MIXING RATE FOR SEMI-DISPERSING BILLIARDS WITH NON-COMPACT CUSPS

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Abstract. Since the seminal work of Sinai one studies chaotic properties of planar billiards tables. Among them is the study of decay of correlations for these tables. There are examples in the literature of tables with exponential and even polynomial decay.

However, until now nothing is known about mixing properties for billiard tables with non-compact cusps. There is no consensual definition of mixing for systems with infinite invariant measure. In this paper we study geometric and ergodic properties of billiard tables with a non-compact cusp. The goal of this text is, using the definition of mixing proposed by Krengel and Sucheston for systems with invariant infinite measure, to show that the billiard whose table is constituted by the x-axis and and the portion in the plane below the graph of $f(x) = \frac{1}{x+1}$ is mixing and the speed of mixing is polynomial.

1. Introduction

The *planar billiard* is the dynamical system defined by the free motion of a particle in the interior of a domain $\mathcal{D} \subset I\!R^2$ (usually called *table*) subjected to elastic collisions to the boundary of $\mathcal{D}$, that is, angle of incidence equals angle of reflexion. In a seminal work, Sinai [22] proved that the billiard map of a system in a two-dimensional torus with finitely many convex obstacles is a K-automorphism.

For billiards with non-compact cusps, that generate a dynamical system with an infinite invariant measure, in [15] Lenci proved an extension of the results of Katok and Strelcyn [11] for the infinite measure case and, as an application, he showed that certain tables with non-compact cusps have hyperbolic structure, that is, existence of absolutely continuous local stable and unstable manifolds. Furthermore, adapting arguments contained in [17], Lenci proved that these billiards maps are ergodic.

About the finite measure case, in [2], Bunimovich and Sinai proved a "stretched" exponential decay of correlations for dispersing billiards. Young [23] showed that

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the decay of correlations is actually exponential. This later result was extended by Chernov [3] for billiards with positive-angle corners.

In [18], Markarian, based on [24], showed that billiards in the Bunimovich stadium has polynomial decay of correlations. More recently, Chernov and Markarian [6] proved that semi-dispersing billiard tables with compact cusps also have polynomial decay of correlations. Improved estimates for correlations in different types of billiard tables were also proved by Chernov and Zhang [7].

We are interested in tables of the form $\mathcal{D} = \{(x, y) \in I\!R^2 : x \geq 0, 0 \leq y \leq f(x)\}$, where $f : I\!R_0^+ \to I\!R^+$ is a three times differentiable bounded convex function, satisfying the hypotheses (H1) to (H5) listed in Section 2.

Theorem A. *The billiard map defined in a table $\mathcal{D}$ with a non-compact cusp is an infinite K-automorphism.*

Following Krengel and Sucheston [13], we say that an endomorphism $\mathcal{F}$ on a σ-finite infinite measure space $(X, \mathcal{B}, \mu)$ is *F-mixing* if for all measurable set A with $\mu(A) < \infty$,

$$\mu(\mathcal{F}^n A \cap A) \to 0 \quad \text{as} \quad n \to \infty.$$

As it was commented before, there is no consensual definition of mixing for systems with infinite measure. A discussion on different definitions of mixing for systems with infinite measure was recently done by Lenci [16].

Parry [19] showed that *"an infinite K-automorphism has countable Lebesgue spectrum"* and in [13], Krengel and Sucheston showed that *"if an endomorphism has countable Lebesgue spectrum then this endomorphism is F-mixing"*. Therefore we get

Corollary 1.1. *The billiard map defined in a table $\mathcal{D}$ with a non-compact cusp is F-mixing.*

For a conservative endomorphism $\mathcal{F}$ on a σ-finite measure space $(X, \mathcal{B}, \mu)$, we define its *entropy* [12] by

$$h(\mathcal{F}) = \sup\{h(\mathcal{F}_E, \mu_E) \mid E \subset X, 0 < \mu(E) < \infty\}.$$

In [12, p. 172], Krengel showed that *"every conservative K-automorphism on a σ-finite measure space has positive entropy"*. Hence

Corollary 1.2. *The entropy of the billiard map defined in a table $\mathcal{D}$ with a non-compact cusp is positive.*

Furthermore we can study the speed of convergence to zero in this definition of F-mixing. We say that an endomorphism $\mathcal{F}$ is *polynomially F-mixing* if

$$\mu(\mathcal{F}^{-n} A \cap A) \geq C \frac{1}{n^\alpha},$$

for some "good" (e.g., with piecewise differentiable boundary) set A with $0 < \mu(A) < \infty$ and some $\alpha > 0$. The constant C depends on A but the exponent α depends only on $\mathcal{F}$.

Using $f(x) = (x+1)^{-1}$ in the definition of the table $\mathcal{D}$ we show

Theorem B. *The billiard map defined in a table $\mathcal{D}$ with a non-compact cusp is polynomially F-mixing.*

2. Definition of the dynamical system

As mentioned in the previous section, we are interested in tables of the form $\mathcal{D} = \{(x,y) \in I\!R^2 : x \geq 0, 0 \leq y \leq f(x)\}$, where $f : I\!R_0^+ \to I\!R^+$ is a three times differentiable bounded convex function.

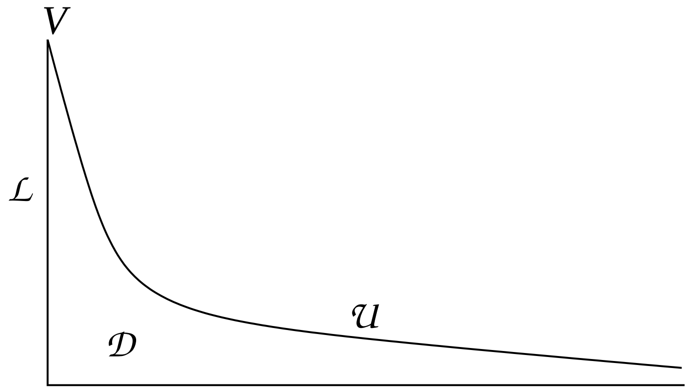


Figure 1. Introducing $\mathcal{D}$, $\mathcal{U}$, $\mathcal{L}$ e V.

We denote by $\mathcal{U}$ the dispersing part of the table $\mathcal{D}$ and by $\mathcal{L}$ the leftmost vertical wall in $\mathcal{D}$. The angle in the vertex $V = (0, f(0))$ is $\pi/2 + \arctan f'(0^+)$ and it can be zero. So the billiard table might have a compact cusp besides the non-compact one on $x = +\infty$.

We present two other tables, that will be used in the definitions below:

$$\mathcal{D}_2 = \{(x,y) \in I\!R_0^+ \times I\!R : |y| \leq f(x)\},$$
$$\mathcal{D}_4 = \{(x,y) \in I\!R \times I\!R : |y| \leq f(|x|)\}.$$

For $f, g : I\!R_0^+ \to I\!R^+$ we use the following notations: $f(x) << g(x)$ indicates that there exists a constant C such that $f(x) \leq Cg(x)$, as $x \to \infty$, analogously for the symbol $>>$ and we denote by $f = o(g)$ if $\frac{f(x)}{g(x)}$ tends to zero, as $x \to \infty$. Moreover, we use the same symbols when $x \to 0$, if there is no ambiguity. Also, we indicate by $A \asymp B$ if there exists a constant $C > 0$ such that $C^{-1} < A/B < C$ and we write $A = O(B)$ if there exists a constant $C > 0$ such that $|A|/B < C$.

Define $x_t = x_t(x)$, for each x on $\mathcal{D}_2$, implicitly by

$$\frac{f(x) + f(x_t)}{x - x_t} = -f'(x_t).$$

One can see that x_t is the x-coordinate of the tangent point on $\mathcal{U}$.

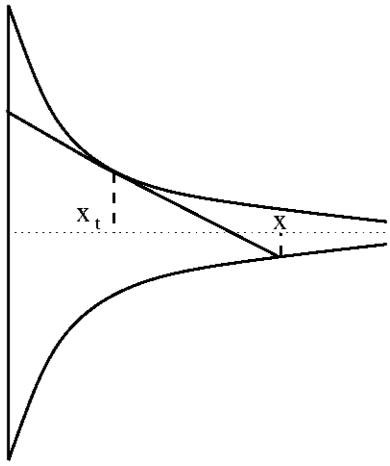


FIGURE 2. The point x_t.

In [15], Lenci studied tables with $f : I\!R_0^+ \to I\!R^+$ satisfying the following assumptions

(H1) $f''(x) \to 0$ as $x \to +\infty$;
(H2) $|f'(x_t)| << |f'(x)|$;
(H3) $\dfrac{f(x)f''(x)}{(f'(x))^2} >> 1$;
(H4) $\dfrac{|f'''(x)|}{f''(x)} << 1$;
(H5) $|f'(x)| >> (f(x))^\theta$, for some $\theta > 0$.

It is not difficult to see that $f(x) = \dfrac{1}{x+1}$ satisfies the conditions above.

Following [15], choosing as cross-section the rebounds against the dispersing part $\mathcal{U}$. we parametrize these line elements as $z = (r, \varphi)$, $r \in (-\infty, 0]$ is the arc length variable along $\mathcal{U}$ (with $r = 0$ for the vertex V) and $\varphi \in [-\pi/2, \pi/2]$ is the angle between the velocity vector and the normal at the point of collision, as in Figure 3. We define the manifold $M = (-\infty, 0) \times (-\pi/2, \pi/2)$ and the return map T defined on M, preserving the measure $d\mu = \cos\varphi dr d\varphi$.

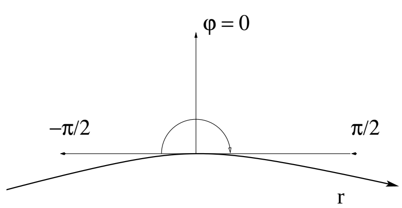


FIGURE 3. the choice of orientation for r and φ

We do not define T on those points that hit tangentially $\mathcal{U}$ or that would end up in the vertex V. That is, we exclude $T^{-1}\partial M$. These points make up the singularity set of T, denoted by S. This set consists of two lines (see [15, p.138]) $S^+ = S^{1+} \cup S^{2+}$ (as shown in Figure 4). The curve S^{1+} corresponds to tangencies on $\partial\mathcal{D}_4$ in the third quadrant (on $\mathcal{D}$, tangencies on $\mathcal{U}$, after a rebound on the vertical side); this curve is as regular as f. As for S^{2+}, its first part corresponds to line elements pointing to V (on $\mathcal{D}$, after a rebound on the horizontal side); as r decreases, these become tangencies on $\partial\mathcal{D}$. the boundary between these two behaviors is the only non-regular point of S^{2+}.

Analogously we define $S^- = S^{1-} \cup S^{2-}$, where $S^{i-}, i = 1, 2$ are the singularity lines of T^{-1}, obtained from S^{i+} using the time-reversal operator $(r, \varphi) \mapsto (r, -\varphi)$. We denote by $S^{\pm}_n = \bigcup_{i=0}^{n} T^{\mp i} S^{\pm}$ e $S^{\pm}_\infty = \lim_{n\to\infty} S^{\pm}_n$.

