dimensional disk around x contained in a fundamental domain of $W^s_{loc}(p,g)$ whose coordinates in the local chart are $(S,s,0)$. Analogously denote by B_u a small 1-dimensional disk contained in $W^u(p,g)$ around x whose coordinates in the local chart are $(0,s,0)$. Note that B_s is characterized by $u = 0$; and B_u is the arc β contained in B_s, parameterized by $s \in [-\delta, 5\delta/4]$. The point x is identified with $(0,0,0)$.

Now, pick another C^∞ bump function φ such that φ vanishes outside a ε neighborhood of β, $\varepsilon \geq 2\varepsilon_1$, and is equal to 1 in the $\varepsilon/2$ neighborhood of β.

Let $h : M \to M$ be given by

$$\big(S,s,t\big) \mapsto \big(S,s,(t+l(s))\varphi(\|Y\|)\big)$$

and $h = id$ outside $B(\beta,\varepsilon)$ where ε is such that the ε-neighborhood of β does not intersect $U \cap g(U) \cap g^{-1}(U)$.

Now, letting $G = h \circ g$, we get, by construction, that G is a small perturbation of g, and, as in Proposition 2.1, it is not difficult to see that $B_s \subset W^s_{loc}(x,G) \subset W^s(p,G)$ and $(0,s,l(s)) \subset$

$W^u_{loc}(x,G) \subset W^u(p,G)$. Furthermore, it is straightforward to show that $W^s(p,G)$ and $W^u(p,G)$ intersect transversely at the points

$$(0,\delta/2,0),\,(0,\delta,0),\,(0,-\delta/4,0),\,(0,0,0),\dots,(0,-\delta(1-3/2^{n+1}),0),\,(0,-\delta(1-1/2^{n-1}),0),\dots$$

and the absolute value of the tangent of the angles at the points

$$(0,-\delta(1-3/2^{n+1}),0),\,(0,-\delta(1-1/2^{n-1}),0) \quad \text{is} \quad \frac{\varepsilon_{n+1}\delta}{5^n 2^{n+1}},\ n\in\mathbb{N}\,.$$

We denote by β' the graph of $l(s)$ in the plane $0uw$. If we choose ε, $\varepsilon_1 \geq \varepsilon_2 \geq \cdots \geq \varepsilon_n \geq \cdots$ with $\varepsilon_n \searrow 0$ and δ small, we may obtain the perturbation $G = h\circ g$ to be C^1 small (see [Nh1]). Moreover, we can also assume that :

(1) $G = g$ on $U\cap g(U)\cap g^{-1}(U)$, where we recall that $U = U(p)$ is a linearizing neighborhood of p.
(2) $W^s_{loc}(p,g) = W^s_{loc}(p,G)$ and $W^u_{loc}(p,g) = W^u_{loc}(p,G)$. Here $loc > 0$ states for a suitable small positive number,
(3) $W^s_{loc}(x,G)\cup W^u_{loc}(x,G) \subset U\setminus G(U)$. In particular $\beta\cup\beta' \subset U\setminus G(U)$.
(4) $G^k(W^s_{loc}(x,G)) \subset U$ for all $k\geq 0$ and there is $T > 0$ such that $G^{-k}(W^u_{loc}(x,G)) \subset U$ for all $k \geq T$,
(5) $G^{-T}(\beta\cup\beta') \subset U\setminus G^{-1}(U)$.

We point out that item (5) above follows from the fact that we may reduce the value of δ, if it were necessary, in order to ensure it.

Lemma 2.4. *There exists a sequence $\varepsilon_n \searrow 0$ such that G is not h-expansive.*

Proof. Recall that we are working in a linearizing neighborhood U of p with respect to g. Set

$$U^u_k = U\cap g(U)\cap\cdots\cap g^k(U) \quad\text{and}\quad U^s_k = U\cap g^{-1}(U)\cap\cdots\cap g^{-k}(U)\,.$$

Let $\gamma' = G^{-T}(\beta') \subset U\setminus G^{-1}(U)$ and denote by $(0,0,d_0)$, $(0,0,d_\infty)$ the coordinates of the end points of γ' corresponding respectively to $s = 5\delta/4$ and $s = -\delta$. In the same way we label all points in γ' corresponding to the *transverse* intersections of β with β': $(0,0,d_1)$ corresponds to $(0,\delta/2,0)$ and $(0,0,d_1')$ corresponds to $(0,\delta,0)$, $(0,0,d_2)$ corresponds to $(0,-\delta/4,0)$ and $(0,0,d_2')$ corresponds to $(0,0,0)$, $(0,0,d_3)$ corresponds to $(0,-5\delta/8,0)$ and $(0,0,d_3')$ corresponds to $(0,-\delta/2,0)$, and so on, labeling the image by G^{-T} of all the points of transverse intersection between β and β'.

