REMARKS ON SOLITARY WAVES OF THE GENERALIZED TWO DIMENSIONAL BENJAMIN-ONO EQUATION

Amin Esfahani[1]

Abstract. In this paper we study the generalized 2D-BO equation in two dimensions:

$$(u_t + \beta \mathscr{H} u_{xx} + u^p u_x)_x + \epsilon u_{yy} = 0, \quad (x, y) \in \mathbb{R}^2, \quad t \geq 0.$$

We classify the existence and non-existence of solitary waves depending on the sign of ϵ, β and on the nonlinearity. We also prove some regularity properties of such waves.

1. Introduction

In this paper we are interested in studying a model which is a natural two-dimensional extension of (1.2), namely, the two-dimensional generalized Benjamin-Ono equation (2D-BO henceforth)

$$u_t + \beta \mathscr{H} u_{xx} + \epsilon v_y + u^p u_x = 0, \quad u_y = v_x, \quad (x, y) \in \mathbb{R}^2, \quad t \geq 0, \tag{1.1}$$

where the constant ϵ measures the transverse dispersion effects and is normalized to ± 1, the constant β is real and $\mathscr{H}$ is the Hilbert transform defined by

$$\mathscr{H} u(x, y, t) = \text{p.v.} \frac{1}{\pi} \int_{\mathbb{R}} \frac{u(z, y, t)}{x - z} \, dz,$$

where p.v. denotes the Cauchy principal value. When $-\beta = p = 1$, equation (1.1) was introduced by Ablowitz and Clarkson in [1] and Ablowitz and Segur in [2], which arises in the study of internal waves in deep stratified fluids (see also [33]). Equation (1.1) is an extension of one-dimensional generalized Benjamin-Ono (BO) equation,

$$u_t - \mathscr{H} u_{xx} + u^p u_x = 0, \qquad x \in \mathbb{R}, \ \ t \in \mathbb{R}^+. \tag{1.2}$$

See also [12]-[17]. The integro-differential equation (1.2), when $p = 1$, serves as a generic model for the study of weakly nonlinear long waves incorporating the lowest order effects of nonlinearity and nonlocal dispersion; in particular, the propagation of one-dimensional internal waves in stratified fluids of great depth. Equation (1.2) has been extensively studied by several authors considering both the initial value problem and the nonlinear stability. The initial value problem associated to equation (1.2) has been studied, recently, for instance in [7, 22, 23, 29, 32, 35], whereas the issue of existence and stability of solitary waves has been studied in [3, 5].

Equation (1.1) is a spacial case of the generalized two dimensional Benjamin equation (2D-B henceforth)

$$u_t + \alpha u_{xxx} + \beta \mathscr{H} u_{xx} + \epsilon v_y + u^p u_x = 0, \quad u_y = v_x, \quad (x, y) \in \mathbb{R}^2, \quad t \geq 0. \tag{1.3}$$

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[1]amin@impa.br

Equation (1.3) contains other classical equations: when $\beta = 0$, $\alpha = p = 1$, (1.3) is known as the KP-II equation ($\epsilon = 1$) or KP-I equation($\epsilon = -1$). Many rigorous results have recently appeared concerning the Cauchy problem for the KP equations [9, 19, 20, 21, 30, 31], whereas the issue of existence and stability of solitary waves has been studied in [10, 26, 27]. Regarding on equation (1.3), Boling and Yongqian investigated the well-posedness of the local solutions for the Cauchy problem in [18]. They used a general local existence theory has been established by Ukai [34] and Saut [33] for KP equations. When $\alpha = 1$, recently, the existence, blow up and instability of solitary waves of (1.3) has been studied by Chen et al. in [8].

In this paper, we shall investigate the existence and nonexistence of the nontrivial solitary waves of equations (1.3) and (1.1), when the exponent p in (1.1) will be a rational number of the form $p = k/m$, where m is odd and m and k are relatively prime.

In order to give a precise definition of our needed spaces, we use the following spaces. We shall denote, $\widetilde{\mathscr{X}}$ and $\mathscr{X}$ the closure of $\partial_x(C_0^\infty(\mathbb{R}^2))$ for the norms

$$\|\partial_x\varphi\|^2_{\widetilde{\mathscr{X}}} = \|\nabla\varphi\|^2_{L^2(\mathbb{R}^2)} + \|\varphi_{xx}\|^2_{L^2(\mathbb{R}^2)} \tag{1.4}$$

and

$$\|\partial_x\varphi\|^2_{\mathscr{X}} = \|\nabla\varphi\|^2_{L^2(\mathbb{R}^2)} + \left\|D_x^{1/2}\varphi_x\right\|^2_{L^2(\mathbb{R}^2)}. \tag{1.5}$$

By a solitary wave solution of 2D-B equation, we mean a solution of (1.1) of the type $u(x - c_1t, y - c_2t)$, where $u \in \mathscr{X}$, $c_1, c_2 \in \mathbb{R}$ are the speeds of propagation of the wave along each direction. So we are looking for *localized* solutions of the systems

$$\begin{cases} -c_1u_x - c_2u_y + \alpha u_{xxx} + \beta\mathscr{H}u_{xx} + \epsilon v_y + u^p u_x = 0 \\ u_y = v_x. \end{cases} \tag{1.6}$$

By a change of variables $\widetilde{x} = x$, $\widetilde{y} = y - \frac{1}{2}\epsilon c_2 x$, after dropping the tilde, we obtain the new system

$$\begin{cases} -cu_x + \alpha u_{xxx} + \beta\mathscr{H}u_{xx} + \epsilon v_y + u^p u_x = 0 \\ u_y = v_x \end{cases} \tag{1.7}$$

with $c = c_1 + \frac{1}{4}\epsilon c_2^2$. A solitary wave solution of 2D-BO is defined in the same vein; namely,

$$-cu + \beta\mathscr{H}u_x + \epsilon\partial_x^{-2}u_{yy} + \frac{1}{p+1}u^{p+1} = 0. \tag{1.8}$$