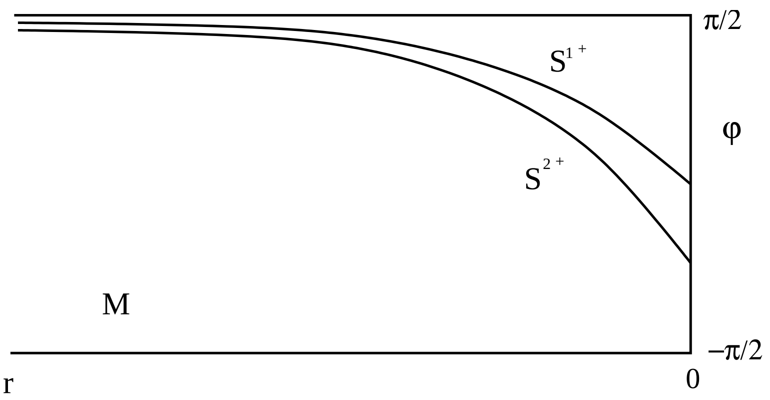


FIGURE 4. Singularity lines.

On TM we define the cone bundles [15, Section 4],

$$\mathcal{C}^u(z) = \{(dr, d\varphi) \in T_z M : dr d\varphi \geq 0\}$$
$$\mathcal{C}^s(z) = \{(dr, d\varphi) \in T_z M : dr d\varphi \leq 0\},$$

which will be denoted *unstable* and *stable* cones, respectively. They are strictly invariant under the action of T.

Remark 2.1. We note that our choice of parametrization is different from the one in [15]. This leads to a different choice of the cone bundles. However, it does not alter the results obtained in that paper.

Let $\mathcal{L}$ be the leftmost wall on $\mathcal{D}$ and M_4 the phase space defined by the vectors based on $\mathcal{L}$. Since $\mathcal{L}$ is a global cross-section we can define a return map T_4 and let μ_{M_4} be the measure μ induced on M_4. Denote by M_3 the region of M located above S^{2+}. From the definition of S^{2+}, the line elements of M_3 are precisely the ones that,on $\mathcal{D}_2$, hits y-axis. We call T_3 the return map to M_3 and one can see that (M_3, T_3, μ_{M_3}) is isomorphic (with respect to μ) to (M_4, T_4, μ_{M_4}).

Lenci [15] showed that the billiard map T has a hyperbolic structure, i.e., existence of local stable and unstable manifolds almost everywhere and these local foliations are absolutely continuous with respect to the invariant measure [15, Theorem 6.2,Theorem 7.5] and adapting the formulation of Liverani and Wojtkowski [17] to the infinite measure case, he proved a local ergodicity property [15, Theorem 8.5] and as consequences its global ergodicity [15, Theorem 8.5]. The definition of ergodicity used in these two results is that the Birkhoff average are constant almost everywhere for all integrable functions, a weaker definition than the usual one, for systems with invariant finite measure. However Lenci also proved that (M_3, T_3, μ) is ergodic [15, Proposition 8.11], which implies that the billiard map is ergodic in the sense that invariant sets are measurably indecomposable.

Remark 2.2. It is not difficult to see that T_3^n is ergodic, for all positive integer n. Indeed, it is just repeated the argument in the proof of [15, Proposition 8.11] .

Next we introduce auxiliary first return maps associated with the billiard system we are considering. Recall that we already defined $T_4 : M_4 \to M_4$ that corresponds for the bouncing at the vertical wall $\mathcal{L}$ and $T : M \to M$ corresponding to the bouncing at the dispersing part $\mathcal{U}$ of $\mathcal{D}$. We point out that a priori, the bouncing at the dispersing part $\mathcal{U}$ contains the most of the chaotic behavior, but this gives a system with an infinity measure, one of the major difficulties in analyzing this billiard system. To bypass this difficulty, we consider as well the bouncing at the vertical wall, that gives a finite measure system. Thus we set $M_5 = M \cup M_4$, that consider the bouncing at the dispersing part and the bouncing at the vertical wall and denote $T_5 : M_5 \to M_5$ the return map to M_5. Note that by construction M_5 comes from bouncing at a global cross section to the billiard, constituted by the union of $\mathcal{U}$ and $\mathcal{L}$, the dispersing and the vertical wall respectively.

Thus T_5 is a billiard map that preserves the infinite measure $d\mu = \cos\varphi dr d\varphi$ defined at M_5. When no ambiguity exists, we will use the same notation for the (infinite) measure invariant by the map $T : M \to M$.

As in [15, Corollary 3.3] we conclude that T_5 is conservative. Moreover, since $T_4 : M_4 \to M_4$ is ergodic and it is induced by T_5, so is T_5.

The map T_5 describes all the dynamics of our system and this is the map we shall deal with from now on.

3. K-automorphisms and proof of Theorem A

Definition 3.1. Let $(X, \mathcal{B}, \mu)$ be an infinite σ-finite measure space and $\mathcal{F} : X \to X$ an automorphism. We say that $\mathcal{F}$ is an *infinite K-automorphism* if there exists a sub-σ-algebra $\mathcal{K} \subset \mathcal{B}$ such that

(i) $\mathcal{F}\mathcal{K} \supset \mathcal{K}$;
(ii) $\bigvee_{n=0}^{\infty} \mathcal{F}^n \mathcal{K} = \mathcal{B} \mod \mu$;
(iii) $\bigcap_{n=0}^{\infty} \mathcal{F}^{-n} \mathcal{K} = \mathcal{N} = \{\emptyset, X\} \mod \mu$.

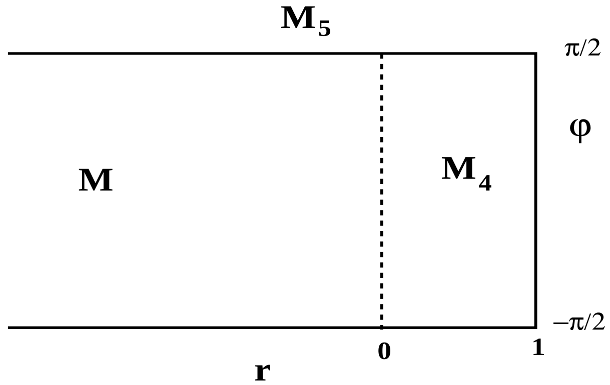


FIGURE 5. The phase space M_5. The line $r = 0$, corresponds to the vertex V and $r = 1$ corresponds to the vertex in the point $(0, 0)$ of $\mathcal{D}$.

The next proposition is an extension for K-automorphisms of a similar result for ergodic maps in spaces with infinite measure. See [1, p. 42].

Proposition 3.2. *Let $\mathcal{F}$ be an ergodic measure preserving map of the σ-finite measure space $(X, \mathcal{B}, \mu)$ and suppose there exists $E \in \mathcal{B}$ such that $\mu(E) > 0$ and $\bigcup_{n=0}^{\infty} \mathcal{F}^{-n}E = X \mod \mu$. If $\mathcal{F}_E$ is a (finite or infinite) K-automorphism then $\mathcal{F}$ is a (finite or infinite) K-automorphism.*

Proof. Since $\mathcal{F}_E$ is a K-automorphism, there exists $\mathcal{K}_E$, sub-σ-algebra of $\mathcal{B}_E = \mathcal{B} \cap E$ such that

(1) $\mathcal{F}_E\mathcal{K}_E \supset \mathcal{K}_E$;
(2) $\bigvee_{n=0}^{\infty} \mathcal{F}_E^n\mathcal{K}_E = \mathcal{B}_E \mod \mu_E$;
(3) $\bigcap_{n=0}^{\infty} \mathcal{F}_E^{-n}\mathcal{K}_E = \mathcal{N}_E = \{\emptyset, E = X \cap E\} \mod \mu_E$.

Let $\mathcal{K} = \bigvee_{i=-\infty}^{0} \mathcal{F}^i\mathcal{K}_E$. Then

(i) $\mathcal{F}\mathcal{K} = \bigvee_{i=-\infty}^{0} \mathcal{F}^{i+1}\mathcal{K}_E = \bigvee_{i=-\infty}^{1} \mathcal{F}^i\mathcal{K}_E \supset \bigvee_{i=-\infty}^{0} \mathcal{F}^i\mathcal{K}_E = \mathcal{K}$.

(ii) $\bigvee_{n=0}^{\infty} \mathcal{F}^n\mathcal{K} = \bigvee_{n=0}^{\infty} \mathcal{F}^n \bigvee_{i=-\infty}^{0} \mathcal{F}^i\mathcal{K}_E = \bigvee_{i=-\infty}^{\infty} \mathcal{F}^i\mathcal{K}_E = \mathcal{B} \mod \mu$; since, by condition (2), $\mathcal{B}_E \subset \displaystyle\bigvee_{i=-\infty}^{\infty} \mathcal{F}^i\mathcal{K}_E$, and given $A \in \mathcal{B}$,

$$A = A \cap X = A \cap \bigcup_{i=0}^{\infty} \mathcal{F}^{-i}E = \bigcup_{i=0}^{\infty}(A \cap \mathcal{F}^{-i}E) = \bigcup_{i=0}^{\infty} \mathcal{F}^{-i}(\mathcal{F}^iA \cap E).$$

Because $\mathcal{F}^iA \cap E \in \mathcal{B}_E$, it follows that $\mathcal{F}^{-i}(\mathcal{F}^iA \cap E) \in \mathcal{F}^{-i}\mathcal{B}_E$, so $A \in \bigvee_{i=-\infty}^{\infty} \mathcal{F}^i\mathcal{K}_E$. Thus $\mathcal{B} \subset \bigvee_{i=-\infty}^{\infty} \mathcal{F}^i\mathcal{K}_E$.

(iii) We must show that $\bigcap_{n=0}^{\infty} \mathcal{F}^{-n}\mathcal{K} = \bigcap_{n=0}^{\infty} \mathcal{F}^{-n} \bigvee_{i=-\infty}^{0} \mathcal{F}^i\mathcal{K}_E = \mathcal{N} \mod \mu$. To do this, we just need to show that $\bigcap_{n=0}^{\infty} \mathcal{F}^{-n}\mathcal{K} \subset \mathcal{N} \mod \mu$.

Let $A \in \bigcap_{n=0}^{\infty} \mathcal{F}^{-n} \bigvee_{i=-\infty}^{0} \mathcal{F}^i \mathcal{K}_E$ and suppose that $\mu(A) > 0$. Furthermore, we may suppose that $\mu(A \cap E) > 0$ because, if not, we take A^c as the set.

Then $A \in \bigcap_{n=0}^{\infty} \mathcal{F}^{-n} \bigvee_{i=-\infty}^{0} \mathcal{F}^i \mathcal{K}_E \Rightarrow A \cap E \in \bigcap_{n=0}^{\infty} \mathcal{F}^{-n} \bigvee_{i=-\infty}^{0} \mathcal{F}^i \mathcal{K}_E \cap E$ which is equal to $\bigcap_{n=0}^{\infty} \mathcal{F}_E^{-n} \bigvee_{i=-\infty}^{0} \mathcal{F}_E^i \mathcal{K}_E$, by the definition of $\mathcal{F}_E$.

