ERGODIC CLOSING AND NEW CRITERIA FOR HYPERBOLICITY BASED ON PERIODIC SETS

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ABSTRACT. We prove some criteria for uniform hyperbolicity based on the periodic points of the transformation. More precisely, if a mild hyperbolicity condition holds for the periodic points of any diffeomorphism in a residual subset of a C^1-open set $\mathcal{U}$ then there exists an open and dense subset $\mathcal{A} \subset \mathcal{U}$ of Axiom A diffeomorphisms. Moreover, we also prove a noninvertible version of Ergodic Closing Lemma which we use to prove a counterpart of this result for local diffeomorphisms.

1. INTRODUCTION

The notion of uniform hyperbolicity was coined in the mid sixties by the pioneering works of Smale and Anosov and constitutes a rich class of dynamical systems. Indeed, many geometrical, topological and ergodic properties have been proved to hold for uniformly hyperbolic dynamical systems in both discrete and continuous time setting. From the eminently topological point of view, one of the greatest goals achieved by mathematicians in describing uniformly hyperbolic behavior was the proof of the C^1-Structural Stability Conjecture, stated by Palis and Smale in [20]. Roughly, a diffeomorphism f is conjugated to every diffeomorphism in a small C^1-neighbourhood of it if and only if f is hyperbolic and satisfies the strong transversality condition. The contributions of Franks [9], Robbin [28], Robinson[29][30], Pliss [24], Mañé[15] and Liao[11], among others, were fundamental to characterize C^1-structural stability by methods that rely on periodic orbits.

It is quite natural to understand that periodic orbits are key ingredients to characterize uniform hyperbolicity and it is natural to ask whether the loss of hyperbolicity can be observed at this level. Moreover, the more recent developments on the theory of nonuniformly hyperbolic transformations can be used to give a positive answer to this question. In fact a recent contribution in this direction was given by Oliveira, Pinheiro and the present author in [7] that, inspired by [5, 6, 23], studied the relation between asymptotic growth rates of the periodic points of diffeomorphisms and local diffeomorphisms on compact manifolds, and its relations with uniform hyperbolicity and uniform expansion. More precisely, the authors proved that if a diffeomorphism satisfies the shadowing property (in particular if it

is conjugated to a hyperbolic one) and the periodic set is non-uniformly hyperbolic (NUH) such diffeomorphism is hyperbolic and an analogous statement for local diffeomorphisms. One of the key ideas of [7] is that the asymptotic hyperbolicity of a NUH periodic set spreads out to any recurrent point using the shadowing property. In particular, if the NUH periodic set exhibits a dominated splitting then any f-invariant probability measure has only nonzero Lyapunov exponents and, using [2], it follows that f is uniformly hyperbolic. However, even though dominated splitting is a generic (residual) feature among robustly transitive maximal invariant sets (see [4]) the strong notion of shadowing seems to be somewhat rare far from uniformly hyperbolic maps as proved in [8]. Nevertheless, [7] gave rise to some ideas to prove uniform hyperbolicity used e.g. in [12].

In the present work we are interested in providing new criteria to obtain uniform hyperbolicity without assuming the strong shadowing property. Our main assumption, in both invertible and non-invertible settings, is just that a C^1-open set $\mathcal{U}$ exhibits a residual subset of transformations whose periodic sets are NUH. Note that, in principle, the class of transformations considered here could be very far from the uniformly hyperbolic ones. Nevertheless, we are able to prove that under this hypothesis there exists a residual subset of $\mathcal{U}$ whose elements present uniform hyperbolicity. We refer the reader to Section 2 for the definitions and precise statements.

One of the novelties of our approach is that, differently from the strong shadowing conditions typically considered when studying the geometrical aspects of dynamical systems, we use weak shadowing properties that hold just for points in total probability sets. This result, known as the Ergodic Closing Lemma, was proved by Mañé's [14] in the invertible context and it was extended by us (see Section 4) to the context of local diffeomorphisms. Since this kind of shadowing by periodic points holds residually in the space of C^1-diffeomophisms and local diffeomorphisms our criteria provides a generic subset of transformations exhibiting uniform hyperbolicity.

