On the Confidence Preferences Model *

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Abstract

In this paper we study the model of decision under uncertainty consistent with confidence preferences. In that model, a decision maker held beliefs represented by a *fuzzy set of priors* and tastes captured by a standard affine utility index on consequences. First, we find some interesting properties concerning the well-known maxmin expected utility model, taking into account the point of view of the confidence preferences model. Further, we provide new examples of preferences that capture ambiguity averse attitudes weaker than ambiguity attitudes featured by maxmin expected utility theory. Finally, we discuss the axiomatic foundations for the confidence preferences model with optimistic behavior. *Journal of Economic Literature Classification Number*: D81.

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1 Introduction

Seminal works in economic theory have pointed out the relevance of unmeasurable uncertainty to economic life. The original distinction comes from Knight [14] and Keynes [13], and the main point is that well specified subjective probabilities, as later proposed in the classical Subjective Expected Utility (SEU) theory of Savage [18], are not behaviorally consistent with vagueness about the relative likelihood of events. Such ambiguity, according to Ellsberg [8], relates to

> a quality depending on the amount, type, reliability and unanimity of information, and giving rise to one´s degree of "confidence" in an estimative of relative likelihood [page 657].

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Indeed, as demonstrated by Ellsberg [8], agents often make choices that cannot be generated by a unique prior, thereby exhibiting an aversion to choices involving ambiguity. In modern decision theory, ambiguity has been an important issue and its main contribution is to provide rigorous foundations for situations where there may be no probabilities on the states of nature that rationalize the decision maker's behavior.

An important line of research replaces the subjective expected utility function with a more general functional capturing *ambiguity aversion*, such as the Choquet expected utility (CEU) of Schmeidler ([19], [20]) or the maxmin expected utility (MEU) of Gilboa and Schmeidler [11][1]. Decision makers with MEU preferences evaluate an act using the minimum expected utility over a given nonempty, convex and (weakly*) compact subset C of the set Δ of all probabilities on the state space, while decision makers with CEU preferences evaluate an act using its expected utility computed according to a capacity (fuzzy measure or nonadditive probability)[2]. Although these models are not the same in general, they coincide in the case of ambiguity aversion, that is, for CEU with a convex capacity. In this case, the Choquet expected utility with respect to a capacity v reduces to the minimum expected value over the set of probability distributions given by the core of capacity v[3].

Chateauneuf and Faro [5] propose a model for which the ambiguity is measured by a (regular*) fuzzy set of priors $\varphi : \Delta \to [0, 1]$, called *confidence function.* Intuitively, for each probability measure p the value of $\varphi(p)$ describes the relative confidence of the decision maker in the probabilistic model p, describing her belief as a fuzzy set in the sense proposed by Zadeh [27]. We note that the multiple priors model might be viewed as a special case of confidence preferences by imposing a binary assessment of confidence in the universe of priors[4]. Concerning its functional representation, confidence preferences feature agents that rank acts f according to the following criterion

$$J(f) = \min_{p \in \{q \in \Delta : \varphi(q) \geq \alpha_0\}} \frac{1}{\varphi(p)} \int u(f) dp,$$

where $\varphi : \Delta \to [0, 1]$ is a mapping representing the agent´s degree of confidence on the possible models p in Δ, α_0 is the threshold level of confidence below which a model is discarded, u is a positive valued affine utility index. In this sense, vagueness about the true probability law in this model is captured by a fully subjective fuzzy set of priors φ, which seems to be a meaningful way for modeling a decision maker who has a relative assessment of probability measures over the states of nature. Note also that we recover the MEU model if, and only

[1] The MEU model is also widely known as multiple priors model. See also, Chateauneuf [4].

[2] An interesting case has been proposed by Wakker [25], providing a behavioral foundation for fuzzy measures given by possibility measures.

[3] See Schmeidler ([19], [20]).

[4] A similar idea of replacing a set of probabilities by a fuzzy set was independently proposed by Walley [26].

if, $\alpha_0 = 0$ or φ is a characteristic function $\mathbf{1}_C$ of some set of priors C[5].

In this paper we study ambiguity averse preferences consistent with the confidence preferences model. First, we study some properties of the MEU model from the point of view of the confidence preferences model showing how we can recover the maximal confidence function related to MEU preferences by a family of maximal confidence functions generated by SEU preferences. Further, we provide new examples of preferences that capture ambiguity-averse attitudes weaker than that captured by MEU preferences. Finally, we discuss axiomatic foundations for ambiguity-loving confidence preferences.

The remainder of the paper is organized as follows. After introducing the notations and setup in Section 2, we recall the axiomatic foundation and the main representation result of [5] in Subsection 3.1. In Subsection 3.2, we show how to recover the maximal confidence function of a MEU preference by a family of SEU preferences. In Subsection 3.3, we introduce a new special case of confidence preferences, namely the ϕ-confidence preferences generated by a perturbation of SEU preferences and we study its ambiguity attitudes in the sense Ghirardato and Marinacci [9]. In Subsection 3.4, we introduce another kind of new confidence preference inspired by the well-known index of relative entropy, namely entropic confidence preference, and also compare the obtained representation with the constraint preferences of Hansen and Sargent [12]. Subsection 3.5 presents a study of monotony continuity properties of confidence preferences. In Section 4 we discuss the axiomatic foundations related to ambiguity-loving confidence functions. Proofs are collected in the Appendix.

2 Notation and Framework

Consider a set S of *states of nature*, endowed with a σ-algebra Σ of subsets called *events*, and a non-empty set X of *consequences*. We denote by $\mathcal{F}$ the set of all (simple) *acts*: finite-valued functions $f : S \to X$ which are Σ-measurable[6]. Moreover, we denote by $B_0(S, \Sigma)$ the set of all real-valued Σ-measurable simple functions $a : S \to \mathbb{R}$. The norm in $B_0(S, \Sigma)$ is given by $\|a\|_\infty = sup_{s\in S} |a(s)|$ (called sup norm) and we can define the space of all bounded and Σ-measurable functions by $B(S, \Sigma) := \overline{B_0(S, \Sigma)}^{\|\cdot\|_\infty}$, i.e., $B(S, \Sigma)$ consists of all uniform limits of finite linear combinations of characteristic functions of sets in Σ (see Dunford and Schwartz [6], page 240). Also, given a set $K \subset \mathbb{R}$, we define $B_0(K) := \{a \in B_0(S, \Sigma) : a(s) \in K, \forall s \in S\}$, $B(K) := \overline{B_0(K)}^{\|\cdot\|_\infty}$, and $B^+ := B(\mathbb{R}_+)$.

