

Subdivision curves on surfaces with arc-length control

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Abstract

In this paper we present a new non-linear interpolatory curve subdivision scheme, suitable for designing curves on surfaces.

We show that the scheme converges and the subdivision curve is continuous. Moreover, starting with a certain natural parametrization of the initial polygon, we obtain a subdivision curve parametrized by a multiple of the arc-length.

1 Introduction

Subdivision curves are very important for several science and engineering applications. In the Euclidean space R^n , they are easy to define by just selecting a small set of control points. A drawing algorithm typically performs some subdivision steps, thus approximating the limit curve by a polygonal line in a matter of milliseconds. In this setting, there are many references on the convergence of these schemes.

Given their good properties and the advantages of using these curves, it is natural to extend them to non-euclidean geometries, such as Riemannian manifolds, Lie groups or triangulations. In [12, 14, 13] one may find some of these extensions.

In this paper we propose a simple method to define a subdivision scheme on a two dimensional manifold S that is easy to be implemented on triangle

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meshes. Under mild conditions, the limit curve of the proposed scheme is a continuous curve on S . Higher order continuity has not been explored in this work, since it depends also on the smoothness of S .

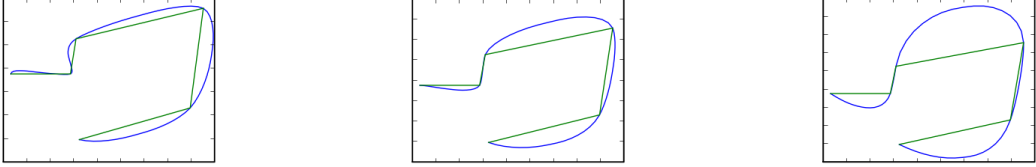


Figure 1: The limit curve of the 4-point subdivision scheme (1) with the initial parametrization (2) with $\beta = 0$ (left), $\beta = 0.5$ (middle) and $\beta = 1$ (right)

The classical 4-point scheme [2, 5] is one of the earliest and most popular interpolatory curve subdivision schemes. It is a member of the Dubuc-Deslauriers family of subdivision schemes [1], where the new points lie on a polynomial interpolating consecutive vertices of the control polygon. More precisely, starting from an initial polygon $P^0 = \{P_i^0, i \in \mathbb{Z}\}$ the 4-point scheme is defined by the equations

$$P_{2i}^{j+1} = P_i^j, \quad P_{2i+1}^{j+1} = g_i^j(t_{2i+1}^{j+1}) \quad (1)$$

where $g_i^j(t)$ is the cubic polynomial interpolating the points P_k^j at uniform parameter values $t_k^j = k/2^j$ for $k = i-1, i, i+1, i+2$, and $t_{2i+1}^{j+1} = (2i+1)/2^{j+1}$. Several authors [7, 4] have noticed that the limit curves of the 4-point scheme fit tightly to the long edges of the initial control polygon and loosely to the short edges, see Figure 1, left. This is a result of the uniform parametrization $t_i^0 = i$ for all i : the same time is used to travel between two consecutive points P_i^0, P_{i+1}^0 of the initial polygon, regardless of their distance. In other words, the limit curve of the uniform 4-point subdivision scheme is far away from being arc-length parametrized.

One way to address this problem is to use a non-uniform parameterization for P^0

$$t_{i+1}^0 = t_i^0 + \|P_{i+1}^0 - P_i^0\|^\beta \quad (2)$$

Then, the parameter values at the step $j+1$, are computed from the parameters of the previous step as

$$t_{2i}^{j+1} = t_i^j, \quad t_{2i+1}^{j+1} = \frac{t_i^j + t_{i+1}^j}{2} \quad (3)$$