The paper is organized as follows. In Section 2 we detail the context and give exact statements of our theorems. The proofs of the criteria to obtain uniform hyperbolicity from the assumption on the periodic set are given along Section 3 in both invertible and noninvertible context. In the last section, we prove the version of Ergodic Closing Lemma for Endomorphisms which is needed for the proof of our Criteria in the Endomorphism case.

2. Setting and Statement of Main results

Throughout, M will always denote a finite dimensional compact Riemannian manifold. Let $NEnd^1(M) \subset C^1(M)$ will denote the set of C^1 nonsingular endomorphisms (or local diffeomorphisms) in M endowed with the C^1 topology and let $Diff^1(M)$ denote the set of C^1 diffeomorphisms on M. First we recall some necessary definitions.

Definition 2.1. Let Λ be a compact invariant set for a C^1 diffeomorphism f of a manifold M. We say that Λ is a hyperbolic set if there is a continuous splitting $T_\Lambda M = E^s \oplus E^u$ which is Df-invariant ($Df(E^s) = E^s, Df(E^u) = E^u$) and for which there are constants $c > 0$, $0 < \varsigma < 1$, such that

$$\|Df^n|_{E^s}\| < c \cdot \varsigma^n, \quad \|Df^{-n}|_{E^u}\| < c \cdot \varsigma^n, \forall n \in \mathbb{N}.$$

Recall also that a diffeomorphism $f : M \to M$ is *Axiom A* if the nonwandering set $\Omega(f)$ is a hyperbolic set and $\Omega(f) = \overline{Per(f)}$. In such case, $\Omega(f)$ admits a decomposition $\Omega(f) = \Omega_1 \cup \cdots \cup \Omega_w$ into closed, disjoint transitive subsets. A *cycle* on $\Omega(f)$ is a sequence $\Omega_{i_1}, \ldots, \Omega_{i_k}$ with points $x_1, y_1 \in \Omega_{i_1}, \ldots x_k, y_k \in \Omega_{i_k}$ such that $W^s(x_1) \cap W^u(y_2) \neq \emptyset,$ $\ldots W^s(x_k) \cap W^u(y_1) \neq \emptyset$.

Definition 2.2. Given $f \in \mathit{Diff}^1(M)$ we say that an f-invariant set $X \subset M$ is a *non uniformly hyperbolic set* (or simply NUH set) if

(1) There is an $Df-$invariant splitting $T_X M = E^{cs} \oplus E^{cu}$;
(2) There exists $\lambda < 0$ and an adapted Riemannian metric for which any point $p \in X$ satisfies

$$\liminf_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Df(f^j(p))|_{E^{cs}(f^j(p))}\| \leq \lambda$$

and

$$\liminf_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|[Df(f^j(p))|_{E^{cu}(f^j(p))}]^{-1}\| \leq \lambda$$

For simplicity sometimes we will say that f is NUH on X meaning that $X \subset M$ is non uniformly hyperbolic for f. Note that if a set X is NUH and X has positive measure for some ergodic measure μ, then μ has only nonzero Liapunov exponents. The properties of such measures for $C^{1+\alpha}$ diffeomorphisms are extensively studied in Pesin Theory (see e.g. [21], [22]).

Definition 2.3. Given $f \in \mathit{Diff}^1(M)$ and an invariant set $X \subset M$, we say that a Df-invariant splitting $T_X M = E \oplus \hat{E}$ is a *dominated splitting* if for some $l \geq 1$ there exists $0 < \eta < 1$ so that

$$\sup_{v \in E, \|v\|=1} \{\|Df^l(x)v\|\} \cdot (\inf_{v \in \hat{E}, \|v\|=1} \{\|Df^l(x)v\|\})^{-1} \leq \eta, \ \forall x \in X.$$

As discussed in the introduction, this paper is devoted to study the consequences for an open set $\mathcal{U}$ of C^1-transformations that exhibits a residual subset $\mathcal{S}$ in which each map exhibits a NUH periodic set admitting a dominated splitting. Our first main results are as follows.