Take small arcs a^s_1 and a'^s_1 contained in $U\setminus G^{-1}(U)$ tangent to the the direction of the eigenvector corresponding to the weakest contracting eigenvalue of $(DG)_p$ at the points $(0,0,d_1)$ and $(0,0,d_1')$. Multiply them by a $(d-2)$-dimensional disk C of diameter c. Analogously take small arcs a^u_1 and a'^u_1 tangent to the direction corresponding to the eigenvector of the expanding eigenvalue of $(DG)_p$ at the points $(0,\delta/2,0)$ and $(0,0,d_1')$ and contained in $U\setminus G(U)$. By the λ-lemma, [PdeM][Lemma 7.1], the forward orbits of a^u_1 and a'^u_1 contain arcs arbitrarily C^1 near $W^u(p,G)$ and the backward orbits of $a^s_1\times C$ and $a'^s_1\times C$ contain $(d-1)$-dimensional disks arbitrarily C^1 near $W^s(p,G)$. By the way we have chosen a^s_1 and a'^s_1 and the assumption about the eigenvalues of $D(G)_p$ (all positive real), we have that there is $k_1 = k_1(\varepsilon_1,\delta)$ such that for $k\geq k_1$ in U we have $\mathrm{dist}(G^{-k}(a^s_1),\beta) < \varepsilon_1\delta^2/32$ and $\mathrm{dist}(G^{-k}(a'^s_1),\beta) < \varepsilon_1\delta^2/32$. Moreover, we may choose $c > 0$

small such that $G^{-k}(a_1^s \times C)$ and $G^{-k}(a_1'^s \times C)$ cut β' but is contained in the $\varepsilon/4$ neighborhood of β and therefore $\varphi = 1$ there.

In the local coordinates we have chosen, we pick a thin rectangle R_1 with top and bottom given by $G^{-k_1}(a_1^s \times C)$ and $G^{-k_1}(a_1'^s \times C)$ and bounded in its sides by segments parallel to the w-axis which is transverse to D_S. Increasing k_1 and reducing c, a_1^s and $a_1'^s$, if it were necessary, we may assume that $G^{k_1}(R)$ is contained in the c-neighborhood of the graph of β' restricted to $[3\delta/8, 9\delta/8]$.

Set $g_1 = G^{k_1}$ and let $g_2 = G^T \big| (U \setminus G^{-1}(U)) : (U \setminus G^{-1}(U)) \to (U \setminus G(U))$ and consider

$$\Lambda_1 = \bigcap_{n \in \mathbb{Z}} (g_2 \circ g_1)^n (R_1).$$

Then Λ_1 contains a horseshoe H_1 (see [Nh1, DN]) and therefore $H_{\varepsilon_1} = \cup_{j=0}^{k_1+T-1} G(H_1)$ has positive topological entropy. Since this horseshoe is arbitrarily small we may assume that there is a periodic point $p_1 \in H_1$ such that $H_1 \subset \Gamma_\varepsilon(p_1)$ see Definition 1.1, where $0 < 2\varepsilon_1 \leq \varepsilon$. Moreover, the periodic point p_1 is homoclinically related to p since by the λ-lemma we have that positive iterates by $(g_2 \circ g_1)^{-1}$ give thin subrectangles crossing all of R_1 and hence the stable manifold of p_1 cuts $W^u_{loc}(x) \subset W^u(p, G)$ and analogously positive iterates by $g_2 \circ g_1$ gives subrectangles close to β' in the Hausdorff metric and therefore the unstable manifold of p_1 cuts $W^s_{loc}(x) \subset W^s(p, G)$.

