STRONGLY REGULAR NONSMOOTH GENERALIZED EQUATIONS (revised)

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ABSTRACT

This note presents an implicit function theorem for generalized equations, simultaneously generalizing Robinson's implicit function theorem for strongly regular generalized equations and Clarke's implicit function theorem for equations with Lipschitz-continuous mappings.

Key words: generalized equation, generalized Jacobian, strong regularity, CD-regularity, inverse function theorem, implicit function theorem.
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1 Introduction and the main result

Consider the generalized equation (GE)

$$\Phi(x) + N(x) \ni 0, \tag{1.1}$$

where $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ is a (single-valued) mapping, and N is a set-valued mapping from $\mathbb{R}^n$ to the subsets of $\mathbb{R}^n$ (i.e., $N(x) \subset \mathbb{R}^n$ for each $x \in \mathbb{R}^n$). This problem setting is a very general framework including the most important cases of variational problems [7]. In particular, the case of usual nonlinear equation

$$\Phi(x) = 0, \tag{1.2}$$

corresponds to (1.1) with $N(\cdot) \equiv \{0\}$.

In this note, we prove a stability result for solutions of GE (1.1), unifying two classical facts of variational analysis. The first fact belongs to Robinson; it is a particular case of the implicit function theorem proved in [11], and it relies on the following fundamental concept assuming that Φ is differentiable at the solution in question (its Jacobian will be denoted by Φ').

Definition 1.1 ([11]) GE (1.1) is said to be *strongly regular* at a solution $\bar{x}$ if there exist neighborhoods U of $\bar{x}$ and V of 0 such that for every $\eta \in V$, the perturbed (partially) linearized GE

$$\Phi(\bar{x}) + \Phi'(\bar{x})(x - \bar{x}) + N(x) \ni \eta$$

has in U a unique solution $x(\eta)$, and the mapping $\eta \to x(\eta) : V \to U$ is Lipschitz-continuous.

Theorem 1.1 ([11]) *Let $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ be differentiable in a neighborhood of $\bar{x} \in \mathbb{R}^n$, with its derivative being continuous at $\bar{x}$. Let N be a multifunction from $\mathbb{R}^n$ to the subsets of $\mathbb{R}^n$. Assume that $\bar{x}$ is a strongly regular solution of the GE* (1.1).

Then there exist neighborhoods U of $\bar{x}$ and V of 0 such that for every $y \in V$ there exists a unique $x(y) \in U$ satisfying the perturbed GE

$$\Phi(x) + N(x) \ni y, \tag{1.3}$$

and the mapping $y \to x(y) : V \to U$ is Lipschitz-continuous.

The second result is Clarke's inverse function theorem [3] (see also [2, Theorem 7.1.1]), which is concerned with the case of a usual nonlinear equation (1.2), but which assumes local Lipschitz continuity of Φ rather than smoothness. In order to state this result, we need to recall the related terminology.

The B-differential of Φ at $\bar{x} \in \mathbb{R}^n$ is the set

$$\partial_B \Phi(\bar{x}) = \{J \in \mathbb{R}^{n \times n} \mid \exists \{x^k\} \subset \mathcal{S}_\Phi \text{ such that } \{x^k\} \to \bar{x},\ \{\Phi'(x^k)\} \to J\},$$

where $\mathcal{S}_\Phi$ is the set of points at which Φ is differentiable. Then the Clarke generalized Jacobian of Φ at $\bar{x}$ is given by

$$\partial \Phi(\bar{x}) = \operatorname{conv} \partial_B \Phi(\bar{x}),$$

where conv stands for the convex hull.

Definition 1.2 ([10]) The mapping $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ is referred to as *CD-regular* at $\bar{x} \in \mathbb{R}^n$ if each matrix $J \in \partial\Phi(\bar{x})$ is nonsingular.

Theorem 1.2 ([3]) *Let $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ be Lipschitz-continuous in a neighborhood of $\bar{x} \in \mathbb{R}^n$. Assume that $\bar{x}$ is a solution of* (1.2)*, and that Φ is CD-regular at $\bar{x}$.*

Then there exist neighborhoods U of $\bar{x}$ and V of 0 such that for every $y \in V$ there exists a unique $x(y) \in U$ satisfying

$$\Phi(x) = y,$$

and the mapping $y \to x(y) : V \to U$ is Lipschitz-continuous.