By condition (1), $\bigvee_{i=-\infty}^{0} \mathcal{F}_E^i \mathcal{K}_E = \mathcal{K}_E$, so $\bigcap_{n=0}^{\infty} \mathcal{F}_E^{-n} \bigvee_{i=-\infty}^{0} \mathcal{F}_E^i \mathcal{K}_E = \bigcap_{n=0}^{\infty} \mathcal{F}_E^{-n} \mathcal{K}_E = \mathcal{N}_E \mod \mu_E$, by condition (3). Then $\mu_E((A \cap E) \bigtriangleup E) = 0$ and $\mu_E(A^c \cap E) = 0$. Thus $\mu(A^c \cap E) = 0$.

Also, $\mu_E(\mathcal{F}_E^k(A^c \cap E)) = \mu_E(\mathcal{F}_E^k A^c \cap E) = 0$, for all $k \geq 0$. Then $\mu_E(\mathcal{F}^j A^c \cap E) = 0$, for all $j \geq 0$, because $\mu_E(\mathcal{F}^j A^c \cap E) \leq \sum_{k=1}^{j} \mu_E(\mathcal{F}_E^k A^c \cap E) = 0$. Since $A^c = A^c \cap X = A^c \cap \bigcup_{j=0}^{\infty} \mathcal{F}^{-j} E = \bigcup_{j=0}^{\infty} \mathcal{F}^{-j}(\mathcal{F}^j A^c \cap E)$, we get $\mu(A^c) = 0$ hence $\bigcap_{n=0}^{\infty} \mathcal{F}^{-n} \mathcal{K} \subset \mathcal{N} \mod \mu$. □

We also need the following theorem due to Pesin and Katok and Strelcyn:

Theorem 3.3. (Pesin [20, Theorem 7.2] , Katok and Strelcyn [11, Theorem 13.1]) *Let $\mathcal{V}$ be a finite union of compact Riemannian manifolds $\mathcal{V}_1, \mathcal{V}_2 \ldots \mathcal{V}_s$ (possibly with boundaries and corners), all of them with dimension $d \geq 2$, glued along finitely many C^1 submanifolds of positive codimension and $\mathcal{F}$ a map on $\mathcal{V}$ preserving a Borel probability measure μ, both satisfying the Katok and Strelcyn conditions indicated in* [11, Section 1.1] *. Suppose that*

$$\Sigma(\mathcal{F}) = \{x \in V : \textit{the Lyapunov exponents on } V \textit{ are non-zero}\}$$

has positive μ-measure. Then there exist sets $\Sigma_i \subset \Sigma(\mathcal{F})$, $i = 0, 1, 2, \ldots$, such that

(1) $\Sigma(\mathcal{F}) = \bigcup_{i \geq 0} \Sigma_i$, $\Sigma_i \cap \Sigma_j = \emptyset$ *for* $i \neq j$, $i, j = 0, 1, 2, \ldots$;
(2) $\mu(\Sigma_0) = 0$, $\mu(\Sigma_i) > 0$, *for* $i > 0$;
(3) *for* $i > 0$: $\mathcal{F}(\Sigma_i) = \Sigma_i$, $\mathcal{F}|\Sigma_i$ *is ergodic;*
(4) *for* $i > 0$, *there exists a splitting* $\Sigma_i = \bigcup_{j=1}^{n_i} \Sigma_i^j, n_i \in \mathbb{Z}^+$ *such that*
   (a) $\Sigma_i^{j_1} \cap \Sigma_i^{j_2} = \emptyset$ *for* $j_1 \neq j_2$;
   (b) $\mathcal{F}(\Sigma_i^j) = \Sigma_i^{j+1}$ *para* $j = 1, 2, \ldots, n_i - 1$, $\mathcal{F}(\Sigma_i^{n_i}) = \Sigma_i^1$;
   (c) $\mathcal{F}^{n_i}|\Sigma_i^1$ *is a finite K-automorphism.*

Returning to the billiard map case:

Lemma 3.4. *Let M_4 be the phase space associated to the rebounds in the vertical wall. Then T_4 is a finite K-automorphism.*

Proof. We know that T_4^n is ergodic for all $n \geq 1$. Also, the Lyapunov exponents for T_4 are non-zero. So we may apply Theorem 3.3. However, by the ergodicity of T_4^n, all the decompositions are trivial and we get that T_4 is a K-automorphism. □

From Proposition 3.2, it follows that (M_5, T_5, μ) an infinite K-system, concluding the proof of Theorem A.

4. Geometric conditions

This section and the next one are inspired in the analysis for trajectories in a finite cusp studied by Chernov and Markarian [6].

We are in the same setting as in the previous sections. Fix $N_0 >> 1$. Let us study the behavior of a trajectory that leaving $\mathcal{L}$ (with coordinates (r, φ) in M_4), enters in the cusp, and comes back after $N > N_0$ rebounds. In order to do this, we shall adopt a new system of coordinates from now on. Let $x_n \in [0, \infty)$, $0 \leq n \leq N$, be the x-coordinate associated to the n-th rebound on $\mathcal{U}$, (where $x_0 = 0$, leaving $\mathcal{L}$),and $\gamma_n \in [0, \pi/2]$, $0 \leq n \leq N$, the positive angle between the trajectory and the tangent at the point of collision with coordinate x_n ($\gamma_0 = \pi/2 - |\varphi|$).

Define

$$x_{N_2} := \max\{x_n : n = 1, 2, \dots, N\},$$

that is, the x-coordinate of the most interior point inside the cusp.

If $n \leq N_2 - 1$ then

$$\gamma_{n+1} = \gamma_n + \tan^{-1}|f'(x_n)| + \tan^{-1}|f'(x_{n+1})| \tag{4.1}$$

$$x_{n+1} = x_n + \frac{f(x_n) + f(x_{n+1})}{\tan(\gamma_n + \tan^{-1}|f'(x_n)|)}. \tag{4.2}$$

If $n \geq N_2$ then

$$\gamma_n = \gamma_{n+1} + \tan^{-1}|f'(x_n)| + \tan^{-1}|f'(x_{n+1})|$$
$$x_n = x_{n+1} + \frac{f(x_n) + f(x_{n+1})}{\tan(\gamma_{n+1} + \tan^{-1}|f'(x_{n+1})|)}.$$

Lemma 4.1. *Using the notation above,* $|N_2 - N/2| = O(1)$.

Proof. Suppose that, without lost of generality, $x_{N_2+1} \geq x_{N_2-1}$. Then

$$\gamma_{N_2} = \gamma_{N_2-1} + \tan^{-1}|f'(x_{N_2-1})| + \tan^{-1}|f'(x_{N_2})|.$$

On the other hand,

$$\gamma_{N_2} = \gamma_{N_2+1} + \tan^{-1}|f'(x_{N_2+1})| + \tan^{-1}|f'(x_{N_2})|.$$

So,

$$\begin{aligned}\gamma_{N_2-1} + \tan^{-1}|f'(x_{N_2-1})| &= \gamma_{N_2+1} + \tan^{-1}|f'(x_{N_2+1})| \\ &\leq \gamma_{N_2+1} + \tan^{-1}|f'(x_{N_2-1})|,\end{aligned}$$

That is,

$$\gamma_{N_2-1} \leq \gamma_{N_2+1}.$$

Now, we must show that $x_{N_2-i} \leq x_{N_2+i}$ and $\gamma_{N_2-i} \leq \gamma_{N_2+i}$, for all $i = 1, 2, \dots$ while the collisions remain inside the cusp. Indeed, suppose that for i it is true and we shall show it for $i + 1$. Then

$$x_{N_2-i} = x_{N_2-(i+1)} + \frac{f(x_{N_2-i}) + f(x_{N_2-(i+1)})}{\tan(\gamma_{N_2-(i+1)} + \tan^{-1}|f'(x_{N_2-(i+1)})|)}$$

$$x_{N_2+i} = x_{N_2+(i+1)} + \frac{f(x_{N_2+i}) + f(x_{N_2+(i+1)})}{\tan(\gamma_{N_2+(i+1)} + \tan^{-1}|f'(x_{N_2+(i+1)})|)}$$
$$\gamma_{N_2-i} = \gamma_{N_2-(i+1)} + \tan^{-1}|f'(x_{N_2-(i+1)})| + \tan^{-1}|f'(x_{N_2-i})|$$
$$\gamma_{N_2+i} = \gamma_{N_2+(i+1)} + \tan^{-1}|f'(x_{N_2+(i+1)})| + \tan^{-1}|f'(x_{N_2+i})|.$$

By the induction hypothesis, the following holds

$$\begin{aligned} & \gamma_{N_2-(i+1)} + \tan^{-1}|f'(x_{N_2-(i+1)})| + \tan^{-1}|f'(x_{N_2-i})| \\ \le\ & \gamma_{N_2+(i+1)} + \tan^{-1}|f'(x_{N_2+(i+1)})| + \tan^{-1}|f'(x_{N_2+i})| \\ \le\ & \gamma_{N_2+(i+1)} + \tan^{-1}|f'(x_{N_2+(i+1)})| + \tan^{-1}|f'(x_{N_2-i})|. \end{aligned}$$

So

$$\gamma_{N_2-(i+1)} + \tan^{-1}|f'(x_{N_2-(i+1)})| \le \gamma_{N_2+(i+1)} + \tan^{-1}|f'(x_{N_2+(i+1)})|. \tag{4.3}$$

Using the hypothesis of induction and by (4.3), we get

$$\begin{aligned} & x_{N_2-(i+1)} + \frac{f(x_{N_2-i}) + f(x_{N_2-(i+1)})}{\tan(\gamma_{N_2-(i+1)} + \tan^{-1}|f'(x_{N_2-(i+1)})|)} \\ \le\ & x_{N_2+(i+1)} + \frac{f(x_{N_2+i}) + f(x_{N_2+(i+1)})}{\tan(\gamma_{N_2+(i+1)} + \tan^{-1}|f'(x_{N_2+(i+1)})|)} \\ \le\ & x_{N_2+(i+1)} + \frac{f(x_{N_2+i}) + f(x_{N_2+(i+1)})}{\tan(\gamma_{N_2-(i+1)} + \tan^{-1}|f'(x_{N_2-(i+1)})|)}. \end{aligned}$$

Thus

$$x_{N_2-(i+1)} + f(x_{N_2-(i+1)}) \le x_{N_2+(i+1)} + f(x_{N_2+(i+1)}).$$

Since $x_i \ge 1$, for all $i = 1, 2, \dots, N$,

$$x_{N_2-(i+1)} \le x_{N_2+(i+1)} \quad \text{e} \quad \gamma_{N_2-(i+1)} \le \gamma_{N_2+(i+1)},$$

as we wish to demonstrate. Thus $|N_2 - N/2| = O(1)$. □

Let us now split the trajectories going through the cusp in three regions. For this, we choose $\bar\gamma$ sufficiently small, that the exact value is not important, e.g. $\bar\gamma = 10^{-10}$. This choice allows us to make estimates in three different regions, defining

$$\begin{aligned} N_1 &= \max\{n < N_2; \gamma_n \le \bar\gamma\} \\ N_3 &= \min\{n > N_2; \gamma_n \le \bar\gamma\}. \end{aligned}$$

We call the series of rebounds between 1 and N_1 the *entering period*, between N_1 and N_3 the *turning period* and between N_3 and N the *exiting period*. Furthermore, consider x_1 large enough, e.g. $x_1 > 10^6$.