A set-function $v : \Sigma \to [0, 1]$ is a capacity if: (i) $v(\emptyset) = 0$, $v(S) = 1$ and (ii) $\forall\ E, F \in \Sigma$ such that $E \subset F \Rightarrow v(E) \le v(F)$. Also, v is convex if for any $A, B \in \Sigma$, $v(A \cup B) + v(A \cap B) \ge v(A) + v(B)$. A capacity p is a (finitely additive) probability when for any $E, F \in \Sigma$ such that $E \cap F = \emptyset$ we have that

[5] That is,

$$\mathbf{1}_C : p \in \Delta \to \mathbf{1}_C(p) \in \{0, 1\},$$

where $\mathbf{1}_C(p) = 1$ iff $p \in C$.

[6] Let $\succsim_0$ be a binary relation on X, we say that a function $f : S \to X$ is Σ-measurable if, for all $x \in X$, the sets $\{s \in S : f(s) \succsim_0 x\}$ and $\{s \in S : f(s) \succ_0 x\}$ belong to Σ.

$p(E\cup F)=p(E)+p(F)$. A countably additive probabilities is a capacity such that, for any family of disjoint events $\{E_k\}_{k\in\mathbb{N}}$, $p\left(\bigcup_{k=1}^{\infty}E_k\right)=\sum_{k=1}^{\infty}p(E_k)$.

We denote by $\Delta:=\Delta(\Sigma)$ the set of all (finitely additive) probability measures $p:\Sigma\to[0,1]$ endowed with the natural restriction of the well-known weak* topology $\sigma(ba,B)$. Moreover, we denote by Δ^σ the set of all countably additive probabilities in Δ. In particular, given $q\in\Delta^\sigma$, we denote by $\Delta^\sigma(q)$ the set of all probabilities in Δ^σ that are absolutely continuous w.r.t. q, *i.e.*, $\Delta^\sigma(q)=\{p\in\Delta^\sigma:p\ll q\}$, where $p\ll q$ means that $\forall A\in\Sigma,\ q(A)=0\Rightarrow p(A)=0$.

Given a function $a\in B$, the Choquet integral of a with respect to v is given by

$$\int a dv=\int_{-\infty}^{0}[v(\{a\geq\lambda\})-1]\,d\lambda+\int_{0}^{+\infty}v(\{a\geq\lambda\})\,d\lambda.$$

Of course, if v is a probability measure we back to the usual additive notion of integration.

We say that a mapping $\varphi:\Delta\to[0,1]$ is normal if $\{\varphi=1\}\neq\emptyset$. Also, recall that φ is weak* upper semicontinuous if $\{\varphi\geq r\}$ is weak* closed for each $r\geq 0$ and φ is quasi-concave if for all $p,q\in\Delta$ and $\alpha\in[0,1]$ it is true that $\varphi(\alpha p+(1-\alpha)q)\geq\min\{\varphi(p),\varphi(q)\}$. A mapping $\varphi:\Delta\to[0,1]$ is a regular* fuzzy set if φ is normal, weak* upper semicontinuous and quasi-concave.

Throughout the paper we will call a regular* fuzzy set φ by a confidence function, which is the central concepty in this paper. Let $\varphi:\Delta\to[0,1]$ be a confidence function and $\alpha\in[0,1]$, we denote the related level set by $L_\alpha\varphi:=\{p:\varphi(p)\geq\alpha\}$. Recall that φ is a regular* fuzzy set (*i.e.*, a confidence functions) if and only if for all $\alpha\in[0,1]$ the level set $L_\alpha\varphi$ is nonempty, convex and weak* closed.

Clearly, note that $u(f)\in B_0(S,\Sigma)$ whenever $u:X\to\mathbb{R}$ and f belongs to $\mathcal{F}$, where the function $u(f):S\to\mathbb{R}$ is the mapping defined by $u(f)(s)=u(f(s))$, for all $s\in S$.

Let x belong to X, define $x\in\mathcal{F}$ to be the constant act such that $x(s)=x$ for all $s\in S$. Hence, we can identify X with the set $\mathcal{F}_c$ of constant acts in $\mathcal{F}$.

Additionally, we assume that X is a convex subset of a vector space. For instance, this is the case if X is the set of all finite-support lotteries on a set of prizes Z, as in the classic setting of Anscombe and Aumann [1].

Using the linear structure of X we can define as usual for every $f,g\in\mathcal{F}$ and $\alpha\in[0,1]$ the act:

$$\begin{aligned}\alpha f+(1-\alpha)g &: S\to X\\ (\alpha f+(1-\alpha)g)(s) &= \alpha f(s)+(1-\alpha)g(s)\end{aligned}$$

The decision maker's preferences are given by a binary relation $\succsim$ on $\mathcal{F}$, whose usual symmetric and asymmetric components are denoted by $\sim$ and $\succ$. Finally, for any $f\in\mathcal{F}$, an element $x_f\in X$ is a *certainty equivalent* of f if $x_f\sim f$.

Following the discussion and terminologies used in the Introduction, a preference relation $\succsim$ is called a CEU preference (Schmeidler, [20]) if there is a function $u : X \to \mathbb{R}$, called utility index, and a capacity v such that for all acts $f, g \in \mathcal{F}$

$$f \succsim g \text{ iff } \int u(f)\, dv \geq \int u(g)\, dv.$$

If the capacity v satisfies $v = p \in \Delta$, then we say that $\succsim$ is a SEU preference (Anscombe and Aumann [1]). Another important class of preferences is defined by: a preference relation $\succsim$ is said to be a MEU preference (Gilboa and Schmeidler, [11]) if there is a utility index $u : X \to \mathbb{R}$ and a weak* closed and convex set of probabilities $C \subset \Delta$ such that for all acts $f, g \in \mathcal{F}$

$$f \succsim g \text{ iff } \min_{p \in C} \int u(f)\, dp \geq \min_{p \in C} \int u(g)\, dp.$$

Finally, given a CEU preference represented by the pair (u, v), if the v is a convex capacity by a well know result (see, for instance, Schmeidler [19]) we have that the set of priors $core(v) := \{p \in \Delta : p(E) \geq v(E) \ \forall E \in \Sigma\}$ is nonempty, convex and weak* closed, and for any act $f \in \mathcal{F}$

$$\int u(f)\, dv = \min_{p \in core(v)} \int u(f)\, dp.$$

The immediate consequence is that CEU preferences with convex capacities is a subclass of MEU preferences.