In order to state our main result, we introduce the following concept extending Definition 1.1 to the nonsmooth case, and at the same time, extending Definition 1.2 to the setting of GE. Observe that the single-valued part of GE (1.4) below is affine, and hence differentiable, with the Jacobian identically equal to J.

Definition 1.3 A solution $\bar{x}$ of GE (1.1) is said to be *CD-regular* if for each $J \in \partial\Phi(\bar{x})$ the GE

$$\Phi(\bar{x}) + J(x - \bar{x}) + N(x) \ni 0 \tag{1.4}$$

is strongly regular at the solution $\bar{x}$.

Theorem 1.3 (main result) *Let $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ be Lipschitz-continuous in a neighborhood of $\bar{x} \in \mathbb{R}^n$. Let N be a multifunction from $\mathbb{R}^n$ to the subsets of $\mathbb{R}^n$. Assume that $\bar{x}$ is a CD-regular solution of GE* (1.1)*.*

Then there exist neighborhoods U of $\bar{x}$ and V of 0 such that for every $y \in V$ there exists a unique $x(y) \in U$ satisfying the perturbed GE (1.3)*, and the mapping $y \to x(y) : V \to U$ is Lipschitz-continuous.*

This theorem extends Theorem 1.1 to the case of a nonsmooth mapping Φ, and at the same time, it extends Theorem 1.2 from usual equations to GEs.

The proof of Theorem 1.1 in [11] relies on the classical contraction mapping principle. Theorem 1.2 can be proven in many ways, but perhaps one of the most prominent is the following: one only needs to show that Φ is locally injective, and then apply Brouwer's invariance of domain theorem (see, e.g., [7, Theorem 2.1.11]) in order to show the existence of solutions. However, apparently, neither the contraction mapping principle nor the invariance of domain theorem are applicable in our general context, and we employ Brouwer's fixed-point theorem in the proof given in the next section.

The property asserted in Theorem 1.3, as well as in Theorems 1.1 and 1.2, is referred to as strong metric regularity of the multifinction $\Phi + N$ at $\bar{x}$ for 0 in the terminology of [5], or the existence of Lipschitzian localization of the solution mapping in the terminology of [12]. Note that strong regularity in Definition 1.1 can be regarded as strong metric regularity for the partially linearized multifunction $\Phi(\bar{x}) + \Phi'(\bar{x})(\cdot - \bar{x}) + N(\cdot)$ at $\bar{x}$ for 0.

Strong metric regularity has multiple applications, e.g., in numerical variational analysis; see [4, Chapter 6], where it appears as an assumption in the analysis of conditioning issues, and of local convergence and rate of convergence of some iterative schemes for GEs and of their particular instances. Therefore, Theorem 1.3 says that in the case of local Lipschitz continuity of Φ, strong metric regularity is implied by CD-regularity. In its turn, the latter property has verifiable characterizations in some more specific problem settings, and in particular, for the Karush–Kuhn–Tucker systems of mathematical programming problems with Lipschitzian first derivatives but possibly without second derivatives. Specifically, in [9, Proposition 3, Remark 1] it was demonstrated that in this case, CD-regularity is implied by the linear independence constraint qualification and the strong second-order sufficient optimality condition for all (infinitely many, in general) matrices in the generalized Hessian (that is, the generalized Jacobian of the gradient) of the Lagrangian. Note, however, that it is evidently sufficient to verify the latter assumption only for matrices in the B-differential of the gradient of the Lagrangian, and the latter set can be much smaller and even finite. For instance, it is always finite in the important case when the derivatives of the problem data are piecewise smooth. In [8], these results are further applied to the very general Newton-type scheme for GEs, and to augmented Lagrangian and linearly constrained Lagrangian methods for optimization problems with specified smoothness properties.

The rest of the paper is organized as follows. In Section 2 we give a proof of Theorem 1.3. In Section 3 we provide the implicit function counterpart of Theorem 1.3, allowing for arbitrary Lipschitzian perturbations rather than right-hand side perturbations only.

2 Proof of the main result

The following observations will be used in the proof below.