Claim 2.5. *There is $\{\varepsilon_n\}_{n=1}^\infty$ such that for every ε_n it is associated a horseshoe H_{ε_n} with $H_{\varepsilon_n} \subset H(p, G)$ and $\lim_{n\to\infty} \operatorname{diam}(H_{\varepsilon_n}) = 0$.*

Proof. Let us choose $\varepsilon_2 > 0$ and construct H_{ε_2}. For this, pick $\varepsilon_2 \leq \varepsilon_1$ such that $G^{-k_1}(a_1^s \times C)$ and $G^{-k_1}(a_1'^s \times C)$ are at a distance greater than ε_2 from $(S, s, 0)$. Since $\varepsilon_n \leq \varepsilon_2$ for all $n \geq 2$ we have that no part of the graph of $l(s)$ for $s \in [-\delta, \delta/4]$ cuts R_1.

We found a new rectangle R_2 disjoint from R_1 contained in $U^s_{k_2} \setminus U^s_{k_2+1}$ with $k_2 > k_1$ applying again the λ-Lemma. Increasing k_2 and reducing the corresponding values of c_2, a_2^s and $a_2'^s$, if it were necessary, we may assume that $G^{k_2}(R_2)$ is contained in the c_2-neighborhood of the graph of β' restricted to $[-5\delta/16, \delta/16]$. By construction when we iterate by G the images of R_1 and R_2 cannot intersect since in $U \setminus G(U)$ there are only one iterate of R_1 and one iterate of R_2 (namely R_1 and R_2). We then have for G two disjoint small horseshoes, H_1, H_2 both with periodic points p_1, p_2 homoclinically related to p (the proof that p_2 is homoclinically related to p is the same than that to p_1). Hence both H_1 and H_2 are included in $H(p, G)$.

Next we choose $\varepsilon_3 \leq \varepsilon_2 \leq \varepsilon_1$ so that $G^{-k_2}(a_2^s \times C_2)$ and $G^{-k_2}(a_2'^s \times C_2)$ are at a distance greater than ε_3 from $(S, s, 0)$. For such ε_3, there is a horseshoe H_{ε_3} disjoint from H_{ε_1} and H_{ε_2} but still contained in $H(p, G)$. This construction follows the same steps as before: first find a thin rectangle R_3 cutting the graph of $l(s)$ only for $s \in [-21\delta/32, -15\delta/32]$, $R_3 \cap R_1 = \emptyset$, $R_3 \cap R_2 = \emptyset$. Then find an appropriate positive real number $k_3 > k_2$ such that $G^{k_3}(R_3)$ is contained in the c_3-neighborhood of the graph of β' restricted to $[-21\delta/32, -15\delta/32]$.

In this way we may pick the sequence ε_n such that for every n it is associated a horseshoe H_{ε_n} satisfying (1) $\lim_{n\to\infty} \operatorname{diam}(H_{\varepsilon_n}) \to 0$, (2) $H_{\varepsilon_j} \cap H_{\varepsilon_i} = \emptyset$ and (3) $H_{\varepsilon_n} \subset H(p, G)$ for all $n \in \mathbb{Z}^+$. This proves Claim 2.5. $\square$

Since the topological entropy of H_{ε_n} is positive for all n, and $H_{\varepsilon_n} \subset H(p,G)$, we conclude that $G/H(p,G)$ is not h-expansive, violating robustness of h-expansiveness. The proof of Lemma 2.4 is complete. □

Then, the final conclusion is that hypothesis (AD) described in the begining of this section can not hold. In another words, we conclude that there exists $m > 0$ such that for all homoclinic point $x \in H(p)$ there is $1 \leq k \leq m$ such that

$$\|Df^k/E(x)\| \, \|Df^{-k}/F(f^k(x))\| \leq \frac{1}{2}.$$

Following [SV, Theorem A], it can be built a dominated splitting for the homoclinic points of $H(p,f)$ as required, and then extend it by continuity to the whole $H(p,f)$ using that the closure of the homoclinic points coincide with $H(p,f)$.

Thus, the proof of Theorem A follows.