Ghirardato and Marinacci [9] proposed a notion of absolute ambiguity aversion by building upon a notion of comparative ambiguity. This comparative ambiguity attitude says that: Given two preferences $\succsim_1$and $\succsim_2$, the preference relation $\succsim_1$ is *more ambiguity averse than* $\succsim_2$, if for all $x \in X$ and $f \in \mathcal{F}$,

$$f \succsim_1 x \Rightarrow f \succsim_2 x.$$

Note that two preference relations $\succsim_1$ and $\succsim_2$ satisfying the comparative ambiguity attitude above induces preferences relation on X that can be represented by the same utility index u on consequences[7]. Hence, ambiguity aversion is comparable across two decision makers only if their risk attitudes coincide.

The absolute notion of ambiguity aversion defined by Ghirardato and Marinacci [9] considers SEU preferences as benchmarks for ambiguity neutrality: We say that a preference relation $\succsim$ is *ambiguity averse* if it is more ambiguity averse than some SEU preference. Moreover, we will use a strictly notion of ambiguity averse saying that a preference relation $\succsim$ is strictly ambiguity averse if $\succsim$ is ambiguity averse but not a SEU preference.

[7] See, for instance, Chateauneuf and Faro [5], page 542, footnote 16.

3 Ambiguity-Averse Confidence Preferences

3.1 Axiomatic foundation and main representation

Chateauneuf and Faro [5] introduced and axiomatized a class of preferences in which ambiguity is measured by a confidence function $\varphi : \Delta \to [0,1]$, *i.e.*, φ is a weak* regular *fuzzy* set of priors. The value of φ describes the decision maker's confidence in the probabilistic model p; in particular $\varphi(p) = 1$ means that the decision maker has full confidence in p. Preferences in this model are represented by:

$$J(f) = \min_{p \in L_{\alpha_0}\varphi} \frac{1}{\varphi(p)} \int u(f) dp$$

where $L_{\alpha_0}\varphi$ is a set of measures with confidence above α_0.

The confidence function φ is not unique and an interesting characterization is obtained for the maximal confidence function $\varphi^* : \Delta \to [0,1]$,

$$\varphi^*(p) = \inf_{f \in \mathcal{F}} \left(\frac{\int u(f) dp}{u(c_f)} \right).$$

Also, by considering the confidence function φ^* we are able to rewrite the formula for the utility functional J as

$$J(f) = \min_{p \in \Delta} \frac{1}{\varphi^*(p)} \int u(f) dp$$

Such mapping φ^* could be viewed as the *maximal confidence function*, specifying upper confidence among priors that the decision maker may face in order to be consistent with the utility functional $J : \mathcal{F} \to \mathbb{R}$.

In order to characterize the confidence preferences model, Chateauneuf and Faro [5] consider a preference relation $\succsim$ on $\mathcal{F}$ and impose a set of axioms as described below[8].

First of all, there exists $x_* \in X$ such that $f \succsim x_*$ for every f belonging to $\mathcal{F}$. The constant act x_* is called the *worst consequence* and in this case we say that $\succsim$ is a bounded below preference relation.

Axiom 1 - Weak order non-degenerate. If $f, g, h \in \mathcal{F}$:

(completeness) either $f \succsim g$ or $g \succsim f$

(transitivity) $f \succsim g$ and $g \succsim h$ imply $f \succsim h$

there exists $(f, g) \in \mathcal{F}^2$ such that $(f, g) \in \succ$

Axiom 2 - Continuity. For all $f, g, h \in \mathcal{F}$ the sets:

$$\{\alpha \in [0,1] : \alpha f + (1-\alpha) g \succsim h\}, \{\alpha \in [0,1] : h \succsim \alpha f + (1-\alpha) g\} \text{ are closed.}$$

Axiom 3 - Monotonicity. For all $f, g \in \mathcal{F}$:

[8] For a discussion on the meanings of the axioms, see [5], page 538.

$$\text{if } f(s) \succsim g(s) \text{ for all } s \in S \text{ then } f \succsim g.$$

Axiom 4 - Uncertainty aversion. If $f, g \in \mathcal{F}$ and $\alpha \in (0,1)$:

$$f \sim g \Rightarrow \alpha f + (1-\alpha) g \succsim f$$

Axiom 5 - Worst independence. For all $f, g \in \mathcal{F}$ and $\alpha \in (0,1)$:

$$f \sim g \Rightarrow \alpha f + (1-\alpha) x_* \sim \alpha g + (1-\alpha) x_*.$$

Axiom 6 - Independence on X. For all $x, y, z \in X$:

$$x \sim y \Rightarrow \tfrac{1}{2}x + \tfrac{1}{2}z \sim \tfrac{1}{2}y + \tfrac{1}{2}z.$$

Axiom 7 - Bounded attraction for certainty. There exists $\delta \geq 1$ such that for all $f \in \mathcal{F}$ and $x, y \in X$:

$$x \sim f \Rightarrow \tfrac{1}{2}x + \tfrac{1}{2}y \succsim \tfrac{1}{2}f + \tfrac{1}{2}(\tfrac{1}{\delta}y + (1 - \tfrac{1}{\delta})x_*).$$

As discussed previously, in the MEU model the beliefs are represented by a nonempty, convex and weak* closed sets of priors. Such model implies in a stronger set of axioms than the list of axioms related to confidence preferences. In fact, a necessary condition for a preference $\succsim$ to be representable by a multiple prior functional $J(f) = \min_{p \in C} \int u(f)\, dp$ is that such preference should satisfy Axioms 1-4, and the certainty independence axiom that says:

Axiom 8 - Certainty independence. For all $f, g \in \mathcal{F}$, $x \in X$ and $\alpha \in (0,1)$:

$$f \sim g \Rightarrow \alpha f + (1-\alpha) x \sim \alpha g + (1-\alpha) x.$$

Next we recall the main results of Chateauneuf and Faro [5] regarding the axiomatic foundation of confidence preferences[9], with a special mention to the MEU model:

Theorem 1 *([5], Theorem 3 and Remark 4) Let $\succsim$ be a binary relation on $\mathcal{F}$, the following conditions are equivalent:*

(i) The preference relation $\succsim$ satisfies Axioms A.1-A.7

(ii) There exist a unique non-constant affine function $u : X \to \mathbb{R}_+$, such that $u(x_*) = 0$, defined up to a positive multiplication, a minimal confidence level $\alpha_0 \in (0,1]$, and a regular fuzzy set $\varphi : \Delta \to [0,1]$ such that, for all $f, g \in \mathcal{F}$*

$$f \succsim g \Leftrightarrow \min_{p \in L_{\alpha_0}\varphi} \frac{1}{\varphi(p)} \int u(f) dp \geq \min_{p \in L_{\alpha_0}\varphi} \frac{1}{\varphi(p)} \int u(g) dp.$$

Moreover, δ in (i) and α_0 in (ii) are linked by the relationship $\alpha_0 = \delta^{-1}$. Importantly it turns out that our model reduces to the MEU model as soon as $\delta = 1$ and in this case the Axiom 8 (certainty independence) holds.

