LORENZ LIKE FLOWS: EXPONENTIAL DECAY OF CORRELATIONS FOR THE POINCARÈ MAP, LOGARITHM LAW, QUANTITATIVE RECURRENCE

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ABSTRACT. In this paper we prove that the Poincarè map associated to a Lorenz like system has exponential decay of correlations with respect to Lipschitz observables. This implies that the hitting time associated to the system satisfies a logarithm law. The hitting time $\tau_r(x, x_0)$ is the time needed for the orbit of a point x to enter for the first time in a ball $B_r(x_0)$ centered at x_0, with small radius r. As the radius of the ball decreases to 0 its asymptotic behavior is a power law whose exponent is related to the local dimension of the SRB measure at x_0: for each x_0 such that the local dimension $d_\mu(x_0)$ exists,

$$\lim_{r\to 0} \frac{\log \tau_r(x, x_0)}{-\log r} = d_\mu(x_0) - 1$$

holds for μ almost each x. In a similar way it is possible to consider a quantitative recurrence indicator quantifying the speed of coming back of an orbit to its starting point. Similar results holds for this recurrence indicator.

CONTENTS

Date: December 8, 2008.

M.J.P. was partially supported by CNPq-Brazil/FAPERJ-Brazil/Pronex Dynamical Systems/CRM Ennio De Giorgi-Scuola Normale Superiore-Pisa.

1. Introduction

It is well known that a chaotic dynamics may share several statistical features with stochastic systems. These statistical features are often described by suitable versions of classical theorems from probability theory: law of large numbers, central limit theorem, large deviation estimations, correlation decay, hitting times, various kind of quantitative recurrence and so on.

In this article we consider a class of flows which contain the celebrated Geometric Lorenz flow and we will study some of its statistical features by a sharp estimation for the decay of correlations of its first return map on a suitable Poincarè section. This will give a quantitative recurrence estimation and an estimation for the scaling behavior for the time which is needed to hit small targets (logarithm law).

Let Φ^t be a C^1 flow in $\mathbb{R}^3$. Quantitative recurrence estimations and logarithm laws can be seen in the following framework: we are interested to a quantitative estimation of the speed of approaching of a certain orbit $\Phi^t(x)$ (starting from the point x) of the system to a given target point x_0. Let $B_r(x_0)$ be a ball with radius r centered at x_0. We consider the time

$$\tau_r(x, x_0) = \inf\{t \in \mathbb{R}^+ : \Phi^t(x) \in B_r(x_0)\}$$

needed for the orbit of x to enter in $B_r(x_0)$ for the first time and the asymptotic behavior of $\tau_r(x, x_0)$ as r decreases to 0. Often this is a power law of the type $\tau_r \sim r^{-d}$ and then it is interesting to extract the exponent d by looking at the behavior of

$$R(x, x_0) = \lim_{r\to 0} \frac{\log \tau_r(x, x_0)}{-\log r}. \tag{1}$$

In this way, we have a hitting time indicator for orbits of the system.[1]

If the orbit Φ^t starts at x_0 itself and we consider the second entrance time in the ball

$$\tau'_r(x_0) = \inf\{t \in \mathbb{R}^+ : \Phi^t(x_0) \in B_r(x_0), \exists i, s.t. \Phi^i(x_0) \notin B_r(x_0)\} \tag{2}$$

(because the orbit trivially starts inside the ball) with the same construction as before, we have a quantitative recurrence indicator. If the dynamics is chaotic enough, often the above indicators converge to a quantity which is related to the local dimension of the invariant measure of the system and in the hitting time case this relation is called *logarithm law*.

[1] Another way to look at the same phenomena is by considering the behavior of the ratio of the distance $\frac{-\log d(\Phi^t(x), x_0)}{\log t}$ as $t \to \infty$ (for the equivalence see [18]).

Hitting time results of this kind (sometime replacing balls with other suitable target sets) have been proved in many continuous time dynamical systems of geometrical interest: geodesic flows, unipotent flows, homogeneous spaces, etc. etc. (see e.g. [6, 20, 23, 38, 30, 32]). For discrete time systems this kind of results hold in general if the system has fast enough decay of correlation ([16]). Mixing is however not sufficient, since this relation does not hold in some slowly mixing system having particular arithmetical properties ([18]). Some further connections with arithmetical properties are shown in interesting examples as rotations and interval exchange maps (see e.g. [14, 21, 22]). This kind of problem is also connected with the so called dynamical Borel Cantelli results (see [17] and e.g. [41, 20, 17, 13]). Moreover, in the symbolic setting, similar results about the hitting time are used in information theory (see e.g. [37, 24]). About quantitative recurrence, our approach follows a set of results connecting a quantitative recurrence estimation with local dimension (see e.g. [36, 35, 7, 9]). We remark that the speed of correlation decay for Lorenz like flows is not yet known (although some are proved to be mixing, see [29]) hence quantitative recurrence and hitting time results cannot be proved directly using this tool, instead of this we will consider a Poincarè section, estimating its correlation decay and work with return times.

1.1. **Statement of results.** Let $I = [-\frac{1}{2}, \frac{1}{2}]$ be a unit interval, we consider a flow X^t on $\mathbb{R}^3$ having a Poincaré section on a square $\Sigma = I \times I$ satisfying the following properties:

- **1):** The flow induces[2] a first return map $F : \Sigma \to \Sigma$ of the form $F(x, y) = (T(x), G(x, y))$ (preserves the natural vertical foliation of the square) and:
- **1.a):** There is $c \in I$ and $k \geq 0$ such that, if x_1, x_2 are such that $c \notin [x_1, x_2]$ then $\forall y \in I : |G(x_1, y) - G(x_2, y)| \leq k \cdot |x_1 - x_2|$
- **1.b):** $F|_\gamma$ is λ-Lipschitz with $\lambda < 1$ (hence is uniformly contracting) on each leaf γ: $|G(x, y_1) - G(x, y_2)| \leq \lambda \cdot |y_1 - y_2|$
- **1.c):** $T : I \to I$ is onto and piecewise monotonic, with two C^1 increasing branches on the intervals $[-\frac{1}{2}, c)$,$(c, \frac{1}{2}]$ and $T' > 1$ where it is defined[3]. Moreover $\lim_{x\to c^-} T(x) = \frac{1}{2}, T(c) = -\frac{1}{2}, \lim_{x\to c} T'(x) = \infty$.
- **1.d):** $\frac{1}{|T'|}$ has bounded variation.

By the statistical properties of the map T, which is piecewise expanding, under the above assumptions, it turns out that F has a unique SRB measure μ_F. We then ask the following property for the flow:

- **2):** The flow X^t is transversal to the section Σ and its return time to Σ is integrable with respect to μ_F.

In Section 2.1 we will describe the geometric Lorenz system and we show that it satisfies these properties.

[2]Up to zero Lebesgue measure sets.

[3]The condition $T' > 1$ can be relaxed to $\lambda[inf_{x\in I}(T'(x))] < 1$ provided that the map T is eventually expanding in the sense of [44], Chapter 3.

The main results of the paper concern some statistical properties of X^t and F, more precisely:

Theorem A (decay of correlation for the Poincarè map) *The unique SRB measure μ_F of F has exponential decay of correlation with respect to Lipschitz observables.*

This result is proved in Section 4 (Theorem 4.7) where the reader can also find a precise definition of correlation decay. The proof also uses a regularity estimation for the invariant measure μ_F which can be found in the Appendix I (Lemma 8.1) and is proved by sort of Lasota-Yorke inequality. We remark that a stretched-exponential bound for the decay of correlation for a two dimensional Lorenz like map was given in [12] and [2].

We say that a point $x_0 \in \mathbb{R}^3$ is *regular* if there are $y_0 \in \Sigma$ and $t_0 \geq 0$ such that X^{t_0} induces a diffeomorphism between a neighborhood of y_0 and a neighborhood of x_0.

In Section 3 we recall how to construct an SRB ergodic invariant measure for the flow X^t which will be denoted by μ_X. It turns out that this measure has the following property

Theorem B (logarithm law for the flow) *For each regular x_0 such that the local dimension $d_{\mu_X}(x_0)$ is defined it holds*

$$\lim_{r\to 0}\frac{\log \tau_r(x,x_0)}{-\log r} = d_{\mu_X}(x_0) - 1 \tag{3}$$

for a.e. starting point x.

This is proved in Section 6 (Theorem 6.3) and uses the above decay of correlation estimation for the first return map F, a result from [16] giving the hitting time estimation for systems having faster than polynomial decay of correlations and finally the integrability of return time is used to get the result for the flow.

Using the main result of [35], by a similar construction, if the flow also satisfies

3): the map T has derivative bounded by a power law near c: there is a $\beta > 0$ s.t. $(x-c)^{\beta}T'(x)$ is limited in a neighborhood of c

in Section 7 (see Corollary 7.4) we prove the following

Theorem C (quantitative recurrence) *If the flow satisfies conditions 1),2), 3) above, then for a.e. x it holds*

$$\limsup_{r\to 0}\frac{\log \tau_r'(x)}{-\log r} = \overline{d}_{\mu_X}(x) - 1, \quad \liminf_{r\to 0}\frac{\log \tau_r'(x)}{-\log r} = \underline{d}_{\mu_X}(x) - 1. \tag{4}$$

In the Appendix II we give an auxiliary result, using a theorem by Steinberger [40] showing that the local dimension is defined a.e. for the Geometric Lorenz system.

2. Geometric Lorenz'model

In this section we will introduce and motivate the so-called Geometric Lorenz system. This is the main example where our results will be applied. Indeed we will see that assumption 1.a),...,1.d) and 2) of the introduction are verified for this model. The results in this section are however not strictly necessary for the proofs of our main theorems. The reader familiar to the construction of such models can skip it and start at Section 3.

In 1963 the meteorologist Edward Lorenz published in the Journal of Atmospheric Sciences ([27]) an example of a parametrized 2-degree polynomial system of differential equations

$$\begin{aligned} \dot{x} &= a(y-x) & a &= 10 \\ \dot{y} &= rx - y - xz & r &= 28 \\ \dot{z} &= xy - bz & b &= 8/3 \end{aligned} \tag{5}$$

as a very simplified model for thermal fluid convection, motivated by an attempt to understand the foundations of weather forecast. Later Lorenz [28] together with other experimental researches showed that the equations of motions of a certain laboratory water wheel are also given by (5).

Numerical simulations performed by Lorenz for an open neighborhood of the chosen parameters suggested that almost all points in phase space tend to a *chaotic attractor*.

An *attractor* is a bounded region in phase-space, invariant under time evolution, such that the forward trajectories of most (positive probability) or, even, all nearby points converge to it. And what makes an attractor *chaotic* is the fact that trajectories converging to the attractor are *sensitive with respect to initial data*: trajectories of two any nearby points get apart under time evolution.

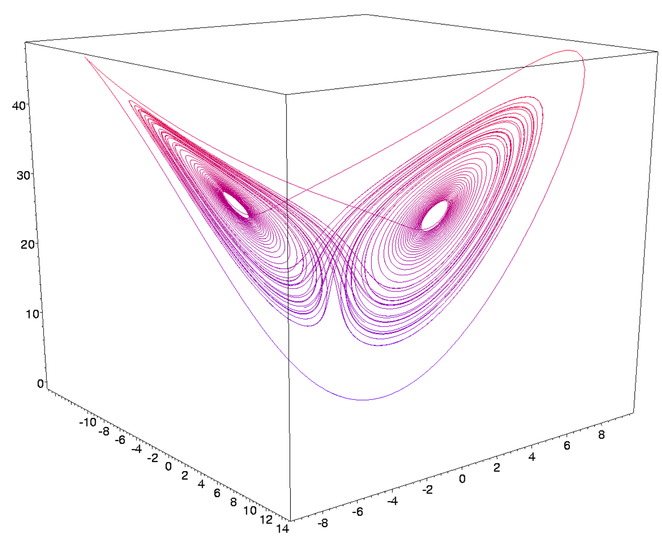

FIGURE 1. Lorenz chaotic attractor

Lorenz's equations proved to be very resistant to rigorous mathematical analysis, and also presented serious difficulties to rigorous numerical study. As an example, the existence of a chaotic attractor for the original Lorenz system where not proved until the year 2000, when Warwick Tucker did it with a computer aided proof (see [43, 42]).

In order to construct a class of flows having properties which are very similar to the Lorenz system and are easier to be studied, Afraimovich, Bykov and Shil'nikov [1], and

Guckenheimer, Williams [19], independently constructed the so-called *geometric Lorenz models* for the behavior observed by Lorenz. These models are flows in 3-dimensions for which one can rigorously prove the existence of a chaotic attractor that contains an equilibrium point of the flow, which is an accumulation point of typical regular solutions. Recall that γ is a regular solution for the flow X^t if $X^t(x) \neq x$ for all $x \in \gamma$. The accumulation of regular orbits near an equilibrium prevents such sets from being hyperbolic [33]. Furthermore, this attractor is robust: it can not be destroyed by any small perturbation of the original flow.

We point out that the robustness of this example provides an open set of flows which are not Morse-Smale, nor hyperbolic, and also non-structurally stable [33, 10]. Recall that a flow is structurally stable if there is a neighborhood of it in the C^1 topology such that the global structure of orbits of any two flows in this neighborhood are the same up to a homeomorphism preserving orientation of the orbits.

2.1. **Construction of the geometric model: near the equilibrium.** In this paper we will consider a class of three dimensional flows which will be defined axiomatically. To show that these axioms are verified in the geometric Lorenz models we give a detailed introduction to this model.

We first analyze the dynamics in a neighborhood of the singularity at the origin, and then we complete the flow, imitating the butterfly shape of the original Lorenz flow (see Figure 1 and compare with Figure 3).