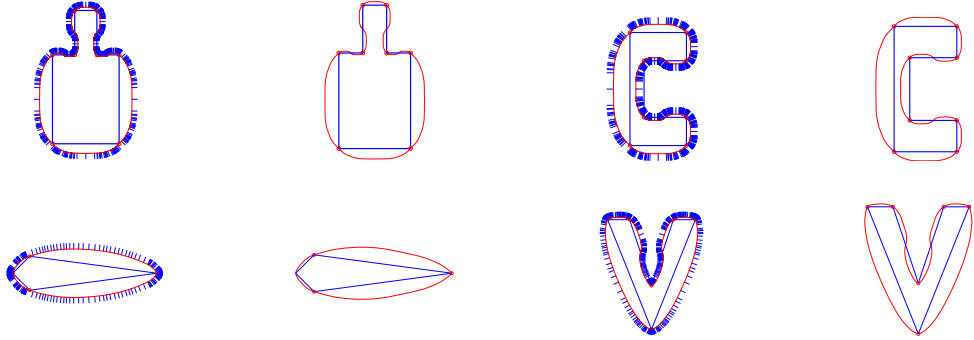


Figure 2: After 6 iterations of the subdivision scheme [6].

Taking this idea one step further, in [4], a reparametrization is introduced in each step, defined by the equation

$$t_0^j = 0, \quad t_{i+1}^j = t_i^j + \|P_{i+1}^j - P_i^j\|^\beta \quad (4)$$

Points in the step $j + 1$ are given by $P_{2i}^{j+1} = P_i^j, P_{2i+1}^{j+1} = f_i^j((t_i^j + t_{i+1}^j)/2)$, where $f_i^j(t)$ is the cubic polynomial interpolating (t_k^j, P_k^j) for $k = i - 1, i, i + 1, i + 2$. The limit curve of this nonlinear scheme is smooth and when the centripetal parametrization is used, it is relatively close to the initial polygon and its shape is pleasing.

In Figure 1 we also show the limit curves of the 4-point subdivision scheme corresponding to different values of β . Notice that the change in the parameterization of the initial polygon affects the shape of the limit curve.

Motivated by the idea of keeping some control on the geometry of the limit curves by means of a simple non-linear subdivision scheme, in [6] is presented a non-stationary, interpolatory, plane curve subdivision scheme, whose limit curve is continuous and parametrized by a multiple of the arc-length. The intermediate polygons P^j are provided with non-uniform parametrizations reflecting the relationship between the length of a side $P_i^j P_{i+1}^j$ of P^j and the length of the subpolygon of P^{j+1} obtained applying the subdivision scheme to $P_i^j P_{i+1}^j$.

Based on this result, it becomes a natural idea to try to extend this scheme to a curved surface S in the most intrinsic way, i.e., using the geometry of the surface S .

The problem of designing curves on smooth manifolds has been addressed

in several works [12, 14, 13], and also on triangulations [8]. Some general frameworks for linear subdivision on smooth and discrete manifolds have been defined [17, 15, 16, 9] in the last years.

In this work, we take another way. We translate the geometric ideas from [6] to the geometry of the manifold. Doing so, we are able to prove some properties of the new curves, which are defined on the manifold, in a way similar to [6]. In the interpolatory scheme proposed in this paper, the new points do not necessarily lie on the cubic polynomial (1). Instead, starting from a parametrization t^0 of the original polygon P^0 computed from the arc-lengths ρ_i^0 of the *geodesic curves* on S joining two consecutive vertices P_i^0 and P_{i+1}^0 , we control the length of the j -th subdivision polygons P^j in such a way that, after k subdivision steps applied to the side $P_i^j P_{i+1}^j$ of polygon P^j , the length of the obtained polygon tends to be proportional, with *the same proportionality factor* for all i , to the length of the parameter interval $t_{i+1}^j - t_i^j$ corresponding to the parametrization t^j assigned to P^j . Furthermore, the limit curve is continuous and it is parametrized by a multiple of the arc-length.

A bound for the Hausdorff distance between the limit curve and the initial polygon is also obtained.

The proposed scheme is nonlinear, hence we cannot study its properties through the Laurent polynomials formalism [3]. Instead, we rely on analytical and geometric arguments that are particular to this type of schemes.

