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SECTIONAL LYAPUNOV EXPONENTS

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Abstract. We define sectional Lyapunov exponents and use it to characterize Sectional Anosov flows in terms of dominated splittings. In particular we improve a result in [10].

1. Introduction

The theory of hyperbolic systems appeared in the sixties, with the seminal works of Smale, introducing the horseshoe. The main feature of this dynamical system was the presence of complementary directions in the tangent bundle, one of them presenting a contractive behaviour and the other presenting an expansive behaviour. Since then the theory of hyperbolic dynamics grew, in particular to understand Anosov diffeomorphisms, which presents this behaviour in the entire manifold, and expanding automorphisms, which presents only expansion in all directions but are non-invertible.

Furthermore the theory evolved in direction to go beyond uniform hyperbolicity, and one of the directions was the study of non-uniformly dynamical systems, with the aid of the so called Lyapounov exponents. These exponents, when they exist and are nonzero, indicates *asymptotic* contraction or expansion along the orbit. The celebrated Oseledets theorem says that these exponents exist, and vary in a measurable way, in a set of full measure, for any invariant probability measure of the system.

It turns out that, for local diffeomorphisms, if all the Lyapunov exponents are positive in a set of *total probability*, i.e. a full measure set for any invariant measure, then the map is expanding, as showed in Alves-Arajo-Saussol [2] and Cao [7]. An analogous statement is true in the case of diffeomorphisms with an invariant dominated splitting, if all the Lyapunov are negative in one subbundle and positive in the other in a set of total probability then the diffeomorphism is Anosov.

In the case of vector fields, the hyperbolic theory also has a good understanding. In particular for Anosov flows, but now, the hyperbolic behaviour occurs transversally to the direction of the vector field. This automatically rules out the presence of singularities when the manifold is closed. Also, the Anosov flows share many features of hyperbolic diffeomorphism, like stability, spectral decomposition, etc.

In the presence of singularities there exist still flows which present some dynamical properties in a robust way and some weak form of hyperbolicity, where the so called *Lorenz attractor* is the paradigmatic example [11], which are called sectional Anosov flows [8]. These flows occur in manifolds with boundary and also posses a dominated splitting, where one direction is uniformly contractive and the other uniformly expands area of 2-planes inside it. There are much efforts to understand dynamical properties of this flows, for instance see [3], [4] [6] and [8].

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In this note, we characterize sectional Anosov flows in terms of the Lyapunov exponents in a set of total probability as in the above cited works, but now we also use sectional Lyapunov exponents to study these flows, these exponents are Lyapunov exponents of the cocycle defined by the second exterior power of the derivative, see the next section for precise definitions. The case of Anosov flows are included in our result, considering closed manifolds instead only compact manifolds. Moreover we also extend a result found in Sataev's paper [10].

2. Statement of the results

Let M be a C^∞ n-dimensional connected compact Riemannian manifold, M may have boundary or not.

Let $\{\varphi_t : M \to M\}_{t\in\mathbb{R}}$ be a flow generated by a C^1-vector field X. We say that a probability measure μ is an invariant measure if $\mu(\varphi_t(A)) = \mu(A)$ for every measurable set A and every $t \in \mathbb{R}$. We say that a subset $Y \subset M$ is a set of *total probability* if $\mu(Y) = 1$ for every invariant measure μ.

We will assume that X is inwardly transverse to the boundary ∂M. The maximal invariant set of the flow is defined as $M(X) = \bigcap_{t\geq 0} \varphi_t(M)$. As usual, we say that a singularity is hyperbolic if the eigenvalues of the derivative of the vector field at the singularity have nonzero real part.

Definition 2.1. We say that the flow is sectional-Anosov if every singularity is hyperbolic and there exists a continuous invariant splitting $T_{M(X)}M = E \oplus F$ over the maximal invariant set and constants $C > 0$ and $\lambda > 0$ such that for every $x \in M(X)$ and $t \geq 0$:

(i) The splitting is not trivial: $E_x \neq \{0\}$ and $F_x \neq \{0\}$.
(ii) The splitting is dominated: $\|D\varphi_t|_{E_x}\|\|D\varphi_{-t}|_{F_{\varphi_t(x)}}\| < Ce^{-\lambda t}$.
(iii) The subbundle E is contracting: $\|D\varphi_t(x)v\| \leq Ce^{-\lambda t}$, for every $v \in E_x - \{0\}$.
(iv) The subbundle F is sectionally expanding: For every 2-plane section $L \subset F$, if we denote $L_x \subset F_x$ the 2-plane in the subspace F_x then

$$|det(D\varphi_t(x)|_{L_x}| > Ce^{\lambda t}.$$

If only the items (i) and (ii) are satisfied we say that the splitting $E \oplus F$ is a dominated splitting over $M(X)$.