REMARK 1.1. *Note that the wave speed c can be normalized to ±1, since the scale change $w(x, y) = |c|^{-1/p}u\left(\frac{x}{|c|}, \frac{y}{|c|^{3/2}}\right)$ transforms the equation (1.8) in u into the same in w with $|c| = 1$.*

REMARK 1.2. *Note that the constant $\beta \neq 0$ can be normalized to ±1, since the scale change $w(\cdot) = u(|\beta|\cdot)$ transforms the equation (1.8) in u into the same in w with $|\beta| = 1$.*

REMARK 1.3. *It is easy to check that there is no scaling for the system (1.6) to normalize the wave speed c.*

We shall prove that the solitary waves of (1.1) are stable in some sense, when $p < 4/5$. We demonstrate that these solution are ground states, namely, they have *minimal energy. It worth remarking that the flow associated to* (1.3) *satisfies the conservation quantities F and E, where $F(u) = \frac{1}{2}\|u\|^2_{L^2(\mathbb{R}^2)}$ and*

$$E(u) = \frac{1}{2}\int_{\mathbb{R}^2}\left[\alpha u_x^2 - \epsilon v^2 - \beta u\mathscr{H}u_x - \frac{2}{(p+1)(p+2)}u^{p+2}\right]\,dxdy,$$

Note that, when $\alpha = 0$, F and E are two invariants of (1.3).

This paper is organized as follows. In Section 2, we shall obtain the conditions of the nonexistence of the solitary wave of 2D-B and 2D-BO equations. Section 3 is devoted to the existence and the regularity properties of solitary waves of 2D-BO equation. Finally, in Section 4, we shall obtain some variational properties of solitary waves obtained in Section 3; and we show that these solutions are ground states.

Before proving the main result let us introduce some notations that will be used throughout this article.

Notations. We shall denote by $\widehat{\varphi}$ the Fourier transform of φ, defined as

$$\widehat{\varphi}(\xi,\eta) = \int_{\mathbb{R}^2}\varphi(x,y)e^{-i(x\xi+y\eta)}\,dxdy.$$

For $s\in\mathbb{R}$, we denote by $H^s(\mathbb{R}^2)$, the nonhomogeneous Sobolev space defined by

$$H^s(\mathbb{R}^2) = \left\{\varphi\in\mathscr{S}'(\mathbb{R}^2)\ :\ \|\varphi\|_{H^s(\mathbb{R}^2)}<\infty\right\},$$

where

$$\|\varphi\|_{H^s(\mathbb{R}^2)} = \left\|\left(1+\xi^2+\eta^2\right)^{\frac{s}{2}}\widehat{\varphi}(\xi,\eta)\right\|_{L^2(\mathbb{R}^2)},$$

and $\mathscr{S}'(\mathbb{R}^2)$ is the space of tempered distributions. We also define $H^s_x(\mathbb{R}^2)$ by

$$H^s_x(\mathbb{R}^2) = \left\{\varphi\in L^2(\mathbb{R}^2);\ \|\varphi\|_{H^s_x(\mathbb{R}^2)}<\infty\right\},$$

where

$$\|\varphi\|_{H^s_x(\mathbb{R}^2)} = \left\|\left(1+\xi^2\right)^{\frac{s}{2}}\widehat{\varphi}(\xi,\eta)\right\|_{L^2(\mathbb{R}^2)}.$$

We also denote $<,>$ as $L^2(\mathbb{R}^2)$-inner product. For any positive numbers a and b, the notation $a\lesssim b$ means that that there exists a positive (harmless) constant c such that $a\le cb$.

2. Nonexistence

In this section, we are going to obtain the conditions of nonexistence of solitary wave solutions of 2D-B and 2D-BO equations.

THEOREM 2.1. *Let $|\alpha|+|\beta|>0$. The equation* (1.7) *admits no nontrivial solitary waves,*

(I) *if $\epsilon = 1$ and one of the following cases occurs:*
  (i) $\alpha,\beta\ge 0$, $c<0$ *and* $p\ge 4/3$,
  (ii) $\alpha\le 0$, $\beta\ge 0$, $c>0$ *and* $p\le 4/3$,
  (iii) $\alpha\le 0$, $\beta\ge 0$, $c<0$ *and* $p\ge 4$,
  (iv) $\alpha\ge 0$, $\beta\le 0$, $c\ne 0$ *and p is arbitrary,*
  (v) $\alpha,\beta\le 0$, $c>0$ *and* $p\le 4$,
  or
(II) *if $\epsilon = -1$ and one of the following cases occurs:*
  (i) $\alpha,\beta\ge 0$, $c<0$ *and* $p\le 4$,

(ii) $\alpha \le 0$, $\beta \ge 0$, $c \ne 0$ *and* p *is arbitrary,*
(iii) $\alpha \ge 0$, $\beta \le 0$, $c > 0$ *and* $p \ge 4$,
(iv) $\alpha \ge 0$, $\beta \le 0$, $c < 0$ *and* $p \le 4/3$,
(v) $\alpha, \beta \le 0$, $c > 0$ *and* $p \ge 4/3$.

Proof.

$$\int_{\mathbb{R}^2} \left(cu^2 + 3\alpha u_x^2 - 2\beta u \mathscr{H} u_x + \epsilon v^2 - \frac{2}{p+2} u^{p+2} \right) \, dxdy = 0. \tag{2.1}$$

Next we multiply the first equation of the system (1.7) by $yv\chi_j$ and we integrate over $\mathbb{R}^2$; similar to above, by using the integration by parts and Lebesgue dominated convergence theorem, we obtain

$$\int_{\mathbb{R}^2} \left(-cu^2 - \alpha u_x^2 + \beta u \mathscr{H} u_x - \epsilon v^2 - \frac{2}{(p+1)(p+2)} u^{p+2} \right) \, dxdy = 0. \tag{2.2}$$

Now we multiply the first equation of the system (1.7) by $u\chi_j$ and we integrate over $\mathbb{R}^2$; similar to above, by using the integration by parts and Lebesgue dominated convergence theorem, we obtain

$$\int_{\mathbb{R}^2} \left(-cu^2 - \alpha u_x^2 + \beta u \mathscr{H} u_x + \epsilon v^2 + \frac{1}{p+1} u^{p+2} \right) \, dxdy = 0. \tag{2.3}$$