2 The subdivision scheme

2.1 General definitions

Let $P^0 = \{P_i^0, i \in Z\}$ be an initial polygon with vertices on a given surface S , where three consecutive vertices are always noncollinear. The equations giving the polygon at step $j + 1$ can be written as

$$P_{2i}^{j+1} = P_i^j, \quad P_{2i+1}^{j+1} = g_i^j(P_{i+1}^j, P_i^j, \alpha^j) \quad (5)$$

where P_{2i+1}^{j+1} is a point on S and the control parameters $\alpha^j > 1$ satisfy the condition

$$\alpha := \prod_j \alpha^j < \infty \quad (6)$$

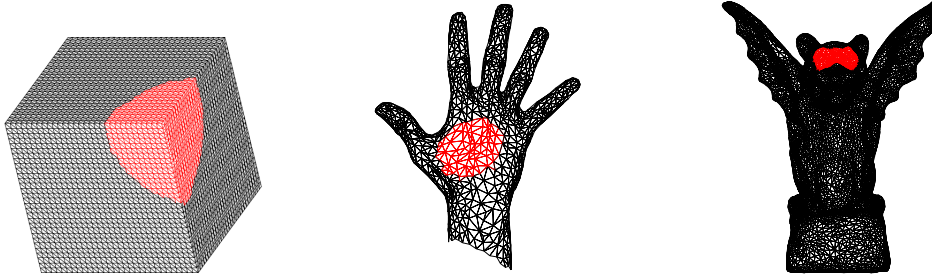


Figure 3: Some *geodesic ellipses*.

For any pair of points $Q_1, Q_2 \in S$ denote by $d_g(Q_1, Q_2)$ the arc-length of the *geodesic* curve on S (we use the term *geodesic* meaning the *locally shortest curve* or *shortest geodesic*, if S is a triangulation, see [11]) with initial point Q_1 and final point Q_2 . In particular, for two consecutive vertices of polygon P^k , $P_{r+1}^k, P_r^k \in S$, we denote $d_g(P_r^k, P_{r+1}^k)$ by ρ_r^k . The new vertex $P_{2i+1}^{j+1} = g_i^j(P_{i+1}^j, P_i^j, \alpha^j)$ is computed in such a way, that for a given parameter $\alpha^j > 1$

$$\rho_{2i}^{j+1} + \rho_{2i+1}^{j+1} = \alpha^j \rho_i^j \quad (7)$$

Condition (7) means that the new point P_{2i+1}^{j+1} is in the set

$$E_g := \{Q \in S / d_g(Q, P_i^j) + d_g(Q, P_{i+1}^j) = \alpha^j \rho_i^j\} \quad (8)$$

due its similarity with the classical definition of an ellipse, we will call this set the *geodesic ellipse* on S with foci P_i^j, P_{i+1}^j and eccentricity $1/\alpha^j$.

2.2 Convergence

To study the convergence of the subdivision scheme, we first define the parametric values corresponding to each point on the subdivision polygon.

For the initial a non-uniform parametrization t^0 of polygon P^0 , we set:

$$t_0^0 = 0, \quad t_{i+1}^0 = t_i^0 + \rho_i^0 \quad (9)$$

For the parametrization t^j of polygon P^j , we keep the parameters of the even indices at level $j+1$ the same as at level j , and set the new parameter t_{2i+1}^{j+1} in the interval $[t_i^j, t_{i+1}^j]$ in such a way that

$$\frac{\rho_{2i}^{j+1}}{t_{2i+1}^{j+1} - t_i^j} = \frac{\rho_{2i+1}^{j+1}}{t_{i+1}^j - t_{2i+1}^{j+1}}$$

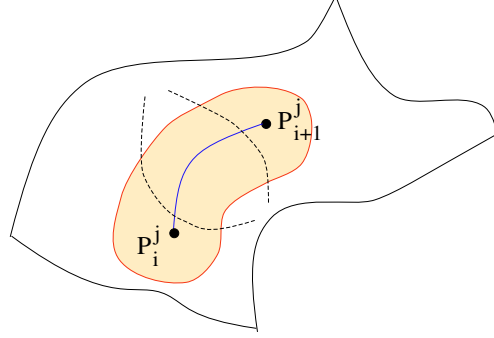


Figure 4: Conditions for convergence.