Theorem A. *Let $\mathcal{U} \subset \mathrm{Diff}^1(M)$ be an open subset of diffeomorphisms and assume that every f in some residual subset $\mathcal{S}$ of $\mathcal{U}$, the set $Per(f) \subset M$ of periodic points of f is non uniformly hyperbolic (NUH), and $T_{Per(g)}M = E^{cs} \oplus E^{cu}$ is a dominated splitting. Then, there exists an open and dense subset of $\mathcal{U}$ whose elements are Axiom A diffeomorphisms*

(with no cycles). In particular, $\mathcal{U}$ is contained in the closure of the Axiom A diffeomorphisms set.

In fact, this result is a consequence of the more general:

Theorem B. *Let $\mathcal{U} \subset \mathrm{Diff}^1(M)$ be an open subset of diffeomorphisms. Suppose that for any f in some residual subset $\mathcal{S}$ of $\mathcal{U}$, the set $Per(f)$ of periodic points of f is non uniformly hyperbolic (NUH), and $T_{Per(f)}M = E^{cs} \oplus E^{cu}$ is a continuous splitting extending to $T_{\overline{Per(f)}}M$. Then, there exists an open and dense subset $\mathcal{A}$ of $\mathcal{U}$ whose elements are Axiom A diffeomorphisms (with no cycles). In particular, $\mathcal{U}$ is contained in the closure of the Axiom A diffeomorphisms set.*

We point out that although Theorem A is a consequence of Theorem B, the hypotheses in Theorem A are easier to verify. An important remark is also that in principle, due to Palis' work (see [18]), Axiom A are not necessarily open if a no-cycles condition is not assumed. We will need a periodic set semicontinuity argument in order to obtain the no-cycles condition and the openness stated in both theorems above. It is also interesting to notice that, at least generically, bifurcations from uniformly hyperbolic diffeomorphisms are verified by some lack of hyperbolicity at periodic points. See e.g. [10] for an example where this is not the case. Furthermore, in the context of the C^1-stability theorem (see [13], [15]), Mañé conjectured that the C^1 interior of the subset of diffeomorphisms whose periodic points are hyperbolic is the set of diffeomorphisms satisfying Axiom A and the no-cycles condition. Such conjecture was proved by N. Aoki [3]. By Kupka-Smale theorem there exists a residual subset of C^1-diffeomorphisms exhibiting only hyperbolic periodic points. This shows that there is a large difference in assuming that a property of hyperbolicity in the periodic set holds residually instead of requiring it for an open set in the space of C^1-diffeomorphisms.

We will say that a system f has the *ergodic closing property* if any f-ergodic measure can be weak* approximated by ergodic measures supported in f-periodic orbits. By the Ergodic Decomposition Theorem, this is obviously equivalent to say that $\mathcal{M}_1(f)$ is the closure of the convex hull of ergodic measures supported in f-periodic orbits.

As a by-product of our techniques we also obtain the following consequence:

Corollary 2.1. *Suppose that $f \in \mathrm{Diff}^1(M)$ exhibits a NUH periodic set with dominated splitting $T_{Per(f)}M = E^{cs} \oplus E^{cu}$. If f has the* ergodic closing property *then, $\overline{Per(f)}$ is hyperbolic.*

In the remaining of this section we deal with the non-invertible setting. As the notion of Axiom A endomorphisms is slightly different and quite elaborated notion we deal first with the case of expanding transformations. For completeness reasons let us recall some necessary definitions.

Definition 2.4. A C^1-map $g : M \to M$ on a compact manifold M is *expanding* if there are constants $C > 0$ and $\sigma > 1$ such that

$$\|[Dg^n(x)]^{-1}\| < C \cdot \sigma^{-n}, \forall n \in \mathbb{N}.$$

The following notion deals with expansion at an asymptotic level and it implies that all Lyapunov exponents of points in X are positive.

Definition 2.5. We say that a map $g : M \to M$ is non uniformly expanding (*NUE*) on a set $X \subset M$ if there exists $\lambda < 0$ such that

$$\liminf_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|[Dg(g^j(x))]^{-1}\| \leq \lambda < 0 \text{ for all } x \in X.$$

We are now in a position to state our criteria to expansion for local diffeomorphisms.