By adding (2.1) and (2.2) we get

$$\int_{\mathbb{R}^2} \left(2\alpha u_x^2 - \beta u \mathscr{H} u_x - \frac{2p}{(p+1)(p+2)} u^{p+2} \right) \, dxdy = 0. \tag{2.4}$$

By subtracting (2.1) from (2.2) we obtain

$$\int_{\mathbb{R}^2} \left(cu^2 + 2\alpha u_x^2 - \frac{3}{2} \beta u \mathscr{H} u_x - \frac{1}{p+1} u^{p+2} \right) \, dxdy = 0. \tag{2.5}$$

Adding (2.4) and (2.3) yields

$$\int_{\mathbb{R}^2} \left(-cu^2 + \frac{\beta}{2} u \mathscr{H} u_x + \epsilon v^2 + \frac{2}{(p+1)(p+2)} u^{p+2} \right) \, dxdy = 0. \tag{2.6}$$

By adding (2.5) and (2.3) we have

$$\int_{\mathbb{R}^2} \left(2\alpha u_x^2 - \beta u \mathscr{H} u_x + 4\epsilon v^2 \right) \, dxdy = 0, \tag{2.7}$$

which rules out (I)(iv) and (II)(ii).
Subtracting (2.7) from (2.4) yields

$$\int_{\mathbb{R}^2} \left(2\epsilon v^2 + \frac{p}{(p+1)(p+2)} u^{p+2} \right) \, dxdy = 0. \tag{2.8}$$

Eliminating u^{p+2} by (2.8) and (2.6) leads to

$$\int_{\mathbb{R}^2} \left[-cu^2 + \frac{\beta}{2} u \mathscr{H} u_x + \epsilon \left(\frac{p-4}{p} \right) v^2 \right] \, dxdy = 0, \tag{2.9}$$

which rules out (I)(iii), (I)(v), (II)(i) and (II)(iii).
Adding (2.1) and 2 times (2.2), and using (2.8), we obtain

$$\int_{\mathbb{R}^2} \left[-cu^2 + \alpha u_x^2 + \epsilon \left(\frac{3p-4}{p} \right) v^2 \right] \, dxdy = 0, \tag{2.10}$$

which rules out (I)(i), (I)(ii), (II)(iv) and (II)(v). □

Corollary 2.2. *The equation* (1.8) *does not admit any nontrivial solitary wave if one of the following cases occurs:*

(i) $\epsilon\beta < 0$, $c \neq 0$ *and* p *is arbitrary,*
(ii) $\epsilon\beta > 0$, $c\epsilon(3p-4) < 0$.

REMARK 2.3. *Note that Theorem* 2.1 *and Corollary* 2.2 *with nonlinearity* u^{p+1} *in* (1.8)*, can be extended to a general nonlinearity* $f(u)$ *satisfying some structure conditions similar to* [36].

3. Existence and Regularity

In this section, we are going to prove the existence and regularity property of solitary wave solutions of 2D-BO Henceforth, we assume that $\alpha = 0$, $c > 0$ and $\epsilon = -1$. For simplicity and without loss of generality, we can also suppose that $\beta = -1$. By Remark 1.1, we can also assume that $c = 1$.

LEMMA 3.1. *Let* $p \leq 4/3$. *Then there is a constant* $C > 0$ *(depending on* p*) such that for any* $u \in \mathscr{X}$,

$$\|u\|_{L^{p+2}}^{p+2} \leq C\|u\|_{L^2}^{(4-3p)/3} \left\|\partial_x^{-1}u_y\right\|_{L^2}^{p/2} \|u\|_{H_x^{1/2}(\mathbb{R}^2)}^{(9p+4)/6}.$$

As a consequence, it follows that there is a constant $C > 0$ *such that for all* $u \in \mathscr{X}$,

$$\|u\|_{L^{p+2}} \leq C\|u\|_{\mathscr{X}},$$

which is to say $\mathscr{X}$ *is embedded in* L^{p+2}.

Proof. The lemma is established for $C_0^\infty(\mathbb{R}^2)$-functions and then limits are taken to complete the proof. Because of Sobolev inequality and interpolation, we have that if $u \in H^{1/2}(\mathbb{R})$, then

$$\|u\|_{L^{p+2}(\mathbb{R})}^{p+2} \lesssim \|u\|_{H^{p/(2(p+2))}(\mathbb{R})}^{p+2} \lesssim \|u\|_{H^{-1/4}(\mathbb{R})}^{4/3}\|u\|_{H^{1/2}(\mathbb{R})}^{(3p+2)/3}.$$

It follows that

$$\begin{aligned}
\|u\|_{L^{p+2}(\mathbb{R}^2)}^{p+2} &\lesssim \int_{\mathbb{R}} \|u(\cdot,y)\|_{H^{p/(2(p+2))}(\mathbb{R})}^{p+2}\, dy \\
&\lesssim \int_{\mathbb{R}} \|u(\cdot,y)\|_{H^{-1/4}(\mathbb{R})}^{4/3}\|u(\cdot,y)\|_{H^{1/2}(\mathbb{R})}^{(3p+2)/3}\, dy \\
&\lesssim \left(\int_{\mathbb{R}} \|u(\cdot,y)\|_{H^{-1/4}(\mathbb{R})}^{2(4-2p)/(4-3p)}\, dy\right)^{(4-3p)/6} \left(\int_{\mathbb{R}} \|u(\cdot,y)\|_{H^{1/2}(\mathbb{R})}^{2}\, dy\right)^{(3p+2)/6} \\
&\lesssim \left(\int_{\mathbb{R}} \|u(\cdot,y)\|_{H^{-1/4}(\mathbb{R})}^{2(4-2p)/(4-3p)}\, dy\right)^{(4-3p)/6} \left(\int_{\mathbb{R}} \|u\|_{H_x^{1/2}(\mathbb{R}^2)}^{2}\, dy\right)^{(3p+2)/3}.
\end{aligned} \tag{3.1}$$