From now on, we use the table $\mathcal{D}$ defined by $f(x) = (x+1)^{-1}$. Until the end of this section, we use the following change of variables:

$$t_n = x_n + 1, \quad \forall 1 \le n \le N.$$

Lemma 4.2. *We have*

$$N_1 \asymp N_2 - N_1 \asymp N_3 - N_2 \asymp N - N_3 \asymp N,$$

so all segments have size of order N. Moreover

$$x_1 \asymp N^{\frac{1}{6}} \quad e \quad x_{N_2} \asymp N^{\frac{1}{2}} \tag{4.4}$$

and

$$x_n \asymp n^{\frac{1}{3}} N^{\frac{1}{6}} \quad \forall n = 2, \ldots, N_1.$$

Also

$$\gamma_1 = O(N^{-1/3}) \quad e \quad \gamma_2 \asymp N^{-1/3} \tag{4.5}$$

and

$$\gamma_n \asymp n^{\frac{1}{3}} N^{-\frac{1}{3}} \quad \forall n = 2, \ldots, N_1.$$

Proof. For each $n = 1, 2, \ldots, N_1$, we define $\omega_n = \dfrac{\gamma_n}{\frac{1}{|f'(t_n)|}}$. Using the definition of f, we obtain $\omega_n = \gamma_n t_n^2$. Also we define $u_n = \dfrac{t_n}{t_{n+1}}$. Multiplying (4.1) by t_{n+1}^2 and expanding $\tan^{-1}$ in its Taylor's series we get

$$\omega_{n+1} = \frac{\omega_n + 1}{u_n^2} + 1 + O(x_n^{-4}). \tag{4.6}$$

From (4.1), we get

$$\gamma_1 + \frac{1}{t_1^2} + \frac{2}{t_2^2} + \ldots + \frac{2}{t_{n-1}^2} + \frac{1}{t_n^2} + O\left(\sum_{i=1}^{n} t_i^{-6}\right) = \gamma_n \leq \frac{\pi}{2}. \tag{4.7}$$

Thus

$$\sum_{i=1}^{n} t_i^{-2} = O(1). \tag{4.8}$$

From equation (4.6),we obtain

$$\omega_n > 2n - 2. \tag{4.9}$$

From (4.2) and using the fact that $\tan x > x$, we have

$$\frac{1}{u_n} < 1 + \frac{2}{\omega_n + 1}(1 + O(t_n^{-6})). \tag{4.10}$$

Replacing (4.10) in (4.6):

$$\begin{aligned} \omega_{n+1} &< 1 + (\omega_n + 1)\left(1 + \frac{2}{\omega_n + 1}(1 + O(t_n^{-6}))\right)^2 + O(t_n^{-4}) \\ &= 6 + \omega_n + \frac{4}{\omega_n + 1} + O(t_n^{-4}) < 6 + \omega_n + \frac{4}{2n - 1} + O(t_n^{-4}). \end{aligned}$$

So

$$\omega_n < 6n + 2\ln n + O(1). \tag{4.11}$$

From (4.9) and (4.11) we conclude that $\omega_n = \gamma_n t_n^2 \asymp n$. Since $\gamma_{N_2} \approx \pi/2$, it follows that $x_{N_2}^2 \asymp N_2 \asymp N$, by Lemma 4.1.

For $n = 1, 2, \ldots, N_1$,

$$t_{n+1} = t_n + \frac{\dfrac{1}{t_n} + \dfrac{1}{t_{n+1}}}{\gamma_n + \dfrac{1}{t_n^2} + O\left(\left(\gamma_n + \dfrac{1}{t_n^2}\right)^3\right)} = t_n + \frac{t_n + \dfrac{t_n^2}{t_{n+1}}}{\omega_n + 1 + t_n^2 O\left(\left(\gamma_n + \dfrac{1}{t_n^2}\right)^3\right)}.$$

Dividing by t_n we obtain

$$\frac{1}{u_n} = 1 + \frac{1+u_n}{w_n+1}\left(\frac{1}{1 + \dfrac{t_n^2}{\gamma_n t_n^2 + 1} O\left(\left(\gamma_n + \dfrac{1}{t_n^2}\right)^3\right)}\right).$$

Since the choices of $\gamma_n < 10^{-10}$ and $x_1 > 10^6$ imply that $O\left(\left(\gamma_n + \dfrac{1}{t_n^2}\right)^3\right) = O(\gamma_n^3)$ and because $\dfrac{t_n^2}{\gamma_n t_n^2 + 1} O\left(\gamma_n^3\right) = O\left(\gamma_n^2\right)$, we obtain

$$\frac{1}{u_n} = 1 + \frac{1+u_n}{w_n+1}\left(\frac{1}{1+O(\gamma_n^2)}\right) = 1 + \frac{1+u_n}{w_n+1}(1+O(\gamma_n^2)),$$

since $O(\gamma_n^2)$ is sufficiently small. From (4.10) we get

$$\begin{aligned}
\frac{1}{u_n} &> 1 + \left(\frac{1}{\omega_n+1} + \frac{1}{\omega_n + 3 + O(t_n^{-6})}\right)(1+O(\gamma_n^2)) > 1 + \frac{2}{\omega_n+3} + O\left(\frac{\gamma_n^2}{n}\right) \\
&> 1 + \frac{2}{6n + 2\ln n + O(1)} + O\left(\frac{\gamma_n^2}{n}\right) \\
&> \exp\left(\frac{2}{6n + 2\ln n + O(1)} - \frac{4}{(6n + 2\ln n + O(1))^2} + O\left(\frac{\gamma_n^2}{n}\right)\right),
\end{aligned}$$

in the last inequality we use the fact that $1 + x > \exp(x - x^2)$ for small x.

Multiplying from $i = 1$ to $n-1$, we get

$$\begin{aligned}
\prod_{i=1}^{n-1} u_i^{-1} &> \exp\left(\sum \frac{2}{6i + 2\ln i + O(1)} - \sum \frac{4}{(6i + 2\ln i + O(1))^2} + O\left(\sum \frac{\gamma_i^2}{i}\right)\right) \\
&> \exp(\ln n^{1/3} - C) = C' n^{1/3},
\end{aligned}$$

because

$$\sum_{i=1}^{n-1}\frac{\gamma_i^2}{i}\le 12\sum_{i=1}^{n-1}\frac{\gamma_i^2}{\omega_i}=12\sum_{i=1}^{n-1}\frac{\gamma_i}{x_i^2}=O(1),$$

since $\gamma_i<\pi/2$ and $\sum_{i=1}^{n} t_i^{-2}=O(1)$, as obtained in (4.8). We obtain

$$\frac{t_n}{t_1}>C'n^{1/3}. \tag{4.12}$$

However

$$\begin{aligned}
\omega_{n+1} &= 1+\frac{\omega_n+1}{u_n^2}+O(t_n^{-4})\\
&> 1+(\omega_n+1)\left(1+\frac{2}{\omega_n+3}+O\left(\frac{\gamma_n^2}{n}\right)\right)^2+O(t_n^{-4})\\
&> 2+\omega_n+4\left(1-\frac{2}{\omega_n+3}\right)+4\left(\frac{1}{\omega_n+3}-\frac{2}{(\omega_n+3)^2}\right)+O(\gamma_n^2)+O(t_n^{-4})\\
&> \omega_n+6-\frac{4}{2n+1}-\frac{4}{(2n+1)^2}+O(\gamma_n^2)+O(t_n^{-4}).
\end{aligned}$$

This implies that

$$\omega_n>6n-2\ln n+O(1).$$

Thus

$$\begin{aligned}
\frac{1}{u_n^2} &= \frac{\omega_{n+1}-1+O(t_n^{-4})}{\omega_n+1}<\frac{\omega_n+1+4+\dfrac{4}{\omega_n+1}+O(t_n^{-4})}{\omega_n+1}\\
&= 1+\frac{4}{\omega_n+1}+\frac{4}{(\omega_n+1)^2}+O\left(\frac{t_n^{-4}}{n}\right)\\
&< 1+\frac{4}{6n-2\ln n+O(1)}+\frac{4}{(6n-2\ln n+O(1))^2}+O\left(\frac{t_n^{-4}}{n}\right).
\end{aligned}$$

Multiplying from $i=1$ to $n-1$,

$$\begin{aligned}
\prod_{i=1}^{n-1}u_i^{-2} &< \exp\left(\sum\frac{4}{6i-2\ln i+O(1)}+\sum\frac{4}{(6i-2\ln i+O(1))^2}+O\left(\sum\frac{x_i^{-4}}{i}\right)\right)\\
&< \exp(\ln n^{2/3}+C)=C'n^{2/3}.
\end{aligned}$$

And we obtain

$$\left(\frac{t_n}{t_1}\right)^2<C'n^{2/3}. \tag{4.13}$$

From (4.12) and (4.13) we get

$$\frac{t_n}{t_1}\asymp n^{1/3}.$$

So

$$n \asymp \gamma_n t_n^2 \asymp \gamma_n n^{2/3} t_1^2.$$

However $\gamma_{N_1} \approx \bar{\gamma} = \text{constant}$, then

$$N_1 \asymp \gamma_{N_1} N_1^{2/3} x_1^2 \Rightarrow x_1 \asymp N_1^{1/6}.$$

And thus

$$t_n \asymp n^{1/3} N_1^{1/6} \quad \text{and} \quad \gamma_n \asymp n^{1/3} N_1^{-1/3},$$

for all $n = 2, \dots, N_1$.

To show that $N_1 \asymp N$, just notice that at the turning period, i.e., $N_1 \le n \le N_2$, the angle γ_n increases from $\bar{\gamma}$ to approximately $\pi/2$ and

$$\frac{1}{t_n^2} = \frac{\gamma_n}{\omega_n} > \frac{\bar{\gamma}}{6n + 2\ln n + C}.$$

It follows from (4.7) and (4.8) that

$$\sum_{n=N_1}^{N_2} (\gamma_n - \gamma_{n-1}) \ge \sum_{n=N_1}^{N_2} \frac{C'}{6n + 2\ln n + C} \ge C'' \ln \frac{N_2}{N_1},$$

for some constants $C', C'' > 0$. This implies that $N_1 < N_2 < C''' N_1$, for some $C''' > 0$. □

In the proof of Lemma 4.2, we obtained the following

$$\omega_n + 1 = 6 + \omega_n + O\left(\frac{1}{n} + \gamma_n^2 + t_n^{-4}\right),$$

$$\omega_n = 6n + O(\ln n),$$

$$\frac{1}{u_n} = 1 + \frac{1}{3n} + O\left(\frac{\ln n}{n^2} + \frac{\gamma_n^2}{n} + \frac{t_n^{-4}}{n}\right),$$

$$u_n = 1 - \frac{1}{3n} + O\left(\frac{\ln n}{n^2} + \frac{\gamma_n^2}{n} + \frac{t_n^{-4}}{n}\right),$$

and we shall use the values from now on.