We observe that if the splitting F decomposes as a continuous splitting $F = \langle X\rangle \oplus G$, where $\langle X\rangle$ is the one-dimensional distribution generated by the vector field then Λ is a hyperbolic set, indeed since in the direction of the vector field the dynamics is an isometry, the expansion of area must come from the G subbundle and this will imply that G is uniformly expanding. In particular, in this case the continuity of the splitting implies that there are no singularities. Moreover, if $\partial M = \emptyset$ then $M(X) = M$ and we recover the definition of an Anosov flow. For more details, see for instance [5].

By Oseledet's theorem [9], for any invariant probability measure μ there exists a subset Y with $\mu(Y) = 1$ such that, for every $x \in Y$ there exists an invariant splitting:

$$T_xM = \langle X\rangle \oplus E_x^1 \oplus \cdots \oplus E_x^{s(x)}$$

And the following limits exists:

$$\lambda_i(x) = \lim_{t\to+\infty} \frac{1}{t} \log \|D\varphi_t(x).v\| \text{ , for every } v \in E_x^i - \{0\} \text{ , } i = 1, \ldots, s(x).$$

Moreover, the functions λ_i and s are measurable and invariant by the flow i.e. $s(\varphi_t(x)) = s(x)$ and $\lambda_i(\varphi_t(x)) = \lambda_i(x)$. Also, the splitting varies measurably. The numbers $\lambda_i(x)$ are called the *Lyapounov exponents* of the flow at the point x. If E is a subbundle of the tangent bundle then by "the Lyapounov exponents of E" we mean the Lyapounov exponents of the nonzero vectors in E.

Let V be a vector space, we denote by $\Lambda^2 V$ the second exterior power of V, defined as follows. If $v_1, \dots, v_n$ is a basis of V then $\Lambda^2 V$ is generated by $\{v_i \wedge v_j\}_{i \neq j}$. Any linear transformation $A : V \to W$ induces a transformation $\Lambda^2 A : \Lambda^2 V \to \Lambda^2 W$. Moreover, $v_i \wedge v_j$ can be viewed as the 2-plane generated by v_i and v_j if $i \neq j$. See for instance [1] for more informations.

Definition 2.2. The *sectional Lyapunov exponents* of x along F are the limits

$$\lim_{t \to +\infty} \frac{1}{t} \log \|\Lambda^2 D\varphi_t(x).\widetilde{v}\|$$

whenever they exists, where $\widetilde{v} \in \Lambda^2 F_x - \{0\}$.

It turns out that if μ is an invariant probability measure, Y is the subset given by Oseledet's theorem and $\{\lambda_i(x)\}_{i=1}^{s(x)}$ are the Lyapunov exponents then the sectional Lyapunov exponents of a point $x \in Y$ are $\{\lambda_i + \lambda_j\}_{1 \leq i < j \leq s(x)}$. Moreover, the (iv) condition in the definition of sectional Anosov flow can be obtained as follows. If L_x is a 2-plane, then it can be saw as $\widetilde{v} \in \Lambda^2(F_x) - \{0\}$ of norm one. Hence, to obtain the sectional expansion we only need to show that for some $\lambda > 0$ and every $t > 0$ the following inequality holds

$$\|\Lambda^2 D\varphi_t(x).\widetilde{v}\| > Ce^{\lambda t}.$$

Our main result is the following.

Theorem 2.3. *Let $\{\varphi_t\}$ be a flow with a dominated splitting $T_{M(X)}M = E \oplus F$ over the maximal invariant set and such that every singularity is hyperbolic. The flow $\{\varphi_t\}$ is a sectional Anosov flow if, and only if, the Lyapunov exponents in the E direction are negative and the sectional Lyapunov exponents in the F direction are positive on a set of total probability. If the manifold has no boundary, the flow has no singularities and it is an Anosov flow.*

We remark that this theorem improves a result found in [10]. First, let us recall the notation used by Sataev and set the scenario.