In the original Lorenz system the origin $p = 0 = (0, 0, 0)$ is an equilibrium of saddle type for the vector field defined by equations (5) with real eigenvalues λ_i, $i \leq 3$ satisfying

$$0 < \frac{\lambda_1}{2} \leq -\lambda_3 < \lambda_1 < -\lambda_2 \tag{6}$$

(in the classical Lorenz system $\lambda_1 \approx 11.83$, $\lambda_2 \approx -22.83$, $\lambda_3 = -8/3$).

If certain nonresonance conditions are satisfied (see [39]) this vector field is smoothly linearizable in a neighborhood of the origin. To construct a model which is similar to the original Lorenz one we start with a linear system $(\dot{x}, \dot{y}, \dot{z}) = (\lambda_1 x, \lambda_2 y, \lambda_3 z)$, with λ_i, $1 \leq i \leq 3$ satisfying relation (6). This vector field will be considered in the cube $[-1, 1]^3$ containing the origin.

For this linear flow, the trajectories are given by

$$X^t(x_0, y_0, z_0) = (x_0 e^{\lambda_1 t}, y_0 e^{\lambda_2 t}, z_0 e^{\lambda_3 t}), \tag{7}$$

where $(x_0, y_0, z_0) \in \mathbb{R}^3$ is an arbitrary initial point near $p = (0, 0, 0)$.

Consider $\Sigma = \big\{(x, y, 1) : |x| \leq 1/2, \quad |y| \leq 1/2\big\}$ and

$$\Sigma^- = \big\{(x, y, 1) \in \Sigma : x < 0\big\}, \qquad \Sigma^+ = \big\{(x, y, 1) \in \Sigma : x > 0\big\} \quad \text{and}$$
$$\Sigma^* = \Sigma^- \cup \Sigma^+ = \Sigma \setminus \Gamma, \qquad \text{where} \quad \Gamma = \big\{(x, y, 1) \in \Sigma : x = 0\big\}.$$

Σ is a transverse section to the linear flow and every trajectory crosses Σ in the direction of the negative z axis.

Consider also $\tilde{\Sigma} = \{(x,y,z) : |x| = 1\} = \tilde{\Sigma}^- \cup \tilde{\Sigma}^+$ with $\tilde{\Sigma}^\pm = \{(x,y,z) : x = \pm 1\}$. For each $(x_0, y_0, 1) \in \Sigma^*$ the time t such that $X^t(x_0, y_0, 1) \in \tilde{\Sigma}$ is given by

$$t(x_0) = -\frac{1}{\lambda_1} \log |x_0| \tag{8}$$

which depends on $x_0 \in \Sigma^*$ only and is such that $t(x_0) \to +\infty$ when $x_0 \to 0$.

Hence, using (8), we get (where $\operatorname{sgn}(x) = x/|x|$ for $x \neq 0$)

$$X^{t(x_0)}(x_0, y_0, 1) = \big(\operatorname{sgn}(x_0), y_0 e^{\lambda_2 \cdot t(x_0)}, e^{\lambda_3 \cdot t(x_0)}\big) = \big(\operatorname{sgn}(x_0), y_0 |x_0|^{-\frac{\lambda_2}{\lambda_1}}, |x_0|^{-\frac{\lambda_3}{\lambda_1}}\big).$$

Since $0 < \frac{\lambda_1}{2} < -\lambda_3 < \lambda_1 < -\lambda_2$, we have $\frac{1}{2} < \alpha = -\frac{\lambda_3}{\lambda_1} < 1 < \beta = -\frac{\lambda_2}{\lambda_1}$.

Consider $L : \Sigma^* \to \tilde{\Sigma}^\pm$ defined by

$$L(x, y, 1) = \big(\operatorname{sgn}(x) y |x|^\beta, |x|^\alpha\big). \tag{9}$$

It is easy to see that $L(\Sigma^\pm)$ has the shape of a cusp triangle without the vertex $(\pm 1, 0, 0)$.

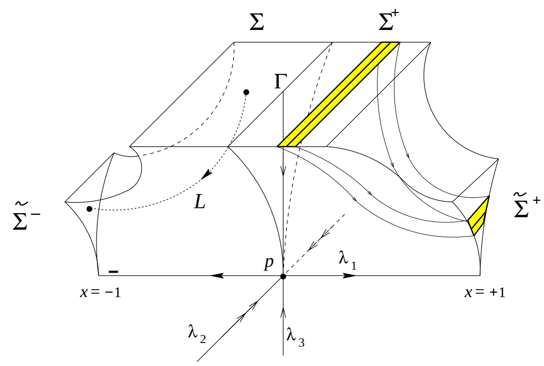

FIGURE 2. Behavior near the origin.

In fact the vertex $(\pm 1, 0, 0)$ are cusp points at the boundary of each of these sets. The fact that $0 < \alpha < 1 < \beta$ together with equation (9) imply that $L(\Sigma^\pm)$ are uniformly compressed in the y-direction.

Clearly each segment $\Sigma^* \cap \{x = x_0\}$ is taken by L to another segment $\tilde{\Sigma}^\pm \cap \{z = z_0\}$ as sketched in Figure 2.

2.2. **The random turns around the origin.** To imitate the random turns of a regular orbit around the origin and obtain a butterfly shape for our flow, as it is in the original Lorenz flow depicted at Figure 1, we proceed as follows.

Recall that the equilibrium p at the origin is hyperbolic and so its stable $W^s(p)$ and unstable $W^u(p)$ manifolds are well defined, [33]. Observe that $W^u(p)$ has dimension one and so, it has two branches, $W^{u,\pm}(p)$, and $W^u(p) = W^{u,+}(p) \cup \{p\} \cup W^{u,-}(p)$.

The sets $\tilde{\Sigma}^\pm$ should return to the cross section Σ through a flow described by a suitable composition of a rotation $R_\pm$, an expansion $E_{\pm\theta}$ and a translation $T_\pm$.

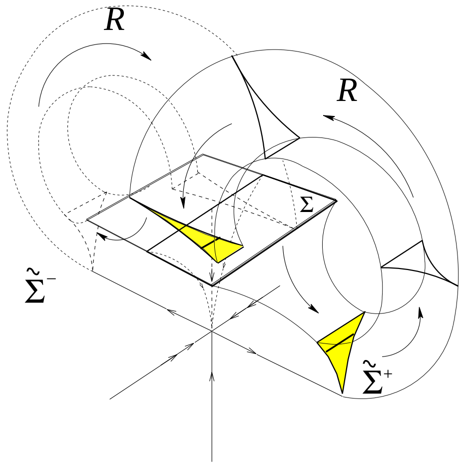

FIGURE 3. $T_\pm \circ R_\pm$ takes $\tilde{\Sigma}^\pm$ to Σ.

The rotation $R_\pm$ has axis parallel to the y-direction, which is orthogonal to the x-direction (which is parallel to the local branches $W^{u,\pm}(p)$). More precisely is such that $(x, y, z) \in \tilde{\Sigma}^\pm$, then

$$R_\pm(x, y, z) = \begin{pmatrix} 0 & 0 & \pm 1 \\ 0 & 1 & 0 \\ \pm 1 & 0 & 0 \end{pmatrix}. \tag{10}$$

The expansion occurs only along the x-direction, so, the matrix of E_θ is given by

$$E_{\pm\theta}(x, y, z) = \begin{pmatrix} \theta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{11}$$

with $\theta \cdot (\frac{1}{2}^{\alpha}) < 1$ and $\theta \cdot \alpha \cdot 2^{1-\alpha} > 1$. The first condition is to ensure that the image of the resulting map is contained in Σ, the second condition makes a certain one dimensional induced map to be piecewise expanding. This point will be discussed below.

$T_\pm : \mathbb{R}^3 \to \mathbb{R}^3$ is chosen such that the unstable direction starting from the origin is sent to the boundary of Σ and the image of both $\tilde{\Sigma}^\pm$ are disjoint. These transformations $R_\pm, E_{\pm\theta}, T_\pm$ take line segments $\tilde{\Sigma}^\pm \cap \{z = z_0\}$ into line segments $\Sigma \cap \{x = x_1\}$ as sketched in Figure 3, and so does the composition $T_\pm \circ E_{\pm\theta} \circ R_\pm$.

This composition of linear maps describes a vector field in a region outside $[-1, 1]^3$ in the sense that one can use the above matrices to define a vector field V such that the time one map of the associated flow realizes $T_\pm \circ E_{\pm\theta} \circ R_\pm$ as a map $\tilde{\Sigma}^\pm \to \Sigma$. This will not be explicit here, since the choice of the vector field is not really important for our purposes.

The above construction allow to describe for each $t \in \mathbb{R}$ the orbit $X^t(x)$ of each point $x \in \Sigma$: the orbit will start following the linear field until $\tilde{\Sigma}^\pm$ and then it will follow V

coming back to Σ and so on. Let us denote with $\mathcal{B} = \{X^t(x), x \in \Sigma, t \in \mathbb{R}^+\}$ the set where this flow acts. The geometric Lorenz flow is then the couple $(\mathcal{B}, X^t)$ defined in this way.

The Poincaré first return map will be hence defined by $F : \Sigma^* \to \Sigma$ as

$$F(x,y) = \begin{cases} T_+ \circ E_{+\theta} \circ R_+ \circ L(x,y,1) & \text{for } x > 0 \\ T_- \circ E_{-\theta} \circ R_- \circ L(x,y,1) & \text{for } x < 0 \end{cases} \tag{12}$$

The combined effects of $T_\pm \circ R_\pm$ and L on lines implies that the foliation $\mathcal{F}^s$ of Σ given by the lines $\Sigma \cap \{x = x_0\}$ is invariant under the return map. In another words, we have

($\star$) *for any given leaf* γ *of* $\mathcal{F}^s$*, its image* $F(\gamma)$ *is contained in a leaf of* $\mathcal{F}^s$.

2.3. **An expression for the first return map and its differential.** Combining equations (9) with the effect of the rotation composed with the expansion and the translation, we obtain that F must have the form

$$F(x,y) = \big(f_{Lo}(x), g_{Lo}(x,y)\big) \tag{13}$$

where $f_{Lo} : I \setminus \{0\} \to I$ and $g_{Lo} : (I \setminus \{0\}) \times I \to I$ are given by

$$f_{Lo}(x) = \begin{cases} f_1(x^\alpha) & x < 0 \\ f_0(x^\alpha) & x > 0 \end{cases} \quad \text{with } f_i = (-1)^i\theta \cdot x + b_i, i \in \{0,1\}, \text{ and} \tag{14}$$

$$g_{Lo}(x,y) = \begin{cases} g_1(x^\alpha, y \cdot x^\beta) & x < 0 \\ g_0(x^\alpha, y \cdot x^\beta) & x > 0, \end{cases} \tag{15}$$

where $g_1 | I^- \times I \to I$ and $g_0 | I^+ \times I \to I$ are suitable affine maps. Here $I^- = (-1/2, 0)$, $I^+ = (0, 1/2)$.

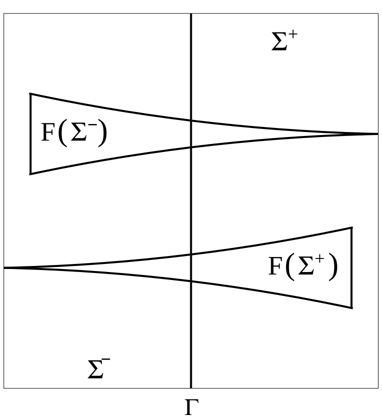

FIGURE 4. $F(\Sigma^*)$.

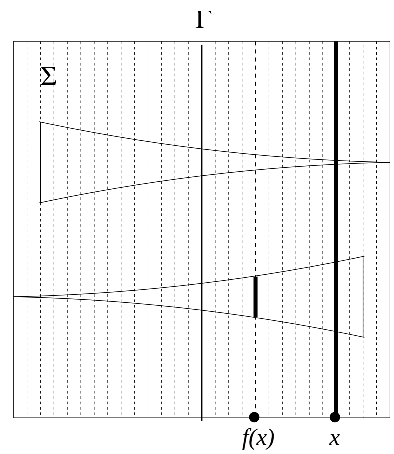

FIGURE 5. Projection on I.

Now, to found an expression for DF we proceed as follows. Recall $F = T_\pm \circ E_{\pm\theta} \circ R_\pm \circ L$, L is as in (9), $DR_\pm$ is as in (10). Given $q = (x,y) \in \Sigma^*$ with $x > 0$, we have

$$DL(x,y,1) = \begin{pmatrix} \beta \cdot y \cdot x^{\beta-1} & x^{\beta} \\ \alpha \cdot x^{\alpha-1} & 0 \end{pmatrix}.$$

Restricting the rotation and the other linear maps to $\tilde{\Sigma}^{\pm}$ and composing the resulting matrices we get

$$DF(x,y) = \begin{pmatrix} \theta \cdot \alpha \cdot x^{(\alpha-1)} & 0 \\ \beta \cdot y x^{(\beta-\alpha)} & x^{\beta} \end{pmatrix}. \tag{16}$$

The expression for DF at $q=(x,y)$ with $x<0$ is similar.

2.4. **Properties of the map g_{Lo}.** Observe that by construction g_{Lo} in equation (12) is piecewise C^2. Moreover, equation (16) implies the following bounds on its partial derivatives :

(a) For all $(x,y) \in \Sigma^*, x>0$, we have $\partial_y g_{Lo}(x,y) = x^{\beta}$. As $\beta > 1$, $|x| \leq 1/2$, there is $0<\lambda<1$ such that

$$|\partial_y g_{Lo}| < \lambda. \tag{17}$$

The same bound works for $x<0$.