For $1 \le n \le N_2$, let τ_n be the time between two consecutive collisions in the billiard table:

$$\tau_n = \frac{f(t_n) + f(t_{n+1})}{\sin(\gamma_n + \tan^{-1}(|f'(t_n)|))}.$$

Then, using the values above,

$$\begin{aligned}
\tau_n &= \frac{\frac{1}{t_n}+\frac{1}{t_{n+1}}}{\left(\gamma_n+\frac{1}{t_n^2}\right)+O\left(\left(\gamma_n+\frac{1}{t_n^2}\right)^3\right)} = \frac{t_n+\frac{t_n^2}{t_{n+1}}}{\omega_n+1+O(t_n^2\gamma_n^3)} \\
&= \frac{t_n(1+u_n)}{\omega_n(1+\omega_n^{-1}+O(\gamma_n^2))} = \frac{t_n}{\omega_n}\frac{2+O(n^{-1})}{1+O(n^{-1})+O(\gamma_n^2)} \\
&= \frac{2t_n}{\omega_n}\frac{1}{\frac{1+O(n^{-1})+O(\gamma_n^2)}{1+O(n^{-1})}} = \frac{2t_n}{\omega_n}(1+O(n^{-1})+O(\gamma_n^2)) \\
&\asymp n^{-2/3}N^{1/6},
\end{aligned}$$

for $1 \le n \le N_2$.

Furthermore, if we denote by K_n the curvature of the dispersing part of the table at the point of collision (r_n, φ_n), we have that, as w enter in the cusp ($n = 1, 2, \dots, N_2$)

$$K_n = \frac{f''(t_n)}{(1+(f'(t_n))^2)^{3/2}} = \frac{\frac{2}{t_n^3}}{\left(1+\left(\frac{1}{t_n^2}\right)^2\right)^{3/2}} = \frac{\frac{2}{t_n^3}}{\left(\frac{t_n^4+1}{t_n^4}\right)^{3/2}} = \frac{2}{t_n^3},$$

since $x_1 > 10^6$.

We also have that

$$\begin{aligned}
\frac{\tau_n K_n}{\sin\gamma_n} &= \frac{2t_n\omega_n^{-1}(1+O(n^{-1})+O(\gamma_n^2))}{\gamma_n+O(\gamma_n^3)}\frac{2}{t_n^3} = \frac{4}{t_n^2}\frac{(1+O(1/n)+O(\gamma_n^2))}{\omega_n\gamma_n(1+O(\gamma_n^2))} \\
&= \frac{4}{t_n^2}(1+O(1/n)+O(\gamma_n^2))(1+O(\gamma_n^2)) = \frac{4}{\omega_n^2}(1+O(n^{-1})+O(\gamma_n^2)) \\
&= \frac{4}{(6n+O(\ln n))^2}(1+O(n^{-1})+O(\gamma_n^2)) \\
&= \frac{4}{36n^2+O(n\ln n)+O((\ln n)^2)}(1+O(n^{-1})+O(\gamma_n^2)) \\
&= \frac{4}{36n^2(1+O(\ln n/n)+O((\ln n/n)^2))}(1+O(n^{-1})+O(\gamma_n^2)) \\
&= \frac{1}{9n^2}\left(1+O\left(\frac{\ln n}{n}\right)\right)(1+O(n^{-1})+O(\gamma_n^2)) \\
&= \frac{1}{9n^2}+O\left(\frac{\ln n}{n^3}++\frac{\gamma_n^2}{n^2}\right). \qquad (4.14)
\end{aligned}$$

Moreover

$$\frac{\tau_{n+1}}{\tau_n} = \frac{f(t_{n+1})+f(t_{n+2})}{f(t_n)+f(t_{n+1})}\frac{\sin(\gamma_n+\tan^{-1}(|f'(t_n)|))}{\sin(\gamma_{n+1}+\tan^{-1}(|f'(t_{n+1})|))} = F_1F_2.$$

To obtain F_1, we notice that

$$\begin{aligned} F_1 &= \frac{\dfrac{1}{t_{n+1}}+\dfrac{1}{t_{n+2}}}{\dfrac{1}{t_{n+1}}+\dfrac{1}{t_n}} = \frac{\dfrac{1}{t_{n+1}}\left(1+\dfrac{t_{n+1}}{x_{n+2}}\right)}{\dfrac{1}{x_n}\left(1+\dfrac{t_n}{t_{n+1}}\right)} \\ &= u_n\frac{(1+u_{n+1})}{(1+u_n)}. \end{aligned}$$

This last one can be computed as

$$\begin{aligned} \frac{1+u_{n+1}}{1+u_n} &= \frac{1+1-\dfrac{1}{3(n+1)}+O\left(\dfrac{\ln(n+1)}{(n+1)^2}+\dfrac{\gamma_{n+1}^2}{n+1}+\dfrac{t_{n+1}^{-4}}{n+1}\right)}{1+1-\dfrac{1}{3n}+O\left(\dfrac{\ln n}{n^2}+\dfrac{\gamma_n^2}{n}+\dfrac{t_n^{-4}}{n}\right)} \\ &= \frac{6(n+1)-1+O\left(\dfrac{\ln(n+1)}{n+1}+\gamma_{n+1}^2+t_{n+1}^{-4}\right)}{6n-1+O\left(\dfrac{\ln n}{n}+\gamma_n^2+t_n^{-4}\right)}\frac{3n}{3(n+1)} \\ &= \left(1-\frac{1}{6(n+1)}+O\left(\frac{\ln n+1}{n+1^2}+\frac{\gamma_{n+1}^2}{n+1}+\frac{t_{n+1}^{-4}}{n+1}\right)\right)\times \\ &\times \left(1+\frac{1}{6n}+O\left(\frac{\ln n}{n^2}+\frac{\gamma_n^2}{n}+\frac{t_n^{-4}}{n}\right)\right) \\ &= 1+\frac{5}{36n^2}\left(1+O\left(\frac{1}{n}\right)\right)+O\left(\frac{\ln n}{n^2}+\frac{\gamma_n^2}{n}+\frac{t_n^{-4}}{n}\right) \\ &= 1+\frac{5}{36n^2}+O\left(\frac{\ln n}{n^2}+\frac{\gamma_n^2}{n}+\frac{t_n^{-4}}{n}\right). \end{aligned}$$

And to obtain F_2,

$$\begin{aligned}
F_2 &= \frac{\gamma_n + \frac{1}{t_n^2} + O(\gamma_n^3)}{\gamma_{n+1} + \frac{1}{t_{n+1}^2} + O(\gamma_{n+1}^3)} = \frac{t_n^2\, t_{n+1}^2}{t_n^2\, t_{n+1}^2} \frac{\gamma_n + \frac{1}{t_n^2} + O(\gamma_n^3)}{\gamma_{n+1} + \frac{1}{t_{n+1}^2} + O(\gamma_{n+1}^3)} \\
&= \frac{t_{n+1}^2}{t_n^2} \frac{\omega_n + 1 + O(\gamma_n^2 \omega_n)}{\omega_{n+1} + 1 + O(\gamma_{n+1}^2 \omega_{n+1})} = \frac{1}{u_n^2} \frac{\omega_n + 1 + O(\gamma_n^2 \omega_n)}{\omega_{n+1} + 1 + O(\gamma_{n+1}^2 \omega_{n+1})} \\
&= \frac{1}{u_n^2} \frac{\omega_n + 1 + O(\gamma_n^2 \omega_n)}{\omega_n + 7 + O(\gamma_{n+1}^2 \omega_{n+1} + n^{-1} + \gamma_n^2 + t_n^{-4})} \\
&= \frac{1}{u_n^2} \left(1 - \frac{1}{n} + \frac{7}{6n^2} + O\left(\frac{\ln n}{n^2} + \frac{\gamma_n^2}{n} + \frac{t_n^{-4}}{n}\right)\right).
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{\tau_{n+1}}{\tau_n} &= u_n \left(1 + \frac{5}{36n^2} + O\left(\frac{\ln n}{n^2} + \frac{\gamma_n^2}{n} + \frac{t_n^{-4}}{n}\right)\right) \times \\
&\times \frac{1}{u_n^2} \left(1 - \frac{1}{n} + \frac{7}{6n^2} + O\left(\frac{\ln n}{n^2} + \frac{\gamma_n^2}{n} + \frac{t_n^{-4}}{n}\right)\right) \\
&= \frac{1}{u_n} \left(1 + \frac{5}{36n^2} + O\left(\frac{\ln n}{n^2} + \frac{\gamma_n^2}{n} + \frac{t_n^{-4}}{n}\right)\right) \times \\
&\times \left(1 - \frac{1}{n} + \frac{7}{6n^2} + O\left(\frac{\ln n}{n^2} + \frac{\gamma_n^2}{n} + \frac{t_n^{-4}}{n}\right)\right) \\
&= \left(1 + \frac{1}{3n}\right)\left(1 + \frac{5}{36n^2}\right)\left(1 - \frac{1}{n} + \frac{7}{6n^2}\right) + O\left(\frac{\ln n}{n^2} + \frac{\gamma_n^2}{n} + \frac{t_n^{-4}}{n}\right) \\
&= 1 - \frac{2}{3n} + O\left(\frac{\ln n}{n^2} + \frac{\gamma_n^2}{n} + \frac{t_n^{-4}}{n}\right). \qquad (4.15)
\end{aligned}$$

Remark 4.3. Due to the reversibility property of the billiard map, all the formulas obtained above hold for the exiting period as well. So

$$x_N \asymp N^{1/6} \quad \text{e} \quad \gamma_N = O(N^{-1/3}). \qquad (4.16)$$

During the exiting period we can use the countdown index $m = N+1-n$ obtaining asymptotic rates for $m = N_3 - 1, \ldots, N_1$, as for example, $x_m \asymp m^{1/3} N^{1/6}$, $\tau_m \asymp m^{-2/3} N^{1/6}$, etc.

5. Hyperbolicity

We use in this section the *p-norm*, defined by

$$\|dx\|_p = \cos\varphi |dr|,$$

for vectors $dx \in T_x M$ of a point $x = (r, \varphi)$. For billiard maps, the expansion rate of unstable vectors (i.e., in an unstable cone) in the p-norm is given by

$$\frac{\|D_x \mathcal{T}^{n+1}(dx)\|_p}{\|dx\|_p} = \prod_{i=0}^{n} |1 + \tau_i B_i|$$

(see [5, p.58]). Here B_i denotes the curvature of a small arc transverse to the wave front. For further details we suggest Chernov and Markarian's book [4, Chapter IV]. Moreover, for semi-dispersing billiards, unstable vectors are expanded monotonically in the p-norm, and this is not necessarily true in the Euclidean norm (see [5, Section 4.4]).