[9] The family of confidence preferences represents an important subclass of ambiguity-averse preferences and it has been studied in some topics of economic theory. See, for instance, Cerreia-Vioglio et al [3], Ghirardato and Siniscalchi [10], Rigotti, Shannon and Strzalecki [17], Strzalecki [23], Strzalecki and Werner [24], Martins-da-Rocha [16].

Also, an interesting conjugation between any confidence function and its related functional representation can be achieved as:

Corollary 2 *([5], Corollary 5) Under the conditions of Theorem 1, there exists a maximal confidence function* φ^* *given by*

$$\varphi^*(p) = \inf_{f \in \mathcal{F}} \left(\frac{\int u(f)dp}{u(c_f)} \right)$$

such that, for all $f, g \in \mathcal{F}$

$$f \succsim g \Leftrightarrow \min_{p \in L_{\alpha_0}\varphi^*} \frac{1}{\varphi^*(p)} \int u(f)dp \geq \min_{p \in L_{\alpha_0}\varphi^*} \frac{1}{\varphi^*(p)} \int u(g)dp.$$

Furthermore, under the maximal confidence function, the confidence level α_0 *is not relevant*[10]*, i.e., for all* $f, g \in \mathcal{F}$

$$f \succsim g \Leftrightarrow \min_{p \in \Delta} \frac{1}{\varphi^*(p)} \int u(f)dp \geq \min_{p \in \Delta} \frac{1}{\varphi^*(p)} \int u(g)dp.$$

Chateauneuf and Faro [5] also investigated the ambiguity attitude featured by the class of confidence preferences . It will be useful to recall the obtained results on this important topic:

Proposition 3 *([5], Proposition 7) Any confidence preference* $\succsim$ *is ambiguity averse.*

Note that any confidence preference can be identified with a pair (u, φ^*) of an affine utility index, such that $u(x_*) = 0$, and a maximal confidence function φ^*. The following result shows that the comparative ambiguity attitudes featured by confidence preferences are determined by the confidence function φ^*.

Proposition 4 *([5], Proposition 8) Given two confidence preferences, the following conditions are equivalent:*
(1) $\succsim_1$ *is more ambiguity averse than* $\succsim_2$*;*
(2) There exist pairs (u, φ_i^*) *that represents* $\succsim_i (i = 1, 2)$*, where* $\varphi_1^* \geq \varphi_2^*$*.*

This proposition says that more ambiguity averse preference relations are characterized, up to index normalization, by greater functions φ^*. In particular, note that if the pair (u_1, φ_1^*) represents a more ambiguity averse preference than the pair (u_2, φ_2^*) , then there exists $\lambda > 0$ such that $u_2 = \lambda u_1$ and $\varphi_1^* \geq \varphi_2^*$.

[10] In fact, the importance of α_0 is implicit in the determination of the maximal confidence function.

3.2 Characterizing a MEU preference through a family of SEU agents.

Subjective expected utility is the most well-known model about decision under uncertainty. The axiomatic foundation proposed by Anscombe and Aumann [1] has as its key axiom a stronger condition than the certainty independence axiom proposed in [11] saying that:

Axiom 9 - Independence. For all $f, g, h \in \mathcal{F}$ and $\alpha \in (0,1)$:

$$f \sim g \Rightarrow \alpha f + (1-\alpha)h \sim \alpha g + (1-\alpha)h.$$

The usual idea concerning a decision maker consistent with subjective expected utility theory is that her behavior is as if she held a unique prior, well-defined in the Anscombe and Aumann´s representation. However, if we consider the maximal confidence function related to a subjective expected utility decision maker, we have that:

Lemma 5 *([5], Corollary 18) Let $\succsim$ be a bounded below preference relation that satisfies Axioms 1-4 and Axiom 9, then $\succsim$ is a expected utility preference and its maximal confidence function $\varphi^* := \varphi_q^*$ satisfies*

$$\varphi_q^*(p) = \inf_{E \in \Sigma} \frac{p(E)}{q(E)}, \forall p \in \Delta$$

for some subjective probability q.

An interesting fact is that a SEU decision maker does not necessarily presents non-null confidence only in a unique prior q, but the confidence among priors other than q implies that such priors are negligible. However, small perturbations in the decision maker´s confidence functions might invalidate the possibility of a SEU representation.

Let φ_C^* be the maximal confidence function for a MEU preference with a set C of multiple priors. Next, we present an interesting characterization of MEU:

Proposition 6 *Let $\succsim$ be a bounded below preference relation that satisfies Axioms 1-4 and Axiom 8, then $\succsim$ has a maxmin expected utility representation and its maximal confidence function φ_C^* is the upper envelope of a family of SEU-maximal confidence functions $\left\{\varphi_q^*\right\}_{q \in C}$, i.e.,*

$$\varphi_C^*(p) = \sup_{q \in C} \varphi_q^*(p), \forall p \in \Delta.$$

This result sounds very natural because it says that the degree of confidence for each plausible prior $p \in C$ for a multiple prior decision maker with confidence function φ_C^* is determined by the maximal level of confidence associated to p across every subjective expected utility decision maker holding its full confidential prior in the set of multiple priors C.

3.3 Approaching subjective probabilities

We now introduce a new class of preference induced by a special confidence function for which the SEU model can be obtained as a special case. In fact, this class of preferences is very close to the standard expected utility case in the sense that the degree of confidence among the universe of probabilities laws could be a small perturbation around the maximal confidence function related to the subjective expected utility case.

We assume there is an underlying probability measure $q \in \Delta$. Given a quasi-concave continuous function $\phi : [0,1] \to [0,1]$ such that $\phi(0) = 0$ and $\phi(1) = 1$, the ϕ-confidence level of a probability $p \in \Delta$ is given by

$$\varphi_q^{\phi}(p) = \inf_{E \in \Sigma} \phi\left(\frac{p(E)}{q(E)}\right).$$

We dub the mapping $\varphi_q^{\phi} : \Delta \to \mathbb{R}$ as a ϕ-confidence function (with respect to q).