That is,

$$t_{2i}^{j+1} = t_i^j, \quad t_{2i+1}^{j+1} = \delta_i^j t_i^j + (1 - \delta_i^j) t_{i+1}^j \quad (10)$$

with $\delta_i^j = \frac{\rho_{2i+1}^{j+1}}{\rho_{2i}^{j+1} + \rho_{2i+1}^{j+1}}$.

Theorem 1 Consider the subdivision scheme (5)-(7) using the parametrization (9)-(10). If the new points P_{2i+1}^{j+1} are selected in such a way that, for all i, j

$$\rho_k^{j+1} \leq \Gamma \rho_i^j \quad \text{for } k = 2i, 2i + 1 \quad \text{with } \Gamma < 1 \quad (11)$$

then the subdivision scheme converges and the limit curve $c(t)$ is continuous.

Proof. Let $P^j(t)$ be the piecewise linear function interpolating (t_i^j, P_i^j) . We will show that $\|P^j(t) - P^{j+1}(t)\|_\infty$ tends uniformly to 0 when $j \rightarrow \infty$. We have

$$\begin{aligned} \|P^j(t) - P^{j+1}(t)\|_\infty &= \\ \max_i \max_{t_i^j \leq t \leq t_{i+1}^j} \|P^j(t) - P^{j+1}(t)\| &= \\ \max_i \|P^j(t_{2i+1}^{j+1}) - P^{j+1}(t_{2i+1}^{j+1})\| & \end{aligned} \quad (12)$$

Since $P^j(t)$ is linear in $[t_i^j, t_{i+1}^j]$ and t_{2i+1}^{j+1} is given by (10) we obtain,

$$P^j(t_{2i+1}^{j+1}) = \delta_i^j P_i^j + (1 - \delta_i^j) P_{i+1}^j \quad (13)$$

Substituting (13) in (12), using the value of δ_i^j in (10), and recalling that $P^{j+1}(t_{2i+1}^{j+1}) = P_{2i+1}^{j+1}$, we obtain

$$\begin{aligned}
\|P^j - P^{j+1}\|_\infty &= \max_i \|P_{2i+1}^{j+1} - (\delta_i^j P_i^j + (1 - \delta_i^j) P_{i+1}^j)\| = \\
&\max_i \|\delta_i^j (P_{2i+1}^{j+1} - P_i^j) + (1 - \delta_i^j) (P_{2i+1}^{j+1} - P_{i+1}^j)\| \leq \\
&\max_i \{\delta_i^j \|P_{2i+1}^{j+1} - P_i^j\| + (1 - \delta_i^j) \|P_{2i+1}^{j+1} - P_{i+1}^j\|\} \leq \\
&\max_i \{\delta_i^j \rho_{2i}^{j+1} + (1 - \delta_i^j) \rho_{2i+1}^{j+1}\} \leq \\
&2 \max_i \left\{ \frac{\rho_{2i+1}^{j+1} \rho_{2i}^{j+1}}{\alpha^j \rho_i^j} \right\} \tag{14}
\end{aligned}$$

since $P_r^k = P_{2r}^{k+1}$ and $\|P_{r+1}^k - P_r^k\|$, the euclidean distance from P_r^k to P_{r+1}^k , is smaller than the geodesic distance (on S) ρ_r^k from P_r^k to P_{r+1}^k . Using (7) and the arithmetic-geometric mean inequality, we get

$$2 \frac{\rho_{2i+1}^{j+1} \rho_{2i}^{j+1}}{\alpha^j \rho_i^j} = 2 \frac{\rho_{2i+1}^{j+1} \rho_{2i}^{j+1}}{\rho_{2i+1}^{j+1} + \rho_{2i}^{j+1}} \leq \frac{\rho_{2i+1}^{j+1} + \rho_{2i}^{j+1}}{2} = \frac{\alpha^j \rho_i^j}{2}$$