The values B_i^+ can be calculated inductively as

$$B_{n+1}^+ = \frac{2K_{n+1}}{\sin \gamma_{n+1}} + \frac{B_n^+}{1 + \tau_n B_n^+}.$$

From the equations (4.5) and (4.16), we know that $\gamma_1 = O(N^{-1/3})$ and $\gamma_N = O(N^{-1/3})$, hence they can be arbitrarily close to zero, which implies that the expansion rate would be extremely high. However B_{n+1}^+ is an increasing function of B_n^+ and $\dfrac{1}{\sin \gamma_{n+1}}$. So if γ_n increases, B_n^+ decreases. In this way, we can obtain an upper bound for the expansion rate taking lower bounds for γ_1 and γ_N. Thus we introduce the following assumption

$$\gamma_1 \asymp N^{-1/3} \quad \text{and} \quad \gamma_N \asymp N^{-1/3}. \tag{5.1}$$

Let

$$E_N = \{x \in M_4 \mid R(x) = N + 1\},$$

$N > N_0$, where

$$R(x) = \inf\{n \geq 1 : T_5^n x \in M_4\},$$

i.e., $R(x) - 1$ indicates the amount of rebounds in the dispersing part before returning to the vertical wall $\mathcal{L}$. So, E_N is the subset of M_4 that return for the first time to M_4 after $N + 1$ iterations of T_5.

The main goal of this section is to prove the following theorem

Theorem 5.1. *For all $x \in E_N$, satisfying $\gamma_1 \asymp N^{-1/3}$ e $\gamma_N \asymp N^{-1/3}$,*

$$\frac{\|D_x T_5^{N+1}(dx)\|_p}{\|dx\|_p} \asymp N.$$

Let us denote $\tau_i B_i$ by λ_i. For $n \geq 1$,

$$\lambda_{n+1} = \frac{2\tau_{n+1} K_{n+1}}{\sin \gamma_{n+1}} + \frac{\tau_{n+1}}{\tau_n} \cdot \frac{\lambda_n}{1 + \lambda_n}. \tag{5.2}$$

Lemma 5.2. *We have that*

$$\lambda_n \asymp \frac{1}{n} \quad , \quad 1 \leq n \leq N_1,$$
$$\lambda_n \asymp \frac{1}{n} \asymp \frac{1}{N} \quad , \quad N_1 \leq n \leq N_3,$$
$$\lambda_n \asymp \frac{1}{(N-n)} \quad , \quad N_3 \leq n < N.$$

Proof. For $1 \leq n \leq N_1$, $\lambda_{n+1} > \dfrac{a}{n^2} + \left(1 - \frac{b}{n}\right) \dfrac{\lambda_n}{1+\lambda_n}$, for some $a, b > 0$. Suppose that $\lambda_n > c/n$. Then

$$\begin{aligned}
\lambda_{n+1} &> \frac{a}{n^2} + \left(1 - \frac{b}{n}\right) \frac{c/n}{1+c/n} = \frac{a}{n^2} + \left(1 - \frac{b}{n}\right) \left(\frac{c}{n+c}\right) \\
&= \frac{a}{n^2} + \frac{c}{n+c} - \frac{bc}{(n+c)n} = \frac{a(n+c) + cn^2 - bcn}{(n+c)n^2} \\
&= \frac{c + (a - bc + ac/n)/n}{n+c}.
\end{aligned}$$

If $c > 0$ is small enough, the expression in parenthesis is positive and $\lambda_{n+1} > \dfrac{c}{n+c} > \dfrac{c}{n+1}$. Similarly $\lambda_{n+1} < \dfrac{A}{n^2} + \left(1 - \frac{B}{n}\right) \dfrac{\lambda_n}{1+\lambda_n}$.

Supposing $\lambda_n < C/n$, we get $\lambda_{n+1} < \dfrac{C + (A - BC + AC/n)/n}{n+C}$.

If $C > 0$ is large enough, the expression in parenthesis is negative for N large and $\lambda_{n+1} < \dfrac{C}{n+C} < \dfrac{C}{n+1}$, completing the induction.

For $N_1 \leq n \leq N_3$, $\lambda_{N_1} \asymp \dfrac{1}{N}$ e $\tau_n \asymp n^{-2/3} N^{1/6} \asymp N^{-1/2}$. So $B^+_{N_1} = \dfrac{\lambda_{N_1}}{\tau_{N_1}} \asymp N^{-1/2}$.

We have that

$$\begin{aligned}
K_{n+1} \asymp N^{-3/2} &\Rightarrow \exists\, a, A > 0 \text{ such that } aN^{-3/2} \leq K_{n+1} \leq AN^{-3/2}, \\
\tau_n \asymp N^{-1/2} &\Rightarrow \exists\, b, B > 0 \text{ such that } bN^{-1/2} \leq \tau_n \leq BN^{-1/2}, \\
B^+_{N_1} \asymp N^{-1/2} &\Rightarrow \exists\, c, C > 0 \text{ such that } cN^{-1/2} \leq B^+_{N_1} \leq CN^{-1/2}.
\end{aligned}$$

Moreover

$$2 \leq \frac{2}{\sin \gamma_{n+1}} \leq \frac{2}{\sin \bar{\gamma}} =: G, \quad \forall N_1 \leq n \leq N_3.$$

So

$$B^+_{N_1+1} = \frac{2K_{N_1+1}}{\sin \gamma_{N_1+1}} + \frac{B^+_{N_1}}{1 + \tau_{N_1} B^+_{N_1}} \leq GAN^{-3/2} + CN^{-1/2}.$$

$$\begin{aligned}
B^+_{N_1+2} &= \frac{2K_{N_1+2}}{\sin \gamma_{N_1+2}} + \frac{B^+_{N_1+1}}{1 + \tau_{N_1+1} B^+_{N_1+1}} \leq GAN^{-3/2} + B^+_{N_1+1} \\
&\leq 2GAN^{-3/2} + CN^{-1/2}.
\end{aligned}$$

Thus

$$\begin{aligned} B_n^+ &= \frac{2K_n}{\sin\gamma_n} + \frac{B_{n-1}^+}{1+\tau_{n-1}B_{n-1}^+} \le (n-N_1)GAN^{-3/2} + CN^{-1/2} \\ &\le DNGAN^{-3/2} + CN^{-1/2} = (DGA+C)N^{-1/2} \\ &= EN^{-1/2}. \end{aligned}$$

On the other hand

$$B_{N_1+1}^+ = \frac{2K_{N_1+1}}{\sin\gamma_{N_1+1}} + \frac{B_{N_1}^+}{1+\tau_{N_1}B_{N_1}^+} \ge 2aN^{-3/2} + \frac{cN^{-1/2}}{1+BEN^{-1}}.$$

$$\begin{aligned} B_{N_1+2}^+ &= \frac{2K_{N_1+2}}{\sin\gamma_{N_1+2}} + \frac{B_{N_1+1}^+}{1+\tau_{N_1+1}B_{N_1+1}^+} \\ &\ge 2aN^{-3/2} + \frac{1}{(1+BEN^{-1})}\left(2aN^{-3/2} + \frac{cN^{-1/2}}{1+BEN^{-1}}\right) \\ &= 2aN^{-3/2}\left(1+\frac{1}{(1+BEN^{-1})}\right) + \frac{cN^{-1/2}}{(1+BEN^{-1})^2}. \end{aligned}$$

$$\begin{aligned} B_n^+ &= \frac{2K_n}{\sin\gamma_n} + \frac{B_{n-1}^+}{1+\tau_{n-1}B_{n-1}^+} \\ &\ge 2aN^{-3/2}\left(\sum_{i=0}^{n-N_1-1}\frac{1}{(1+BEN^{-1})^i}\right) + \frac{cN^{-1/2}}{(1+BEN^{-1})^{n-N_1}}. \end{aligned}$$

There exist constants $f>0$ and $h>0$ such that $\sum_{i=0}^{n-N_1-1}\frac{1}{(1+BEN^{-1})^i} \ge fN$ and also $\frac{1}{(1+BEN^{-1})^{n-N_1}} \ge h$, because $\sum_{i=0}^{n-N_1-1}\frac{1}{(1+BEN^{-1})^i} \ge \sum_{i=0}^{n-N_1-1}\frac{1}{(1+BEN^{-1})^{N_3-N_1}} \asymp N$, and $\frac{1}{(1+BEN^{-1})^{n-N_1}}$ is a bounded sequence.

So

$$\begin{aligned} B_n^+ &\ge 2afN^{-1/2} + chN^{-1/2} \\ &= (2af+ch)N^{-1/2} \\ &= eN^{-1/2}. \end{aligned}$$

Thus $B_n^+ \asymp N^{-1/2}$, and therefore $\lambda_n = B_n^+\tau_n \asymp N^{-1/2}N^{-1/2} = N^{-1}$.

For $N_3 \le n < N$, using the reversibility property of the billiard map,

$$\lambda_{m-1} = \frac{2\tau_{m-1}K_{m-1}}{\sin\gamma_{m-1}} + \frac{\tau_{m-1}}{\tau_m}\cdot\frac{\lambda_m}{1+\lambda_m},$$

for $m = N+1-n$. In particular,

$$\frac{a}{m^2} < \frac{2\tau_{m-1}K_{m-1}}{\sin\gamma_{m-1}} < \frac{A}{M^2} \quad \text{e} \quad 1+\frac{b}{m} < \frac{\tau_{m-1}}{\tau_m} < 1+\frac{B}{m},$$

for some $0<a<A<\infty$ and $0<b<B<\infty$.

Supposing $\lambda_m > c/m$,

$$\begin{aligned}\lambda_{m-1} &> \frac{a}{m^2}+\left(1+\frac{b}{m}\right)\frac{c/m}{1+c/m}\\ &= \frac{c+[a+bc-c-c^2+(ac-a-bc-ac/m)/m]/(m+c)}{m-1}\end{aligned}$$

If $c>0$ is small enough, the expression between brackets is positive, for m large, and we obtain that $\lambda_{m-1} > c/(m-1)$.

Supposing that $\lambda_m < C/m$,

$$\begin{aligned}\lambda_{m-1} &< \frac{A}{m^2}+\left(1+\frac{B}{m}\right)\frac{C/m}{1+C/m}\\ &= \frac{C+[A+BC-C-C^2+(AC-A-BC-AC/m)/m]/(m+C)}{m-1}\end{aligned}$$

If $C>0$ is large enough, the expression between brackets is negative, for m large, and we obtain $\lambda_{m-1} < C/(m-1)$, completing the proof. □

Lemma 5.2 implies that $\sum_{n=1}^{N-1}\lambda_n^2 = O(1)$. Therefore, for $1 \le N' < N'' \le N$,

$$\prod_{n=N'}^{N''-1}(1+\lambda_n) = \exp\left(\sum_{n=N'}^{N''-1}\ln(1+\lambda_n)\right) \asymp \exp\left(\sum_{n=N'}^{N''-1}\lambda_n\right). \tag{5.3}$$

in the turning period, we have that $\sum_{n=N_1}^{N_3-1}\lambda_n \asymp 1$, showing that the expansion during this period is negligible.

Lemma 5.3. *For all $x\in E_N$ satisfying (5.1), $\prod_{n=1}^{N_1}(1+\lambda_n)\asymp N^{2/3}$.*

Proof. According to the equation (5.3), it is sufficient to show that

$$\lambda_n = \frac{2}{3n}+\chi_n; \quad \text{where} \quad \sum_{n=1}^{N_1}\chi_n = O(1).$$

We have that, by (5.2)

$$\lambda_{n+1} = \frac{2}{9n^2}+a_n+\left(1-\frac{2}{3n}+b_n\right)\frac{\lambda_n}{1+\lambda_n},$$

where

$$a_n = O\left(\frac{\ln n}{n^3}+\frac{\gamma_n^2}{n^2}\right) \quad \text{e} \quad b_n = O\left(\frac{\ln n}{n^2}+\frac{\gamma_n^2}{n}+\frac{x_n^{-4}}{n}\right),$$

are relative to the equations (4.14) and (4.15).