(b) For all $(x,y) \in \Sigma^*, x \neq 0$, we have $\partial_x g_{Lo}(x,y) = \beta \cdot x^{\beta-\alpha}$. As $\beta - \alpha > 0$ and $|x| \leq 1/2$, we get

$$|\partial_x g_{Lo}| < \infty. \tag{18}$$

Item (a) above implies that the map $F=(f_{Lo}, g_{Lo})$ is uniformly contracting on the leaves of the foliation $\mathcal{F}^s$: there is $C>0$ such that

($\star\star$) if γ is a leaf of $\mathcal{F}^s$ and $x,y \in \gamma$ then

$$\operatorname{dist}\big(F^n(x), F^n(y)\big) \leq \lambda^n \cdot C \cdot \operatorname{dist}(x,y)$$

where λ can be chosen as the one given by equation (17).

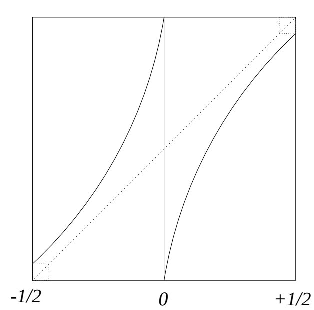

FIGURE 6. The Lorenz map f_{Lo}.

2.5. **Properties of the one-dimensional map f_{Lo}.** Now let us outline the main properties of f_{Lo}. We recall that we chosen θ such that $\theta \cdot \alpha \cdot 2^{1-\alpha} > 1$.

The following properties are easily implied from the construction of X^t:

(f1) By equation (14) and the way $T_\pm$ is defined, f_{Lo} is discontinuous at $x = 0$. The lateral limits $f_{Lo}(0^\pm)$ do exist, $f_{Lo}(0^\pm) = \pm\frac{1}{2}$,

(f2) f_{Lo} is C^2 on $I \setminus \{0\}$. By the choice of θ $f'_{Lo}(1/2) > 1$. By the convexity properties of f_{Lo} we then obtain that

$$f'_{Lo}(x) > 1 \qquad \text{for all} \qquad x \in I \setminus \{0\}. \tag{19}$$

(f3) The limits of f'_{Lo} at $x = 0$ are $\lim_{x\to 0} f'_{Lo}(x) = +\infty$.

We obtain that f_{Lo} is a piecewise expanding map. Moreover f_{Lo} has a dense orbit, which in its turn implies that the closure of the maximal invariant set by f_{Lo} is the whole interval I, see [5, Lemma 2.11].

Now recall that the *variation* $\operatorname{var}\phi$ of a function $\phi : [0,1] \to \mathbb{R}$ is defined by

$$\operatorname{var}\phi = \sup \sum_{i=1}^{n} |\phi(x_{i-1}) - \phi(x_i)|$$

where the supremum is taken over all finite partitions $0 = x_0 < x_1 < \cdots < x_n = 1$, $n \geq 1$, of $[0,1]$. The variation $\operatorname{var}_J \phi = \operatorname{var}(\phi|J)$ of ϕ over an arbitrary interval $J \subset [0,1]$ is defined by a similar expression, with the supremum taken over all the $x_0, x_1, \cdots, x_n \in J$, with $\inf J \leq x_0 < x_1 < \cdots < x_n \leq \sup J$. One says that ϕ has *bounded variation*, or ϕ is BV for short, if $\operatorname{var}\phi < \infty$.

The one dimensional map has the following property, which is important to obtain the existence of an SRB invariant measure and its statistical properties.

Lemma 2.1. *Let X^t a C^2 geometric Lorenz flow as before and f_{Lo} be the one-dimensional map associated to X^t. Then $\frac{1}{f'_{Lo}}$ is BV.*

Proof. Each branch of f_{Lo} is the composition of an affine map with x^α then it is a convex function. Hence, the derivative f'_{Lo} is monotonic on each branch, implying that $(f'_{lo})^{-1}$ is also monotonic. On the other hand, $(f'_{Lo})^{-1}$ is bounded because $f'_{lo} > 1$. Thus $(f'_{Lo})^{-1}$ is monotonic and bounded and hence is BV. □

We have seen that f_{Lo} is a topologically transitive piecewise expanding map with $\frac{1}{f'_{Lo}}$ BV. The statistical properties of such maps are well known. Next we state a result about it, which will be used later:

Proposition 2.2. *([44], Prop.3.8) The one-dimensional f_{Lo} admits a unique invariant probability $\mu_{f_{Lo}}$ which is absolutely continuous with respect to Lebesgue measure m, it is ergodic and so a SRB measure for the map. Moreover $d\mu_{f_{Lo}}/dm$ is a BV function and in particular it is bounded. Furthermore f_{Lo} has exponential decay of correlations for L^1 and BV observables and any a.c.i.m. converges exponentially fast to the invariant measure: there are constants $C > 0$ and $\lambda > 0$, depending on the system such that for each n and* *observables* f, g:

$$\left|\int g(F^n(x))f(x)dm - \int g(x)d\mu \int f(x)dm\right| \leq C \cdot \|g\|_{L_1} \cdot \|f\|_{BV} \cdot e^{-\lambda n}.$$

Summarizing, for what it was said above, the study of the 3-flow can be reduced to the study of a bi-dimensional map F and, moreover, the dynamics of this map can be further reduced to a one-dimensional map, f_{Lo}, since the invariant contracting foliation enables us to identify two points on the same leaf, see Figure 5. The quotient map f_{Lo} obtained through this identification will be called one dimensional *Lorenz map*. Figure 6 shows the graph of this one-dimensional transformation, and Figure 4 sketches $F(\Sigma^*)$.

3. A SRB measure for a Lorenz like flow

In this section, following [44] we construct a SRB measure for a flow X^t which satisfies the assumptions 1a),...,1d),2) in the introduction. As noticed in the previous section, these assumptions are satisfied by the Geometric Lorenz system.

Properties 1a),...,1d) implies that the flow Poincaré map has an invariant foliation and the one dimensional induced map T is piecewise expanding. Piecewise expanding maps (see Proposition 2.2) admits a unique invariant probability measure μ_T which is absolutely continuous with respect to Lebesgue measure m.

From μ_T we may construct a SRB measure μ_F, for the first return map F through the following general procedure ([11, 44]). Since μ_T is defined on the interval I which can be identified to the space of leaves of the contracting foliation $\mathcal{F}^s$, we may also think of it as a measure on the σ-algebra of Borel subsets of Σ which are union of entire leaves of $\mathcal{F}^s$. Using the fact that F is uniformly contracting on leaves of $\mathcal{F}^s$ we conclude that the sequence

$$F^{*n}(\mu_T), \quad n \geq 1,$$

of push-forwards of μ_T under F is weak*-Cauchy: given any continuous $\psi : \Sigma \to \mathbb{R}$

$$\int \psi d(F^{n*}\mu) = \int (\psi \circ F^n) d\mu, \quad n \geq 1,$$

is a Cauchy sequence in $\mathbb{R}$, see [44, pp.173]. Define μ_F to be the weak*-limit of this sequence, that is,

$$\int \psi d\mu_F = \lim \int \psi d(F^{*n}\mu)$$

for each continuous ψ. Then μ_F is invariant under F, and it is an ergodic SRB measure for F. The last statement follows from the fact that μ_T is an ergodic SRB measure for T, together with the fact that asymptotic time-averages of continuous functions $\psi : \Sigma \to \mathbb{R}$ are constant on the leaves of $\mathcal{F}^s$.

Given any point x whose orbit sooner or later will cross Σ we denote with $t(x)$ the first strictly positive time such that $X^{t(x)}(x) \in \Sigma$ (the *return time* of x to Σ). Coherently with the Geometric Lorenz system, we will denote by Σ^* the (full measure, by the assumption 1 in the introduction) subset of Σ where t is defined.

Now we show how to construct an SRB invariant measure for the flow, when the return time is integrable:

$$\int_{\Sigma^*} t d\mu_F < \infty. \tag{20}$$

Denote by $\sim$ the equivalence relation on $\Sigma \times \mathbb{R}$ given by $(w, t(w)) \sim (F(w), 0)$.

Let $N = (\Sigma^* \times \mathbb{R})/\sim$ and $\nu = \pi_*(\mu_F \times dt)$, where $\pi : \Sigma^* \times \mathbb{R} \to N$ is the quotient map and dt is a Lebesgue measure in $\mathbb{R}$. Equation (21) gives that ν is a finite measure. Let $\phi : N \to \mathbb{R}^3$ be defined by $\phi(w, t) = X^t(w)$. Let $\mu_X = \phi_*\nu$. The measure μ_X is a SRB for the flow X^t:

$$\frac{1}{T}\int_0^T \psi(X^t(w))dt \to \int \psi d\mu_X \quad \text{as} \quad T \to \infty$$

for every continuous function $\psi : \mathbb{R}^3 \to \mathbb{R}$, and Lebesgue almost every point $w \in \phi(N)$.

We end the subsection remarking that the Geometric Lorenz flow has integrable return time, hence the above construction for the invariant measure can be applied to it. As before denote by $t : \Sigma \setminus \Gamma \to (0, \infty)$ the return time to Σ. Then, recalling Equation (8) there are $K, C > 0$ such that

$$-K^{-1}\log(d(w, \Gamma)) - C \le t(x) \le -K\log(d(w, \Gamma)) + C.$$

Combining this with the definition of μ_F and the remark made above that $d\mu_{f_{Lo}}/dm$ is a bounded function, we conclude that

Proposition 3.1. *The return time is integrable*

$$t_0 = \int t d\mu_F < \infty. \tag{21}$$

3.1. **Local dimension.** Let us recall the definition of local dimension and fix some notations for what follows.

Let (M, d) be a metric space and assume that μ is a Borel probability measure on M. Given $x \in M$, let $B_r(x) = \{y \in M; d(x, y) \le r\}$ be the ball centered at x with radius r. The *local dimension* of μ at $x \in M$ is defined by

$$d_\mu(x) = \lim_{r\to\infty} \frac{\log \mu(B_r(x))}{\log r}$$

if this limit exists. In this case $\mu(B_r(x)) \sim r^{d_\mu(x)}$.

This notion characterizes the local geometric structure of an invariant measure with respect to the metric in the phase space of the system, see [45] and [34].

We can always define the *upper* and the *lower* local dimension at x as

$$d_\mu^+(x) = \lim\sup_{r\to\infty} \frac{\log \mu(B_r(x))}{\log r}, \qquad d_\mu^-(x) = \lim\inf_{r\to\infty} \frac{\log \mu(B_r(x))}{\log r}.$$

If $d^+(x) = d^-(x) = d$ almost everywhere the system is called *exact dimensional.* In this case many properties of dimension of a measure coincide. In particular, d is equal to the dimension of the measure μ: $d = \inf\{\dim_H Z; \mu(Z) = 1\}$. This happens in a large class of systems, for example, in C^2 diffeomorphisms having non zero Lyapunov exponents almost everywhere, [34].

3.2. **Relation between local dimension for F and for X^t.** Let us establish a relation between d_{μ_F} and d_{μ_X} which will be used in the following.

Proposition 3.2. *Let $x \in \mathbb{R}^3$ and $\pi(x)$ be the projection on Σ given by $\pi(x) = y$ if x is on the orbit of $y \in \Sigma$ and the orbit from y to x does not cross Σ. For all regular point $x \in \mathbb{R}^3$*

$$d_{\mu_X}{}^+(x) = d_{\mu_F}{}^+(\pi(x)) + 1, \quad d_{\mu_X}{}^-(x) = d_{\mu_F}{}^-(\pi(x)) + 1. \tag{22}$$

Proof. First observe that for product measures as $\mu_X = \mu_F \times dt$, where dt is the Lebesgue measure at the line, the formula is trivially verified. But, by construction $\mu_X = \phi_*(d\mu_F \times dt)$, where $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ is a local bi-Lipschitz map at each regular point. Since the local dimension is invariant by local bi-Lipschitz maps, it follows the required equation (22). □

4. Decay of correlations for two dimensional Lorenz maps

In this section we estimate the decay of correlations for a class of Lorenz like maps containing the first return map of the geometric Lorenz system described above. Inspired by a remark of R. S. Mc Kay (see [31], p. 8), this will be done by estimating the speed of approaching of iterates of suitable measures (corresponding to Lipschitz observables) to the invariant measure. For this purpose we will consider the space of measures on Σ as a metric space, endowed with the Wasserstein-Kantorovich distance, whose basic properties we are going to describe.

Notations. Let us introduce some notations: we will consider the sup distance on $\Sigma = [-\frac{1}{2}, \frac{1}{2}]^2$, so that the diameter, $diam(\Sigma) = 1$. This choice is not essential, but will avoid the presence of many multiplicative constants in the following making notations cleaner.

As before, the square Σ will be foliate by stable, vertical leaves. We will denote the leaf with x coordinate by γ_x or, with a small abuse of notation, when no confusion is possible we will denote both the leaf and its coordinate with γ.

Let $f\mu$ be the measure μ_1 such that $d\mu_1 = fd\mu$. Moreover, let us sometime for short denote the integral by $\mu(f) = \int fd\mu$. Let μ a measure on Σ. In the following, such measures on Σ will be often disintegrated in the following way: for each Borel set A

$$\mu(A) = \int_{\gamma \in I} \mu_\gamma(A \cap \gamma) d\mu_x \tag{23}$$

with μ_γ being probability measures on the leaves γ and μ_x is the marginal on the x axis which will be an absolutely continuous probability measure. We will also denote by ϕ_x its density.