Remark 2.6. *Let us point out that even though we can assume that g, the diffeomorphism with a segment of homoclinic tangencies, is C^∞, the bump function $l(s)$, used to perturb it, is just C^2. Hence it seems that a similar construction can be used to prove the stronger result that $G/H(p)$ is not asymptotically h-expansive. Recall,* [Bu, BFF]*, that C^∞- diffeomorphisms are asymptotically h-expansive and so a C^∞ perturbation of a C^∞ diffeomorphism does not disprove asymptotically h-expansiveness.*

3. Proof of Theorems B and C

In this section we prove both Theorems B and C. For this, let us first remark that after [ABCDW, §2.1], C^1-generically the finest dominated splitting has a very special form. Thus, before we continue, let us first put f in that context.

Generic assumptions. There exists a residual subset $\mathcal{G}$ of $\mathrm{Diff}^1(M)$ such that if $f: M \to M$ is a diffeomorphisms belonging to $\mathcal{G}$ then

(1) f is Kupka-Smale, (i.e.: all periodic points are hyperbolic and their stable and unstable manifolds intersect transversally)
(2) for any pair of saddles p, q, either $H(p,f) = H(q,f)$ or $H(p,f) \cap H(q,f) = \emptyset$.
(3) for any saddle p of f, $H(p,f)$ depends continuously on $g \in \mathcal{G}$.
(4) The periodic points of f are dense in $\Omega(f)$.
(5) The chain recurrent classes of f form a partition of the chain recurrent set of f.
(6) every chain recurrent class containing a periodic point p is the homoclinic class associated to that point.

Taking into account [Go, Corollary, 6.6.2, Theorem 6.6.8], that guarantees that the homoclinic tangency can be associated to a saddle inside the homoclinic class, the next result is proved in [ABCDW, Corollary 3]:

Theorem 3.1. *(*[ABCDW, Corollary 3]*) There is a residual subset $\mathcal{I} \subset \mathcal{G}$ of* $\mathrm{Diff}^1(M)$ *such that if $f \in \mathcal{I}$ has a homoclinic class $H(p,f)$ which contains hyperbolic saddles of indices $i < j$ then either*

(1) *For any neighborhood U of $H(p,f)$ and any C^1-neighborhood $\mathcal{U}$ of f there is a diffeomorphism $g \in \mathcal{U}$ with a homoclinic tangency associated to a saddle of the homoclinic class $H(p_g,g)$, where p_g is the continuation of p. or*

(2) *There is a dominated splitting*

$$T_{H(p,f)}M = E \oplus F_1 \oplus \cdots \oplus F_{j-i} \oplus G$$

with $\dim(E) = i$ *and* $\dim(F_h) = 1$ *for all h and* $\dim(G) = \dim(M) - j$. *Moreover, the sub-bundles F_h are not hyperbolic.*

Proof of Theorem B. Let $H(p) \subset M$ be a homoclinic class robustly entropy expansive, i.e., there is a neighbourhood $\mathcal{U} \subset \mathrm{Diff}^1(M)$ such that $f \in \mathcal{U}$, there is a continuation $H(p_g)$ of $H(p)$ for all $g \in \mathcal{U}$ and $H(p_g)$ is h-expansive. By Theorem A we have a dominated splitting defined on $T_{H(p)}M$. Moreover, by [Go, Theorem 6.6.8], we have that in $H(p_g)$ there is a finest dominated splitting which has the form

$$T_{H(p_g,g)}M = E \oplus F_1 \oplus \cdots \oplus F_{j-i} \oplus G \tag{2}$$

with E, G and F_h Df-invariant sub-bundles, $h = 1, \ldots, j-i$, and all F_h one-dimensional, and

$$E \prec F_1 \prec F_2 \cdots \prec F_{j-i} \prec G\,.$$

Otherwise, by the theorem of [Go] cited above, we may create with an arbitrarily small C^1-perturbation a tangency *inside* the perturbed homoclinic class. After that we repeat the arguments of 2.1.1 contradicting h-expansiveness. Theorem B is proved.