Therefore, from (14) we obtain,

$$\|P^j(t) - P^{j+1}(t)\|_\infty \leq \frac{\alpha^j}{2} \max_i \{\rho_i^j\}$$

Assuming (11), we get

$$\begin{aligned}
\|P^j(t) - P^{j+1}(t)\|_\infty &\leq \Gamma \left(\frac{\alpha^j}{2} \max_i \{\rho_i^{j-1}\} \right) \leq \Gamma^2 \left(\frac{\alpha^j}{2} \max_i \{\rho_i^{j-2}\} \right) \\
&\leq \dots \leq \Gamma^j \left(\frac{\alpha^j}{2} \max_i \{\rho_i^0\} \right)
\end{aligned}$$

Observe that $\Gamma < 1$ and that (11) implies $\alpha^j \leq 2$, therefore passing to the limit we obtain, $\lim_{j \rightarrow \infty} \|P^j - P^{j+1}\|_\infty = 0$. The last expression means that the sequence $\{P^j(t)\}$ is a Cauchy sequence in the sup norm and in consequence it *converges*. Since we have proved that $\{P^j(t)\}$ converges uniformly, the limit function $c(t)$ has to be *continuous*.

Remark 1 Notice that (11) is sufficient but not necessary condition. In particular, if the hypothesis holds only after a certain step j_0 , the scheme still converges to a continuous curve as we can see by applying the same proof to the polygon P^{j_0} .

3 Properties of the limit curve

3.1 Distance from the subdivision curve to the polygon

In this section a bound for the Hausdorff distance between the limit curve and the initial polygon P^0 is obtained.

Lemma 1 *Any vertex P_k^j of the j -th subdivision of the edge $P_i^0 P_{i+1}^0$ belongs to the set*

$$P_k^j \in \{Q \in S / d_g(Q, P_i^0) + d_g(Q, P_{i+1}^0) \leq (\alpha^0 \alpha^1 \cdots \alpha^{j-1}) \rho_i^0\} \quad (15)$$

i.e. , P_k^j is inside the geodesic ellipse with foci P_i^0, P_{i+1}^0 and eccentricity $(\alpha^0 \alpha^1 \cdots \alpha^{j-1})^{-1}$.

Proof We have $P_i^0 = P_{2^j i}^j$ and $P_{i+1}^0 = P_{2^j(i+1)}^j$. The vertices in j -th step corresponding to the edge $P_i^0 P_{i+1}^0$ are P_k^j , for $k = 2^j i, \dots, 2^j(i+1)$ and

$$\begin{aligned} d_g(P_i^0, P_k^j) + d_g(P_k^j, P_{i+1}^0) &= \\ d_g(P_{2^j i}^j, P_k^j) + d_g(P_k^j, P_{2^j(i+1)}^j) &\leq \\ \sum_{l=2^j i}^{k-1} \rho_l^j + \sum_{l=k}^{2^j(i+1)-1} \rho_l^j &= \\ \sum_{l=2^j i}^{2^j(i+1)-1} \rho_l^j &= \\ \alpha^{j-1} \sum_{l=2^{j-1} i}^{2^{j-1}(i+1)-1} \rho_l^{j-1} &= \\ \dots & \\ = \alpha^{j-1} \alpha^{j-2} \cdots \alpha^0 \rho_i^0 & \end{aligned}$$

Hence, the sum of the geodesic distances from P_k^j to $P_l^0, l = i, i+1$ is smaller or equal to $\alpha^{j-1} \alpha^{j-2} \cdots \alpha^0$ times the geodesic distance from P_i^0 to P_{i+1}^0, ρ_i^0 .

Using Lemma 1, we obtain an upper bound of the Hausdorff distance d_H between the segment of the limit curve $\{c(t), t \in [t_i^j, t_{i+1}^j]\}$ and the edge $P_i^0 P_{i+1}^0$.

Theorem 2 Let $c(t)$ be the limit curve of the subdivision scheme (5). Assume that $\alpha = \prod_{j=0}^{\infty} \alpha^j$ is finite. Then

$$d_H(\{c(t), t \in [t_i^0, t_{i+1}^0]\}, P_i^0 P_{i+1}^0) \leq \frac{\|P_{i+1}^0 - P_i^0\| \sqrt{\omega^2 - 1}}{2} \quad (16)$$

with $\omega := \frac{\alpha \rho_i^0}{\|P_{i+1}^0 - P_i^0\|}$.