Lemma 7 *A ϕ-confidence function is a regular* fuzzy set. Furthermore, if ϕ is concave then φ_q^{ϕ} is concave.*

Thanks to Lemma 7, preferences represented by the functional

$$J(f) = \min_{p \in L_{\alpha_0}(\varphi_q^{\phi})} \frac{1}{\varphi_q^{\phi}(p)} \int u(f)dp,$$

where $u : X \to \mathbb{R}_+$ is an affine function, $\alpha_0 \in (0,1)$ is the minimal level of confidence[11], and $\phi : [0,1] \to [0,1]$ is a concave and continuous function s.t. $\phi(0) = 0$ and $\phi(1) = 1$, belongs to the class of preferences that satisfies the Axioms A1-A7 as in our main result.

A ϕ-confidence preference (w.r.t. q and α_0) is the preference relation $\succsim_{\phi}$ on $\mathcal{F}$ such that

$$f \succsim_{\phi} g \Leftrightarrow \min_{p \in L_{\alpha_0}(\varphi_q^{\phi})} \frac{1}{\varphi_q^{\phi}(p)} \int u(f)dp \geq \min_{p \in L_{\alpha_0}(\varphi_q^{\phi})} \frac{1}{\varphi_q^{\phi}(p)} \int u(g)dp.$$

Proposition 8 *The ϕ-confidence function preferences $\succsim_{\phi}$ are preferences satisfying the Axioms A1-A7 with confidence function given by φ_q^{ϕ}. In particular, these preferences are*

a) ambiguity neutral if $\phi(x) = x$ on $[0,1]$;

b) strictly ambiguity averse if $\phi(x_0) > x_0$ for some $x_0 \in (0,1)$[12].

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[11] The case $\alpha_0 = 1$ is trivial: It entails the SEU representation with subjective probability q if $[\phi(x) = 1$ iff $x = 1]$; otherwise, we obtain the MEU model with set of priors given by

$$\left\{p \in \Delta : \varphi_q^{*}(p) \in \{\phi = 1\}\right\}.$$

[12] Since $\phi(0) = 0$, $\phi(1) = 1$, and ϕ is concave, it is easy to see that the existence of some x_0 such that $\phi(x_0) > x_0$ implies that $\phi(x) > x$ for any $x \in (0,1)$. Also, strictly concavity in ϕ implies that $\phi(x) < 1$ for any $x \in (0,1)$.

We conclude this subsection by reporting that if we consider the CEU model with convex capacity $v : \Sigma \to [0,1]$ we might define a ϕ-confidence function with respect to v by

$$\varphi_v^{\phi}(p) = \inf_{E\in\Sigma} \phi\left(\frac{p\,(E)}{v\,(E)}\right).$$

We note that analogue results as obtained for probabilities are true when we consider convex capacities. Also, in a more general way it is possible to consider the distortion of the MEU model generated by confidence functions given by

$$\varphi_C^{\phi}(p) = \sup_{q\in C} \varphi_q^{\phi}(p).$$

3.4 Entropic confidence and Hansen and Sargent´s constraint preferences.

Relative entropy is a classical measure of "distance" between two probability measures also known as Kullback-Leibler divergence. Formally, the relative entropy with respect to a probability $q \in \Delta^{\sigma}$ is the mapping $R\,(\cdot \parallel q)$ from Δ^{σ} to $[0,\infty]$ satisfying

$$R\,(p \parallel q) = \begin{cases} \int \ln\left(\frac{dp}{dq}\right) dp \text{ if } p \in \Delta^{\sigma}\,(q)\,, \\ \infty \text{ otherwise.} \end{cases}$$

Hansen and Sargent [12] proposed a criterion for decision making building on the notion of relative entropy where the preference relation, called multiplier preference, is induced by the following functional

$$V\,(f) = \left(\min_{p\in\Delta^{\sigma}} \int u(f)dp + \theta R\,(p \parallel q)\right).$$

Such criterion captures in a particular way the lack of trust in a single prior: the decision maker thinks that q is the most plausible true probability distribution, but he considers that other probabilities p are also plausible and such plausibility is proportional to their respective relative entropy w.r.t. q. One important fact is that multiplier preferences are a particular case of variational preferences as introduced by Maccheroni, Marinacci and Rustichini [15]. Variational preferences are represented by the functional

$$V\,(f) = \min_{p\in\Delta^{\sigma}} \left(\int u(f)dp + c\,(p)\right),$$

where $c : \Delta \to [0,+\infty]$ is interpreted as an ambiguity index[13]. So, variational preferences recover the multiplier preferences model when $c\,(p) = \theta R\,(p \parallel q)$.

[13] An ambiguity index satisfies: the set $\{c = 0\}$ is nonempty, c is convex and weak* lower semicontinuous. The multiple prior model is also a particular case of variational preferences, and it is obtained by taking the ambiguity index given by the indicator function

$$\delta_C : p \in \Delta \to \delta_C\,(p) \in \{0,+\infty\}\,,$$

where $\delta_C\,(p) = 0$ iff $p \in C$.

Strzalecki [22] provided a set of behaviorally meaningful axioms as foundations for the multiplier preferences.

Hence, as a particular case of variational preferences with behavior foundations incompatible with the multiple prior model, we have that multiplier preferences cannot be viewed as a particular case of preferences as in our main result. However, the intuitive appeal of multiplier preferences goes in the same direction as the notion of confidence functions. So, what is the corresponding confidence function that can recover the same type of subjective plausibility as captured by the ambiguity index given by the relative entropy $R(p \parallel q)$? Next, we propose the notion of entropic confidence:

Definition 9 *Given an underlying probability measure $q \in \Delta^\sigma$ we denote by $\varphi_e(p)$ the entropic confidence where for any $p \in \Delta$*

$$\varphi_e(p) = \begin{cases} \exp\left(-R\left(p \parallel q\right)\right), & \text{if } p \in \Delta^\sigma(q) \\ 0, & \text{otherwise.} \end{cases}$$

Remark 10 *Building on the results obtained by Dupuis and Ellis ([7], Lemma 1.4.3.), we have the following: Denote by Π the class of all finite measurable partitions of S, given a probability measure q it is easy to show that the entropic confidence of p satisfying $p \ll q$ is also given by*

$$\varphi_e(p) = \inf_{\pi\in\Pi} \prod_{E\in\pi} \left(\frac{q(E)}{p(E)}\right)^{p(E)}.$$

Moreover, we can rewrite the entropic confidence as

$$\varphi_e(p) = \begin{cases} \inf_{a\in B^+} \left\{\exp\left(\int a dp\right) \int \exp(-a)\, dq\right\}, & \text{if } p \in \Delta^\sigma(q) \\ 0, & \text{otherwise.} \end{cases}$$