Proof of Theorem C. By [CMP] there is residual subset $\mathcal{R}_0$ of $\mathrm{Diff}^1(M)$ such that, for every $f \in \mathcal{R}_0$, any pair of homoclinic classes of f are either disjoint or coincide. For $f \in \mathcal{R}_0$, by [Ab], the number of different homoclinic classes of f is locally constant in $\mathcal{R}_0$. We split the proof into two cases: (1) this number is finite (and in this case f is *tame*) or (2) there are infinitly many distinct homoclinic classes (and in this case f is *wild*.

f **is tame** In this case $H(p)$ is isolated. Before we continue, recall that if $V \subset M$ and $\Lambda_f(V)$ is the maximal invariant set of f in V, i.e.: $\Lambda_f(V) + \cap_{n\in\mathbb{Z}} f^n(V)$, then set $\Lambda_f(V)$ is robustly transitive if there is a C^1-neighbourhood $\mathcal{U}$ of f such that $\Lambda_g(\overline{V}) = \Lambda_g(V)$ and $\Lambda_g(V)$ is transitive for all $g \in \mathcal{U}$ (i.e.: $\Lambda_g(V)$ has a dense orbit).

Lemma 3.2. *Assume $f : M \to M$ is tame and that $T_{H(p)}M$ has a dominated splitting of the form (2). Then E is contracting and G is expanding.*

Proof. Since $H(p)$ is isolated it is a robustly transitive set maximal invariant in a neighbourhood $U \subset M$ and hence, according to [BDPR][Theorem D], the extremal sub-bundles E and G are contracting and expanding respectively. □

Under the same hypothesis of the previous lemma either we have that in a C^1-robust way the index of periodic points in $H(p_g)$, g near f, are the same and equal to index(p) or there are g arbitrarily C^1-close to f such that in $H(p_g)$ there are periodic points of different index. In the first case we have

Lemma 3.3. *There is a dense open subset $\mathcal{U}_1$ of $\mathcal{U}(f)$ in the C^1 topology such that for all $g \in \mathcal{U}_1$ we have that $H(p_g)$ is hyperbolic.*

Proof. We follow the lines of the proof at [BDi, Section 6]. Since $H(p)$ is isolated by [BC, Corollaire 1.13] or [Ab, Theorem A] it is robustly isolated. Let E and F be sub bundles such that $T_{H(p_g)}M = E \oplus F$ is m-dominated, for all $g \in \mathcal{U}(f)$, with $\dim(E) = \text{index}(p)$. We need to prove that $\|Df^n_{/E(x)}\| \to 0$ as $n \to +\infty$ and $\|Df^{-n}_{/F(x)}\| \to 0$ as $n \to +\infty$ for any $x \in H(p_g)$ in order to prove that $H(p_g)$ is hyperbolic. Let us show only that $\|Df^n_{/E(x)}\| \to 0$ as $n \to +\infty$, the other one being similar. For this, it is enough to show that for any $x \in H(p_g)$ there exists $k = k(x)$ such that $\prod_{i=0}^{k} \|Dg^m_{/E(g^{im}(x))}\| < \frac{1}{2}$.

Arguing by contradiction, assume this does not hold. Then, there exist $z \in H(p_g)$ such that $\prod_{i=0}^{k} \|Df^m_{/E(f^{im}(z))}\| \geq \frac{1}{2}$ $\forall k \geq 0$.

As in the proof of [Ma2, Theorem B] we may find $y \in H(p_g) \cap \Sigma(g)$, where $\Sigma(g)$ is a set of total probability measure, such that

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log \|D_{g^{mi}(y)} g^m \, / \, E(g^{mi}(y)\| \geq 0$$

Thus there is a perturbation h of g such that h has a non hyperbolic periodic point in $H(p_h)$. After a new perturbation we obtain periodic points P and Q contained in a small neighborhood U of $H(p_h)$ and with different indeces. Since $H(p)$ is C^1-robustly isolated $P, Q \in H(p_h)$ contradicting our assumption that in a C^1-robust way the index of periodic points in $H(p)$ are the same and equal to index(p). This finishes the proof of Theorem C in this case. □

In the second case, that is, there are g arbitrarily C^1-close to f such that in $H(p_g)$ there are periodic points of different indeces, by [GW], C^1-generically the diffeomorphism g, and hence f, can be C^1 approximated by diffeomorphisms exhibiting a heterodimensional cycle. Next we show that in this case the eigenvalues of periodic points are robustly in $\mathbb{R}$.