Proof From Lemma 1, we know that all points P_k^j obtained at the j -th subdivision of the edge $P_i^0 P_{i+1}^0$ are contained in the *geodesic ellipse* with foci P_i^0, P_{i+1}^0 and eccentricity $(\alpha^0 \alpha^1 \cdots \alpha^{j-1})^{-1}$, given by

$$\{Q \in S / d_g(Q, P_i^0) + d_g(Q, P_{i+1}^0) = (\alpha^0 \alpha^1 \cdots \alpha^{j-1}) \rho_i^0\}$$

Let be Q a point on the curve segment $c[t_i^0, t_{i+1}^0]$, and denote by Π_Q the plane spanned by Q, P_i^0 and P_{i+1}^0 . Since the euclidean distance is smaller than the geodesic distance, then Q is in the interior of the *euclidean ellipse* on Π_Q with foci P_i^0, P_{i+1}^0 and eccentricity $1/\omega$, defined by

$$E_Q := \{R \in \Pi_Q / \|R - P_i^0\| + \|R - P_{i+1}^0\| = \alpha \rho_i^0\}$$

with $\omega := \frac{\alpha \rho_i^0}{\|P_{i+1}^0 - P_i^0\|}$.

Observe that the length of the semiminor axis of the *euclidean ellipse* E_Q is $\frac{\|P_{i+1}^0 - P_i^0\| \sqrt{\omega^2 - 1}}{2}$, while the euclidean distance from each focus to the closest intersection point between the semimajor axis of E_Q and the euclidean ellipse E_Q is $\frac{\|P_{i+1}^0 - P_i^0\|(\omega - 1)}{2}$. Since $\omega \geq 1$, the first one is bigger than the second one. Therefore, the Hausdorff distance from the section of the limit curve corresponding to the parameter interval $[t_i^0, t_{i+1}^0]$ to the edge $P_i^0 P_{i+1}^0$ is bounded above by $\frac{\|P_{i+1}^0 - P_i^0\| \sqrt{\omega^2 - 1}}{2}$.

3.2 Parametrization

In this section we prove properties of the subdivision scheme, when t^0 is the parametrization defined in (9).

Theorem 3 Consider the subdivision scheme (5)-(7) using the parametrization (9)-(10) and assume that conditions (6) and (11) hold. Denote by

$l^j(P_0^j, P^j(t))$ the length of the subdivision curve P^j in the step j between points P_0^j and $P^j(t)$. Then, for any $\varepsilon > 0$ exists j_0 , such that

$$|l^{j_0+k}(P_0^{j_0+k}, P^{j_0+k}(t)) - \frac{R(j_0)}{R(j_0+k)} t| < \varepsilon$$

holds, for all $k > 0$, where $R(k) := \prod_{j \geq k} \alpha^j$.

Proof Assume that $t_i^{j_0} < t \leq t_{i+1}^{j_0}$. Then, it is not difficult to show that using the recurrence (7) we obtain

$$l^{j_0+k}(P_0^{j_0+k}, P^{j_0+k}(t_i^{j_0})) = \frac{R(j_0)}{R(j_0+k)} t_i^{j_0}$$

Therefore, the equality

$$\begin{aligned} & |l^{j_0+k}(P_0^{j_0+k}, P^{j_0+k}(t)) - \frac{R(j_0)}{R(j_0+k)} t| = \\ & |l^{j_0+k}(P^{j_0+k}(t_i^{j_0}), P^{j_0+k}(t)) - \frac{R(j_0)}{R(j_0+k)} (t - t_i^{j_0})| \end{aligned} \quad (17)$$

holds for $t_i^{j_0} < t \leq t_{i+1}^{j_0}$.