Now we are going to control the norm $\|u(\cdot,y)\|_{H^{-1/4}(\mathbb{R})}$. Let Λ^s be a standard Bessel potential given by

$$\widehat{\Lambda^s u}(\xi) = (1+|\xi|^2)^{s/2}\widehat{u}(\xi).$$

Use Fubini's theorem to derive

$$\begin{aligned}
\frac{1}{2}\|u(\cdot,y)\|^2_{H^{-1/4}(\mathbb{R})} &= \int_{-\infty}^{y}\int_{\mathbb{R}} \Lambda^{-1/4}u(x,z)\Lambda^{-1/4}u_z(x,z)\;dxdz\\
&= -\int_{-\infty}^{y}\int_{\mathbb{R}} \Lambda^{-1/4}u(x,z)\partial_x\partial_x^{-1}\Lambda^{-1/4}u_z(x,z)\;dxdz\\
&= -\int_{-\infty}^{y}\int_{\mathbb{R}} \Lambda^{-1/4}u_x(x,z)\partial_x^{-1}\Lambda^{-1/4}u_z(x,z)\;dxdz\\
&= -\int_{-\infty}^{y}\int_{\mathbb{R}} \Lambda^{-1/2}u_x(x,z)\partial_x^{-1}u_z(x,z)\;dzdx\\
&\lesssim \int_{\mathbb{R}} \|u(\cdot,y)\|_{H^{1/2}(\mathbb{R})}\|\partial_x^{-1}u_y(\cdot,y)\|_{L^2(\mathbb{R})}\;dy\\
&\lesssim \|u\|_{H_x^{1/2}(\mathbb{R}^2)}\left\|\partial_x^{-1}u_y\right\|_{L^2(\mathbb{R}^2)};
\end{aligned}$$

and hence

$$\sup_{y\in\mathbb{R}}\|u(\cdot,y)\|^2_{H^{-1/4}(\mathbb{R})} \lesssim \|u\|_{H_x^{1/2}(\mathbb{R}^2)}\left\|\partial_x^{-1}u_y\right\|_{L^2(\mathbb{R}^2)}. \tag{3.2}$$

On the other hand,

$$\begin{aligned}
\int_{\mathbb{R}}\|u(\cdot,y)\|^{2(4-2p)/(4-3p)}_{H^{-1/4}(\mathbb{R})}\;dy &\le \sup_{y\in\mathbb{R}}\|u(\cdot,y)\|^{2p/(4-3p)}_{H^{-1/4}(\mathbb{R})}\int_{\mathbb{R}}\|u(\cdot,y)\|^2_{H^{-1/4}}\;dy\\
&= \sup_{y\in\mathbb{R}}\|u(\cdot,y)\|^{2p/(4-3p)}_{H^{-1/4}(\mathbb{R})}\|u\|^2_{L^2(\mathbb{R}^2)}.
\end{aligned}\tag{3.3}$$

Plugging (3.2) and (3.3) into (3.1), we obtain that

$$\|u\|^{p+2}_{L^{p+2}} \le C\|u\|^{(4-3p)/3}_{L^2}\left\|\partial_x^{-1}u_y\right\|^{p/2}_{L^2}\|u\|^{(9p+4)/6}_{H_x^{1/2}(\mathbb{R}^2)}.$$

□

REMARK 3.2. *Unfortunately, we do not know whether the following inequality holds:*

$$\|u\|^{p+2}_{L^{p+2}} \le C\|u\|^{(4-3p)/3}_{L^2}\left\|\partial_x^{-1}u_y\right\|^{p/2}_{L^2}\left\|D_x^{1/2}u\right\|^{(9p+4)/6}_{L^2(\mathbb{R}^2)}.$$

LEMMA 3.3. *Let $p<4/3$. Then the following embedding is compact:*

$$\mathscr{X} \hookrightarrow\hookrightarrow L^{p+2}_{\mathrm{loc}}(\mathbb{R}^2).$$

Proof. Suppose that $\{u_n\}_{n\in\mathbb{N}}\subset\mathscr{X}$ be a bounded sequence in $\mathscr{X}$. Without loss of generality, assume that there exists $\{\varphi_n\}\subset L^2_{\mathrm{loc}}(\mathbb{R}^2)$ such that $u_n=\partial_x\varphi_n$. Let $v_n=\partial_y\varphi_n\in L^2(\mathbb{R}^2)$.

Multiplying φ_n by a function $\psi\in C_0^\infty(\mathbb{R}^2)$ such that $0\le\psi\le1$, $\psi\equiv1$ on $B_R(0)$ and supp $\psi\subset \mathrm{B}_{2\mathrm{R}}(0)$, we may assume that supp $\varphi_n\subset\mathrm{B}_{2\mathrm{R}}(0)$. Selecting if necessary to a subsequence, we may assume that $u_n\rightharpoonup u=\varphi_x$ in $\mathscr{X}$ and replacing φ_n by $\varphi_n-\varphi$, we may assume that $\varphi=0$. Let

$$\begin{aligned}
&Q_0=\{(\xi,\eta)\in\mathbb{R}^2;\ |\xi|\le\rho^2,\ |\eta|\le\rho^3\}\\
&Q_1=\{(\xi,\eta)\in\mathbb{R}^2;\ |\xi|\le\rho^2,\ |\eta|\ge\rho^3\}\qquad\text{and}\\
&Q_2=\{(\xi,\eta)\in\mathbb{R}^2;\ |\xi|\ge\rho^2\}.
\end{aligned}$$

For $\rho > 0$, there holds

$$\int_{B_{2R}(0)} |u_n|^2 = \int_{\mathbb{R}^2} |\widehat{u}_n|^2 = \sum_{i=0}^{2} \int_{Q_i} |\widehat{u}_n|^2.$$

It is clear that

$$\int_{Q_1} |\widehat{u}_n|^2 = \int_{Q_1} \frac{|\xi|^2}{|\eta|^2} \left| \left(\partial_x^{-1}(u_n)_y\right)^{\wedge} \right|^2 \leq \frac{1}{\rho^2} \left\|\partial_x^{-1}(u_n)_y\right\|_{L^2}$$

and there holds

$$\int_{Q_2} |\widehat{u}_n|^2 = \int_{Q_2} \frac{1}{|\xi|} \left| \left(D_x^{1/2} u_n\right)^{\wedge} \right|^2 \leq \frac{1}{\rho^2} \left\| D_x^{1/2} u_n \right\|_{L^2}.$$

