Hypersurface of sphere with two principal curvatures

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[1]

Abstract

In this paper we obtained a classification of hypersurfaces in the Euclidean sphere with two principal curvatures and nonnegative curvature.

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1 Introduction and statements of results

Let M^n be an oriented Riemannian n-manifold, $n \geq 3$, K the sectional curvature of M^n and Ric its Ricci curvature. $f : M^n \longrightarrow \mathbb{S}^{n+1}$ is a hypersurface, where $\mathbb{S}^{n+1}$ is the unity sphere. Let $\mathbb{S}^k_c$ be the sphere with constant sectional curvature c and let $\tilde{M}^n$ be the universal covering of M^n. Consider $\Lambda = \{r \in \mathbb{R} \mid \exists x \in M^n, \exists \lambda \mid \lambda(x) = r\}$, where λ is a principal curvature of f and $\Lambda^{\pm} = \Lambda \cap \mathbb{R}^{\pm}$. Note that if M^n is compact then $\Lambda = \overline{\Lambda}$ is compact.
In [S], Brian Smith proved that if $\Lambda^+ = \emptyset$ or $\Lambda^- = \emptyset$ then M^n is difeomorphic to $\mathbb{S}^n$. Moreover, if $\Lambda^{\pm} \neq \emptyset$ Smith considered $\alpha = \inf \Lambda^- \leq a = \sup \Lambda^- \leq 0 \leq b = \inf \Lambda^+ \leq \beta = \sup \Lambda^+$ and proved the following :

"*Let* $f : M^n \longrightarrow \mathbb{S}^{n+1}$, $n \geq 3$, *complete oriented such that* $\alpha\beta \geq -1$ *or* $ab \leq -1$ *and* $0 \notin \Lambda$. *Then* $\tilde{M}^n$ *is homeomorphic to* $\mathbb{S}^n$ *or* $f(M^n) = \mathbb{S}^r_{c_1} \times \mathbb{S}^{n-r}_{c_2}$."

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The condition $\alpha\beta \geq -1$ implies that M^n has sectional curvature $K \geq 0$ in M^n. On the other hand, the condition $ab \leq -1$ implies the following :

(*) *If there is $x \in M^n$ and exist eigenvalues λ, μ of f such that $\lambda(x) \in \Lambda^+$ and $\mu(x) \in \Lambda^-$, then $1 + \lambda(x)\mu(x) \leq 0$*

We could improve the result of Smith:

Theorem 1.1 - *Let $f : M^n \longrightarrow \mathbb{S}^{n+1}$, $n \geq 3$, be a complete oriented hypersurface, K the sectional curvature of M^n and $\tilde{M}^n$ be the universal covering of M^n. If $K \geq 0$ in M^n or $0 \notin \Lambda \cap \overline{\Lambda}$ and the condition (*) holds, then $\tilde{M}^n$ is homeomorphic to $\mathbb{S}^n$, $\tilde{M}^n$ is homeomorphic to $\mathbb{R}^n$, $f(M^n) = \mathbb{S}^r_{c_1} \times \mathbb{S}^{n-r}_{c_2}$ or $f(M^n) = \mathbb{R} \times \mathbb{S}^{n-1}_{c_2}$.*

If $f : M^n \longrightarrow \mathbb{S}^{n+1}$ has nonnegative Ricci curvature, we have

Theorem 1.2 - *Let $f : M^n \longrightarrow \mathbb{S}^{n+1}$, $n \geq 3$, complete oriented where M^n has Ricci curvature $Ric \geq 0$ in M^n. If M^n is compact and has infinity fundamental group then $f(M^n) = \mathbb{S}^1_{c_1} \times \mathbb{S}^{n-1}_{c_2}$. If M^n is non compact and has at least two ends then $f(M^n) = \mathbb{R} \times \mathbb{S}^{n-1}_{c_2}$.*

Let us consider hypersurfaces with two principal curvatures. In view of the Classification theorem (see [LLWZ, p. 438]), the Corollary 3.6 of [LLWZ] and the Theorem 2.2 of [HL], we have

Theorem of local classification- *Let $f : M^n \longrightarrow \mathbb{S}^{n+1}$, $n \geq 3$, be a hypersurface , where M^n is a n-dimensional Riemannian manifold. If f has two and distinct principal curvatures then M^n is difeomorphic to an open part of one the following manifolds :*

i) $\mathbb{S}^r \times \mathbb{S}^{n-r}$,

ii) $\mathbb{S}^r \times \mathbb{R}^{n-r}$,

iii) $\mathbb{S}^r \times \mathbb{H}^{n-r}$,

where $\mathbb{H}^{n-r}$ is the hyperbolic space.