Let us consider the projection π_y on the y coordinate. Let us denote the "restriction" of μ on the leaf γ by

$$\mu|_\gamma = \pi_y^*(\phi_x(\gamma)\mu_\gamma).$$

This is a measure on I and it is not normalized. We remark that $\mu|_\gamma(I) = \phi_x(\gamma)$. If Y is a metric space, we denote by $PM(Y)$ the set of Borel probability measures on Y. Let us finally denote by $L(g)$ be the best Lipschitz constant of $g : L(g) = \sup_{x,y} \frac{|g(x)-g(y)|}{|x-y|}$ and set $\|g\|_{lip} = \|g\|_\infty + L(g)$.

4.1. **The Wasserstein-Kantorovich distance.** Let us consider a bounded metric space Y. Let us consider the following notion of distance between measures: given two probability measures μ_1 and μ_2 on Y

$$W_1(\mu_1, \mu_2) = \sup_{g \in 1lip(Y)} (|\int_Y g \ d\mu_1 - \int_Y g \ d\mu_2|)$$

where $1lip(Y)$ is the space of 1-Lipschitz functions on Y. We remark that adding a constant to the test function g does not change the above difference $\int g \ d\mu_1 - \int g \ d\mu_2$. The above defined W_1 has moreover the following basic properties.

Proposition 4.1. [3, Prop 7.1.5] *The following properties hold*

(1) *W_1 is a distance and if Y is separable and complete, then $PM(Y)$ with this distance is a separable and complete metric space.*
(2) *If Y is bounded, a sequence is convergent for the W_1 metrics if and only if it is convergent for the weak topology.*

Remark 4.2. (distance and convex combinations) If $a + b = 1, a \geq 0, b \geq 0$ then

$$W_1(a\mu_1 + b\mu_2, a\mu_3 + b\mu_4) \leq a \cdot W_1(\mu_1, \mu_3) + b \cdot W_1(\mu_2, \mu_4). \tag{24}$$

Indeed

$$W_1(a\mu_1 + b\mu_2, a\mu_3 + b\mu_4) = \sup_{g \in 1lip(Y)} (|\int g \ d(a \cdot \mu_1 + b \cdot \mu_2) - \int g \ d(a \cdot \mu_3 + b \cdot \mu_4)|) =$$

$$= \sup_{g \in 1lip(Y)} (|a \cdot \int g \ d\mu_1 + b \cdot \int g \ d\mu_2 - a \cdot \int g \ d\mu_3 - b \cdot \int g \ d\mu_4|)$$

$$\leq \sup_{g \in 1lip(Y)} (|a \int g \ d\mu_1 - a \cdot \int g \ d\mu_3| + |b \cdot \int g \ d\mu_2 - b \cdot \int g \ d\mu_4|) =$$

$$\sup_{g \in 1lip(Y)} (a \cdot |\int g \ d\mu_1 - \int g \ d\mu_3| + b \cdot |\int g \ d\mu_2 - \int g \ d\mu_4|) \leq a \cdot W_1(\mu_1, \mu_3) + b \cdot W_1(\mu_2, \mu_4).$$

We also remark that the same kind of estimation can be done if the convex combination has more than 2 summand.

Remark 4.3. If g is ℓ-Lipschitz and μ_1, μ_2 are probability measures then

$$|\int_Y g \ d\mu_1 - \int_Y g \ d\mu_2| \leq \ell \cdot W_1(\mu_1, \mu_2).$$

4.2. **Wassertein distance and decay of correlations over Lipschitz observables.** We give some general facts on the relation between W_1 distance and decay of correlations.

Let (Y, F, μ) be a dynamical system on a metric space with invariant probability measure μ. The transfer operator associated to F will be indicated with F^*.

Proposition 4.4 (decay in function of distance)**.** *Let μ_1 be a probability measure which is absolutely continuous with respect to μ, and $d\mu_1 = f(x)d\mu$ (hence $\int f(x)d\mu = 1$). Let $g \in lip(Y)$ then*

$$|\int g(F^n(x))\ f(x)d\mu - \int g(x)d\mu| \leq L(g) \cdot W_1((F^*)^n(\mu_1), \mu).$$

Proof. Dividing by $L(g)$ we can suppose $g \in 1lip(Y)$. As $\int g(F(x))f(x)d\mu = \int g(x)d(F^*(\mu_1))$ then the decay of correlations between f and g can be estimated in function of the distance between $(F^*)^n(\mu_1)$ and μ as:

$$|\int g(F^n(x))\ f(x)d\mu - \int g(x)d\mu| = |\int g(x)d(F^{*n}(\mu_1)) - \int g(x)d\mu|$$

$$\leq \sup_{g\in 1lip(Y)} (|\int g\ d(F^{*n}(\mu_1)) - \int g\ d\mu|) = W_1((F^*)^n(\mu_1), \mu).$$

□

Conversely,

Proposition 4.5 (distance in function of decay)**.** *If for each $f \in L^1(\mu)$, $f \geq 0$ and $g \in lip(Y)$ it holds*

$$|\int g(F^n(x))\ f(x)d\mu - \int f(x)d\mu \int g(x)d\mu| \leq C \cdot L(g) \cdot \|f\|_{L^1} \cdot \Phi(n)$$

then taking $d\mu_1 = \frac{f(x)}{||f||_{L^1}} d\mu$ it holds

$$W_1((F^*)^n(\mu_1), \mu) \leq C \cdot \Phi(n).$$

Proof. Consider $g \in 1lip$. Hence

$$\begin{aligned} \frac{C \cdot L(g)\|f\|_{L^1} \cdot \Phi(n)}{||f||_{L^1}} &\geq \frac{|\int g(F^n(x))\ f(x)d\mu - \int f(x)d\mu \int g(x)d\mu|}{||f||_{L^1}} = \\ &= |\int g(x)d(F^{*n}(\mu_1)) - \int g(x)d\mu| \end{aligned}$$

since this hold for each g hence $W_1(F^{*n}(\mu_1), \mu) \leq C \cdot \Phi(n)$. □

4.3. **Disintegration and Wasserstein distance.** We will consider maps having an invariant foliation, as we have seen in the Lorenz map. The invariant measure will then be disintegrated as in Equation (23) into a family of measures μ_γ on almost each stable leaf γ and an absolutely continuous measure μ_x on the unstable direction.

If μ^1 and μ^2 are two disintegrated measures as above, their W_1 distance can be estimated in function of some distance between their respective marginals on the x axis and measures on the leaves:

Proposition 4.6. *Let μ^1, μ^2 be measures on Σ as above, such that for each Borel set A*

- $\mu^1(A) = \int_{\gamma\in I} \mu^1_\gamma(A\cap\gamma)d\mu^1_x$
- $\mu^2(A) = \int_{\gamma\in I} \mu^2_\gamma(A\cap\gamma)d\mu^2_x$

with μ^i_x absolutely continuous with respect to the Lebesgue measure, moreover let us suppose

(1) *for almost each vertical leaf γ, $W_1(\mu^1_\gamma, \mu^2_\gamma) \le \varepsilon$ and*
(2) $\sup_{\|h\|_\infty\le 1} |\int h d\mu^1_x - \int h d\mu^2_x| \le \delta$

then $W_1(\mu^1, \mu^2) \le \varepsilon + \delta$.

Proof. Considering the W_1 distance and disintegrating μ^1 and μ^2:

$$W_1(\mu^1,\mu^2) \le \sup_{g\in 1lip} |\mu^1(g) - \mu^2(g)| = \tag{25}$$

$$= \sup_{g\in 1lip} |\int_{\gamma\in I}\int_\gamma g(*)d\mu^1_\gamma d\mu^1_x - \int_{\gamma\in I}\int_\gamma g(*)d\mu^2_\gamma d\mu^2_x|.$$

Adding and subtracting $\int\int_\gamma g(*)d\mu^2_\gamma d\mu^1_x$ the last expression is equivalent to

$$\sup_{g\in 1lip} |\int_I\int_\gamma g(*)d\mu^1_\gamma d\mu^1_x - \int_I\int_\gamma g(*)d\mu^2_\gamma d\mu^1_x +$$
$$+\int_I\int_\gamma g(*)d\mu^2_\gamma d\mu^1_x - \int_I\int_\gamma g(*)d\mu^2_\gamma d\mu^2_x|.$$

This becomes

$$\sup_{g\in 1lip} |\int_I(\int_\gamma g(*)d\mu^1_\gamma - g(*)d\mu^2_\gamma)d\mu^1_x + \int_I\int_\gamma g(*)d\mu^2_\gamma d\mu^1_x - \int_I\int_\gamma g(*)d\mu^2_\gamma d\mu^2_x| \le$$

$$\le \sup_{g\in 1lip} |\int_I \varepsilon d\mu^1_x + \int_I\int_\gamma g(*)d\mu^2_\gamma d\mu^1_x - \int_I\int_\gamma g(*)d\mu^2_\gamma d\mu^2_x| \le$$

$$\le \varepsilon + |\int_I\int_\gamma g(*)d\mu^2_\gamma d\mu^1_x - \int_I\int_\gamma g(*)d\mu^2_\gamma d\mu^2_x| \tag{26}$$

Since $g \in 1lip$ and $diam(\Sigma) = 1$, then by adding a constant to g (which does not change $\int g d\mu^1_\gamma - \int g d\mu^2_\gamma$) we can suppose without loss of generality that $g \le 1$ and then for almost each γ it holds $h(\gamma) = |\int_\gamma g(*)d\mu^2_\gamma| \le 1$. Hence, by assumption (2) the statement is proved. □

4.4. **Exponential decay of correlations.** Now we are ready to prove that a Lorenz like two dimensional map F has exponential decay of correlations with respect to its SRB measure μ. We recall (see Proposition 2.2) that for a piecewise expanding map of the interval T, there are constants $C>0$ and $\lambda>0$, depending on the system such that, if g and f are respectively L_1 and BV (bounded variation) observables on I for each n it holds:

$$|\int g(T^n(x))f(x)dm - \int g(x)d\mu \int f(x)dm| \leq C\cdot \|g\|_{L_1}\cdot \|f\|_{BV}\cdot e^{-\lambda n} \tag{27}$$

(recall that m is the Lebesgue measure above). Next, we consider systems behaving as the first return map of the Geometric Lorenz system and prove

Theorem 4.7. *Let $F:\Sigma\to\Sigma$ a Borel function such that $F(x,y)=(T(x),G(x,y))$. Let μ be an invariant measure for F with marginal μ_x on the x-axis (which is invariant for $T:I\to I$). Let us suppose that*

1. *(T,μ_x) satisfies the above equation 27 and $T^{-1}(x)$ is finite for each $x\in I$.*
2. *F is a contraction on each vertical leaf: G is λ-Lipschitz in y for each x with $\lambda<1$.*
3. *μ is regular enough that for each ℓ-Lipschitz function $f:\Sigma\to\mathbb{R}$ the projection $\pi_x^*(f\mu)$ has bounded variation density $\overline{f}$ [4], with*

$$||\overline{f}||_{BV}\leq K\ell \tag{28}$$

where K is not depending on f.

Then (F,μ) has exponential decay of correlation (with respect to Lipschitz observables).

We already saw that the first two points in the above proposition are satisfied by the first return map of the Geometric Lorenz system. In the Appendix I we will prove that also the above point 3 is satisfied by a systems containing the Geometric Lorenz one.

We point out that this is the hard part of the proof that Lorenz like maps have exponential decay of correlations and this will be done by a sort of Lasota-Yorke inequality. Putting together all the necessary assumptions this prove Theorem A in the introduction.

Before the proof of Theorem 4.7 we also make the following remark which is a simple but important fact implied by the uniform contraction on stable leaves

Remark 4.8. Under the above assumptions, let us consider a leaf γ and two probability measures μ, ν on it. Then

$$W_1(F^*(\mu),F^*(\nu))\leq \lambda W_1(\mu,\nu).$$

Proof. This is because the map is uniformly contracting on each leaf. If g is 1-Lipschitz on $F(\gamma)$ then $g(F(*))$ is λ-Lipschitz on γ. This implies that

$$|\int_{F(\gamma)} g\ d(F^*\mu) - \int_{F(\gamma)} g\ d(F^*\nu)| = |\int_\gamma g\circ F\ d\mu - \int_\gamma g\circ F\ d\nu| \leq \lambda W_1(\mu,\nu)$$

finishing the proof. □

[4] which can also be expressed as $\overline{f}(x)=\int f(x,y)\ d\mu|_{\gamma_x}$.

Proof. (of Theorem 4.7) Let us consider $\nu = f\mu$ with $f \geq 0$ being $\ell-$Lipschitz.and $\int f d\mu = 1$ The strategy is to use Proposition 4.6 and find exponentially decreasing bounds for ε and δ so that we can estimate the Wasserstein distance between μ and iterates of $f\mu$ and then apply Proposition 4.4 to deduce decay of correlations. Let us consider the leaf γ_x with coordinate x. The density $\overline{f}$, by item 3 has bounded variation and $||\overline{f}||_{BV} \leq K\ell$. Let $\nu_x = \overline{f}m$ (as before m is the Lebesgue measure). Let us consider the quotient map T. Let $g \in L^1([-\frac{1}{2}, \frac{1}{2}])$. Since $|\int g\ d(T^{*n}(\nu_x)) - \int g\ d\mu_x| = |\int g(T^n(x))\overline{f}(x)dm - \int g(x)d\mu_x|$, by equation (27)

$$|\int gd(T^{*n}(\nu_x)) - \int gd\mu_x| \leq \|g\|_{L_1} \cdot \|\overline{f}\|_{BV} \cdot C \cdot e^{-\lambda n},$$

implying that $\sup_{\|g\|_\infty \leq 1} |\int gdT^{*n}(\nu_x) - \int gd\mu_x| \leq \|\overline{f}\|_{BV} \cdot C \cdot e^{-\lambda n} \leq K\ell C \cdot e^{-\lambda n}$ and hence we see that item (2) at Proposition 4.6 is satisfied with an exponential bound depending on the Lipschitz constant ℓ of f.