Remark 2.1 Assuming Lipschitz continuity of Φ in a neighborhood of $\bar{x}$, due to the fact that the generalized Jacobian of a mapping with this property is compact [2, Proposition 2.6.2], and that strong metric regularity is stable subject to small Lipschitzian perturbations of Φ (see, e.g., [5, Theorem 1.4]), CD-regularity of the solution $\bar{x}$ implies the following: there exist neighborhoods O of $\bar{x}$ and W of 0, and $\ell > 0$, such that for all $J \in \partial\Phi(\bar{x})$ and all $\eta \in W$ there exists the unique $\varphi_J(\eta) \in O$ satisfying the GE

$$\Phi(\bar{x}) + J(x - \bar{x}) + N(x) \ni \eta, \tag{2.1}$$

and the mapping $\varphi_J(\cdot) : W \to O$ is Lipschitz-continuous on W with the constant ℓ. This is demonstrated, e.g., in [9, Proposition 2]. Observe that by necessity

$$\varphi_J(0) = \bar{x} \quad \forall\, J \in \partial\Phi(\bar{x}). \tag{2.2}$$

We complete this remark with an argument demonstrating that $\varphi_J(\cdot)$ is locally Lipschitz-continuous with respect to J as well[1]. This can be shown in many ways. One possibility is to

[1]Recently it was pointed out by R. Cibulka and A.L. Dontchev in [1] that the earlier versions of this paper were missing the argument demonstrating that $\varphi_J(\cdot)$ is continuous with respect to J. By that time the author was also informed about this gap in the proof by E.I. Uskov. This missing fact is quite simple, but the author agrees that it cannot be considered completely evident. That is why this version of the paper provides a fix for this gap, quite similar to the one suggested at Step 3 of the argument in [1], but demonstrating not merely continuity but Lipschitz-continuity. The author is grateful to the colleagues mentioned above for pointing out the need of such fix, and to A.S. Kurennoy and M.V. Solodov for useful discussions of this issue.

apply the implicit function theorem from [11] to the GE (2.1) with a smooth base mapping, considering $(J,\, \eta)$ as a parameter. However, below we present a more direct argument.

Specifically, we will next directly verify that for the objects defined above in this remark, there exists a neighborhood $\tilde{W} \subset W$ of 0 such that the mapping $(J,\, \eta) \to \varphi_J(r)$ is Lipschitz-continuous on $\partial\Phi(\bar{x}) \times \tilde{W}$.

Indeed, for every $J_1,\, J_2 \in \partial\Phi(\bar{x})$ and every $\eta^1,\, \eta^2 \in W$ we have

$$\Phi(\bar{x}) + J_1(\varphi_{J_1}(\eta^1) - \bar{x}) + N(\varphi_{J_1}(\eta^1)) \ni \eta^1,$$

implying that

$$\Phi(\bar{x}) + J_2(\varphi_{J_1}(\eta^1) - \bar{x}) + N(\varphi_{J_1}(\eta^1)) \ni \eta^1 - (J_1 - J_2)(\varphi_{J_1}(\eta^1) - \bar{x}). \tag{2.3}$$

Furthermore, according to (2.2) and by Lipschitz continuity of $\varphi_{J_1}(\cdot)$ with a constant ℓ, it holds that

$$\|\varphi_{J_1}(\eta^1) - \bar{x}\| = \|\varphi_{J_1}(\eta^1) - \varphi_{J_1}(0)\| \le \ell\|\eta^1\|. \tag{2.4}$$

Since $\partial\Phi(\bar{x})$ is bounded, the latter implies the existence of a neighborhood $\tilde{W} \subset W$ of 0 such that $\varphi_{J_1}(\eta^1) \in O$ and $\eta^1 - (J_1 - J_2)(\varphi_{J_1}(\eta^1) - \bar{x}) \in W$ provided $\eta^1 \in \tilde{W}$. Therefore, according to (2.3),

$$\varphi_{J_1}(\eta^1) = \varphi_{J_2}(\eta^1 - (J_1 - J_2)(\varphi_{J_1}(\eta^1) - \bar{x})),$$

and hence,

$$\begin{aligned} \|\varphi_{J_1}(\eta^1) - \varphi_{J_2}(\eta^2)\| &= \|\varphi_{J_2}(\eta^1 - (J_1 - J_2)(\varphi_{J_1}(\eta^1) - \bar{x})) - \varphi_{J_2}(\eta^2)\| \\ &\le \ell\|\eta^1 - \eta^2 - (J_1 - J_2)(\varphi_{J_1}(\eta^1) - \bar{x})\| \\ &\le \ell(\|\eta^1 - \eta^2\| + \ell\|\eta^1\|\|J_1 - J_2\|), \end{aligned}$$

where the last equality is by (2.4). The needed property is now evident.