The theorem of local classification is essential in the proof the following theorem :

Theorem 1.3 - *Let $f : M^n \longrightarrow \mathbb{S}^{n+1}$, $n \geq 3$, complete oriented such that f has two principal curvatures λ, μ. Let K be the sectional curvature of M^n and Ric its Ricci curvature.*

a) If M^n is compact, $K \geq 0$ in M^n and $\lambda \neq \mu$ in M^n then $f(M^n) = \mathbb{S}^r_{c_1} \times \mathbb{S}^{n-r}_{c_2}$.

b) If M^n is compact and for all $x \in M^n$ exists a two plane $P \subset T_xM$ such that $K(P) \leq 0$ then $f(M^n) = \mathbb{S}^r_{c_1} \times \mathbb{S}^{n-r}_{c_2}$.

c) If M^n is compact, Ric ≥ 0 in M^n and λ and μ has multiplicity 1 and $n-1$, respectively then $f(M^n) = \mathbb{S}^1_{c_1} \times \mathbb{S}^{n-1}_{c_2}$.

d) If Ric ≥ 0 in M^n and for all $x \in M^n$ exists $v \in T_xM$, $\mid v \mid = 1$, such that Ric$(v) = 0$ then $f(M^n) = \mathbb{S}^1_{c_1} \times \mathbb{S}^{n-1}_{c_2}$ or $f(M^n) = \mathbb{R} \times \mathbb{S}^{n-1}_{c_2}$.

e) If Ric ≥ 0, f has constant m^{th} mean curvature H_m and λ and μ has multiplicities 1 and $n-1$, then $f(M^n) = \mathbb{S}^1_{c_1} \times \mathbb{S}^{n-1}_{c_2}$ or $f(M^n) = \mathbb{R} \times \mathbb{S}^{n-1}_{c_2}$.

Corollary 1.4 - *It does not exists compact hypersurface $f : M^n \longrightarrow \mathbb{S}^{n+1}$, $n \geq 3$, with only two principal curvatures in each point of M^n such that M^n has scalar curvature $\tau \leq 0$ in M^n.*

Remark 1.5

a) Corollary 1.4 fails if f has more that two principal curvatures. In fact, the Cartan hypersurface in $\mathbb{S}^4$ has tree principal curvatures and has constant scalar curvature $\tau = 0$.

b) If f has constant m^{th} mean curvature H_m, and f has two principal curvatures of multiplicity $r > 1$ and $n - r > 1$ it easy proved that $f(M^n) = \mathbb{S}^r_{c_1} \times \mathbb{S}^{n-r}_{c_2}$.

c) The theorems 1.2, 1.3 and 1.4 of Guoxin Wei [W] are consequences of Theorem 1.3(a). In fact, the condition (1.1) of [W] implies that the Gauss map of f is a hypersurface with nonnegative sectional curvature and two distinct principal curvatures. So, Theorem 1.2 of [W] follows from Theorem 1.3(a). Similarly, the condition (1.2) of [W] and the condition of Theorem 1.4 of [W] implies that M^n has nonnegative Ricci curvature.

d) Theorem 1.3 of Q. Wang and C. Xia [WX] and the Theorem 1.4 of G. Wei and Y. Jin Suh [SW] are consequences from Theorem 1.3(a). In fact, if $f : M^n \longrightarrow \mathbb{S}^{n+1}$ has only two principal curvatures λ and μ of multiplicty 1 and $n-1$ such that $\lambda\mu \leq -1$ then the Gauss map of f is a hypersurface with nonnegative sectional curvature and two distinct principal curvatures. So, Theorem 1.3 of [WX] and Theorem 1.4 of [SW] follow from Theorem 1.3(a).

e) Theorem 1.1 of Q. Wang and C. Xia [WX1] follows from Theorem 1.3 (c). In fact, the condition (1.1) of [WX1] implies that M^n has nonnegative Ricci curvature.

We could improve some previous results if will consider hypersurfaces of spheres with nonnegative isotropic curvature (see [MM]).

Theorem 1.6 - *Let $f : M^n \longrightarrow \mathbb{S}^{n+1}$, $n \geq 4$, be a compact oriented hypersurface with nonnegative isotropic curvature. Then we have*

a) If M^n is reducible then $f(M^n) = \mathbb{S}^r_{c_1} \times \mathbb{S}^{n-r}_{c_2}$.

b) If M^n is irreducible then universal covering $\tilde{M}^n$ is homeomorphic to $\mathbb{S}^n$ or M^n has infinite fundamental group and the Betti numbers of M^n are $b_i(M^n, \mathbb{Z}) = 0$, for $i = 2, ..., n-2$.

c) If f has only two principal curvatures of multiplicity $r > 1$ and $n - r > 1$ then $f(M^n) = \mathbb{S}^r_{c_1} \times \mathbb{S}^{n-r}_{c_2}$.