Let us consider $\nu^n = F^{*n}\nu$ again. Since, as said before the map F sends vertical leaves into vertical ones then there is a family of probability measures ν_γ^n on vertical leaves such that

$$(F^{*n}\nu)(g) = \int_{\gamma \in I} \int_\gamma g(*)d\nu_\gamma^n d((T^{*n}(\nu_x))).$$

To satisfy item (1) at Proposition 4.6 and hence conclude the statement we only have to prove that there are C_2, λ_2 s.t.

$$\forall\gamma \ \ W_1(\nu_\gamma^n, \mu_\gamma) \leq C_2 \cdot e^{-\lambda_2 n}$$

this is by uniform contraction on stable leaves.

Indeed, by remark 4.8, if ν_γ and ρ_γ are the two probability measures on the leaf γ then the measures $F^*(\nu_\gamma), F^*(\rho_\gamma)$ on the contracting leaf $F(\gamma)$ are such that

$$W_1(F^*(\nu_\gamma), F^*(\rho_\gamma)) \leq \lambda \cdot W_1(\nu_\gamma, \rho_\gamma).$$

Now let us consider $F^{-1}(\gamma) = \gamma_1 \cup \gamma_2 ... \cup \gamma_k$ and apply the above inequality to estimate the distance of iterates of the measure on the leaves. For simplicity let us show the case where the pre-image of a leaf consists of two leaves as it happen in the Geometric Lorenz system, the case where the pre-image consists of more leaves is analogous: let hence $F^{-1}(\gamma) = \gamma_1 \cup \gamma_2$, after one iteration of F^* on the measures ν and μ the "new" measures $\nu_\gamma^1 = (F^*(\nu))_\gamma$ and μ_γ (which is equal to $(F^*(\mu))_\gamma$ because μ is invariant) on the leaf γ will be a convex combination of the images of the "old" measures on γ_1 and γ_2

$$\nu_\gamma^1 = a \cdot F^*(\nu_{\gamma_1}) + b \cdot F^*(\nu_{\gamma_2}),$$

$$\mu_\gamma = a \cdot F^*(\mu_{\gamma_1}) + b \cdot F^*(\mu_{\gamma_2}) \tag{29}$$

with $a + b = 1, a, b \geq 0$ (the second equality is again because μ is invariant). By the triangle inequality (remark 4.2)

$$W_1(\nu_\gamma^1, \mu_\gamma) \leq a \cdot W_1(F^*(\nu_{\gamma_1}), F^*(\mu_{\gamma_1})) + b \cdot W_1(F^*(\nu_{\gamma_2}), F^*(\mu_{\gamma_2}))$$

and by remark 4.8

$$W_1(\nu_\gamma^1, \mu_\gamma) \leq \lambda(a \cdot W_1(\nu_{\gamma_1}, \mu_{\gamma_1}) + b \cdot W_1(\nu_{\gamma_2}, \mu_{\gamma_2}))$$

hence

$$W_1(\nu_\gamma^1, \mu_\gamma) \leq \lambda \sup_\gamma (W_1(\nu_\gamma, \mu_\gamma)).$$

The same can be done in the case when the pre-image $F^{-1}(\gamma) = \gamma_1$ is only one leaf or more than two, hence by induction $W_1(\nu_\gamma^n, \mu_\gamma) < \lambda^n$, and the exponential bound on the distance of iterates on the leaves (item 1 of Proposition 4.6) is provided. □

5. Hitting time: flow and section

We now consider again a Lorenz like flow, with integrable return time, i.e. a flow X^t having a transversal section Σ whose first return map satisfies the assumptions of Theorem 4.7 and the return time is integrable, as before. As before $F : \Sigma \setminus \Gamma \to \Sigma$ is the first return map associated.

Let $x_0 \in \mathbb{R}^3$ and

$$\tau_r^{X^t}(x, x_0) = \inf\{t \geq 0 | X^t(x) \in B_r(x_0)\}$$

be the time needed for the X-orbit of a point x to enter for the *first time* in a ball $B_r(x_0)$. The number $\tau_r^{X^t}(x, x_0)$ is the *hitting time associated to* the flow X^t and $B_r(x_0)$.

If $x_0 \in \Sigma$ and $B_r^\Sigma(x_0) = B_r(x_0) \cap \Sigma$, we define $\tau_r^\Sigma(x, x_0) = \min\{n \in \mathbb{N}^+; F^n(x) \in B_r^\Sigma(x_0)\}$: the *hitting time associated to* the discrete system F.

Given any x we recall that we denoted with $t(x)$ the first strictly positive time, such that $X^{t(x)}(x) \in \Sigma$ (the *return time* of x to Σ). A relation between ${\tau_r}^X(x, x_0)$ and $\tau_r^\Sigma(x, x_0)$ is given by

Proposition 5.1. *If $\int_\Sigma t(x)\ d\mu_F < \infty$, then, there is $K \geq 0$ and a set $A \subset \Sigma$ having full μ_F measure such that for each $x_0 \in \Sigma$, $x \in A$*

$$c(r) \cdot \tau_{Kr}^\Sigma(x, x_0) \cdot \int_\Sigma t(x)\ d\mu_F \leq \tau_r^{X^t}(x, x_0) \leq c(r) \cdot \tau_r^\Sigma(x, x_0) \cdot \int_\Sigma t(x)\ d\mu_F \tag{30}$$

with $c(r) \to 1$ as $r \to 0$.

Proof. Let us assume that $x, x_0 \in \Sigma$. Since the flow cannot hit the section near x_0 without entering in a small ball of the space centered at x_0 before, there is $e(r) \to 0$ as $r \to 0$ such that $\tau_r^\Sigma(x, x_0)$ and $\tau_r^{X^t}(x, x_0)$ are related by

$$\tau_r^{X^t}(x, x_0) \leq e(r) + \sum_{i=0}^{\tau_r^\Sigma(x,x_0)} t(F^i(x)). \tag{31}$$

Moreover the transversality condition implies that there is a K such that

$$\tau_r^{X^t}(x, x_0) \geq [\sum_{i=0}^{\tau_{Kr}^\Sigma(x,x_0)} t(F^i(x))] - e(r). \tag{32}$$

The last inequality follows by the fact that if the flow at some time crosses the ball centered at x_0 then it will cross the section at a distance less than Kr, where K depends on the angle between the flow and the section (when r is small approximate locally the flow by a constant one).

The above sums are Birkhoff sum of the observable t on the F-orbit of x and μ_F is ergodic. Then there is a full measure set A such that

$$\frac{1}{n}\sum_{i=0}^{n} t(F^i(x)) \longrightarrow \int_\Sigma t(x)\ d\mu_F, \quad \text{as} \quad n\to\infty$$

for $x\in A$. Hence

$$\frac{1}{\tau_r^\Sigma(x,x_0)}\sum_{i=0}^{\tau_r^\Sigma(x,x_0)} t(F^i(x)) \longrightarrow \int_\Sigma t(x)\ d\mu_F, \quad \text{as} \quad n\to\infty$$

for $x\in A$. Thus we get that for each $x\in A$

$$\sum_{i=0}^{\tau_r^\Sigma(x,x_0)} t(F^i(x)) = c(r)\cdot\tau_r^\Sigma(x,x_0)\cdot\int_\Sigma t(x)\ d\mu_F \tag{33}$$

with $c(r)\to 1$ as $r\to 0$. Combining Equations (31,32) and (33) we get (30). □

Let π be the projection on Σ defined in Proposition 3.2. The above statement implies the following

Proposition 5.2. *There is a full measure set $B\subset\mathbb{R}^3$ such that if $x_0\in\mathbb{R}^3$ is regular and $x\in B$ it holds*

$$\lim_{r\to 0}\frac{\log\tau_r^{X^t}(x,x_0)}{-\log r} = \lim_{r\to 0}\frac{\log\tau_r^\Sigma(x,\pi(x_0))}{-\log r} \tag{34}$$

Proof. The above Proposition implies that if $x_0, x\in\Sigma$ and $x\in A$ then

$$\lim_{r\to 0}\frac{\log\tau_r^{X^t}(x,x_0)}{-\log r} = \lim_{r\to 0}\frac{\log\tau_r^\Sigma(x,x_0)}{-\log r} = d_{\mu_F}(x_0). \tag{35}$$

This is also true for each $x\in B=\pi^{-1}(A)$. If $x_0\in\mathbb{R}^3$ is a regular point, the flow X induces a bilipschitz homeomophism from a neighborhood of x_0 to a neighborhood of $\pi(x_0)\in\Sigma$.

Hence there is $K\geq 1$ such that

$$\tau_{K^{-1}r}^X(x,\pi(x_0)) + Const \leq \tau_r^X(x,x_0) \leq \tau_{Kr}^X(x,\pi(x_0)) + Const$$

where $Const$ represents the time which is needed to go from $\pi(x_0)$ to x_0 by the flow. Extracting logarithms and taking the limits we get the required result. □

We recall that (see Section 3) the assumption $\int_\Sigma t(x)\ d\mu_F<\infty$ is verified for the geometric Lorenz flow. Hence these results applies for this example.

6. A logarithm law for the hitting time

In this section we give the main result for the behavior of the hitting time on Lorenz like flows. First let us recall a result on discrete time systems.

Let (Y, T, μ) be a measure preserving (discrete time) dynamical system. We say that (X, T, μ) has super-polynomial decay of correlations with respect to Lipschitz observables if

$$\left|\int \varphi \circ T^n \psi \cdot d\mu - \int \varphi \cdot d\mu \cdot \int \psi \cdot d\mu\right| \leq \|\varphi\| \cdot \|\psi\| \cdot \theta_n,$$

with $\lim_n \theta_n \cdot n^p = 0$ for all $p > 0$ and $\|\cdot\|$ is the Lipschitz norm.

In [15] it is proved the following fact for discrete time systems:

Theorem 6.1. *Let (Y, T, μ) a dynamical system having superpolynomial decay of correlations as above. For each $x_0 \in Y$ such that $d_\mu(x_0)$ is defined*

$$\lim_{r \to 0} \frac{\log \tau_r(x, x_0)}{-\log r} = d_\mu(x_0)$$

for μ-almost each $x \in Y$.

We can apply this to a 2-dimensional systems (Σ, F, μ_F) satisfying the assumptions of Theorem 4.7 since we proved that the system has exponential decay of correlations. We hence conclude the following

Corollary 6.2. *Let $F : \Sigma \to \Sigma$ be a map with an invariant measure μ_F satisfying the assumptions of Theorem 4.7. For each $x_0 \in \Sigma$ such that $d_{\mu_F}(x_0)$ exists then*

$$\lim_{r \to 0} \frac{\log \tau_r^{\Sigma}(x, x_0)}{-\log r} = d_{\mu_F}(x_0).$$

for μ_F-almost $x \in \Sigma$.

Now, if we consider a flow having such a map as its Poincaré section and integrable return time, we can construct as in Section 3 an SRB invariant measure μ_X for the flow. By Proposition 5.2, Corollary 6.2 and Proposition 3.2 we can estimate the hitting time to balls for the flow by the corresponding estimation for the Poincaré map and we get our main result, which corresponds to Theorem B in the introduction (where all the necessary assumptions on the map are listed):

Theorem 6.3. *If X^t is a Lorenz like flow, that is a flow having a transversal section, with a Poincaré map satisfying the assumptions of proposition 4.7 and integrable return time, then for each regular $x_0 \in \mathbb{R}^3$ such that $d_{\mu_X}(x_0)$ exists, it holds*

$$\lim_{r \to 0} \frac{\log \tau_r^{X^t}(x, x_0)}{-\log r} = d_{\mu_X}(x_0) - 1$$

for μ_X-almost each $x \in \mathbb{R}^3$.

7. Quantitative recurrence for Lorenz like systems

We now recall a general result proved by Saussol in [35] about quantitative recurrence in order to apply it to a Lorenz like flow. The result shows that the power law behavior of the recurrence rate can be estimated in function of the local dimension if the system has fast enough decay of correlations.

Let (Y, μ, T) be discrete time dynamical system. Given a set $A \subset Y$, we denote the boundary of A as ∂A.

Theorem 7.1. [35, Thm 4, Lemma 13]. *Let (Y, T, μ) be a measure preserving dynamical system. Assume that the entropy $h_\mu(T) > 0$ and T is such that there exists a partition $\mathcal{A}$ (modulo μ) into open sets such that for each $A \in \mathcal{A}$ the map T is Lipschitz with constant $L_T(A)$. Furthermore, suppose that*

1. *the set $\mathcal{S}(\mathcal{A}) = \cup\{\partial A \in \mathcal{A}\}$ is such that there are constants $c > 0$ and $a > 0$ so that*
$$\mu\left(\{x \in X : \operatorname{dist}(x, \mathcal{S}(\mathcal{A})) < \varepsilon\}\right) < c \cdot \varepsilon^a.$$
2. *the average Lipschitz exponent*
$$\sum_{A \in \mathcal{A}} \mu(A) \log^+ L_T(A)$$
is finite,
3. *the decay of correlation of T is super-polynomial.*

Then

$$\liminf_{r \to 0} \frac{\log \tau_r(x,x)}{-\log r} = d_\mu^-(x) \ , \quad \textit{and} \quad \limsup_{r \to 0} \frac{\log \tau_r(x,x)}{-\log r} = d_\mu^+(x) \ \textit{a.e.} \tag{36}$$

Let us first show that the above theorem can be applied to the Geometric Lorenz system.

Lemma 7.2. *The first return map (F, Σ, μ_F) of the Geometric Lorenz system (described in Section 2) satisfies the hypothesis of Theorem 7.1 above.*

Proof. Since we have proved that the system (F, Σ, μ_F) is exponentially mixing, item (3) at Theorem 7.1 is satisfied.