Proposition 11 *The entropic confidence is a regular* fuzzy set.*

An entropic confidence function preference (w.r.t. q and α_0) is the preference relation $\succsim_e$ on $\mathcal{F}$ such that

$$f \succsim_e g \Leftrightarrow \min_{p\in L_{\alpha_0}(\varphi_e)} \frac{1}{\varphi_e(p)} \int u(f)dp \geq \min_{p\in L_{\alpha_0}(\varphi_e)} \frac{1}{\varphi_e(p)} \int u(g)dp.$$

A direct consequence of Theorem 1 follows,

Proposition 12 *The entropic confidence function preferences $\succsim_e$ are preferences satisfying the Axioms A1-A7, where $\delta = \alpha_0^{-1}$, with a confidence function given by φ_e.*

This class of preferences models decision makers that behaves as if evaluating each distorced expected utility $\varphi_e(p)^{-1} \int u(f)\, dp$ by considering only priors p beloging to the set of confidence priors $\{p \in \Delta^\sigma(q) : \varphi_e(p) \geq \alpha_0\} =$

$\{p \in \Delta^\sigma(q) : R(p \parallel q) \leq \ln \delta\}$. Note that the lower confidence level α_0 is related to the constant given in Axiom 7[14].

Hansen and Sargent [12] also proposed a subclass of multiple priors preference called *constraint preferences* where the set of multiple priors C depends on the relative entropy $R(p \parallel q)$ and on a positive constant r_0:

$$C := \{p \in \Delta : R(p \parallel q) \leq r_0\},$$

so, the functional form satisfies

$$V(f) = \min_{\overline{p} \in \{p \in \Delta : R(p \| q) \leq r_0\}} \int u(f)\, d\overline{p}.$$

Since $\varphi_e(p) = \exp(-R(p \parallel q))$, we obtain that entropic confidence function preferences can be rewritten as

$$J(f) = \min_{\overline{p} \in \{p \in \Delta : R(p \| q) \leq r_0\}} \exp(R(\overline{p} \parallel q)) \int u(f)\, d\overline{p},$$

where $r_0 := \ln \delta$, which is similar to constraint preferences because in both cases we have the same set of multiple priors, but while in the Hansen and Sargent´s constraint preferences every prior in C holds the same degree of plausibility, in our case the level of relative entropic matters for choice. Also, note that Proposition 4 indicated that an entropic confidence preference cannot be more ambiguity averse than the related multiple priors preference with the same utility index u and same set of plausible priors C.

3.5 Monotone continuity

In order to derive subjective expected utility representations with countable additive priors, Arrow [2] introduced the monotone continuity axiom saying,

Axiom 10 - For any acts $f, g \in F$, $x \in X$, and a sequence of events $\{E_n\}_{n \geq 1}$ such that $E_n \downarrow \emptyset$ (*i.e.*, $E_1 \supseteq E_2 \supseteq ...$ and $\cap_{n \geq 1} E_n = \emptyset$), then $f \succ g$ implies that there exists $n_0 \geq 1$ such that $x E_{n_0} f \succ g$.

An important question concerning the representation of confidence preferences (through its maximal confidence function φ^*) is whether the monotone continuity axiom is a sufficient condition for each confidence level $\{\varphi^* \geq \alpha > 0\}$ to be contained in the set of countable additive priors.

For instance, consider an expected utility agent with belief $q \in \Delta^\sigma$, *i.e.*, an agent that satisfies conditions of Lemma 5 and the monotone continuity axiom. Now, let p be a probability measure given by $p := \frac{1}{2}q + \frac{1}{2}\overline{p}$, where $\overline{p} \in \Delta \backslash \Delta^\sigma$. Hence, p is not countable additive and

$$\begin{aligned} \varphi_q^*(p) &= \inf_{E \in \Sigma} \frac{\frac{1}{2}q(E) + \frac{1}{2}\overline{p}(E)}{q(E)} \\ &= \inf_{E \in \Sigma} \left\{ \frac{1}{2} + \frac{1}{2} \frac{\overline{p}(E)}{q(E)} \right\} = \frac{1}{2} + \frac{1}{2}\varphi_q^*(\overline{p}) > 0. \end{aligned}$$

[14] For a discussion concerning the meaning of such constant see Chateauneuf and Faro [5], page 539

Hence, the maximal confidence function of a SEU monotone continuous preferences does not discard finitely additive priors, which could be important if we consider distortions like φ_q^{ϕ} as discussed in Lemma 7. On the other hand, by Cerreia-Vioglio et al [3], Corollary 22, for a monotone continuous confidence preference represented by (u, φ, α_0), it follows that for any $\alpha \geq \alpha_0$ we have $\{\varphi \geq \alpha\} \subset \Delta^{\sigma}$.

Another interesting fact is presented in the following proposition:

Proposition 13 *Consider a SEU preference with maximal confidence function φ_q^*, where q is not countably additive. Then for any $\alpha \in (0,1]$, the set of priors $\left\{\varphi_q^* \geq \alpha\right\}$ belongs to $\Delta \backslash \Delta^{\sigma}$.*

A consequence of Proposition 13 is that if we consider $q \in \Delta \backslash \Delta^{\sigma}$and the confidence function φ_q^{ϕ} as in the Proposition 8, we obtain that for any $\alpha \in (0,1]$ the set of priors $\left\{\varphi_q^{\phi} \geq \alpha\right\}$ belongs to $\Delta \backslash \Delta^{\sigma}$, and this kind of confidence preferences may be useful in the study of some problems in economic theory where the lack of countably additive has, for instance, meaningful consequences for equilibrium prices.

4 Ambiguity-Loving Confidence Preferences

Since the beginning we have been concentrate on the pessimistic attitudes toward ambiguity, now we change our focus to optimist attitude. Indeed, in a similar way of uncertainty averse preferences it is possible to provide axiomatic foundations for ambiguity-loving behavior without extremely optimistic attitudes. In fact, first we need to replace the uncertainty-aversion axiom of Gilboa and Schmeidler [11] by *uncertainty-loving* as introduced by Schmeidler [20],

Axiom $4^{\#}$ - For any acts $f, g \in \mathcal{F}$ and $\alpha \in [0,1]$,

$$f \sim g \Rightarrow f \succsim \alpha f + (1-\alpha) g.$$

Also, we should replace Axiom 7 (bounded attraction for certainty) by the following axiom,

Axiom $7^{\#}$ - *Bounded attraction for uncertainty*: There exists a constant $\delta \geq 1$ such that for any act $f \in \mathcal{F}$, and any consequences $x, y \in X$:

$$f \sim x \Rightarrow \frac{1}{2} f + \frac{1}{2} y \succsim \frac{1}{2} x + \frac{1}{2}(\frac{1}{\delta} y + (1-\frac{1}{\delta}) x_*).$$

Building on Axiom 1,2, 3, $4^{\#}$, 5, 6 and $7^{\#}$, we obtain a preference relation with a functional representation $J : \mathcal{F} \rightarrow \mathbb{R}$ such that

$$J(f) = \max_{p \in L_{\alpha_0} \varphi} \varphi(p) \int u(f) dp.$$

More precisely, we obtain the following result[15]:

Theorem 14 *Let $\succsim$ be a binary relation on $\mathcal{F}$, the following conditions are equivalent:*

(i) The preference relation $\succsim$ satisfies Axioms 1,2, 3, $4^{\#}$, 5, 6 and $7^{\#}$.