Lemma 3.4. *Let us assume that there is a periodic point $q \in H(p)$ with expanding complex eigenvalues such that index $(q) <$ index (p). Then there is an arbitrarily C^1-small perturbation of f creating a tangency inside the perturbed homoclinic class $H(p_g)$.*

Proof. C^1 generically we may assume that there is a robust heterodimensional cycle between p and q and that $W^s(p) \cap W^u(q)$ contains a compact arc l homeomorphic to $[0,1]$, (see [BDi]). Let us consider a disk D of the same dimension s of $W^s(p)$ and contained in $W^s(p)$ such that D is homeomorphic to $[0,1] \times [-1,1]^{s-1}$ by a homeomorphism h such that $h([0,1] \times \{0\}^{s-1} = l$. Iterating by $f^{-\pi(q)}$ this arc l spiralizes around q while D stretches approaching $W^s(q)$. Since $W^s(q) \cap W^u(p) \neq \emptyset$ there is a C^1 small perturbation of f creating a tangency between $W^s(p_g)$ and $W^u(p_g)$. □

Corollary 3.5. *If there is a periodic point $q \in H(p)$ with expanding complex eigenvalues such that index $(q) <$ index (p) then $H(p)$ is not C^1 robustly h-expansive.*

Proof. Under the hypothesis of the lemma we may create tangencies inside $H(p)$ and by another C^1- perturbation an arbitrarily small horseshoe in the intersection between $W^s_{loc}(p)$ and $W^u_{loc}(p)$ contradicting h-expansiveness. □

Thus Corollary 3.5 implies that the eigenvalues of periodic points in $H(p)$ are real numbers in a robust way. By [ABCDW] for C^1-generic diffeomorphisms the set of indices of the (hyperbolic) periodic points in a homoclinic class form an interval in $\mathbb{N}$. Thus by [BDi][Theorem 2.1] there are diffeomorphisms arbitrarily C^1-close to f with C^1-robust heterodimensional cycles.

As a consequence we obtain in both cases the following result:

Theorem 3.6. *If $f/H(p)$ is C^1 robustly h-expansive and $H(p)$ is an isolated homoclinic class then for a dense open subset $\mathcal{U}' \subset \mathcal{U}(f)$ either $f/H(p)$ is hyperbolic and we have $T_{H(p)}M = E^s \oplus E^u$ or there is a robust heterodimensional cycle in $H(p_g)$ for g arbitrarily close to f.*

Proof. If we have that in a C^1-robust way the index of periodic points in $H(p_g)$ are the same and equal to index(p_g) by Lemma 3.3 there is an open dense subset $\mathcal{V}$ of $\mathcal{U}(g)$ such that $H(p_g)$ is hyperbolic for $g \in \mathcal{V}$. Hence we are done. Otherwise we have an open subset $\mathcal{U}(g)$ in any neighborhood $\mathcal{V} \subset \mathcal{U}(f)$ of any diffeomorphism $g \in \mathcal{U}(f)$ exhibiting a heterodimensional cycle, [BDi]. This finishes the proof Theorem 3.6, which in its turn gives the proof of Theorem C. □

f **is wild** Now let us assume that $H(p)$ is not isolated. Either there is a small C^1-perturbation g of f such that $H(p_g)$ is isolated or $H(p)$ is persistently not isolated, i.e.: $H(p_g)$ is not isolated for any g close to f. In the first case we are done by Theorem 3.6.

In the second case the following result of [Cr] (see also [W]) is valid assuming that f is far from homoclinic cycles.

Remark 3.7. *Since $f/H(p)$ is h-expansive we are far from homoclinic tangencies.*

Theorem 3.8 (Crovisier)**.** *There exists a dense G_δ subset of* $\mathrm{Diff}^1(M)\setminus\overline{\mathrm{Tang}\cup\mathrm{Cycles}}$ *such that each homoclinic class H has a dominated splitting $T_HM = E^s \oplus E^c_1 \oplus E^c_2 \oplus E^u$ which is partially hyperbolic and such that each central bundle E^c_1, E^c_2 has dimension 0 or 1.*

Thus Theorem D is a consequence of Theorem 3.8 and the previous remark.

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