Fix $\varepsilon > 0$; then choosing $\rho > 0$ sufficiently large leads to

$$\int_{Q_1} |\widehat{u}_n|^2 + \int_{Q_2} |\widehat{u}_n|^2 \leq \varepsilon/2.$$

Since $u_n \rightharpoonup 0$ in $L^2(\mathbb{R}^2)$, then $\widehat{u}_n$ tends to zero as $n \to \infty$ and $|\widehat{u}_n(\xi, \eta)| \leq \|u_n\|_{L^2}$. Lebesgue's dominated convergence theorem implies that

$$\int_{Q_0} |\widehat{u}_n|^2 \to 0,$$

as $n \to \infty$. Thus we have proved that $u_n \to 0$ in $L^2_{\mathrm{loc}}(\mathbb{R}^2)$. By Lemma 3.1 and interpolation theorem, there holds $u_n \to 0$ in $L^p_{\mathrm{loc}}(\mathbb{R}^2)$ if $2 \leq p < 10/3$. □

LEMMA 3.4. *Let $r > 0$. If $\{u_n\}_{n\in\mathbb{N}}$ be a bounded sequence in $\mathscr{X}$ and*

$$\sup_{\zeta\in\mathbb{R}^2} \|u_n\|_{L^2(B_r(\zeta))} \to 0,$$

as $n \to \infty$, then $u_n \to 0$ in $L^p(\mathbb{R}^2)$ for $2 < p < 10/3$.

Proof. Let $2 < s < \widetilde{p} = 10/3$ and $u \in \mathscr{X}$. By Hölder inequality and Lemma 3.1, there holds

$$\begin{aligned} \|u\|_{L^s(B_r(\zeta))} &\leq \|u\|^{1-\theta}_{L^2(B_r(\zeta))} \|u\|^{\theta}_{L^{\widetilde{p}}(B_r(\zeta))} \\ &\leq C\|u\|^{1-\theta}_{L^2(B_r(\zeta))} \left(\int_{B_r(\zeta)} u^2 + |D_x^{1/2}u|^2 + |\partial_x^{-1}u_y|^2 \right)^{\theta/2}, \end{aligned}$$

where $\frac{1}{2} = \frac{1-\theta}{2} + \frac{\theta}{\widetilde{p}}$. Choosing s such that $s\theta = 2$ yields

$$\|u\|^s_{L^s(B_r(\zeta))} \leq C^s \|u\|^{(1-\theta)s}_{L^2(B_r(\zeta))} \int_{B_r(\zeta)} u^2 + |D_x^{1/2}u|^2 + |\partial_x^{-1}u_y|^2.$$

Now, covering $\mathbb{R}^2$ by balls of radius r in such a way that each point of $\mathbb{R}^2$ is contained in at most 3 balls, then there holds

$$\|u\|^s_{L^s(B_r(\zeta))} \leq 3C^s \sup_{\zeta\in\mathbb{R}^2} \|u\|^{(1-\theta)s}_{L^2(B_r(\zeta))} \int_{\mathbb{R}^2} u^2 + |D_x^{1/2}u|^2 + |\partial_x^{-1}u_y|^2.$$

Plugging the assumption of the lemma into the last inequality, we obtain that $u_n \to 0$ in $L^s(\mathbb{R}^2)$. By Hölder inequality and Lemma 3.1, there holds $u_n \to 0$ in $L^p(\mathbb{R}^2)$ for all $2 < p < \widetilde{p}$. □

We recall the following Mountain Pass Lemma without Palais-Smale condition as our Lemma 3.5 (cf. [4]).

LEMMA 3.5 (Mountain Pass Lemma)**.** *Let X be a Banach space. Let M_0 be a closed subspace of the metric space M and $\Gamma_0 \subset C(M_0, X)$. Define*

$$\Gamma := \{\gamma \in C(M, X) \ \ : \ \ \gamma|_{M_0} \in \Gamma_0\}.$$

If $\varphi \in C^1(X, \mathbb{R})$ satisfies

$$\infty > b := \inf_{\gamma\in\Gamma} \sup_{u\in M} \varphi(\gamma(u)) > a := \sup_{\gamma_0\in\Gamma_0} \sup_{u\in M_0} \varphi(\gamma_0(u)),$$

then, for every $\varepsilon \in (0, (b-a)/2)$, $\delta > 0$ and $\gamma \in \Gamma$ such that $\sup_M \varphi \circ \gamma \leq b + \varepsilon$, there exists $u \in X$ such that

- $b - 2\varepsilon \leq \varphi(u) \leq b + 2\varepsilon$,
- $\text{dist}(u, \gamma(M)) \leq 2\delta$,
- $\|\varphi'(u)\| \leq 8\varepsilon/\delta$.

The weak solutions of (1.8) are the critical points of the functional S defined on $\mathscr{X}$ as

$$S(u) = E(u) + F(u).$$

LEMMA 3.6. *There exists $e \in \mathscr{X}$ and $r > 0$ such that $\|e\|_{\mathscr{X}} \geq r$ and*

$$b := \inf_{\|u\|_{\mathscr{X}}=r} S(u) > S(0) = 0 \geq S(e).$$

Proof. From the definition of the norm of $\mathscr{X}$, there holds

$$S(u) \geq \frac{\|u\|_{\mathscr{X}}^2}{2} - \|u\|_{L^{p+2}(\mathbb{R}^2)}^{p+2}.$$

By Lemma 3.1, there exists $r > 0$ such that

$$b := \inf_{\|u\|_{\mathscr{X}}=r} S(u) > S(0) = 0.$$

It is also easy to see that $S(\lambda v) \to -\infty$ as λ tends to $+\infty$. Hence there exists $\lambda_0 > 0$ such that $e = \lambda_0 v$ satisfies $\|e\|_{\mathscr{X}} > r$ and $S(e) \leq 0$. □

Now we define

$$\Gamma := \{\gamma \in C([0,1], \mathscr{X});\ \gamma(0) = 0,\ \gamma(1) = e\}$$

and

$$d := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} S(\gamma(t)).$$

Clearly, $d \geq b > 0$. Applying Lemma 3.5, there exists a sequence $\{u_n\} \subset \mathscr{X}$ such that $S(u_n) \to d$ and $E'(u_n) \to 0$ as $n \to \infty$.