Remark 2.2 If Φ is Lipschitz-continuous in a neighborhood of $\bar{x}$, then the following approximation property is valid: for any $\varepsilon > 0$ there exists $\delta > 0$ such that the inequality

$$\min_{J \in \partial\Phi(\bar{x})} \|\Phi(x^1) - \Phi(x^2) - J(x^1 - x^2)\| \le \varepsilon\|x^1 - x^2\| \tag{2.5}$$

holds for all $x^1,\, x^2 \in B_\delta(\bar{x})$. Here and throughout $B_r(x)$ stands for the closed ball of radius $r > 0$ centered at $x \in \mathbb{R}^n$.

Indeed, by the mean-value theorem [2, Proposition 2.6.5], for any $x^1,\, x^2 \in \mathbb{R}^n$ close enough to $\bar{x}$ there exists

$$M \in \operatorname{conv} \bigcup_{t \in [0,\,1]} \partial\Phi(tx^1 + (1-t)x^2) \tag{2.6}$$

such that

$$\Phi(x^1) - \Phi(x^2) = M(x^1 - x^2). \tag{2.7}$$

By upper semicontinuity of generalized Jacobian [2, Proposition 2.6.2 (c)] it follows that for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\partial\Phi(x) \subset \partial\Phi(\bar{x}) + B_\varepsilon(0) \quad \forall\, x \in B_\delta(\bar{x}),$$

and hence,

$$\partial\Phi(tx^1 + (1-t)x^2) \subset \partial\Phi(\bar{x}) + B_\varepsilon(0) \quad \forall\, x^1,\, x^2 \in B_\delta(\bar{x}),\ \forall\, t \in [0,\, 1].$$

By convexity of generalized Jacobian (and hence, of the set $\partial\Phi(\bar{x}) + B_\varepsilon(0)$) it further follows that

$$\operatorname{conv} \bigcup_{t\in[0,\,1]} \partial\Phi(tx^1 + (1-t)x^2) \subset \partial\Phi(\bar{x}) + B_\varepsilon(0) \quad \forall\, x^1,\, x^2 \in B_\delta(\bar{x}).$$

Therefore, according to (2.6), for any $x^1,\, x^2 \in B_\delta(\bar{x})$ there exists $J \in \partial\Phi(\bar{x})$ such that $\|M - J\| \leq \varepsilon$, and according to (2.7),

$$\|\Phi(x^1) - \Phi(x^2) - J(x^1 - x^2)\| = \|(M - J)(x^1 - x^2)\| \leq \varepsilon\|x^1 - x^2\|.$$

Since ε is arbitrary, this proves (2.5), taking into account that $\partial\Phi(\bar{x})$ is compact [2, Proposition 2.6.2 (a)] (and hence, minimum in (2.5) is attained).

When for some J it holds that for every $\varepsilon > 0$ there exists $\delta > 0$ such that (2.5) holds without the minimum, then Φ is strictly differentiable at $\bar{x}$, and J is its true Jacobian, by necessity. The point here is that in (2.5), J moves with x^1 and x^2, as it depends on them through this minimum operation, and of course, (2.5) does not subsume any differentiability of Φ at $\bar{x}$.

Apparently, the property discussed in this remark was for the first time explicitly observed in [6]. This property might suggest to pick up any $J = J_x \in \Phi(\bar{x})$ for which minimum in the left-hand side of (2.5) is attained, and to consider the mapping $h : \mathbb{R}^n \to \mathbb{R}^n$,

$$h(x) = \Phi(\bar{x}) + J_x(x - \bar{x}),$$

as an estimator of Φ at $\bar{x}$ (see [4, p. 37] for the definition of an estimator). However, unfortunately, Theorem 2B.7 in [4] or similar results cannot be applied with this h. The problem is that this theorem requires the estimator to be strict, that is, the error mapping $\Phi - h$ must be Lipschitz-continuous in a neighborhood of $\bar{x}$ with a small Lipschiz constant, which is not guaranteed by the properties of J_x.