Let be $s_i^{j_0+k}$ the sum of the geodesic distances between two consecutive vertices of the subpolygon of P^{j_0+k} that is obtained after k subdivision steps of the side $P_i^{j_0} P_{i+1}^{j_0}$ of P^{j_0} . Clearly, holds

$$l^{j_0+k}(P^{j_0+k}(t_i^{j_0}), P^{j_0+k}(t)) \leq s_i^{j_0+k} \quad (18)$$

On the other hand, if we take P^{j_0} as initial polygon and t^{j_0} as initial parametrization, we get the same limit curve $c(t)$ as when we take P^0 as initial polygon and t^0 as initial parametrization, thus after a similar argument to the one used in Lemma 1, we get that $s_i^{j_0+k}$ has the upper bound

$$s_i^{j_0+k} \leq \frac{R(j_0)}{R(j_0+k)} \rho_i^{j_0} \quad (19)$$

Therefore, substituting (18) and (19) in (17), we get the following inequalities,

$$|l^{j_0+k}(P_0^{j_0+k}, P^{j_0+k}(t)) - \frac{R(j_0)}{R(j_0+k)} t| =$$

$$\begin{aligned}
|l^{j_0+k}(P^{j_0+k}(t_i^{j_0}), P^{j_0+k}(t)) - \frac{R(j_0)}{R(j_0+k)}(t - t_i^{j_0})| &\leq \\
\frac{R(j_0)}{R(j_0+k)}(\rho_i^{j_0} - (t - t_i^{j_0})) &< \\
\frac{R(j_0)}{R(j_0+k)}\rho_i^{j_0} &
\end{aligned}$$

Recall that (6) implies that $R(j_0+k) \xrightarrow{k \rightarrow \infty} 1$. On the other hand, after Hence, if we choose j_0 sufficiently large, such that

$$\frac{R(j_0)}{R(j_0+k)} < 2 \quad \text{and} \quad \max_i \{\rho_i^{j_0}\} < \varepsilon/2$$

hold, we get the desired result.

Remark 2 *The above result means that for sufficiently large j , the piecewise linear function $P^j(t)$ interpolating (t_i^j, P_i^j) is approximately parametrized by a multiple of the arc-length. Indeed, let $c(t)$ be the limit curve and assume that conditions (6) and (11) hold. Defining $L(0, t) := \lim_{k \rightarrow \infty} l^{j_0+k}(P_0^0, P^{j_0+k}(t))$ as the arc-length of the section of $c(t)$ between points $c(0) = P_0^0$ and $c(t)$, we get that $L(0, t) \simeq R(j_0) t$ holds. Hence, $c(t)$ is parametrized approximately by a multiple of the arc-length.*

4 Conclusions

We described a subdivision scheme with control over the length of the limit curve, suitable for designing curves on surfaces.

At each subdivision step, similarly to the classic 4-point scheme, the existing vertices are retained, making the scheme interpolatory.

Despite the parametrization is not uniform, it is possible to compute a sequence of m points approximately on the subdivision curve, with approximately uniform arc-length distribution. Recall that even when these points are not exactly on the subdivision curve, a bound for the Hausdorff distance between any point of the polygon in the last step and the subdivision curve can be computed.

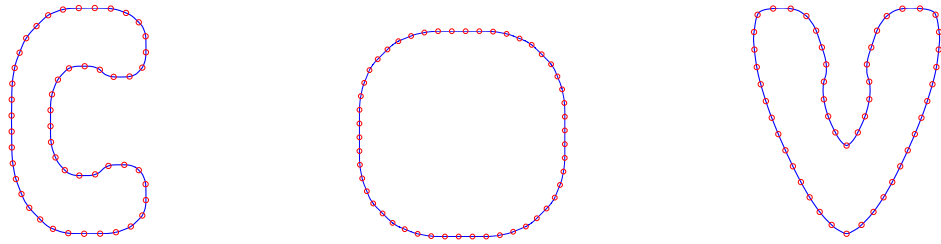


Figure 5: Sequences of points approximately on the subdivision curve, with approximately uniform arc-length distribution.

The formulation of the scheme is presented for general two dimensional manifolds. In the particular case of triangulated surfaces, we believe that the results obtained in [8, 10] will make it possible to obtain efficient implementations. That should be discussed in detail in a future work.

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