Theorem C. *Let $\mathcal{S} \subset \mathcal{U}$ a residual subset of an open set $\mathcal{U}$ contained in $\mathrm{NEnd}^1(M)$. Suppose that each $g \in \mathcal{S}$ is non uniformly expanding on the set $Per(g) \subset M$ of periodic points. Then, there exists an open and dense subset of $\mathcal{U}$ whose elements are expanding maps.*

We also obtain a result analogous to Corollary 2.1 above:

Corollary 2.2. *Suppose that $g \in \mathrm{NEnd}^1(M)$ exhibits a NUE periodic set. If g has the ergodic closing property then f is an expanding map.*

Now we deal with the more subtle case of Axiom A endomorphisms and prove a result analogous to the one of Theorem B. One of the main difficulties when dealing with non-invertible maps is the possible existence of positively invariant sets (that is, sets which are equal to their images) for an endomorphism that are not negatively invariant. In fact, the single existence of an unstable space is not uniquely determined: if the dimension of unstable space $E^u(x)$ of a point x is not the same of the ambient manifold (which is the expanding map case), for each choice of negative branch in the pre-orbit of x we may obtain a different unstable space for x. This motivates to the use of unstable cone fields.

Let $X \subset M$ and let E be a subbundle of the tangent bundle TM restricted to X. Given a point $p \in X$, the cone $C_a(p)$ of width $0 < a = a(p) < 1$ around $E(p)$ by

$$C_a(p) := \{v \in T_pM, \min_{w \in E(p)} \{\angle(v, w) \leq a\}\}$$

We define the *cone field C_a* of width $a : X \to (0, 1)$ around E as the map $X \ni p \mapsto C_a(p)$. The cone field is continuous if both a and $p \mapsto E(p)$ are continuous.

We say that a cone field C_a around E and a subbundle $\hat{E}$ are complementary, if $E(p) \oplus \hat{E}(p) = T_pM$, and $C_a(p) \cap \hat{E}(p)$ is trivial, for all p in the intersection of the domains of E and $\hat{E}$.

We now present the definition of hyperbolic set for non-singular endomorphisms.

Definition 2.6. Let Λ be a positively invariant compact set for a C^1 nonsingular endomorphism g of a manifold M and set $X := \cup_{n=0}^{+\infty} g^{-n}(\Lambda)$. We say that Λ is a *hyperbolic set* if there are complementary invariant subbundle E^s of $T_\Lambda M$ and a positively invariant cone field C_a^u defined on $T_X M$, such that:

- E^s is Dg-invariant, that is, $Dg(E^s(y)) \subset E^s(g(y))$, $\forall y \in \Lambda$;
- C_a^u is a Dg-invariant, ie, $Dg(x) \cdot C_a^u(x) \subset C_a^u(g(x))$, $\forall x \in X$;

- The angle between $E^s(y)$ and $C_a^u(y)$ is greater than a positive constant, $\forall y \in \Lambda$.
- There are constants $c > 0$, $0 < \varsigma < 1$, such that $\forall n \in \mathbb{N}$

$$\|Dg^n(y)|_{E^s}\| < c \cdot \varsigma^n, \forall y \in \Lambda \qquad \text{and}$$

$$\|[Dg^n(x)|_{C_a^u(x)}]^{-1}\| := \inf_{v \in C_a^u(x), \|v\|=1} \{\|Dg^n(x) \cdot v\|^{-1}\} < c \cdot \varsigma^n, \forall x \in X.$$

We introduce the following definition of non uniformly hyperbolic set for endomorphisms.

Definition 2.7. Let $g : M \to M$ be a nonsingular endomorphism on a compact manifold M. We say that a positively invariant set $\Lambda \subset M$ is a *non uniformly hyperbolic set* (or simply NUH set) if

(1) There are Dg-invariant, complementary subbundle E^{cs} whose domain is Λ and a cone field C^{cu} on $X := \cup_{n=0}^{+\infty} g^{-n}(\Lambda)$;
(2) There exists $\lambda < 0$ and an adapted Riemannian metric for which any point $p \in \Lambda$ satisfies

$$\liminf_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Dg(g^j(p))|_{E^{cs}(g^j(p))}\| \leq \lambda$$

and

$$\liminf_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|[Dg(g^j(p))|_{C^{cu}(g^j(p))}]^{-1}\| \leq \lambda$$

Roughly, comparing to the invertible context this condition replaces the existence of a uniquely determined expanding direction by means of the existence of an unstable complementary cone field exhibiting non-uniform expansion.