We proceed with the proof of Theorem 1.3.

For any $J \in \partial\Phi(\bar{x})$ and any $y \in \mathbb{R}^n$, GE (1.3) is equivalent to (2.1) with $\eta = \eta_J(x,\, y)$, where the mapping $\eta_J : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is defined by

$$\eta_J(x,\, y) = -(\Phi(x) - \Phi(\bar{x}) - J(x - \bar{x})) + y. \tag{2.8}$$

Let $O,\, W,\, \ell > 0$ and the family of mappings $\varphi_J(\cdot) : W \to O,\ J \in \partial\Phi(\bar{x})$, be defined according to Remark 2.1. Since $\partial\Phi(\bar{x})$ is compact, there exist $\delta > 0$ and $\rho > 0$ such that for any $x \in B_\delta(\bar{x}) \subset O$ and any $y \in B_\rho(0)$ it holds that $\eta_J(x,\, y) \in W$ and η_J is continuous on

$B_\delta(\bar{x}) \times \mathbb{R}^n$, for all $J \in \partial\Phi(\bar{x})$, and therefore, for such x, y and J, (1.3) is further equivalent to the usual equation

$$x = \varphi_J(\eta_J(x, y)). \tag{2.9}$$

The main part of the proof consists of showing that with an appropriate choice of J (as a function of x!), equation (2.9) is solvable.

Fix any $\varepsilon \in (0, 1/(3\ell)]$, and define the function $\omega : \mathbb{R}^n \to \mathbb{R}_+$,

$$\omega(x) = \min_{J \in \partial\Phi(\bar{x})} \|\Phi(x) - \Phi(\bar{x}) - J(x - \bar{x})\|. \tag{2.10}$$

Then from Remark 2.2 it follows that by further reducing $\delta > 0$, if necessary, we can ensure that

$$\omega(x) \leq \varepsilon\delta \quad \forall\, x \in B_\delta(\bar{x}). \tag{2.11}$$

Now for each $x \in B_\delta(\bar{x})$ we will select the specific $J = J_x \in \partial\Phi(\bar{x})$ as follows. Consider the parametric optimization problem

$$\begin{array}{ll} \text{minimize} & \|\Phi(x) - \Phi(\bar{x}) - J(x - \bar{x})\| + \alpha\|J\|_*^2 \\ \text{subject to} & J \in \partial\Phi(\bar{x}), \end{array} \tag{2.12}$$

where $x \in \mathbb{R}^n$ and $\alpha > 0$ are parameters, and $\|\cdot\|_*$ is any norm defined by an inner product in $\mathbb{R}^{n\times n}$ (e.g., the Frobenius norm). Let $v : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}_+$ be the optimal-value function of this problem:

$$v(x, \alpha) = \min_{J \in \partial\Phi(\bar{x})} (\|\Phi(x) - \Phi(\bar{x}) - J(x - \bar{x})\| + \alpha\|J\|_*^2). \tag{2.13}$$

Then, according to (2.10), $v(x, 0) = \omega(x)$ for all $x \in \mathbb{R}^n$, and since $\partial\Phi(\bar{x})$ is compact, for any fixed $\bar{\alpha} > 0$ the function v is continuous on the compact set $B_\delta(\bar{x}) \times [0, \bar{\alpha}]$. Since a continuous function on a compact set is uniformly continuous, this further implies the existence of $\alpha > 0$ such that

$$v(x, \alpha) \leq \omega(x) + \varepsilon\delta \quad \forall\, x \in B_\delta(\bar{x}). \tag{2.14}$$

Furthermore, with this positive α fixed, the objective function of problem (2.12) is strongly convex, and therefore, this problem with a convex feasible set has the unique solution J_x for any $x \in \mathbb{R}^n$. This implies that the mapping $x \to J_x : B_\delta(\bar{x}) \to \partial\Phi(\bar{x})$ is continuous. Moreover, according to (2.11), (2.13) and (2.14),

$$\|\Phi(x) - \Phi(\bar{x}) - J_x(x - \bar{x})\| \leq v(x, \alpha) \leq \omega(x) + \varepsilon\delta \leq 2\varepsilon\delta \quad \forall\, x \in B_\delta(\bar{x}). \tag{2.15}$$