Assume that $M(X)$ has a dominated splitting $T_{M(X)}M = E \oplus F$, where the subbundle E is uniformly contractive, and suppose that $dimF = 2$. Moreover, we assume that the fibers of E are oriented.

Let Ω be the volume form of the manifold, now we define a 2-form ω given by:

$$\omega(w_1, w_2) = \Omega(w_1, \dots, w_n) \text{ for } w_1, w_2 \in T_x M$$

where $w_3, \dots, w_n$ form a parallelepiped of E_x with unit volume and positive orientation. Hence the rank space of ω is the orthogonal complement of E.

Using the invariance of the subbundle E, Sataev proves that there exists a function $\theta(x)$ such that

$$\dot{\omega}(x) = -\omega(x) D_2(x) + \theta(x)\omega(x)$$

where D_2 acts as follows. Denote by $D(x)$ the matrix with elements $\{\frac{\partial X_i}{\partial x_j}\}$, where X_i are the coordinates of the vector field X in the local coordinates $\{x_j\}$. Now, given two one-forms p_1 and p_2 and $p = p_1 \wedge p_2$ we set

$$pD_2 = p_1 D \wedge p_2 + p_1 \wedge p_2 D$$

finally D_2 is defined by linearity.

As noticed in [10][p.54], it turns out that if K_x is a small cone field containing the F subbundle then there exists constants C_1 and C_2, such that for any v_1 and v_2 in the cone, if we denote $A(L)$ as the area, defined by the Riemannian metric, of the plane generated by this two vectors then we have

$$C_1|\omega(v_1, v_2)| \leq A(L) \leq C_2|\omega(v_1, v_2)|.$$

Let $x \in M(X)$, the previous remark shows that the linear growth of the integral of the function θ along the orbit of x is equivalent to the function $\log \|\Lambda^2 D\varphi_t(x)\widetilde{v}\|$, where $\widetilde{v} \in \Lambda^2 F_x$ has norm equal to one.

Indeed, Sataev proves that the following equality holds:

$$\omega(\varphi_t(x)) = e^{\int_0^t \theta(\varphi_s(x))ds} D_{*2}\varphi_t(\omega(x)),$$

where D_{*2} is the action of the differential on 2-forms. But the expansion of area is equivalent to the existence of constants $C > 0$ and $\gamma > 0$ such that:

$$\omega(\varphi_t(x))(D\varphi_t(x)v_1, D\varphi_t(x)v_2) > Ce^{\gamma t}\omega(v_1, v_2).$$

Hence our theorem implies:

Corollary 2.4 (Lemma 2.12 of [10])**.** *In this setting, the flow is sectional Anosov if and only if all singularities are hyperbolic and there exists constants $C \in \mathbb{R}$ and $D > 0$ such that for every orbit $\varphi_t(x)$ and every $T > 0$ the following inequality holds:*

$$\int_0^T \theta(\varphi_t(x))dt > C + DT.$$

Proof. The last statement obviously is a necessarily condition to the flow be sectional Anosov. Now, if the inequality holds for every orbit, then dividing by T and taking limits we have that the sectional Lyapunov exponents in the F direction are positive for every orbit, in particular in a set of total probability hence theorem 2.3 applies. □

We also remark that our results also holds to sectional hyperbolic *sets*, i.e. with the same definition using a compact invariant set Λ instead of the maximal invariant set $M(X)$.

3. Proof of the Theorem

In this section we prove theorem 2.3, following the lines in [7].

Lemma 3.1. *Let $f : M(X) \to \mathbb{R}$ be a continuous function such that $\int f d\mu < \lambda$ for any invariant probability measure μ, then for every $x \in M(X)$ there exists $t(x) > 1$ such that:*

$$\frac{1}{t(x)} \int_0^{t(x)} f(\varphi_s(x))ds < \lambda.$$

Proof. If not, there exists x such that $\frac{1}{t}\int_0^t f(\varphi_s(x))ds \geq \lambda$ for every $t > 0$. Hence we define the measures $\mu_t = \frac{1}{t}\int_0^t \delta_{\varphi_s(x)}ds$, where δ_x is the Dirac measure at x. Now, we take $\mu = \lim_{k\to\infty} \mu_{t_k}$, as a cluster point of this sequence, with $t_k \to \infty$.