(ii) There exist a unique non-constant affine function $u : X \to \mathbb{R}_+$, such that $u(x_*) = 0$, defined up to a positive multiplication, a minimal confidence level $\alpha_0 \in (0, 1]$, and a regular fuzzy set $\varphi : \Delta \to [0, 1]$ such that, for all $f, g \in \mathcal{F}$*

$$f \succsim g \Leftrightarrow \max_{p \in L_{\alpha_0}\varphi} \varphi(p) \int u(f)dp \geq \max_{p \in L_{\alpha_0}\varphi} \varphi(p) \int u(g)dp.$$

Moreover, δ in (i) and α_0 in (ii) are linked by the relationship $\alpha_0 = \delta^{-1}$.

Also, similarly to ambiguity-averse confidence preferences, we obtain the following conjugation between any confidence function and its related functional representation[16],

Corollary 15 *Under the conditions of Theorem 14, there exists a maximal confidence function $\varphi^{\#}$ given by*

$$\varphi^{\#}(p) = \inf_{f \in \mathcal{F}} \frac{u(c_f)}{\int u(f)dp}$$

such that, for all $f, g \in \mathcal{F}$

$$f \succsim g \Leftrightarrow \max_{p \in L_{\alpha_0}\varphi^{\#}} \varphi^{\#}(p) \int u(f)dp \geq \max_{p \in L_{\alpha_0}\varphi^{\#}} \varphi^{\#}(p) \int u(g)dp.$$

Furthermore, for all $f, g \in \mathcal{F}$

$$f \succsim g \Leftrightarrow \max_{p \in \Delta} \varphi^{\#}(p) \int u(f)dp \geq \max_{p \in \Delta} \varphi^{\#}(p) \int u(f)dp.$$

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[15] We note that the functional $I : B(u(X)) \to \mathbb{R}$ defined by $I(a) = J(f)$, where $a = u(f)$, satisfies the key condition: $\forall a$, $\forall k$

$$I(a + k1_S) \geq I(a) + \delta^{-1}k.$$

And the proof follows in a similar way of ambiguity-averse confidence preferences.

[16] We note that $1/\varphi^{\#} = \sup_{a \in B^+} \left\{ \int adp/I(a) \right\}$ on $\left\{ \varphi^{\#} > 0 \right\}$ and this mapping is weak* upper semicontinuous and convex. Hence, $\varphi^{\#}$ is lower semicontinous and quasi-concave. Also, a similar argument as done in [5] shows that $\varphi^{\#}$ is a normal fuzzy set. The rest is very similar to the case of ambiguity-averse confidence preferences.

5 APPENDIX

First, we recall the following important result that we will use in the next proof:

Theorem 16 *(Sion [21], Corollary 3.3)Let M and N be convex spaces one of which is compact, and f a function on $M \times N$, quasi-concave and u.s.c. on M and quasi-convex and l.s.c. on N, then $\sup_M \inf_N f = \inf_N \sup_M f$.*

PROOF OF PROPOSITION 6: Denoting by φ_C^* the maximal confidence function related to the multiple prior model, we know that there exists a functional $I : B^+ \to \mathbb{R}$, where $I(a) = \min_{p \in C} \int adp$, such that

$$\varphi_C^*(p) = \inf_{a \in B^+} \frac{\int adp}{I(a)} = \inf_{a \in B^+} \frac{\int adp}{\min_{q \in C} \int adq} = \inf_{a \in B^+} \sup_{q \in C} \frac{\int adp}{\int adq}.$$

Now, we define the mapping

$$\begin{aligned} \Phi_p(\cdot \wr \cdot) &: \Delta \times B^+ \to \mathbb{R} \\ (q, a) &\mapsto \Phi_p(q \wr a) := \int adp / \int adq. \end{aligned}$$

We note that

1. For any $a \in B^+$, the mapping $\Phi_p(\cdot \wr a) : \Delta \to \mathbb{R}$ is quasi–convex and weakly* lower semi-continuous: For $a \in B^+$ such that $\int adp = 0$ the assertion is trivial. So, consider the case where $\int adp > 0$. Recall that in this case if $\int adq = 0$ then $\int adp / \int adq = \infty$. But, in fact we can ignore this possibility because $\inf_{a \in B^+} \left\{ \int adp / I(a) \right\} = \inf_{\{I > 0\}} \left\{ \int adp / I(a) \right\}$. Also, note that $a \in \{I > 0\}$ implies that $\int adq > 0$ for any $q \in C$. For quasi-convexity consider a real number c and the related $\Gamma_c := \{q \in C : \Phi_p(q \wr a) \leq c\}$. Let $q^1, q^2 \in \Gamma_c$, so for $i = 1, 2$

   $$\frac{\int adp}{\int adq^i} \leq c \Rightarrow \int adp \leq c \int adq^i,$$

   hence for any $\lambda \in [0, 1]$,

   $$\lambda \int adp \leq \lambda c \int adq^1 \text{ and } (1 - \lambda) \int adp \leq (1 - \lambda) c \int adq^2,$$

   so,

   $$\begin{aligned} \int adp &\leq \lambda c \int adq^1 + (1 - \lambda) c \int adq^2, \\ \textit{i.e.,} \quad \frac{\int adp}{\int ad(\lambda q^1 + (1 - \lambda) q^2)} &\leq c \Rightarrow \lambda q^1 + (1 - \lambda) q^2 \in \Gamma_c. \end{aligned}$$

   For lower semicontinuity, given a real number c consider the set Γ_c. If $c \leq 0$ then $\Gamma_c = \emptyset$. Now, if $c > 0$ and denoting by $\eta_a(q) := \int adq$ we obtain that $\Gamma_c = \left\{ q \in C : \eta_a(q) \geq c^{-1} \int adp \right\} = \eta_a^{-1}\left([c^{-1} \int adp, \infty) \right) \cap C$ which is $\sigma(B, \Delta)$-compact because $\eta_a(\cdot)$ is $\sigma(B, \Delta)$-continuous and C is $\sigma(B, \Delta)$-compact.