The partition $\mathcal{A} = \{A_i\}$, with

$$A_i = \left(\frac{1}{i+1}, \frac{1}{i}\right) \times \mathring{I}, \ i \in \mathbb{Z} \tag{37}$$

where $\mathring{I}$ denotes the interior of I, satisfies (1) and (2) at Theorem 7.1. Here we note that F is not globally Lipschitz, but from Eq. (16) we get $L_T(A_i) \leq K \cdot i^\beta$, with $\beta > 1$ and $K > 0$.

Moreover, the fact that μ_F has a bounded density marginal (the density will be denoted by f_0 as before) on the x direction implies that the measure of the sets A_i can be estimated by

$$\mu(A_i) \leq \frac{2 \cdot \sup(f_0)}{i^2}.$$

Thus,

$$\sum_{A\in\mathcal{S}(\mathcal{A})} \log^+ L_F(A)\cdot\mu(A) = \sum_{A\in\mathcal{S}(\mathcal{A})} \log^+(K\cdot i^\beta)\cdot\frac{2\cdot\sup(f_0)}{i^2} < \infty.$$

This finishes the proof. □

In the same way, replacing Equation 16 with assumption 3) in the introduction it can be proved that the above theorem applies to Lorenz like flows:

Lemma 7.3. *If the system (F,Σ,μ_F) is the first return map of a flow satisfying assumptions 1.a),...1.d),2),3) of the introduction, then it satisfies the hypothesis of Theorem 7.1.*

Applying Theorem 7.1 to such system, then we get

Corollary 7.4. *For the system (F,Σ,μ_F) it holds*

$$\liminf_{r\to 0}\frac{\log\tau_r^\Sigma(x,x)}{-\log r} = d^-_{\mu_F}, \qquad \limsup_{r\to 0}\frac{\log\tau_r^\Sigma(x,x)}{-\log r} = d^+_{\mu_F}, \quad \mu_F - a.e..$$

Finally, remarking that regular points have full measure, with the same arguments as in Proposition 5.2 by Theorem 3.2, we get

Corollary 7.5. *For the Geometric Lorenz flow and for Lorenz like flows as above it holds*

$$\liminf_{r\to 0}\frac{\log\tau'_r(x)}{-\log r} = d^-_{\mu_X} - 1, \qquad \limsup_{r\to 0}\frac{\log\tau'_r(x)}{-\log r} = d^+_{\mu_X} - 1, \quad \mu_X - a.e..$$

where τ' is the recurrence time for the flow, as defined in the introduction.

This is the content of Theorem C in the introduction.

8. Appendix I: about regularity of the measure μ_F

In this section we are going to prove that the SRB measure of a Lorenz like map satisfies item 3 of Theorem 4.7. We remark that this is a kind of regularity assumption for the measure μ_F (a certain projection is BV). The proof is done in several steps and it will be completed at the end of the section. The statement we are going to prove is:

Lemma 8.1. *Let $F(x,y) = (T(x), G(x,y))$ be a Borel map preserving the vertical foliation, such that:*

1. *There is $k \geq 0$ such that, if x_1, x_2 are such that $0 \notin [x_1,x_2]$ then $\forall x \in I$:*
$$|G(x_1,y) - G(x_2,y)| \leq k\cdot|x_1 - x_2|$$
2. *$F|_\gamma$ is λ-Lipschitz with $\lambda < 1$ (hence is uniformly contracting) on each leaf γ.*
3. *$T : I \to I$ is onto and, piecewise monotonic, with two C^1 increasing branches on the intervals $[-\frac{1}{2}, c)$,$(c, \frac{1}{2}]$ and $T' > 1$ where it is defined. Moreover $\lim_{x\to c^-} T(x) = \frac{1}{2}$, $\lim_{x\to c^+} T(x) = -\frac{1}{2}$, $T(c) = -\frac{1}{2}$, $\lim_{x\to c} T'(x) = \infty$.*
4. *$\frac{1}{T'}$ has bounded variation.*

then (Σ, F) has an unique invariant SRB measure which satisfies item 3 of Theorem 4.7.

We recall that the existence and the uniqueness of the SRB measure can be obtained by the general arguments exposed in Section 3. To proceed to prove the above statement, we need to introduce some concepts.

To deal with non normalized measures as the measures $\mu|\gamma$ on the leaves are, we consider the following modification of the Wasserstein distance: let $b1lip(I)$ be the set of 1-Lipschitz functions on I having L_∞ norm less or equal than 1 ($b1lip(I) = 1lip(I) \cap \{g, \|g\|_\infty \leq 1\}$).

Let us consider two finite measures μ, ν and the distance

$$W_1^0(\mu,\nu) = \sup_{g \in b1lip(I)} |\int g \, d\mu - \int g \, d\nu|.$$

Remark 8.2. We remark that choosing $g = 1$ we obtain $W_1^0(\mu,\nu) \geq |\mu(I) - \nu(I)|$.

Let us consider the space $M(I)$ of Borel finite measures over I with the distance W_1^0. Given a function $G : I \to (M(I), W_1^0)$ we define the variation of G as follows: let $x_1, ..., x_n$ be an increasing finite sequence in I (which induces a subdivision in small intervals) let Sub be the set of such subdivisions. We define the variation of G as:

$$\begin{aligned} Var(G, x_1, ..., x_n) &= \sum_{i \leq n} W_1^0(G(x_i), G(x_{i+1})) \\ Var(G) &= \sup_{(x_i) \in Sub} Var(G, x_1, ..., x_n). \end{aligned}$$

We will consider the Lebesgue measure on the section Σ and its iterates by F. The strategy is to disintegrate along stable leaves and estimate the variation of the induced function $I \to (M(I), W_1^0)$ proving that this is uniformly bounded. Le us precise this point: if μ is a finite measure on Σ, by disintegration this induces a function $G_\mu : I \to M(I)$ defined almost everywhere by

$$G_\mu(\gamma) = \mu|_\gamma.$$

Suppose that G_μ is defined everywhere. The BV norm of G_μ will be an estimation of the regularity of μ. For example, the Lebesgue measure on the square Σ induces a function G_m which is constant everywhere and its value is the Lebesgue measure on the interval. The variation in this case is obviously null. We remark that each iterate of the Lebesgue measure by F^* induces a $G_{F^{*n}(m)}$ which is defined everywhere (see eq. 38). We will give an estimation of the variation for these iterates in our system.

Definition 8.3. We say that a measure μ on Σ is K-good if the function $G_\mu : I \to M(I)$, with $G_\mu(\gamma) = \mu|_\gamma$ as above is well defined and s.t. $Var(G_\mu) \leq K$.

Some preliminary lemmata and remarks.

Remark 8.4. We remark that if μ is K-good then $\sup_\gamma(\mu|_\gamma(I)) \leq 1 + K$.

Proof. Since μ is a probability measure then for some γ, $\mu|_\gamma(I) \leq 1$ if for some ξ it was $\mu|_\xi(I) > 1 + K$ then by Remark 8.2 this would contradict the assumption.

This elementary remark about real sequences will be used in the following. □

Lemma 8.5. *If a sequence a_n is such that $a_{n+1} \leq \lambda a_n + k$ for some $\lambda < 1, k > 0$, then*

$$\sup(a_n) \leq \max(a_0, \frac{k}{1-\lambda})$$

Proof. If for some m, $a_m > \frac{k}{1-\lambda}$ then there is $\delta > 0$ such that $a_m = \frac{k+\delta}{1-\lambda}$. Hence, $a_{m+1} \leq \lambda \cdot \frac{k+\delta}{1-\lambda} + k = \frac{k+\lambda\delta}{1-\lambda} < a_m$. Similarly $a_n \leq \frac{k}{1-\lambda} \implies a_{n+1} \leq \frac{k}{1-\lambda}$. □

The following is analogous to remark 4.8 for the distance W_1^0, and also follows by uniform contraction on stable leaves.

Remark 8.6. Let F be λ contracting as above. Let us consider a leaf γ and two finite (non necessarily normalized) measures μ, ν on it. Then

$$W_1^0(F^*(\mu), F^*(\nu)) \leq |\mu(\gamma) - \nu(\gamma)| + \lambda \cdot W_1^0(\mu, \nu).$$

Proof. If g is in $b1lip$ on $F(\gamma)$ then $g(F(*))$ is λ-Lipschitz on γ, moreover since $|g| \leq 1$then $|g \circ F - \theta| \leq \lambda$ for some $\theta \leq 1$. This implies that

$$|\int_{F(\gamma)} g \ d(F^*\mu) - \int_{F(\gamma)} g \ d(F^*\nu)| = |\int_\gamma g \circ F \ d\mu - \int_\gamma g \circ F \ d\nu| \leq$$

$$\theta \cdot |\mu(I) - \nu(I)| + |\int_\gamma (g \circ F) - \theta \ d\mu - \int_\gamma (g \circ F) - \theta \ d\nu| \leq$$

$$|\mu(I) - \nu(I)| + \lambda \cdot W_1^0(\mu, \nu).$$

□

Now we are ready to prove the main technical lemma estimating the regularity of the iterates $F^{*n}(m)$. We will explicit the assumptions we need on F.

Lemma 8.7. *Let $F(x, y) = (T(x), G(x, y))$ be a Borel map preserving the vertical foliation such that:*

1. *There is $k \geq 0$ such that, if x_1, x_2 are such that $0 \notin [x_1, x_2]$ then $\forall x \in I : |G(x_1, y) - G(x_2, y)| \leq k \cdot |x_1 - x_2|$*
2. *$F|_\gamma$ is λ-Lipschitz with $\lambda < 1$ on each vertical leaf γ.*

Let γ_1 and γ_2 two close leaves with that $F^{-1}(\gamma_1) = \{\alpha_1, \alpha_2\}, F^{-1}(\gamma_2) = \{\beta_1, \beta_2\}$ and suppose that T' is defined in the points α_i and β_i and at these points $T' \geq 1$. Let $\mu_0$be a probability measure on Σ such that $\mu_0|_\gamma$ is defined everywhere and

$$\mu_0|_\gamma(I) = \overline{f}_0(\gamma)$$

for a bounded density function $\overline{f}_0$. Then

$$W_1^0(F^*(\mu_0)|_{\gamma_1}, F^*(\mu_0)|_{\gamma_2}) \leq |\overline{f}_0(\alpha_1) - \overline{f}_0(\beta_1)| + \lambda W_1^0(\mu_0|_{\alpha_1}, \mu_0|_{\beta_1}) +$$
$$+ |\overline{f}_0(\alpha_2) - \overline{f}_0(\beta_2)| + \lambda W_1^0(\mu_0|_{\alpha_2}, \mu_0|_{\beta_2})) +$$
$$+2 \cdot k \cdot \sup \overline{f}_0(|\alpha_1 - \beta_1| + |\alpha_2 - \beta_2|) + \sup \overline{f}_0 |\frac{1}{T'(\alpha_1)} - \frac{1}{T'(\beta_1)}| + \sup \overline{f}_0 |\frac{1}{T'(\alpha_2)} - \frac{1}{T'(\beta_2)}|.$$

Proof. Let $F^*(\mu_0)|_{\gamma_1}$ be the restriction of $F^*(\mu_0)$ to the leaf γ. Remark that

$$F^*(\mu_0)|_{\gamma_1} = \frac{1}{T'(\alpha_1)} F^*_{\alpha_1}(\mu_0|_{\alpha_1}) + \frac{1}{T'(\alpha_2)} F^*_{\alpha_2}(\mu_0|_{\alpha_2}) \tag{38}$$

where $F_{\alpha_i} : I \to I$ is given by $F_{\alpha_i}(y) = \pi_y(F(y,\alpha_i))$ and

$$F^*(\mu_0)|_{\gamma_2} = \frac{1}{T'(\beta_1)} F^*_{\beta_1}(\mu_0|_{\beta_1}) + \frac{1}{T'(\beta_2)} F^*_{\beta_2}(\mu_0|_{\beta_2})$$

with similar notation for F_{β_1}. Now the remaining part of the proof is a (long) straightforward calculation:

$$W_1^0(F^*(\mu_0)|_{\gamma_1}, F^*(\mu_0)|_{\gamma_2}) = \sup_{g\in 1blip} |\int g\ d(F^*(\mu_0)|_{\gamma_1}) - \int g\ d(F^*(\mu_0)|_{\gamma_2})|$$

and

$$\begin{aligned} \int g\ d(F^*(\mu_0)|_{\gamma_1}) &= \int g\ d(\frac{1}{T'(\alpha_1)} F^*_{\alpha_1}(\mu_0|_{\alpha_1}) + \frac{1}{T'(\alpha_2)} F^*_{\alpha_2}(\mu_0|_{\alpha_2})), \\ \int g\ d(F^*(\mu_0)|_{\gamma_2}) &= \int g\ d(\frac{1}{T'(\beta_1)} F^*_{\beta_1}(\mu_0|_{\beta_1}) + \frac{1}{T'(\beta_2)} F^*_{\beta_2}(\mu_0|_{\beta_2})). \end{aligned}$$

let us estimate these two terms:

$$\begin{aligned} \int gd(F^*(\mu_0)|_{\gamma_1}) &= \int gd(\frac{1}{T'(\alpha_1)} F^*_{\alpha_1}(\mu_0|_{\alpha_1}) + \frac{1}{T'(\alpha_2)} F^*_{\alpha_2}(\mu_0|_{\alpha_2})) = \\ &= \frac{1}{T'(\alpha_1)} \int g(F_{\alpha_1}(y)) d(\mu_0|_{\alpha_1}) + \frac{1}{T'(\alpha_2)} \int g(F_{\alpha_2}(y)) d(\mu_0|_{\alpha_2}) \end{aligned}$$

and similarly

$$\int gd(F^*(\mu_0)|_{\gamma_2}) = \frac{1}{T'(\beta_1)} \int g(F_{\beta_1}(y)) d(\mu_0|_{\beta_1}) + \frac{1}{T'(\beta_2)} \int g(F_{\beta_2}(y)) d(\mu_0|_{\beta_2}).$$