It is well know that μ is an invariant measure, moreover since f is continuous we have that

$$\int f d\mu = \lim_{k\to\infty} \int f d\mu_{t_k} = \lim_{k\to\infty} \frac{1}{t_k}\int_0^{t_k} f(\varphi_s)ds \geq \lambda.$$

□

Corollary 3.2. *If $f : M(X) \to \mathbb{R}$ is a continuous function such that $\int f d\mu < \lambda$ for any invariant probability measure μ, then there exists $T > 0$ such that for any $t \geq T$ we have:*

$$\frac{1}{t}\int_0^t f(\varphi_s(x))ds < \lambda \textit{ for all } x \in M(X).$$

Proof. The previous lemma says that for every x there exists some $t(x) > 1$ and $\varepsilon(x) > 0$ such that $\int_0^{t(x)} f(\varphi_s(x))ds < t(x)(\lambda - \varepsilon(x))$.

Since f is continuous, there exists some neighborhood U_x such that for any $y \in U_x$ we have that $\int_0^{t(x)} f(\varphi_s(y))ds < t(x)(\lambda - \varepsilon(x))$.

By compactness, we can cover M by a finite number of such neighborhoods, say $U_{x_1}, \ldots, U_{x_n}$, and define $T_0 = \max\{t(x_1), \ldots t(x_n)\}$ and $\varepsilon = \min\{\varepsilon(x_1), \ldots \varepsilon(x_n)\}$.

Now, we define by induction a sequence of functions $T_k : M \to \{1, \ldots, T_0\}$ for $k \geq 1$ as follows;

$$\begin{aligned} T_0(x) &= 0 \\ T_1(x) &= \min\{t(x_i); x \in U_{x_i},\ i = 1, \ldots n\} \\ T_{k+1}(x) &= T_k(x) + T_1(\varphi_{T_k(x)}(x)). \end{aligned}$$

Hence, for any $x \in M$ and $t > 0$ there exists k such that $T_k(x) \leq t \leq T_{k+1}(x)$, since $T_k(x) \to \infty$. In particular,

$$\int_0^t f(\varphi_s(x))ds \leq T_k(x)(\lambda - \varepsilon) + \|f\|_0 T_0.$$

Hence if $T = \frac{2\|f\|_0 T_0}{\varepsilon}$ we have that, for any $x \in M$ and $t \geq T$:

$$\frac{1}{t}\int_0^t f(\varphi_s(x))ds < \lambda.$$

□

We say that a family of functions $\{f_t : M(X) \to \mathbb{R}\}_{t\in\mathbb{R}}$ is sub-additive if for every $x \in M$ and $t, s \in \mathbb{R}$ we have that $f_{t+s}(x) \leq f_s(x) + f_t(\varphi_s(x))$. The Subadittive Ergodic theorem shows that the function $\overline{f}(x) = \liminf_{t\to+\infty} \frac{f_t(x)}{t}$ coincides with $\widetilde{f}(x) = \lim_{t\to+\infty} \frac{1}{t} f_t(x)$ in a set of total probability.

Remark 3.3. For any invariant measure μ we have that $\int \widetilde{f}d\mu = \lim_{t\to+\infty} \int \frac{f_t}{t} d\mu$.

Proposition 3.4. *Let $\{t \mapsto f_t : M(X) \to \mathbb{R}\}_{t\in\mathbb{R}}$ be a continuous family of continuous function which is sub-additive and suppose that $\overline{f}(x) < 0$ in a set of total probability. Then there exists constants $C > 0$ and $\lambda < 0$ such that for every $x \in M$ and every $t > 0$:*

$$e^{f_t(x)} \leq C^{-1} e^{\frac{\lambda t}{2}}.$$

Proof. The hypothesis says that $\widetilde{f}(x) < 0$ in a set of total probability. Hence, $\int \widetilde{f}d\mu < 0$ for every invariant measure μ. By the previous remark, for any invariant measure μ there exist $t(\mu) \in \mathbb{R}$ such that for every $t \geq t(\mu)$:

$$\int \frac{f_t}{t} d\mu < \frac{1}{2}\int \widetilde{f}d\mu.$$

Hence, there exists a neighborhood U_μ of μ in the weak-* topology such that if $\eta \in U_\mu$ then:

$$\int \frac{f_{t(\mu)}}{t(\mu)} d\eta < \frac{1}{4}\int \widetilde{f}d\mu.$$

By weak compactness, the set of invariant measures can be cover by a finite number of such neighborhoods, say $U_{\mu_1}, \ldots, U_{\mu_n}$. Let $t(i) = t(\mu_i)$ for $i = 1, \ldots n$ (we can suppose $t(i) > 1$) and define