For any $y \in B_\rho(0)$, define the mapping $\chi_y : B_\delta(\bar{x}) \to \mathbb{R}^n$,

$$\chi_y(x) = \varphi_{J_x}(\eta_{J_x}(x, y)). \tag{2.16}$$

By further reducing $\delta > 0$ if necessary, so that $\varepsilon\delta \leq \rho$, for any $y \in B_{\varepsilon\delta}(0)$, from (2.2), (2.8) and (2.15) we derive

$$\begin{array}{rcl} \|\chi_y(x) - \bar{x}\| & = & \|\varphi_{J_x}(\eta_{J_x}(x,\, y)) - \varphi_{J_x}(0)\| \\ & \leq & \ell\|\eta_{J_x}(x,\, y)\| \\ & \leq & \ell(\|\Phi(x) - \Phi(\bar{x}) - J_x(x - \bar{x})\| + \|y\|) \\ & \leq & \ell(2\varepsilon\delta + \varepsilon\delta) \\ & = & 3\ell\varepsilon\delta \\ & \leq & \delta \quad \forall\, x \in B_\delta(\bar{x}), \end{array}$$

where the last inequality holds because $\varepsilon \leq 1/(3\ell)$. Therefore, recalling again Remark 2.1, χ_y continuously maps $B_\delta(\bar{x})$ into itself, and hence, by Brouwer's fixed-point theorem (see, e.g., [7, Theorem 2.1.18]), there exists $x(y) \in B_\delta(\bar{x})$ such that

$$x(y) = \chi_y(x(y)).$$

According to (2.16), this means that for any $y \in B_{\varepsilon\delta}(0) \subset B_\rho(0)$ the point $x(y) \in B_\delta(\bar{x})$ satisfies (2.9) with $J = J_{x(y)}$, and as discussed above, this is equivalent to saying that $x(y)$ solves GE (1.3).

We thus proved that for any $y \in B_{\varepsilon\delta}(0)$ GE (1.3) has a solution $x(y) \in B_\delta(\bar{x})$. It remains to show that this solution is unique, and the mapping $x(\cdot)$ is Lipschitz-continuous on $B_{\varepsilon\delta}(0)$, provided $\delta > 0$ is small enough. Then setting $U = B_\delta(\bar{x})$ and $V = B_{\varepsilon\delta}(0)$, we will obtain the needed conclusion.

We first show uniqueness. Suppose that there exist sequences $\{x^{1,\,k}\} \subset \mathbb{R}^n$, $\{x^{2,\,k}\} \subset \mathbb{R}^n$ and $\{y^k\} \subset \mathbb{R}^n$ such that both $\{x^{1,\,k}\}$ and $\{x^{2,\,k}\}$ converge to $\bar{x}$, $\{y^k\}$ converges to 0, and for any k it holds that $x^{1,\,k} \neq x^{2,\,k}$, and the points $x^{1,\,k}$ and $x^{2,\,k}$ solve (1.3) with $y = y^k$.

According to Remark 2.2, for any k we can select $J_k \in \partial\Phi(\bar{x})$ such that

$$\|\Phi(x^{1,\,k}) - \Phi(x^{2,\,k}) - J_k(x^{1,\,k} - x^{2,\,k})\| = o(\|x^{1,\,k} - x^{2,\,k}\|). \tag{2.17}$$

Since $\partial\Phi(\bar{x})$ is compact, without loss of generality we can assume that $\{J_k\}$ converges to some $J \in \partial\Phi(\bar{x})$, and then (2.17) implies the estimate

$$\|\Phi(x^{1,\,k}) - \Phi(x^{2,\,k}) - J(x^{1,\,k} - x^{2,\,k})\| = o(\|x^{1,\,k} - x^{2,\,k}\|). \tag{2.18}$$