THEOREM 3.7. *Let $p < 4/3$. The equation (1.8) admits a nontrivial solitary wave solution $u \in \mathscr{X}$.*

Proof. First we will show the boundedness of the sequence $\{u_n\}$ derived above. As $m \to \infty$, we have that

$$d + o(1) + o(1)\|u_n\|_{\mathscr{X}} \geq S(u_n) - \frac{1}{p+2}\langle S'(u_n), u_n\rangle_{\mathscr{X}} = \left(\frac{1}{2} - \frac{1}{p+2}\right)\|u_n\|_{\mathscr{X}}^2,$$

where $\langle , \rangle_{\mathscr{X}}$ is the inner product induced by the norm $\|\cdot\|_{\mathscr{X}}$. Hence $\{u_n\}$ is bounded in $\mathscr{X}$. On the other hand,

$$\delta := \limsup_{n\to\infty} \sup_{\zeta\in\mathbb{R}^2} \int_{B_1(\zeta)} \|u_n\|^2 \neq 0.$$

Indeed, in contrary, if $\delta = 0$, by Lemma 3.4, we obtain that $u_n \to 0$ in $L^s(\mathbb{R}^2)$ for $2 < s < 10/3$. Therefore

$$0 < d = S(u_n) - \frac{1}{2}\langle S'(u_n), u_n\rangle_{\mathscr{X}} + o(1) = \frac{p}{2(p+1)(p+2)} \int_{\mathbb{R}^2} u_n^{p+2} + o(1) = O(1),$$

which is a contradiction; thus $\delta \neq 0$.

Now selecting if necessary a subsequence, we can assume that there exists a sequence $\{(x_n, y_n)\} \subset \mathbb{R}^2$ such that

$$\int_{B_1(x_n,y_n)} |u_n|^2 > \delta/2.$$

Define $w_n(x,y) := u_n(x + x_n, y + y_n)$ so that

$$\int_{B_1(0)} |w_n|^2 > \delta/2.$$

Selecting if necessary a subsequence, we can assume that there exists a $u \in \mathscr{X}$ such that $w_n \rightharpoonup u$ in $\mathscr{X}$ as $n \to \infty$. By Lemma 3.3, w_n tends to u in L^2_{loc} and so $u \neq 0$, and for every $w \in \mathscr{X}$, there holds

$$\int_{\mathbb{R}^2} \left(w_n^{p+2} - u^{p+2}\right) w = \int_{B_R(0)} \left(w_n^{p+2} - u^{p+2}\right) w + \int_{\mathbb{R}^2\setminus B_R(0)} (w_n^{p+2} - u^{p+2}) w.$$

Since $w \in \mathscr{X}$, then $w \in L^p(\mathbb{R}^2)$ and $\{w_n\}$ is bounded in $\mathscr{X}$, hence $\{w_n\}$ is bounded in $L^p(\mathbb{R}^2)$, thus for any $\varepsilon > 0$, there exists $R = R(\varepsilon)$ large enough and independent on n such that

$$\int_{\mathbb{R}^2\setminus B_R(0)} \left(w_n^{p+2} - u^{p+2}\right) w < \varepsilon, \ \forall n.$$

On the other hand, for this $R > 0$, from Lemma 3.3, there holds

$$\int_{B_R(0)} \left(w_n^{p+2} - u^{p+2}\right) w \to 0,$$

as $n \to \infty$. Thusly, we have

$$\int_{\mathbb{R}^2} w_n^{p+2} w \to \int_{\mathbb{R}^2} u^{p+2} w,$$

as $n \to \infty$; which implies

$$\langle S'(u), w\rangle_{\mathscr{X}} = \lim_{n\to\infty} \langle S'(w_n), w\rangle_{\mathscr{X}} = 0.$$

Hence $S'(u) = 0$ and u is a nontrivial solution. □

Now, we are going to prove that any solitary wave of (1.8) is a $C^\infty(\mathbb{R}^2)$ function, provided $p = 1$. More precisely, we have

THEOREM 3.8. *Any solitary wave solution u of (1.1) is continuous and belongs to $C^1_\infty(\mathbb{R}^2)$, the space of functions with continuous derivatives vanishing at infinity. In the case $p = 1$, u belongs to $H^\infty(\mathbb{R}^2)$; moreover $\partial_x^{-1} u_y$ belongs to $H^\infty(\mathbb{R}^2)$.*

Proof. We are reduced to prove the to regularity result for the nonlinear equation

$$u_{xx} + \mathscr{H} u_{xxx} + u_{yy} = \partial_x^2 \left(\frac{u^{p+1}}{p+1} \right). \tag{3.4}$$

First we consider the case $p = 1$. By Lemma 3.1, one has $\mathscr{X} \subset L^{10/3}(\mathbb{R}^2)$ and therefore $u^2 \in L^{5/3}(\mathbb{R}^2)$. It can be checked out that $\frac{\xi^2}{\xi^2+\xi^2|\xi|+\eta^2}$, $\frac{\xi^3}{\xi^2+\xi^2|\xi|+\eta^2}$ and $\frac{\xi^2\eta^{2/3}}{\xi^2+\xi^2|\xi|+\eta^2}$ satisfy the Lizorkin's Theorem of the Hörmander-Mikhlin multipliers [28]. Hence u, u_x, $D_y^{2/3} u \in L^{5/3}(\mathbb{R}^2)$. By using [24, Theorem 1], one has $u \in L^5(\mathbb{R}^2)$; which implies that $u^2 \in L^{5/2}(\mathbb{R}^2)$. Another application of Lizorkin's Theorem for $\frac{\xi^2}{\xi^2+\xi^2|\xi|+\eta^2}$, $\frac{\xi^3}{\xi^2+\xi^2|\xi|+\eta^2}$ and $\frac{\xi^2\eta^{1/2}}{\xi^2+\xi^2|\xi|+\eta^2}$ leads to u, u_x, $D_y^{1/2} u \in L^{5/2}(\mathbb{R}^2)$; and by aforementioned result of [24], $u \in L^{15}(\mathbb{R}^2)$.