Proof of Theorem 1.1-

If M^n is compact and has sectional curvature $K \geq 0$, the Theorem 1.1 is essentially the theorem B of [C]. Consider now M^n complete such $0 \notin \Lambda \cap \overline{\Lambda}$ and the condition (*) holds. Then the Gauss map $N : M^n \longrightarrow \mathbb{S}^{n+1}$ admits a complete metric (see [S]). Moreover, N is hypersurface with principal curvatures $1/\lambda_i$, i=1,...n, where the λ_i are the principal curvatures of f. Let

$i \neq j$ and $x \in M^n$. Then we have only the possibilities :

i) $\lambda_i(x) > 0$ and $\lambda_j(x) > 0$,

ii) $\lambda_i(x) < 0$ and $\lambda_j(x) < 0$,

iii) $\lambda_i(x) < 0$ and $\lambda_j(x) > 0$.

So, by the condition (*) we have that $\frac{1}{\lambda_i(x)\lambda_j(x)} + 1 \geq 0$ an this proof that N has nonnegative sectional curvature. In this case, Theorem 1.1 reduces to previous case.

Proof of Theorem 1.2-

If M^n is compact and has infinite fundamental group, then the Theorem 1.2 follows from Theorem 1 of [BBCL]. Suppose now that M^n is complete non compact with nonnegative Ricci curvature. By the splitting theorem of Cheeger-Gromoll, $M^n = \mathbb{R} \times M_1^{n-1}$. So, M^n is reducible and by the Lemma 1.2 of [C] we have that f has two principal curvatures λ and μ of multiplicity r and $n-r$ such that $\lambda\mu = -1$. Let A the Weingarten operator of f and consider the distributions $D_\lambda = \{X : AX = \lambda X\}$ and $D_\mu = \{X : AX = \mu X\}$. Suppose that r= dim $D_\lambda > 1$ and $n - r =$ dim $D_\mu > 1$. Since that $\lambda\mu = -1$, by Codazzi equation we have that λ and μ are constants and $f(M^n) = \mathbb{S}^r_{c_1} \times \mathbb{S}^{n-r}_{c_2}$, which contradicts that $M^n = \mathbb{R} \times M_1^{n-1}$. So, r= dim $D_\lambda = 1$ and $n - 1 =$ dim $D_\mu > 1$. Using the condition $\lambda\mu = -1$ and the Codazzi equation, we have that $< \nabla_X^X, Y >= 0$, for $X \in D_\lambda$ and $Y \in D_\mu$ and this proves that the orthogonal distribution $D_\mu^\perp = D_\lambda$ is totally geodesic. Since that M^n has nonnegative Ricci curvature, then it follows from Corollary 2 of [BW] that the distribution D_μ is totally geodesic. Again, using the Codazzi equation, we have that $X(\mu) = 0$ for all $X \in D_\lambda$, λ and μ are constants and so $f(M^n) = \mathbb{R} \times \mathbb{S}^{n-1}_{c_2}$.

Proof of Theorem 1.3

a) Let M^n compact with $K \geq 0$. Since that M^n is compact, by the Theorem of local classification (see Introduction) we deduce that M^n is difeomorphic to $\mathbb{S}^r \times \mathbb{S}^{n-r}$. Using the theorem 1.1, we have that $f(M^n) = \mathbb{S}^r \times \mathbb{S}^{n-r}$.

b) Let M^n compact such that $\forall x \in M^n$, exist a two-plane $P \subset T_xM$ with $K(P) \leq 0$. Let $x \in M^n$, λ and μ the principal curvatures of f such that $\lambda(x) = \lambda$ and $\mu(x) = \mu$, where λ and μ has multiplicity r and $n-r$. The principal sectional curvatures (in x) are λ^2+1, $\lambda\mu+1 > 0$ and $\mu^2+1 > 0$. If $\lambda\mu+1 > 0$, then M^n has positive sectional curvature in x (contradiction). So, $\lambda\mu+1 \leq 0$ and holds the condition (*). Moreover $\lambda \neq 0$, $\mu \neq 0$ and $\lambda \neq \mu$. Then the Gauss map $N : M^n \longrightarrow \mathbb{S}^{n+1}$ is compact and has nonnegative sectional curvature. By the same arguments of proof of (a) we have that $f(M^n) = \mathbb{S}^r \times \mathbb{S}^{n-r}$.

c) Let M^n compact with $Ric \geq 0$ such that f has two principal curvatures of multiplicity r and n-r. Is easy see that M^n has nonnegative sectional curvature. Using (a) we have that $f(M^n) = \mathbb{S}^1 \times \mathbb{S}^{n-1}$.

d) Let M^n complete with $Ric \geq 0$. Consider λ and μ principal curvatures of f, of multiplicity r and $n-r$, respectively. The principal Ricci curvatures are

$(\lambda^2+1)(r-1) + (\lambda\mu+1)(n-r) \geq 0$ and

$(\mu^2+1)(n-r-1) + (\lambda\mu+1)r \geq 0$.