Since $x^{1,\,k} \in B_\delta(\bar{x})$, $x^{2,\,k} \in B_\delta(\bar{x})$ and $y^k \in B_\rho(0)$ for all k large enough, we have that for such k both points $x^{1,\,k}$ and $x^{2,\,k}$ satisfy (2.9) with $y = y^k$. Employing (2.8) and (2.18), we then have that

$$\begin{array}{rcl} \|x^{1,\,k} - x^{2,\,k}\| & = & \|\varphi_J(\eta_J(x^{1,\,k},\, y^k)) - \varphi_J(\eta_J(x^{2,\,k},\, y^k))\| \\ & \leq & \ell\|\eta_J(x^{1,\,k},\, y^k) - \eta_J(x^{2,\,k},\, y^k)\| \\ & \leq & \ell\|\Phi(x^{1,\,k}) - \Phi(x^{2,\,k}) - J(x^{1,\,k} - x^{2,\,k})\| \\ & = & o(\|x^{1,\,k} - x^{2,\,k}\|), \end{array}$$

giving a contradiction.

We proceed with demonstrating Lipschitz continuity of the solution mapping. Suppose that there exist sequences $\{x^{1,\,k}\} \subset \mathbb{R}^n$, $\{x^{2,\,k}\} \subset \mathbb{R}^n$, and $\{y^{1,\,k}\} \subset \mathbb{R}^n$, $\{y^{2,\,k}\} \subset \mathbb{R}^n$, such that both $\{x^{1,\,k}\}$ and $\{x^{2,\,k}\}$ converge to $\bar{x}$, both $\{y^{1,\,k}\}$ and $\{y^{2,\,k}\}$ converge to 0, and for all k it holds that $y^{1,\,k} \neq y^{2,\,k}$, the point $x^{i,\,k}$ solves (1.3) with $y = y^{i,\,k}$ for $i = 1,\, 2$, and

$$\frac{\|x^{1,\,k} - x^{2,\,k}\|}{\|y^{1,\,k} - y^{2,\,k}\|} \to \infty \text{ as } k \to \infty. \tag{2.19}$$

Repeating the argument used to establish uniqueness, we then obtain the estimate

$$\begin{aligned} \|x^{1,\,k} - x^{2,\,k}\| &= \|\varphi_J(\eta_J(x^{1,\,k},\, y^{1,\,k})) - \varphi_J(\eta_J(x^{2,\,k},\, y^{2,\,k}))\| \\ &\leq \ell\|\eta_J(x^{1,\,k},\, y^{1,\,k}) - \eta_J(x^{2,\,k},\, y^{2,\,k})\| \\ &\leq \ell(\|\Phi(x^{1,\,k}) - \Phi(x^{2,\,k}) - J(x^{1,\,k} - x^{1,\,k})\| + \|y^{1,\,k} - y^{2,\,k}\|) \\ &= \ell\|y^{1,\,k} - y^{2,\,k}\| + o(\|x^{1,\,k} - x^{2,\,k}\|), \end{aligned}$$

giving a contradiction with (2.19). This completes the proof.

3 The implicit function theorem

The implicit function theorem in [11] is formally more general than Theorem 1.1: it allows for more general parametric perturbations. Theorem 1.2 also allows for an implicit function counterpart; see [2, Corollary of Theorem 7.1.1]. In this section, we present the corresponding extension of Theorem 1.3, covering both implicit function theorems by Robinson and Clarke.

Following [2], for a mapping $\Phi : \mathbb{R}^s \times \mathbb{R}^n \to \mathbb{R}^n$ and a point $(\bar{\sigma},\, \bar{x}) \in \mathbb{R}^s \times \mathbb{R}^n$, by $\pi_x\partial\Phi(\bar{\sigma},\, \bar{x})$ we denote the projection of the set $\partial\Phi(\bar{\sigma},\, \bar{x})$ in $\mathbb{R}^{n\times s} \times \mathbb{R}^{n\times n}$ onto $\mathbb{R}^{n\times n}$: the set $\pi_x\partial\Phi(\bar{\sigma},\, \bar{x})$ consists of matrices $J \in \mathbb{R}^{n\times n}$ such that the matrix $(S \;\; J)$ belongs to $\partial\Phi(\bar{\sigma},\, \bar{x})$ for some $S \in \mathbb{R}^{n\times s}$.