Hence

$$|\int gd(F^*(\mu_0)|_{\gamma_1}) - \int gd(F^*(\mu_0)|_{\gamma_2})| =$$

$$|\frac{1}{T'(\alpha_1)} \int g(F_{\alpha_1}(y)) d(\mu_0|_{\alpha_1}) + \frac{1}{T'(\alpha_2)} \int g(F_{\alpha_2}(y)) d(\mu_0|_{\alpha_2})$$
$$-\frac{1}{T'(\beta_1)} \int g(F_{\beta_1}(y)) d(\mu_0|_{\beta_1}) - \frac{1}{T'(\beta_2)} \int g(F_{\beta_2}(y)) d(\mu_0|_{\beta_2})|.$$

To estimate the last expression by the triangle inequality, let us add and subtract

$$\frac{1}{T'(\beta_1)} \int g(F_{\beta_1}(y)) d(\mu_0|_{\alpha_1}) + \frac{1}{T'(\beta_2)} \int g(F_{\beta_2}(y)) d(\mu_0|_{\alpha_2})$$

obtaining

$$|\int gd(F^*(\mu_0)|_{\gamma_1}) - \int gd(F^*(\mu_0)|_{\gamma_2})| \leq |A| + |B|,$$

where

$$\begin{aligned} A &= \frac{1}{T'(\alpha_1)}\int g(F_{\alpha_1}(y))d(\mu_0|_{\alpha_1}) + \frac{1}{T'(\alpha_2)}\int g(F_{\alpha_2}(y))d(\mu_0|_{\alpha_2}) \\ &\quad -\frac{1}{T'(\beta_1)}\int g(F_{\beta_1}(y))d(\mu_0|_{\alpha_1}) - \frac{1}{T'(\beta_2)}\int g(F_{\beta_2}(y))d(\mu_0|_{\alpha_2}) \end{aligned}$$

and

$$\begin{aligned} B &= \frac{1}{T'(\beta_1)}\int g(F_{\beta_1}(y))d(\mu_0|_{\alpha_1}) + \frac{1}{T'(\beta_2)}\int g(F_{\beta_2}(y))d(\mu_0|_{\alpha_2}) \\ &\quad -\frac{1}{T'(\beta_1)}\int g(F_{\beta_1}(y))d(\mu_0|_{\beta_1}) - \frac{1}{T'(\beta_2)}\int g(F_{\beta_2}(y))d(\mu_0|_{\beta_2}) \end{aligned}$$

Estimation of A. Now let us estimate A:

$$\begin{aligned} |A| &\leq |\frac{1}{T'(\alpha_1)}\int g(F_{\alpha_1}(y))d(\mu_0|_{\alpha_1}) - \frac{1}{T'(\beta_1)}\int g(F_{\beta_1}(y))d(\mu_0|_{\alpha_1})| + \\ &\quad |\frac{1}{T'(\alpha_2)}\int g(F_{\alpha_2}(y))d(\mu_0|_{\alpha_2}) - \frac{1}{T'(\beta_2)}\int g(F_{\beta_2}(y))d(\mu_0|_{\alpha_2})| = I + II \end{aligned}$$

let us analyze the first term in the sum (the estimation of the other term is similar)

$$\begin{aligned} I = |\frac{1}{T'(\alpha_1)}\int g(F_{\alpha_1}(y))d(\mu_0|_{\alpha_1}) - \frac{1}{T'(\beta_1)}\int g(F_{\beta_1}(y))d(\mu_0|_{\alpha_1})| = \\ = |\int \frac{1}{T'(\alpha_1)}g(F_{\alpha_1}(y)) - \frac{1}{T'(\beta_1)}g(F_{\beta_1}(y))\ d(\mu_0|_{\alpha_1})| \end{aligned}$$

adding and subtracting $\frac{1}{T'(\alpha_1)}g(F_{\beta_1}(y))$ we obtain

$$|\int \frac{1}{T'(\alpha_1)}g(F_{\alpha_1}(y)) - \frac{1}{T'(\beta_1)}g(F_{\beta_1}(y)) + \frac{1}{T'(\alpha_1)}g(F_{\beta_1}(y)) - \frac{1}{T'(\alpha_1)}g(F_{\beta_1}(y))\ d(\mu_0|_{\alpha_1})| \leq$$

$$\begin{aligned} &\leq |\int \frac{1}{T'(\alpha_1)}g(F_{\alpha_1}(y)) - \frac{1}{T'(\alpha_1)}g(F_{\beta_1}(y))\ d(\mu_0|_{\alpha_1})| + \\ &\quad +|\int \frac{1}{T'(\alpha_1)}g(F_{\beta_1}(y)) - \frac{1}{T'(\beta_1)}g(F_{\beta_1}(y))\ d(\mu_0|_{\alpha_1})|. \end{aligned}$$

Now, since $\overline{f}_0$ is bounded $\mu_0|_{\alpha_1}(I) \leq \sup(\overline{f}_0)$ and then

$$|\int \frac{1}{T'(\alpha_1)}g(F_{\beta_1}(y)) - \frac{1}{T'(\beta_1)}g(F_{\beta_1}(y))\ d(\mu_0|_{\alpha_1})| \leq \sup \overline{f}_0 |\frac{1}{T'(\alpha_1)} - \frac{1}{T'(\beta_1)}|.$$

The other summand is

$$|\int \frac{1}{T'(\alpha_1)}g(F_{\alpha_1}(y)) - \frac{1}{T'(\alpha_1)}g(F_{\beta_1}(y))\ d(\mu_0|_{\alpha_1})| \leq |\int g(F_{\alpha_1}(y)) - g(F_{\beta_1}(y))\ d(\mu_0|_{\alpha_1})|$$

By assumption (1) then $|F(y,\alpha_1) - F(y,\beta_1)| \leq k \cdot |\alpha_1 - \beta_1|$ and hence

$$|\int g(F_{\alpha_1}(y)) - g(F_{\beta_1}(y))\ d(\mu_0|_{\alpha_1})| \leq k \cdot |\alpha_1 - \beta_1| \cdot \sup \overline{f}_0.$$

summarizing

$$I \leq \sup \overline{f}_0 |\frac{1}{T'(\alpha_1)} - \frac{1}{T'(\beta_1)}| + k \cdot |\alpha_1 - \beta_1| \cdot \sup \overline{f}_0. \tag{39}$$

Considering in the same way the summand II in the expression of A, this gives

$$|A| \leq k \cdot \sup \overline{f}_0 (|\alpha_1 - \beta_1| + |\alpha_2 - \beta_2|) +$$

$$\sup \overline{f}_0 |\frac{1}{T'(\alpha_1)} - \frac{1}{T'(\beta_1)}| + \sup \overline{f}_0 |\frac{1}{T'(\alpha_2)} - \frac{1}{T'(\beta_2)}|.$$

Estimation of B. The upper bound on B follows by contraction on stable leaves. Indeed,

$$\begin{aligned} |B| \ \leq \ & |\frac{1}{T'(\beta_1)} \int g(F_{\beta_1}(y)) d(\mu_0|_{\alpha_1}) - \frac{1}{T'(\beta_1)} \int g(F_{\beta_1}(y)) d(\mu_0|_{\beta_1})| + \\ & + |\frac{1}{T'(\beta_2)} \int g(F_{\beta_2}(y)) d(\mu_0|_{\alpha_2}) - \frac{1}{T'(\beta_2)} \int g(F_{\beta_2}(y)) d(\mu_0|_{\beta_2})| \end{aligned}$$

now, since F contracts all the leaves by a factor at least λ by Remark 8.6 it holds

$$\begin{aligned} |B| \ \leq \ & \frac{1}{T'(\beta_1)} (|\mu_0|_{\alpha_1}(I) - \mu_0|_{\beta_1}(I)| + \lambda W_1(\mu_0|_{\alpha_1}, \mu_0|_{\beta_1})) + \\ & + \frac{1}{T'(\beta_2)} (|\mu_0|_{\alpha_2}(I) - \mu_0|_{\beta_2}(I)| + \lambda W_1(\mu_0|_{\alpha_2}, \mu_0|_{\beta_2})). \end{aligned}$$

Summarizing, $\forall\ g \in b1lip$

$$\left| \int g d(F^*(\mu_0)|_{\gamma_1}) - \int g d(F^*(\mu_0)|_{\gamma_2}) \right| \leq \tag{40}$$

$$\leq \frac{1}{T'(\beta_1)} (|\mu_0|_{\alpha_1}(I) - \mu_0|_{\beta_1}(I)| + \lambda W_1(\mu_0|_{\alpha_1}, \mu_0|_{\beta_1})) +$$

$$+ \frac{1}{T'(\beta_2)} (|\mu_0|_{\alpha_2}(I) - \mu_0|_{\beta_2}(I)| + \lambda W_1(\mu_0|_{\alpha_2}, \mu_0|_{\beta_2})) +$$

$$+ 2k \sup \overline{f}_0 (|\alpha_1 - \beta_1| + |\alpha_2 - \beta_2|) + \sup \overline{f}_0 |\frac{1}{T'(\alpha_1)} - \frac{1}{T'(\beta_1)}| + \sup \overline{f}_0 |\frac{1}{T'(\alpha_2)} - \frac{1}{T'(\beta_2)}|$$

finishing the proof. □

Remark 8.8. We remark that this last step in the proof (equations 40 and following) is the only one where the expansivity of T is explicitly used. In fact, an equivalent result can be obtained with the weaker assumption $\lambda(\inf_{x \in I} T'(x)) < 1$, instead of $T' > 1$.

A similar lemma holds for the case where the pre-image of γ_1 and γ_2 is only one leaf. The proof is similar to the previous one.

Lemma 8.9. *Let $F : \Sigma \to \Sigma$ be as above, satisfying points (1)–(3) of Lemma 8.7. Let γ_1 and γ_2 be two leaves and suppose that $F^{-1}(\gamma_1) = \{\alpha_1\}, F^{-1}(\gamma_2) = \{\beta_1\}$. Let us consider* *a probability measure* μ_0 *on* Σ *such that* $\mu_0|\gamma(I) = \overline{f}_0(\gamma)$ *for a bounded function* $\overline{f}_0$ *, then*

$$W_1^0(F^*(\mu_0)|_{\gamma_1}, F^*(\mu_0)|_{\gamma_2}) \leq |\overline{f}_0(\alpha_1) - \overline{f}_0(\beta_1)| + \lambda \cdot W_1(\mu_0|_{\alpha_1}, \mu_0|_{\beta_1}) +$$
$$+2 \cdot k \cdot \sup(\overline{f}_0)(|\alpha_1 - \beta_1|) + \sup(\overline{f}_0)|\frac{1}{T'(\alpha_1)} - \frac{1}{T'(\beta_1)}|.$$

The above lemmata give the following result, in the spirit of the Lasota Yorke inequality (see in the following proof, eq. 42 and compare with [25], [26] e.g.) giving an upper bound on the variation of iterates $F^{*n}(\mu_0)$.

We recall that by the classical Lasota-Yorke inequalities, for piecewise expanding maps of the interval, iterating a bounded variation density g_0 we get a sequence of uniformly bounded variation densities,

$$Var(T^{*n}(g_0 m)) \leq C_{g_0} \tag{41}$$

where C_{g_0} depends on g_0 and on the dynamics T.

Theorem 8.10. *Let* $F : \Sigma \to \Sigma$ *be as above, satisfying assumptions (1)–(4) of Lemma 8.1. Let* $\mu_n = F^{*n}(\mu_0)$ *where* μ_0 *is* K*-good and has* BV *density on the* x *axis,* $\overline{f}_0$*. Then, each* μ_n *is* K'*-good, where* $K' = \max(K, \frac{3+C_{\overline{f}_0}+(C_{\overline{f}_0}+1)Var(\frac{1}{T'})+2k(C_{\overline{f}_0}+1)}{1-\lambda})$.

Proof. Let us consider a subdivision $\gamma_1, ..., \gamma_n$ made of small intervals and set $s_i = T^{-1}([\gamma_i, \gamma_{i+1}))$. If we are in the case of Lemma 8.7 s_i consists of two small intervals, if we are in the case of Lemma 8.9 s_i consists of one small interval and in the remaining case we have one small interval and an interval of the type $(\alpha_2, \frac{1}{2})$ or $(-\frac{1}{2}, \beta_1)$ (this can happen only in two intervals containing the points $T(-\frac{1}{2})$ and $T(\frac{1}{2})$) of the subdivision). The endpoints of all these pre-image intervals $(s_i)_{i\in(1,...,n)}$ constitute another subdivision $\gamma_1^*, ..., \gamma_m^*$ of I.