2. For any $q \in C$, the mapping $\Phi_p(q \wr \cdot) : \left\{a \in B^+ : \int adq > 0\right\} \to \mathbb{R}$ is quasi-concave and continuous: For continuity, note that each mapping $a \to \int adp$ and $a \to \int adq$ is $\|\cdot\|_\infty$-continuous, so the mapping $a \to \int adp / \int adq$ is $\|\cdot\|_\infty$-continuous on $\left\{a \in B^+ : \int adq > 0\right\}$. For quasi-concavity, consider a real number c and the set $\Theta_c := \{a \in B^+ : \Phi_p(q \wr a) \geq c\}$. For a^1, a^2 such that $\int a^i dp / \int a^i dq \geq c$ we obtain for any $\lambda \in [0,1]$

$$\begin{aligned} \int \lambda a^1 dp &\geq c \int \lambda a^1 dq \text{ and} \\ \int (1-\lambda) a^2 dp &\geq c \int (1-\lambda) a^2 dq, \end{aligned}$$

so,

$$\int \left(\lambda a^1 + (1-\lambda) a^2\right) dp \geq c \int \left(\lambda a^1 + (1-\lambda) a^2\right) dq,$$

i.e., $\lambda a^1 + (1-\lambda) a^2 \in \Theta_c$.

Hence, the mapping $\Phi_p(\cdot \wr \cdot)$ satisfies the conditions of the Minimax theorem as shown by Sion [21] and presented before this proof. So,

$$\varphi_C^*(p) = \inf_{a \in B^+} \sup_{q \in C} \frac{\int adp}{\int adq} \overset{(\text{min max theorem})}{=} \sup_{q \in C} \inf_{a \in B^+} \frac{\int adp}{\int adq} = \sup_{q \in C} \varphi_q^*(p).$$

PROOF OF LEMMA 7: φ_q^ϕ is normal because $\varphi_q^\phi(q) = 1$[17]. Now, we note that given an event E, the mapping

$$p \mapsto \phi\left(\frac{p(E)}{q(E)}\right)$$

is weakly* continuous. So, the mapping

$$p \mapsto \inf_{E \in \Sigma} \phi\left(\frac{p(E)}{q(E)}\right)$$

is weakly* upper semicontinuous.

Consider $p_1, p_2 \in \Delta$ and $\beta \in [0,1]$, and let $p^\beta = \beta p_1 + (1-\beta) p_2$. For all $E \in \Sigma$, $p^\beta(E) = \beta p_1(E) + (1-\beta) p_2(E)$ and

$$\begin{aligned} \phi\left(\frac{p^\beta(E)}{q(E)}\right) &= \phi\left(\frac{\beta p_1(E) + (1-\beta) p_2(E)}{q(E)}\right) \\ &\geq \min\left\{\phi\left(\frac{p_1(E)}{q(E)}\right), \phi\left(\frac{p_2(E)}{q(E)}\right)\right\}. \end{aligned}$$

Hence,

$$\begin{aligned} \varphi_q^\phi(p^\beta) &= \inf_{E \in \Sigma} \phi\left(\frac{p^\beta(E)}{q(E)}\right) \geq \inf_{E \in \Sigma} \min\left\{\phi\left(\frac{p_1(E)}{q(E)}\right), \phi\left(\frac{p_2(E)}{q(E)}\right)\right\} \\ &= \min\left\{\inf_{E \in \Sigma} \phi\left(\frac{p_1(E)}{q(E)}\right), \inf_{E \in \Sigma} \phi\left(\frac{p_2(E)}{q(E)}\right)\right\} = \min\left\{\varphi_q^\phi(p^1), \varphi_q^\phi(p^2)\right\}. \end{aligned}$$

[17] Note that under the additional assumption saying that $\phi(x) = 1$ iff $x = 1$, we obtain that $\varphi_q^\phi(p) = 1$ iff $p = q$.

By a similar reasoning, if ϕ is concave then φ_q^{ϕ} is concave.

PROOF OF PROPOSITION 8: By Lemma 7 and Theorem 1, it is immediate that ϕ-confidence function preferences satisfy Axioms A1-A7, hence this class of preferences characterizes a subclass of ambiguity-averse preferences by Propositions 3 and 4. Item (a) follows from Lemma 5 about the characterization of SEU maximal confidence functions. For item (b), note that $\succsim_{\phi}$ is more ambiguity averse than the SEU preference induced by the probability q (with the same utility index) and denoted by $\succsim_q$. If the preference $\succsim_q$is not the unique SEU preference for which $\succsim_{\phi}$ can be comparable in the sense of the relation "more ambiguity averse than", it is clear that $\succsim_{\phi}$ is not a SEU preference. Otherwise, since $\phi(x) > x$ for any $x \in (0,1)$, we obtain that for any p such that $\varphi_q^{\phi}(p) \in [\alpha_0, 1)$

$$\varphi_q^*(p) = \inf_{E\in\Sigma} \frac{p(E)}{q(E)} < \phi\left(\inf_{E\in\Sigma} \frac{p(E)}{q(E)}\right) = \inf_{E\in\Sigma} \phi\left(\frac{p(E)}{q(E)}\right) = \varphi_q^{\phi}(p),$$

hence, $\succsim_{\phi}$ is not a SEU preference.

PROOF OF PROPOSITION 11: Note that, since $R(\cdot \parallel q)$ is convex and weakly* lower semicontinous, we obtain that $\exp(R(\cdot \parallel q))$ is also convex and weakly* lower semicontinous, which entails that $\exp(-R(\cdot \parallel q)) = [\exp(R(\cdot \parallel q))]^{-1}$ is weakly* upper semicontinous and quasiconcave.

PROOF OF PROPOSITION 13: Since q is not countably additive there exists some sequence of events $\{A_n\}_{n\geq 1}$ such that $A_n \downarrow \emptyset$ where $q(A_n) \downarrow \beta$ for some $\beta > 0$. If $p \in \Delta^{\sigma}$ we have that $p(A_n) \downarrow 0$ and

$$\inf_{E\in\Sigma} \frac{p(E)}{q(E)} \leq \frac{p(A_n)}{q(A_n)} \leq \frac{p(A_n)}{\beta},\ \forall n \geq 1.$$

Hence, $0 \leq \varphi_q^*(p) \leq \frac{p(A_n)}{\beta} \to 0$.

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