Definition 3.1 A solution $\bar{x}$ of GE

$$\Phi(\sigma,\, x) + N(x) \ni 0 \tag{3.1}$$

for $\sigma = \bar{\sigma}$ is said to be *parametrically CD-regular* if for each $J \in \pi_x\partial\Phi(\bar{\sigma},\, \bar{x})$ the solution $\bar{x}$ of the GE

$$\Phi(\bar{\sigma},\, \bar{x}) + J(x - \bar{x}) + N(x) \ni 0$$

is strongly regular.

Theorem 3.1 *Let $\Phi : \mathbb{R}^s \times \mathbb{R}^n \to \mathbb{R}^n$ be Lipschitz-continuous in a neighborhood of $(\bar{\sigma},\, \bar{x}) \in \mathbb{R}^s \times \mathbb{R}^n$. Let N be a multifunction from $\mathbb{R}^n$ to the subsets of $\mathbb{R}^n$. Assume that $\bar{x}$ is a parametrically CD-regular solution of GE* (3.1) *for $\sigma = \bar{\sigma}$.*

Then there exist neighborhoods $\mathcal{U}$ of $\bar{\sigma}$ and U of $\bar{x}$ such that for every $\sigma \in \mathcal{U}$ there exists a unique $x(\sigma) \in U$ satisfying the GE

$$\Phi(\sigma,\, x) + N(x) \ni 0, \tag{3.2}$$

and the mapping $\sigma \to x(\sigma)$ is Lipschitz-continuous in $\mathcal{U}$.

Theorem 3.1 can be derived from Theorem 1.3 by means of a well-known trick commonly used to derive implicit function theorems from inverse function theorems. In particular, this trick was employed in [2].

Define the auxiliary mapping $\Psi : \mathbb{R}^s \times \mathbb{R}^n \to \mathbb{R}^s \times \mathbb{R}^n$ by

$$\Psi(u) = (z,\, \Phi(z,\, x)),$$

and the multifunction M from $\mathbb{R}^s \times \mathbb{R}^n$ to the subsets of $\mathbb{R}^s \times \mathbb{R}^n$ by

$$M(u) = \{-\bar{\sigma}\} \times N(x),$$

where $u = (z,\, x)$. Then $\bar{u} = (\bar{\sigma},\, \bar{x})$ is a solution of the GE

$$\Psi(u) + M(u) \ni 0. \tag{3.3}$$

Moreover, the perturbed GE

$$\Psi(u) + M(u) \ni v$$

with $v = (\sigma - \bar{\sigma},\, 0)$, $\sigma \in \mathbb{R}^s$, takes the form of the system

$$z = \sigma, \quad \Phi(z,\, x) + N(x) \ni 0,$$

which is further equivalent to (3.2). Therefore, Theorem 3.1 will readily follow from Theorem 1.3 if we will show that $\bar{u}$ is a CD-regular solution of GE (3.3).

Evidently, $\partial\Psi(\bar{u})$ consists of matrices of the form

$$\Lambda = \begin{pmatrix} I & 0 \\ S & J \end{pmatrix},$$

where $I \in \mathbb{R}^{s\times s}$ is the unit matrix, and $(S \;\; J)$ belongs to $\partial\Phi(\bar{\sigma},\, \bar{x})$. The GE

$$\Psi(\bar{u}) + \Lambda(u - \bar{u}) + M(u) \ni w$$

with such matrix Λ, and with $w = (\zeta,\, \nu) \in \mathbb{R}^s \times \mathbb{R}^n$, takes the form of the system

$$\sigma = \bar{\sigma} + \zeta, \quad \Phi(\bar{\sigma},\, \bar{x}) + S(\sigma - \bar{\sigma}) + J(x - \bar{x}) + N(x) \ni \nu,$$

which is further equivalent to the GE

$$\Phi(\bar{\sigma},\, \bar{x}) + J(x - \bar{x}) + N(x) \ni \eta$$

with $\eta = \nu - S\zeta$. From Definitions 1.1, 1.3 and 3.1, and from the inclusion $J \in \pi_x\partial\Phi(\bar{\sigma},\, \bar{x})$ it now evidently follows that parametric CD-regularity of the solution $\bar{x}$ of GE (3.1) for $\sigma = \bar{\sigma}$ implies CD-regularity of the solution $\bar{u}$ of GE (3.3). This completes the proof.

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