Let us estimate the variation of $\mu_1 = F^*(\mu_0)$ on the subdivision $\gamma_1, ..., \gamma_n$. Let us suppose that the intervals of $\gamma_1, ..., \gamma_n$ which are of the third type are $(\gamma_{j_1}, \gamma_{j_1+1})$ and $(\gamma_{j_2}, \gamma_{j_2+1})$. In this case we bound trivially from above the variation: $W_1^0(\mu_1|_{\gamma_{j_i}}, \mu_1|_{\gamma_{j_i+1}}) \leq \sup \overline{f}_0$ (for $i = 1, 2$). Lemma 8.7 and Lemma 8.9 imply

$$Var(G_{\mu_1}, \gamma_1, ..., \gamma_n) =$$
$$\sum_{i\leq n} W_1(\mu_1|_{\gamma}, \mu_1|_{\gamma_{i+1}}) \leq 2\sup \overline{f}_0 + \sum_{i\leq m} |\overline{f}_0(\gamma_i^*) - \overline{f}_0(\gamma_{i+1}^*)| + \lambda W_1(\mu_0|_{\gamma_i^*}, \mu_0|_{\gamma_{i+1}^*})) +$$
$$+ \sum_{i\leq m} 2k \sup \overline{f}_0(|\gamma_i^* - \gamma_{i+1}^*|) + \sup \overline{f}_0 |\frac{1}{T'(\gamma_i^*)} - \frac{1}{T'(\gamma_{i+1}^*)}|.$$

Hence

$$Var(G_{\mu_1}, \gamma_1, ..., \gamma_n) \leq 2\sup \overline{f}_0 + Var(\overline{f}_0) + \sup \overline{f}_0 Var(\frac{1}{T'}) + \sup \overline{f}_0 2k + \lambda Var(G_{\mu_0})$$

and we conclude that

$$Var(G_{\mu_1}) \leq 2\sup \overline{f}_0 + Var(\overline{f}_0) + \sup \overline{f}_0 Var(\frac{1}{T'}) + \sup \overline{f}_0 2k + \lambda Var(G_{\mu_0}). \tag{42}$$

If $\overline{f}_i$are the marginals of μ_i then as recalled before $Var(\overline{f}_i) \leq C_{\overline{f}_0}$. This allows to Iterate the above inequality and obtain, by Lemma 8.5

$$\sup_i(Var(G_{\mu_i})) \leq \max(Var(G_{\mu_0}), \frac{2 + 3C_{\overline{f}_0} + (C_{\overline{f}_0} + 1)Var(\frac{1}{T'}) + 2k(C_{\overline{f}_0} + 1)}{1 - \lambda}),$$

(remark that $\sup \overline{f}_0 \leq Var(\overline{f}_0) + 1$) finishing the proof. □

If μ is a good measure, the measure $f\mu$ associated to a Lipschitz observable f is also a good measure:

Lemma 8.11. *If μ_n is a sequence of $K-$good measures on Σ and $\nu_n = f\mu_n$ with f be ℓ-Lipschitz and $\|f\|_\infty \leq \ell$ then each ν_n is a $(3\ell K + \ell)$-good measure.*

Proof. let γ_1 and γ_2 be two close leaves

$$W_1^0(\nu_n|_{\gamma_1}, \nu_n|_{\gamma_2}) = \sup_{g \in b1lip} \left| \int_{\gamma_1} g(*)f(*, \gamma_1)d(\mu_n|_{\gamma_1}) - \int_{\gamma_2} g(*)f(*, \gamma_2)d(\mu_n|_{\gamma_2}) \right|$$

we recall that $|g| \leq 1$, hence

$$\begin{aligned} &\left| \int g(*)f(*, \gamma_1)\ d(\mu|_{\gamma_1}) - \int g(*)f(*, \gamma_2)\ d(\mu|_{\gamma_2}) \right| \leq \\ &\leq \left| \int g(*)f(*, \gamma_1)\ d(\mu|_{\gamma_1}) - \int g(*)f(*, \gamma_1)\ d(\mu|_{\gamma_2}) \right| + \\ &+ \left| \int g(*)f(*, \gamma_1)\ d(\mu|_{\gamma_2}) - \int g(*)f(*, \gamma_2)\ d(\mu|_{\gamma_2}) \right| \leq \\ &\leq 2\ell \cdot W_1^0(\mu_n|_{\gamma_1}, \mu_n|_{\gamma_2}) + \ell|\gamma_1 - \gamma_2| \sup_\gamma(\mu|_\gamma(I)) \end{aligned}$$

hence $Var(G_{\nu_n}) \leq 2\ell K + \ell(K + 1)$. □

Remark 8.12. If μ_n is K-good for each n and $\overline{g}_n : I \to I$ is the marginal, such that

$$\overline{g}_n(\gamma) = \mu_n|_\gamma(I),$$

since $|\overline{g}_n(\gamma_1) - \overline{g}_n(\gamma_2)| \leq W_1^0(\mu_n|_{\gamma_1}, \mu_n|_{\gamma_2})$ then it holds

$$Var(\overline{g}_n) \leq K \tag{43}$$

for each n.

Remark 8.13. If $\mu_n \to \mu$ and $\nu_n = f\mu_n$, $\nu = f\mu$ with f be ℓ-Lipschitz then $\nu_n \to \nu$.(this is easily obtained because $\int hf\ d\mu_n \to \int hf\ d\mu$, for each continuous h since hf is continuous).

We are finally ready to end the proof of the main proposition of the section.

Proof. (of Lemma 8.1) We prove that $\overline{f}$ as defined at item 3 of Theorem 4.7 has bounded variation and $Var(\overline{f}) \leq 3\ell K' + \ell$, where ℓ is the Lipschitz constant of f and K' is given in Theorem 8.10 and does not depend on f. Let $\mu_n = F^{*n}(m)$ be the sequence of iterates of the Lebesgue measure. By Theorem 8.10 these are K'-good. Since the invariant measure μ is a SRB with full basin, hence $\mu_n \to \mu$ in the weak topology. Then for each continuous

h it holds $\mu_n(h) \to \mu(h)$. In particular this holds for the functions which are constant on each contracting leaf. Let h be such a function. Then $\int_\Sigma h\ d\mu_n = \int_I h\overline{g}_n dx$ where $\overline{g}_n(x) = \mu_n|_{\gamma_x}(I)$ are the densities of μ_n on the x axis as in Remark 8.12.

Let f be ℓ-Lipschitz, $\nu_n = f\mu_n$ and $\nu = f\mu$ as required by Lemma 8.1. Since h is constant along the leaves, again $\int_\Sigma h\ d\nu = \int_I h\overline{f}dx$ and $\int_\Sigma h\ d\nu_n = \int_I h\overline{f}_n dx$ where $\overline{f}_n(\gamma) = \int_\gamma f\ d(\mu_n|_\gamma)$ as above. By Remark 8.13

$$\int_\Sigma h\ d\nu_n \to \int_\Sigma h\ d\nu$$

hence

$$\int_I h\overline{f}_n dx \to \int_I h\overline{f}dx.$$

We have to prove that $\overline{f}$ is BV. By Lemma 8.11 the measures ν_n are $(3\ell K' + \ell)$-good. Now by Remark 8.12, $Var(\overline{f}_n) \leq 3\ell K' + \ell$. By the Helly theorem there is a sub-sequence $\overline{f}_{n_i}$ converging in the L^1 norm to some bounded variation function $\tilde{f}$ such that $Var(\tilde{f}) \leq 3\ell K' + \ell$.

Hence $\int h\overline{f}_{n_i}dx \to \int h\tilde{f}dx$ for each h as above and so $\int h\overline{f}dx = \int h\tilde{f}dx$ for each continuous h and then this implies that they coincide a.e.. Hence $\overline{f}$ can be supposed to be BV and having $Var(\overline{f}) \leq 3\ell K' + \ell$. □

9. Appendix II: Exact dimensionality

In several of the above results we used the local dimension of the system at certain points. In this section we recall a result of of Steinberger ([40]) about the local dimension of Lorenz like systems and prove that for the geometric Lorenz system the local dimension is defined at almost every point.

Let us consider a map $F : [0,1]^2 \to [0,1]^2$ $F(x,y) = (T(x), g(x,y))$ where

(1) $T : [0,1] \to [0,1]$ is piecewise monotonic. This means that there are $c_i \in [0,1]$ for $0 \leq i \leq N$ with $0 = c_0 < \cdots < c_N = 1$ such that $T|(c_i, c_{I+1})$ is continuous and monotone for $0 \leq i < N$. Furthermore, for $0 \leq i < N$, $T|(c_i, c_{i+1})$ is C^1 and that $\inf_{x\in\mathcal{S}} |T'(x)| > 0$ holds where $\mathcal{P} = [0,1] \setminus \cup_{0\leq i<N} c_i$.
(2) $g : [0,1]^2 \to (0,1)$ is C^1 on $\mathcal{P} \times [0,1]$. Furthermore, $\sup|\partial g/\partial x| < \infty$, $\sup|\partial g/\partial y| < 1$ and $|(\partial g/\partial y)(x,y)| > 0$ for $(x,y) \in \mathcal{P} \times [0,1]$.
(3) $F((c_i, c_{i+1}) \times [0,1]) \cap F((c_i, c_{i+1}) \times [0,1]) = \emptyset$ for distinct i,j with $0 \leq i,j < N$.

Now consider the projection $\pi_x : I^2 \to I$, set $\mathcal{V} = \{(-1/2, 0), (0, 1/2)\}$ and $\mathcal{V}_k = \bigvee_{i=0}^k f^{-i}\mathcal{V}$, which is a partition of $E = \bigcap_{i=0}^\infty H^{-1}(I \setminus \{0\})$ into open intervals. For $x \in E$ let $J_k(x)$ be the unique element of $\mathcal{V}_k$ which contains x. We say that $\mathcal{V}$ *is a generator* if the length of the intervals $J_k(x)$ tends to zero for $n \to \infty$ for any given x. Set

$$\psi(x,y) = \log|T'(x)| \quad \text{and} \quad \varphi(x,y) = -\log|(\partial g/\partial y)(x,y)|.$$

The result of Steinberger that we shall use is the following

Theorem 9.1. [40, Theorem 1] *Let F be a two-dimensional map as above and μ an ergodic, F-invariant probability measure on I^2 with the entropy $h_\mu(F) > 0$. Suppose $\mathcal{V}$ is a generator, $\int \psi \cdot d\mu_F < \infty$ and $0 < \int \varphi d\mu_F < \infty$. If the maps $y \mapsto \psi(x,y)$ are uniformly equicontinuous for $x \in I \setminus \{0\}$ and $1/|f'|$ is BV then*

$$d_\mu(x,y) = h_\mu(F)\Big(\frac{1}{\int \psi \cdot d\mu} + \frac{1}{\int \varphi \cdot d\mu}\Big)$$

for μ-almost all $(x,y) \in I^2$.

Now we verify that the Lorenz geometric system as defined in Section 3 is exact dimensional. First we observe that for the first return map $F : \Sigma \setminus \Gamma \to \Sigma$ associated to the Lorenz geometric flow its entropy $h_\mu(F) > 0$, see [5, 4, pp.188]. Next, equations (17), (18), and the properties of f_{Lo} described in Subsections 2.4 and 2.5 guaranty that $F = (f_{Lo}, g_{Lo})$ is a two-dimensional transformation satisfying the above points (1–3). So, all we need to prove that $(\Sigma, F, d\mu_F)$ is exact dimensional is to verify that $F(x,y) = (f_{Lo}(x), g_{Lo}(x,y))$ satisfies the hypothesis of Theorem 9.1. For this, let

$$\psi(x,y) = \log|f'_{Lo}(x)| \quad \text{and} \quad \varphi(x,y) = -\log|(\partial g_{Lo}/\partial y)(x,y)|.$$

Then the following result holds:

Proposition 9.2. *For $q = (x,y) \in \Sigma^*$, let $\varphi(q) = -\log|\partial g_{Lo}/\partial y(q)|$ and $\psi(q) = \log|f'_{Lo}(x)|$. Then*

(1) $\int \varphi d\mu_F < \infty$,
(2) $0 < \int \psi d\mu_F < \infty$, *and*
(3) *the maps $y \mapsto \varphi(x,y)$ are uniformly equicontinuous for $x \in I \setminus \{0\}$.*

where μ_F is the invariant ergodic SRB measure described in Subsection 3.

Proof. Given $q = (x,y) \in [-1/2, 1/2]^2$, we provide the calculations for $x > 0$, the other case being analogous.

By equation (16) we have

$$DF(x,y) = \begin{pmatrix} \partial_x f_{Lo} & \partial_y f_{Lo} \\ \partial_x g_{Lo} & \partial_y g_{Lo} \end{pmatrix} = \begin{pmatrix} M \cdot \alpha \cdot x^{(\alpha-1)} & 0 \\ \sigma \cdot \beta \cdot y x^{(\beta-\alpha)} & \sigma x^\beta \end{pmatrix}.$$

Proof of (1): By the expression above we have $\partial g_{Lo}/\partial y(q) = \sigma \cdot x^\beta$ and so $\log|\partial g_{Lo}/\partial y(q)| = \log|\sigma \cdot x^\beta|$ does not depend on y. Since the measure μ_F is constant at each leaf $\ell \in \mathcal{F}$ and the projection of μ_F on the x-axis, $\mu_{f_{Lo}}$, is absolutely continuous with respect to Lebesgue measure (and even has a finite density), see Proposition 2.2, we immediately conclude that

$$\int \log|\partial g_{Lo}/\partial y(q)| d\mu_F < \infty.$$

proving (1).
Proof of (2): Again from the expression for $DF(x,y)$ above we get $f'_{Lo}(x) = M \cdot \alpha \cdot x^{(\alpha-1)}$, recall $0 < \alpha < 1$. Hence, for $0 < x < 1/2$, $\log(f'_{Lo}(x)) = \log(M \cdot \alpha \cdot x^{(\alpha-1)})$. Thus

$$0 < \int \log(f'_{Lo}(x)) d\mu_F \le K_0 + (\alpha - 1)[x \cdot \log(x) - x] \le K_0 + (\alpha-1)K_1,$$

proving (2).
Proof of (3)

Note that $\varphi(x,y) = -\log|(\partial g_{Lo}/\partial y)(x,y)| = \log(\sigma) + \beta \cdot \log(|x|)$ and so the maps $y \mapsto \varphi(x,y)$ are obviously uniformly equicontinuous for $x \neq 0$.

All together finishes the proof of Proposition 9.2 establishing that μ_F is exact dimensional. □

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