Note that $|a_n| \le c/n^2$ and $|b_n| \le c/n$, for some $c>0$ small enough.

Take

$$\lambda_n = 2\frac{1+Z_n}{3n}.$$

We get that

$$\begin{aligned} 2\frac{1+Z_{n+1}}{3(n+1)} &= \frac{2}{9n^2} + a_n + \left(1 - \frac{2}{3n} + b_n\right) \times \\ &\times \left(\frac{2}{3n} + \frac{2Z_n}{3n}\right)\left(1 - \frac{2}{3n} - \frac{2Z_n}{3n} + O\left(\frac{1}{n^2} + \frac{Z_n^2}{n^2}\right)\right) \\ &= \frac{2}{9n^2} + a_n + X_1 \cdot X_2 \cdot X_3, \end{aligned}$$

$$\begin{aligned} X_2 \cdot X_3 &= \frac{2}{3n} + \frac{2Z_n}{3n} - \frac{4}{9n^2} - \frac{8Z_n}{9n^2} - \\ &- \frac{4Z_n^2}{9n^2} + O\left(\frac{1}{n^3} + \frac{Z_n}{n^3} + \frac{Z_n^2}{n^3} + \frac{Z_n^3}{n^3}\right) \end{aligned}$$

$$\begin{aligned} X_1 \cdot X_2 \cdot X_3 &= \frac{2}{3n} - \frac{8}{9n^2} + \frac{2b_n}{3n} + \frac{2Z_n}{3n} - \frac{12Z_n}{9n^2} + \\ &+ \frac{2b_nZ_n}{3n} - \frac{4Z_n^2}{9n^2} + O\left(\frac{1}{n^3} + \frac{Z_n}{n^3} + \frac{Z_n^2}{n^3} + \frac{Z_n^3}{n^3}\right), \end{aligned}$$

Therefore

$$\begin{aligned} Z_{n+1} &= R_n + Z_n \times \\ & \left(1 - \frac{1}{n} + b_n + O\left(\frac{1}{n^2}\right) - Z_n\left(\frac{2}{3n} + O\left(\frac{1}{n^2}\right)\right) + O\left(\frac{Z_n^2}{n^2}\right)\right), \end{aligned}$$

where

$$R_n = \frac{3}{2}na_n + b_n + O\left(\frac{1}{n^2}\right).$$

If we fix a small $\delta > 0$, then for n large enough

$$|Z_{n+1}| \leq |R_n| + |Z_n|\left(1 - \frac{\delta}{n}\right).$$

Without affecting the asymptotic behavior of Z_n, we can assume that the upper bound holds for all n. Using it recurrently we get that

$$\begin{aligned} |Z_n| &\leq |R_n| + \sum_{k=1}^{n-1}|R_k|\prod_{i=k}^{n-1}\left(1 - \frac{\delta}{i+1}\right) \\ &\leq \text{const}\sum_{k=1}^{n}\left(|R_k|\exp\left(-\sum_{i=k}^{n}\frac{\delta}{(i+1)}\right)\right) \\ &\leq \text{const}\sum_{k=1}^{n}|R_k|(k/n)^{\delta}. \end{aligned}$$

Then

$$\begin{aligned}\sum_{n=1}^{N_1}|\chi_n| &\leq \sum_{n=1}^{N_1}|Z_n|/n \\ &\leq \text{const}\sum_{n=1}^{N_1}\sum_{k=1}^{n}|R_k|k^{\delta}/n^{\delta+1} \\ &\leq \text{const}\sum_{k=1}^{N_1}|R_k|\sum_{n=k}^{N_1}k^{\delta}/n^{\delta+1} \\ &\leq \text{const}\sum_{k=1}^{N_1}|R_k|.\end{aligned}$$

The last sum is uniformly bounded on N, which completes the proof. □

Lemma 5.4. *For all $x \in E_N$ satisfying (5.1)* $\prod_{n=N_3}^{N}(1+\lambda_n) \asymp N^{1/3}$.

Proof. It is sufficient to show that, for $m = N - n + 1$,

$$\lambda_m = \frac{1}{3m} + \chi_m; \quad \text{where} \quad \sum_{m=2}^{N-N_3}\chi_m = O(1).$$

We have that

$$\lambda_{m-1} = \frac{2}{9m^2} + a_m + \left(1 + \frac{2}{3m} + b_m\right)\frac{\lambda_m}{1+\lambda_m},$$

where

$$a_m = O\left(\frac{\ln m}{m^3} + \frac{\gamma_m^2}{m^2}\right) \text{ e } b_m = O\left(\frac{\ln m}{m^2} + \frac{\gamma_m^2}{m} + \frac{x_m^{-4}}{m}\right).$$

Note that $|a_m| \leq c/m^2$ and $|b_m| \leq c/m$, for some $c > 0$ small enough.

Take

$$\lambda_m = \frac{1+Z_m}{3m}.$$

We have that

$$\begin{aligned}\frac{1+Z_{m-1}}{3(m-1)} &= \frac{2}{9m^2} + a_m + \left(1 + \frac{2}{3m} + b_m\right)\times \\ &\quad \times\left(\frac{1}{3m} + \frac{Z_m}{3m}\right)\left(1 - \frac{1}{3m} - \frac{Z_m}{3m} + O\left(\frac{1}{m^2}\right) + O\left(\frac{Z_m^2}{m^2}\right)\right) \\ &= \frac{2}{9m^2} + a_m + X_1 \cdot X_2 \cdot X_3.\end{aligned}$$

$$\begin{aligned} X_2 \cdot X_3 &= \frac{1}{3m} + \frac{Z_m}{3m} - \frac{1}{9m^2} - \frac{2Z_m}{9m^2} - \\ &\quad - \frac{Z_m^2}{9m^2} + O\left(\frac{1}{m^3} + \frac{Z_m}{m^3} + \frac{Z_m^2}{m^3} + \frac{Z_m^3}{m^3}\right), \end{aligned}$$

$$\begin{aligned} X_1 \cdot X_2 \cdot X_3 &= \frac{1}{3m} - \frac{1}{9m^2} + \frac{b_m}{3m} + \frac{Z_m}{3m} + \frac{b_m Z_m}{3m} - \\ &\quad - \frac{Z_m^2}{9m^2} + O\left(\frac{1}{m^3} + \frac{Z_m}{m^3} + \frac{Z_m^2}{m^3} + \frac{Z_m^3}{m^3}\right), \end{aligned}$$

Therefore

$$\begin{aligned} Z_{m-1} &= R_m + Z_m \times \\ &\quad \left(1 - \frac{1}{m} + b_m + O\left(\frac{1}{m^2}\right) - Z_m\left(\frac{1}{3m} + O\left(\frac{1}{m^2}\right)\right) + O\left(\frac{Z_m^2}{m^2}\right)\right), \end{aligned}$$

where

$$R_m = 3ma_m + b_m + O\left(\frac{1}{m^2}\right).$$

If we fix a small $\delta > 0$, then for n large enough

$$|Z_{m-1}| \leq |R_m| + |Z_m|\left(1 - \frac{\delta}{m}\right).$$

Without affecting the asymptotic behavior of Z_m, we can assume that the bound above holds for all $m \geq 3$. Using it recurrently we get

$$\begin{aligned} |Z_m| &\leq \sum_{k=m}^{N-N_3} |R_k| \prod_{i=m}^{k}\left(1 - \frac{\delta}{i}\right) \\ &\leq \text{const} \sum_{k=m}^{N-N_3}\left(|R_k| \exp\left(-\sum_{i=m}^{k} \frac{\delta}{(i)}\right)\right) \\ &\leq \text{const} \sum_{k=m}^{N-N_3} |R_k|(m/k)^{\delta}. \end{aligned}$$

Then

$$\begin{aligned}\sum_{m=2}^{N-N_3} |\chi_m| &\leq \sum_{m=2}^{N-N_3} |Z_m|/m \\ &\leq \text{const} \sum_{m=2}^{N-N_3} \sum_{k=m}^{N-N_3} |R_k| m^{\delta-1}/k^{\delta} \\ &\leq \text{const} \sum_{k=2}^{N-N_3} |R_k| \sum_{m=2}^{k} m^{\delta-1}/k^{\delta} \\ &\leq \text{const} \sum_{k=2}^{N} |R_k|.\end{aligned}$$

The last sum is uniformly bounded on N, which completes the proof. □

Proof of Theorem 5.1. Let dx be an unstable vector. At the exiting period, $\lambda_m \asymp 1/m$ and $\tau_m \asymp m^{-2/3}N^{1/6}$, for $m = 2, \ldots, N - N_3$. Therefore $B_m^+ = \frac{\lambda_m}{\tau_m} \asymp m^{-1/3}N^{-1/6}$, $m = 2, \ldots, N - N_3$. When $m = 1$, $\tau_N \asymp N^{1/6}$.

For $m = 1$ (or $n = N$),

$$B_N^+ \asymp B_{N-1}^+ \asymp N^{-1/6}.$$

Hence, from Lemma 5.3 and Lemma 5.4, we get

$$\frac{\|D_x T^{N_5+1}(dx)\|_p}{\|dx\|_p} \asymp N^{2/3} \times N^{1/3} \asymp N,$$

for all $x \in E_N$ satisfying $\gamma_1 \asymp N^{-1/3}$ and $\gamma_N \asymp N^{-1/3}$. □

6. Proof of Theorem B

Let $E_N = \{x \in M_4 \mid R(x) = N+1\}$. This set is as in Figure 6. It is bounded by curves, denoted by S^*, S^*_{N-1} and S^*_N, and by the line $r = 1$. The curve S^* is made up of points from M_4 that, leaving $\mathcal{L}$, they hit the dispersing part $\mathcal{U}$ tangentially at the first collision. This is a decreasing curve, since it is a singularity line for T_5 for positive values of φ until $(1, 0) \in M_5$; and it is not hard to calculate the slope of this line, obtaining that it has an horizontal tangency at $(1, 0) \in M_5$ Besides, the lines S^*_N separating E_N and E_{N+1} are constituted of trajectories which the last collision in $\mathcal{U}$ leaving out the cusp is tangent. So, they are singularity lines for T_5^N, and then , decreasing lines and regular as consequence of the results in [5, Chapter 4].

The images $F_N = T_4(E_N) = T_5^{N+1}(E_N)$ are domains bounded by singularity lines for T_5^{-i}, $i = 1, 2, \ldots, N$, which are curves with positive slope. Moreover, by the property of time-reversing of the billiard map, $(r, \varphi) \in E_N$ if, and only if,, $(r, -\varphi) \in F_N$. So F_N is obtained reflecting E_N along the line $\varphi = 0$.