$$\lambda = \max_{i=1,\ldots,n}\{\frac{1}{4}\int \widetilde{f}d\mu_i\} < 0.$$

In particular, for every invariant measure μ there exists some i such that $\int f_{t(i)}d\mu < \lambda t(i)$ and

$$\begin{aligned}\frac{1}{kt(i)}\int f_{kt(i)}(x)d\mu &\leq \frac{1}{k}\sum_{j=0}^{k-1}\frac{1}{t(i)}\int f_{t(i)}(\varphi_{jt(i)}(x))d\mu \\ &= \frac{1}{k}\sum_{j=0}^{k-1}\frac{1}{t(i)}\int f_{t(i)}(x)d\mu \\ &< \lambda\end{aligned}$$

Hence, if $L = t(1)t(2)\dots t(n)$ then $\int f_L d\mu < \lambda L$.

Now, by the previous corollary applied to the continuous function $\frac{f_L}{L}$, there exists some T_0 such that, for any $t \geq T_0$ and any $x \in M$ we have:

$$\frac{1}{t}\int_0^t \frac{1}{L}f_L(\varphi_s(x))ds < \lambda.$$

Since $f_{kL}(x) \leq \sum_{j=0}^{k-1} f_L(\varphi_{jL}(x))$, then for every $0 \leq s < L$ we obtain:

$$f_{kL}(x) \leq f_s(x) + \left(\sum_{j=0}^{k-2} f_L(\varphi_{jL+s}(x))\right) + f_{L-s}(\varphi_{(k-1)L+s}(x)).$$

Integrating,

$$Lf_{kL}(x) \leq \int_0^L \sum_{j=0}^{k-2} f_L(\varphi_{jL+s}(x))ds + \left(\int_0^L f_s(x) + f_{L-s}(\varphi_{(k-1)L+s}(x))ds\right).$$

Since $\{f_t\}$ is a continuous family of continuous functions over a compact manifold, we have that $B = \sup_{t\in[0,L]}\sup_{x\in M}|f_t(x)| < \infty$. Thus,

$$f_{kL}(x) \leq \int_0^{(k-1)L}\frac{1}{L}f_L(\varphi_s(x))ds + 2B.$$

In particular,

$$f_{kL}(x) \leq L(k-1)\lambda + 2B, \text{ if } (K-1)L > T_0.$$

Now, take $t \geq T_0 + 2L$ and write $t = kL + j$ where $0 \leq j < L$. So, $f_t(x) \leq f_{kL}(x) + f_j(\varphi_{kL}(x))$. Hence, $f_t(x) \leq L(k-1)\lambda + 3B$. Since $(k-1)L < t$, we have:

$$\frac{1}{t}f_t(x) \leq \lambda + \frac{3B}{t}.$$

Take $K = \max\{T_0 + 2L, \frac{6B}{-\lambda}\}$, and we obtain that for every $x \in M$ and $t \geq K$:

$$\frac{1}{t}f_t(x) \leq \frac{\lambda}{2}.$$

Finally, define $C^{-1} = \sup_{s\in[0,K]}\{e^{f_s(x)}, 1\}$. And note that,

$$e^{f_t(x)} \leq C^{-1}e^{\frac{\lambda t}{2}} \text{ for every } t > 0.$$

□

Proof of theorem 2.3. Let $\phi_t(x) = \log \|D\varphi_t|_{E_x}\|$ and $\psi_t(x) = \log \|\Lambda^2 D\varphi_{-t}|_{F_x}\|$, both defined on $M(X)$ and continuous, since the subbundles are continuous.

Now, applying the proposition to the function ϕ_t we obtain that there exists some constant $C > 0$ and $\lambda < 0$ such that $e^{\phi_t(x)} \leq C^{-1}e^{\lambda t}$, hence $\|D\varphi_t|_{E_x}\| \leq C^{-1}e^{\lambda t}$ for every x and this shows that E is a contractive subbundle. Analogously, using ψ_t we obtain a constant $D > 0$ and $\eta < 0$ such that $\|\Lambda^2 D\varphi_t|_{F_x}\| \geq De^{-\eta t}$ and this shows that F is a sectionally expanding subbundle.

The proof is now complete.

□

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