Similarly Lizorkin's Theorem implies that u, u_x, $D_y^{1/7} u \in L^{15/2}(\mathbb{R}^2)$ which implies [24], $u \in L^{60}(\mathbb{R}^2)$. By reiteration of the process we obtain that

$$u, u_x \in L^q(\mathbb{R}^2), \qquad \forall q,\ 5/3 \le q < \infty.$$

This implies that $\left(\frac{u^2}{2}\right)_x = u u_x \in L^q(\mathbb{R}^2)$, $\forall q$, $1 \le q < \infty$. By using (3.4) and Lizorkin's Theorem, we obtain that u, u_y, $u_{xx} \in L^q(\mathbb{R}^2)$, $\forall q$, $1 < q < \infty$. Now by applying [6, Theorem 10.2] , one obtains that u, $u_x \in L^\infty(\mathbb{R}^2)$. The proof is now obtained, in this case, by reiteration.

This leads us to

$$\left(\frac{u^2}{2}\right)_{xx} = u_x^2 + u u_{xx} \in L^q(\mathbb{R}^2), \qquad \forall q,\ 1 \le q < \infty.$$

We use again Lizorkin's Theorem to obtain that u_{xxx}, $u_{yy} \in L^q(\mathbb{R}^2)$, $\forall q$, $1 < q < \infty$; yielding u_y, $u_{xx} \in L^\infty(\mathbb{R}^2)$ by using [6]. The regularity of $\partial_x^{-1} u_y$ follows from the regularity of u and (1.8).

The first part of the theorem, in the case $p < 4/3$, can be proved similarly by using the fact $u^{p+1} \in L^{p_0/(p+1)}(\mathbb{R}^2)$, Lizorkin's Theorem, the embedding of anisotropic Sobolev–Liouville spaces [24], Theorem 10.2 of [6] and a reiteration argument. □

REMARK 3.9. *Note that in this case, $u^p u_x$ is not a C^∞ function so that we have the finite order regularity of the solitary wave.*

4. Variational Characterizations

In this section, we are going to obtain some variational properties of the solution constructed in last section which plays an important role in our stability/instability analysis (cf.[11]). In particular, we shall show that these solutions are ground states, i.e. solutions with minimal energy. Let

$$I(u) = \langle S'(u), u \rangle = \|u\|_{\mathscr{X}}^2 - \frac{1}{p+1} \|u\|_{L^{p+2}(\mathbb{R}^2)}^{p+2}.$$

Then a solution of (1.8) with least energy is a solution of the following minimization problem:

$$m = \inf\{S(u);\ u \in \mathscr{N}\}, \quad \text{where } \mathscr{N} = \{u \in \mathscr{X};\ u \ne 0,\ I(u) = 0\}. \tag{4.1}$$

THEOREM 4.1. *There is a nontrivial minimizer for (4.1) which is a solution of the equation (1.8).*

Before proving Theorem (4.1), we need to find a uniform lower bound for the functions in the manifold $\mathscr{N}$.

LEMMA 4.2. *There exist ε_1, $\varepsilon_2 > 0$ such that for every nontrivial function $u \in \mathscr{N}$, we have $\|u\|_{\mathscr{X}} \geq \varepsilon_1$ and $S(u) \geq \varepsilon_2$.*

Proof. Given $\varepsilon > 0$. Then, by Lemma 3.1, there exists C_ε such that

$$\|u\|_{\mathscr{X}}^2 = \|u\|_{L^{p+2}(\mathbb{R}^2)}^{(p+2)} \leq \varepsilon \|u\|_{\mathscr{X}}^2 + C_\varepsilon \|u\|_{\mathscr{X}}^p, .$$

which shows that

$$\|u\|_{\mathscr{X}} \geq \left(\frac{1-\varepsilon}{C_\varepsilon}\right)^{1/p}.$$

On the other hand,

$$S(u) \geq \frac{1}{2}\|u\|_{\mathscr{X}}^2 - \frac{1}{p+2}\|u\|_{L^{p+2}(\mathbb{R}^2)}^{p+2} = \frac{p}{p+2}\|u\|_{\mathscr{X}}^2.$$

This, together with the first estimate, gives the desired lower bound and the proof is complete. □

Proof of Theorem 4.1. Let $u_n \in \mathscr{N}$ be minimizing sequence of (4.1). From Lemma 4.2, we obtain that $\|u_n\|_{\mathscr{X}} \geq \varepsilon_1$ and

$$S(u_n) = \frac{p}{2(p+1)(p+2)}\|u_n\|_{L^{p+2}(\mathbb{R}^2)}^{p+2} = \frac{p}{2(p+1)(p+2)}\|u_n\|_{\mathscr{X}}^2,$$

which shows that $\{u_n\}_n$ is bounded in $\mathscr{X}$. To show that there is a convergent subsequence, with a limit $u \in \mathscr{X}$, we use the concentration-compactness of Lions [25], applied to the sequence

$$\rho_n = u_n^2 + \left|D_x^{1/2}u_n\right|^2 + \left|\partial_x^{-1}(u_n)_y\right|^2.$$

First, by using Lemma 3.4, the evanescence case is excluded. To rule out the dichotomy case, one shows that

$$m < m_\sigma := \inf\left\{S(u) - \frac{1}{2}I(u);\ I(u) = \sigma\right\},$$

for all $\sigma < 0$. Now if the dichotomy would occur, i.e. u_n splits into a sum $u_n^1 + u_n^2$ and the distance of the supports of these functions tends to $+\infty$, then one shows that $I\left(u_n^1\right) \to \sigma$, $I\left(u_n^2\right) \to -\sigma$, $\sigma \in \mathbb{R}$ and $m \geq m_\sigma + m_{-\sigma} > m$ which is a contradiction.