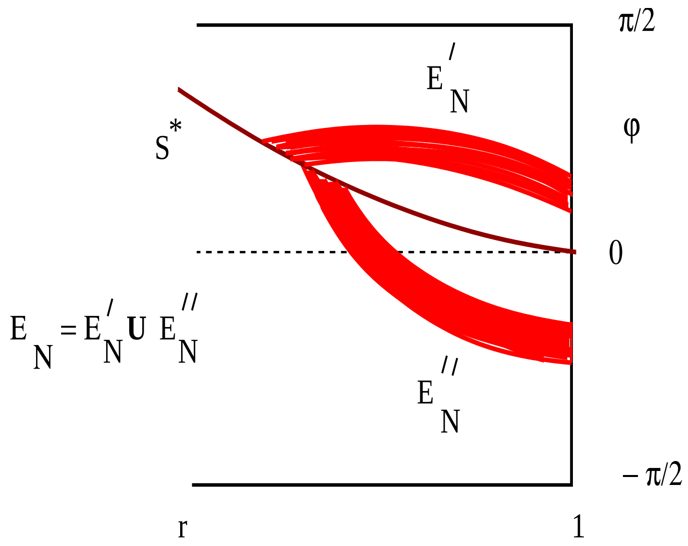
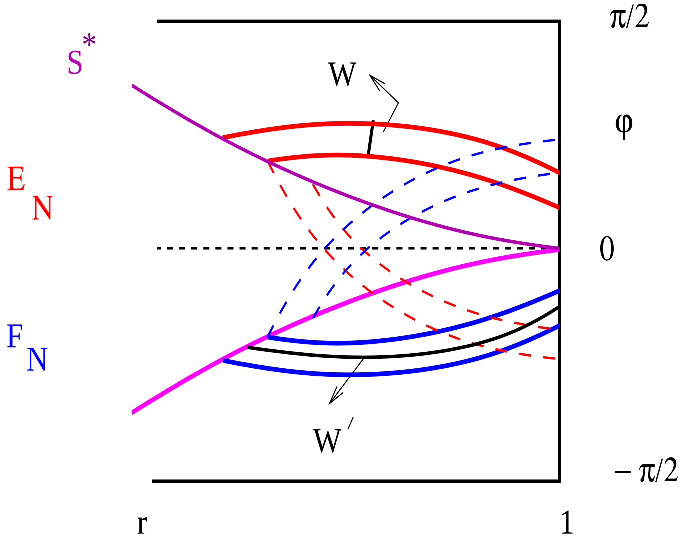


FIGURE 6. The sets E'_N and E''_N.

The domain E_N close to $(1,0) \in M_5$ is made up of two strips: the inferior strip E''_N, which consists of points that leave the vertical wall and hit the cusp directly; and the superior strip E'_N, which consists of points that hit the cusp after a rebound in the horizontal part of the table $\mathcal{D}$. The sets E_N, $N > N_0$, make up a nested structure that shrink to (1,0) as N goes to infinity (it is enough to note that in order to achieve more rebounds inside the cusp, on $\mathcal{D}$, we must begin closer to the point $(0,0)$ and the particle must be thrown almost parallel with respect to the axis x).

The point from E_N farthest from $(1,0) \in M_5$ over S^* is at a distance $\asymp N^{-1/6}$ because $x_1 \asymp N^{1/6}$ on $\mathcal{D}$ (given by equation (4.4)). Over $r = 1$, using the values of x_1 and γ_1 obtained in (4.4) and (4.5), respectively, and a simple geometric construction, we get that the distance of the strip E''_N to the point $(1,0)$ is $\asymp N^{-1/3}$. Since the lines S^*, S^*_{N-1} and S^*_N are decreasing and S^* has horizontal tangent on$(1,0)$, the "length" of each strip of E_N is $\asymp N^{-1/6}$.

Now consider an unstable curve W inside one of the strips of E_N, transverse to the direction of S^*_N, by the relation between cones and singularity lines given by condition (C6) from [15, Section 8]. Using the symmetry of $T_5^{N+1}(E_N)$, the set $T_5^{N+1}(W)$ is a line stretching "from top to bottom" one of the strips of $F_N = T_5^{N+1}(E_N)$, therefore, it has "length" $\asymp N^{-1/6}$. Using the fact that the derivative of T_5^{N+1} has an expansion rate of $\asymp N$ for unstable vectors, given by Theorem 5.1, we get that $|W| \asymp N^{-1/6}/N = N^{-7/6}$. This is the "width" of each of the strips of E_N.



Since the sets E_N are away from $\varphi = \pm\pi/2$, the measure μ is equivalent to the Lebesgue measure on $\mathbb{R}^2$. Thus,

$$\mu(E_N) \asymp N^{-1/6} \times N^{-7/6} = N^{-4/3},$$

so

$$\mu\left(\bigcup_{n=N}^{\infty} E_n\right) \asymp N^{-1/3},$$

hence $A = \bigcup_{n=N}^{\infty} E_n$ has finite measure.

Also the measure of the intersection $E_m \cap T_5^m E_m$ can be computed using the symmetry of the sets E_m and F_m,

$$\mu\left(E_m \cap T_5^m E_m\right) \asymp m^{-7/6} \times m^{-7/6} = m^{-7/3}.$$

Thus

$$\mu\left(A \cap T_5^m A\right) = \mu\left(\bigcup_{n=N}^{\infty} E_n \cap T_5^m\left(\bigcup_{n=N}^{\infty} E_n\right)\right) \geq \mu\left(E_m \cap T_5^m E_m\right) \asymp m^{-7/3},$$

showing that the speed of decay is at most polynomial.

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Appendix A. Mixing systems and another proof of Corollary 1.1

In this section we explain a sufficient condition for a system to be F-mixing. It is based on the work of Coudene [8] adapted to our definition of F-mixing. From now on X is a metric space, $\mathcal{A}$ the Borel σ-algebra of X, μ an infinite σ-finite regular measure on X and $\mathcal{F} : X \to X$ a μ-measure preserving transformation.

Definition A.1. ([8, Definition 1]) We define the *stable distribution* of $\mathcal{F}$ of a point $x \in X$ as

$$W^s(x) = \{y \in X : d(\mathcal{F}^n(x), \mathcal{F}^n(y)) \to 0 \text{ as } n \to \infty\}.$$

A measurable function $f : X \to I\!\!R$ is called W^s*-invariant* when there exists a set $\Omega \subset X$ with full measure such that for all $x, y \in \Omega$, $y \in W^s(x)$ implies $f(x) = f(y)$.

If $\mathcal{F}$ is invertible we define the *unstable distribution* $W^u(x)$ of a point x for $\mathcal{F}$ as the stable distribution for $\mathcal{F}^{-1}$. In a similar way , we define a W^u-invariant function.

We say that the stable distribution W^s is ergodic if every W^s-invariant function is μ-almost everywhere.

The propositions below are slight modifications of Theorem 2 and Theorem 3 from [8]. Indeed, since the main elements used in the proof are the Banach-Alaoglu Theorem and Banach-Saks Theorem, which are true in Hilbert spaces, there will be few changes in the proofs. We also assume that the measure is regular because we use the fact that continuous functions with compact support are dense in $L^2_\mu(X)$ [21, p.69].

Proposition A.2. (Based on [8, Theorem 2]) *Let X be a metric space, μ a regular infinite σ-finite measure on X, $\mathcal{F} : X \to X$ a μ-measure-preserving transformation and $f \in L^2_\mu(X)$. Then any weak limit of $f \circ \mathcal{F}^n$ is W^s-invariant.*

Proof. Let g be a weak limit of $f \circ \mathcal{F}^{n_i}$. First assume that f is continuous with compact support (thus uniformly continuous). The Banach-Saks theorem guarantees that there exist subsequences m_l and n_{i_k} such that

$$\Psi_l(x) = \frac{1}{m_l} \sum_{k=1}^{m_l} f \circ \mathcal{F}^{n_{i_k}} \stackrel{l\to\infty}{\longrightarrow} g \quad \mu - q.t.p..$$

If $y \in W^s(x)$, then

$$|\Psi_l(x) - \Psi_l(y)| \le \frac{1}{m_l} \sum_{k=1}^{m_l} |f \circ \mathcal{F}^{n_{i_k}}(x) - f \circ \mathcal{F}^{n_{i_k}}(y)| \stackrel{l\to\infty}{\to} 0.$$

So g is W^s-invariant.

Let $f \in L^2_\mu$. For all $\varepsilon > 0$, there exists a continuous function f_0 with compact support such that $\|f - f_0\|_2 < \varepsilon$. Passing to a subsequence, by Banach-Alaoglu Theorem, we can assume that $f_0 \circ \mathcal{F}^{n_i}$ converges weakly to a function g_0 which is W^s-invariant. It follows that $(f - f_0) \circ \mathcal{F}^{n_i} \longrightarrow g - g_0$ weakly, which implies that

$$\|g - g_0\|_2 \leq \liminf \|(f - f_0) \circ \mathcal{F}^{n_i}\|_2 \leq \|f - f_0\|_2 < \varepsilon.$$

Thus there exists a sequence of W^s-invariant functions that converges to g in the L^2_μ-norm and, passing to a subsequence, almost everywhere. Hence, for a set Ω with full measure, if $y, x \in \Omega$, $y \in W^s(x)$, we get that

$$g(y) = \lim g_n(y) = \lim g_n(x) = g(x).$$

This shows that g is W^s-invariant. □

Using the proposition above, the next one is proved as in [8].

Proposition A.3. (Based on [8, Theorem 3]) *Let X be a metric space, μ a regular infinite σ-finite measure on X, $\mathcal{F} : X \to X$ an invertible μ-measure-preserving transformation and $f \in L^2_\mu(X)$. Then any weak limit of $f \circ \mathcal{F}^n$ is W^s-invariant and W^u-invariant.*

Corollary A.4. *If W^s is ergodic then $\mathcal{F}$ is F-mixing.*

Proof. Let $f \in L^2_\mu$. If $f \circ \mathcal{F}^n$ has a weak limit, by Proposition A.2 and hypothesis, this limit is constant at almost every point; thus it is equal to zero at almost every point. Therefore F is F-mixing.

Suppose that $f \circ \mathcal{F}^n$ does not converge weakly to zero. Thus there exist an $\varepsilon > 0$, a subsequence n_i and a function $h \in L^2_\mu$ such that

$$\lim_{i\to\infty} \int (f \circ \mathcal{F}^{n_i}) h \, d\mu > 0.$$

However by Banach-Alaoglu Theorem there exists a subsequence n_{i_k} such that $f \circ \mathcal{F}^{n_{i_k}}$ converges weakly to a function W^s-invariant, by Proposition A.2, which is constant by hypothesis and hence must be zero almost everywhere. This contradiction shows that $\mathcal{F}$ is F-mixing. □

Proof of Corollary 1.1. Take a function $\psi : X \to I\!R$ which is W^s-invariant and W^u-invariant. By the property of absolute continuity of the local stable and unstable manifolds [15], Theorem 7.5, W^s-invariance and W^u-invariance imply that this function must be constant almost everywhere in the ergodic component of T_5. However, since T_5 has only one ergodic component ψ is constant almost everywhere, that is, W^s and W^u are ergodic. Thus, by Corollary A.4, T_5 is F-mixing □

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