Since that $\forall x \in M^n$ $\exists v, \mid v \mid = 1$ with $Ric(v) = 0$, then

$(\lambda^2+1)(r-1) + (\lambda\mu+1)(n-r) = 0$ [1.1]

or

$(\mu^2+1)(n-r-1) + (\lambda\mu+1)r = 0$ [1.2].

Let $r > 1$ and $(n-r) > 1$. Note that $\lambda \neq 0$, $\mu \neq 0$ and $\lambda \neq \mu$ in M^n and consider the sets

M_1= int $\{x \in M^n \mid [1.1]\text{holds}\}$ and M_2= int $\{x \in M^n \mid [1.2]\text{holds}\}$.

Let A the Weingarten operator of f, $X \in D_\lambda = \{X \in TM \mid AX = \lambda X\}$ and $Y \in D_\mu = \{Y \in TM \mid AY = \mu Y\}$. Note that $X(\lambda) = 0$, $Y(\mu) = 0$. Moreover, follows from [1.1] and [1.2] that $X(\mu) = Y(\lambda) = 0$ in $M_1 \cup M_2$.

Since that $M^n = M_1 \cup M_2 \cup M_3$, where $\mathrm{int}(M_3) = \emptyset$, by continuity we have that λ and μ are constants in M^n and in this case, $f(M^n) = \mathbb{S}^r_{c_1} \times \mathbb{S}^{n-r}_{c_2}$, where $r > 1$ and $n - r > 1$, which contradicts the fact of that $Ric(v) = 0$. So, $r = 1$ or $n-r = 1$ and $1+\lambda\mu = 0$. Using the same arguments of the proof of Theorem 1.2 we deduce that $f(M^n) = \mathbb{S}_{c_1} \times \mathbb{S}^{n-1}_{c_2}$ or $f(M^n) = \mathbb{R} \times \mathbb{S}^{n-1}_{c_2}$.

e) Assumes that f has two principal curvatures λ and μ of multiplicities 1 and $n-1$ and has constant mean curvature. The proof in the general case where f has constant m^{th} mean curvature H_m is similar. Let M^n with $Ric \geq 0$ such that $\lambda + (n-1) = nH$ is constant. Using a similar argument of the proof of Theorem 1.3(b), we obtain that $f(M^n) = \mathbb{S}_{c_1} \times \mathbb{S}^{n-1}_{c_2}$ or $f(M^n) = \mathbb{R} \times \mathbb{S}^{n-1}_{c_2}$.

Proof of Corollary 1.4 - Assumes that M^n has scalar curvature $\tau \leq 0$. Then $\forall x \in M^n$ exists a two plane $P \subset T_xM$ such that $K(P) \leq 0$ and by Theorem 1.3(b), $f(M^n) = \mathbb{S}^r_{c_1} \times \mathbb{S}^{n-r}_{c_2}$ which has M^n has constant scalar curvature $\tau > 0$ (contradiction).

Proof of Theorem 1.6-

a) Let M^n reducible. By Lemma 1.2 of [C] f has two principal curvatures λ and μ such that $\lambda\mu = -1$. As in the proof of Theorem 1.2 we can deduce that $f(M^n) = \mathbb{S}^r \times \mathbb{S}^{n-r}$.

b) Let M^n irreducible. By [DN], we have that the Betti numbers of M^n are $b_i(M^n, \mathbb{Z}) = 0$ for $2 \leq n \leq n-2$. Using Theorem 1.1 of [H], we have the possibilities

b_1) M^n is homeomorphic to CP^n. Since that $b_2(\mathbb{CP}^n, \mathbb{Z}) = 1$, this case can't occur.

b_2) $\tilde{M}^n$ is a compact irreducible symmetric space. In this case, follows from Proposition 4.2 of [DMN] that $\tilde{M} = \mathbb{S}_c$.

b_3) M^n admits a metric with positive isotropic curvature. Then M^n has finite fundamental group and follows from [MM] that $\tilde{M}^n$ is homeomorphic to $\mathbb{S}^n$ or M^n has infinite fundamental group.

c) Assumes that f has two principal curvatures with multiplicity $r > 1$

and $n - r > 1$. Since that M^n has nonnegative isotropic curvature is easy see that M^n has nonnegative sectional curvature. By Theorem 1.3(a) $f(M^n) = \mathbb{S}^r_{c_1} \times \mathbb{S}^{n-r}_{c_2}$.

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