Therefore the sequence u_n concentrates and the limit u satisfies $I(u) \leq 0$. The case $I(u) < 0$ can be excluded by the same reasoning as above and we obtain $u \in \mathscr{X}$, a minimizer for (4.1).

Now since u is minimizer for (4.1), there exists a Lagrange multiplier θ such that $S'(u) = \theta I'(u)$. Since $\langle S'(u), u\rangle = 0$ and

$$\langle I'(u), u\rangle = \|u\|_{\mathscr{X}}^2 - \frac{1}{p+1}\|u\|_{L^{p+2}(\mathbb{R}^2)}^{p+2} \leq -C\|u\|_{\mathscr{X}}^2 < 0,$$

we claim that $\theta = 0$, i.e. u is a solution of (1.8). □

Next, we are going to obtain another characterization of the minimum m in (4.1). Indeed, we consider another minimization problem, *viz.*

$$d' = \inf_{u\in\mathscr{Z}} \sup_{t>0} S(tu), \qquad \text{where} \quad \mathscr{Z} = \left\{ u \in \mathscr{X},\ \int_{\mathbb{R}^2} u^{p+2} > 0 \right\}. \tag{4.2}$$

REMARK 4.3. *It is worth noting that for every $u \in \mathscr{Z}$, there exists a unique $t = t_u$ such that $tu \in \mathscr{N}$,*

$$S(tu) = \max_{s>0} S(su)$$

and t_u depends continuously only $u \in \mathscr{Z}$. In fact, it is easy to see that the function

$$\frac{d}{dt} S(tu) = I(tu) = t^2 \left(\|u\|_{\mathscr{X}}^2 - \frac{t^p}{p+1} \int_{\mathbb{R}^2} u^{p+2} \right)$$

vanishes at only one point $t = t_u > 0$. Thusly S is positive on $\mathscr{N}$. Since $S(0) = 0$, we see that t is a point of maximum for $S(tu)$.

THEOREM 4.4. *$d' = m = d$, where d is as we defined in Section 3, viz.*

$$d = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} S(\gamma(t)), \qquad \textit{where} \quad \Gamma = \{\gamma \in C([0,1], \mathscr{X});\ \gamma(0) = 0,\ S(\gamma(1)) < 0\}.$$

Proof. $d \geq m$: since u^{p+2} is subquadratic at zero, we see that $I(u) > 0$ in a neighborhood of the origin, except of zero. Hence $I(\gamma(t)) > 0$, $\gamma \in \Gamma$, for some $t > 0$. Now for $u \in \mathscr{Z}$, we have

$$2S(u) = \|u\|_{\mathscr{X}}^2 - \frac{2}{(p+1)(p+2)} \int_{\mathbb{R}^2} u^{p+2} > I(u).$$

Hence $I(\gamma(t)) < 0$, therefore $\gamma(t)$ crosses $\mathscr{N}$ and this implies that $d \geq m$.

$d' \geq d$: for $u \in \mathscr{Z}$, we have $(tu)^{p+2} \geq Ct^\mu$, for some $C, \mu > 0$, if $t > 0$ is large enough. This implies that $S(tu) < 0$ for every $u \in \mathscr{Z}$ for sufficiently large $t > 0$. Hence the half-axis $\{tu;\ t > 0\}$ generates, in a natural way, an element of Γ . This leads us to the inequality $d \leq d'$.

$m \geq d'$: let $u \in \mathscr{N}$. By definition of I we have $\sigma = \int_{\mathbb{R}^2} u^{p+2} > 0$ and

$$\frac{d}{dt} \int_{\mathbb{R}^2} \frac{t^{p+2} u^{p+2}}{(p+1)(p+2)} = \int_{\mathbb{R}^2} \frac{t^{p+1} u^{p+2}}{p+1} \geq \sigma > 0,$$

provided $t \geq (p+1)^{1/(p+1)}$. Hence for $t > 0$ large enough

$$\int_{\mathbb{R}^2} \frac{t^{p+2} u^{p+2}}{(p+1)(p+2)} > 0.$$

By the definition of d' and m, we see that $d' \geq m$. This completes the proof. □

The following theorem characterize the minimax value d. Let

$$L(u) = \frac{p}{2(p+1)(p+2)} \int_{\mathbb{R}^2} u^{p+2}.$$

THEOREM 4.5. *For a nontrivial $u \in \mathscr{X}$, the following statements are equivalent:*

(i) *u is a ground state of (1.8),*
(ii) *$I(u) = 0$ and $L(u) = m = \inf\{L(v);\ v \in \mathscr{N}\}$,*
(iii) *$I(u) = 0 = \inf\{I(v);\ v \in \mathscr{X},\ L(v) = m\}$.*

Proof. Implication (i)$\Longrightarrow$(ii) is proved in Theorem 4.1.

(ii)$\Longrightarrow$(i): assume that $u \in \mathscr{X}$ satisfying (ii). Since $S = L$ on $\mathscr{N}$, there exists a Lagrange multiplier θ such that $S'(u) = \theta I'(u)$. Then

$$\theta \langle I'(u), u \rangle = \langle S'(u), u \rangle = I(u) = 0.$$

On the other hand,

$$\langle I'(u), u \rangle = -\frac{p}{p+1} \int_{\mathbb{R}^2} u^{p+2}.$$

However $L(u) > 0$ on $\mathscr{N}$, so that $\theta = 0$ and u is a ground state.

(ii)$\Longrightarrow$(iii): if u satisfies (ii), $I(u) = 0$. Assume that there exists $v \in \mathscr{X}$ such that $L(v) = m$ and $I(v) < 0$. Then $L(v) > 0$ and there exists $t_0 \in (0,1)$ such that $I(t_0 v) = 0$. This contradicts $L(t_0 v) < L(v) = m$.

(iii)$\Longrightarrow$(ii): let $u \in \mathscr{X}$ satisfying (iii). Then $L(u) \geq m$. Assume that $L(u) > m$. Again, we have $L(u) > 0$. Thusly there exists $t_0 \in (0,1)$ such that $L(t_0 u) = m$. However $I(t_0 u) > 0$ and this contradicts (iii). ☐

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