Theorem D. *Let $\mathcal{U} \subset \mathrm{NEnd}^1(M)$ be an open subset of nonsingular endomorphisms on a compact manifold M. Assume that for any g in some residual subset $\mathcal{S}$ of $\mathcal{U}$, the set $Per(g)$ of periodic points of g is non uniformly hyperbolic, that the subbundle E^{cs} extends continuously to $\overline{Per(g)}$ and that the cone field C^{cu} extends continuously to $\overline{\cup_{n=0}^{\infty} g^{-n}(Per(g))}$. Then, there exists a residual subset $\mathcal{A}$ of $\mathcal{U}$ whose elements are Axiom A endomorphisms. In particular, $\mathcal{U}$ is contained in the closure of the set of Axiom A endomorphisms.*

For the proof of the theorems in the endomorphisms case, we use

Theorem E. *(Ergodic Closing Lemma for nonsingular endomorphisms.) There exists a residual subset $\mathcal{R} \subset \mathrm{NEnd}^1(M)$ such that for any $f \in \mathcal{R}$, the set of $f-$invariant probabilities $\mathcal{M}_1(f)$ is the closed convex hull of ergodic measures supported on periodic orbits of f.*

3. Proof of the Criteria for generic Hyperbolicity

Throughout this section we assume that f is a diffeomorphism in a C^1-open set $\mathcal{U}$ and that the periodic set $Per(f)$ is non uniformly hyperbolic.

Remark 3.1. It is not hard to check that the set $Per(f)$ is non uniformly hyperbolic if and only if there exists $\lambda < 0$ such that every periodic point p with period $t(p)$ satisfies

$$\sum_{j=0}^{t(p)-1} \log(\|[Df|_{E^{cu}}(f^j(p))]^{-1}\|) \leq \lambda \cdot t(p)$$

and

$$\sum_{j=0}^{t(p)-1} \log(\|Df|_{E^{cs}}(f^j(p))\|) \leq \lambda \cdot t(p).$$

Before the proof of Theorem B we recall some notations and preliminary results. The first of these results is the C^1-Closing Lemma proved by Pugh [25].

Theorem 3.2. *There is a residual subset $\tilde{\mathcal{R}}$ of $\mathrm{Diff}^1(M)$ such that $\Omega(f) = \overline{Per(f)}, \forall f \in \tilde{\mathcal{R}}$.*

Using this version of Pugh's Closing Lemma, Mañé [14] stated the following result, often known as Ergodic Closing Lemma, which plays an important role in our proof.

Theorem 3.3. *There is a residual $\mathcal{R} \subset \mathrm{Diff}^1(M)$ such that for each $f \in \mathcal{R}$, the set $\mathcal{M}_1(f)$ of f-invariant probability measures is the closed convex hull of the ergodic measures supported at hyperbolic periodic orbits of f.*

Given an f-invariant probability measure μ it is well known from the Oseledets Theorem [17] that the *Lyapunov exponent* at x in the direction $v \in T_xM$

$$\lambda(x,v) = \lim_{n\to\infty} \frac{1}{n} \log \|Df^n(x)v\|,$$

is well defined in a set of total probability that is, a set that of full measure for every invariant probability measure. In [2], Cao proved that every nonsingular endomorphism g for which all invariant measures have only positive Lyapunov exponents is an uniformly expanding map. An analogous result for diffeomorphisms admitting continuous splitting also follows. We will use the following technical lemma:

Lemma 3.4. [2, Lemma 2] *Let $g : K \to K$ be a continuous map defined in a compact metric space K. Let $\mathcal{M}_1(g)$ be the space of g-invariant probabilities and let ϕ be a continuous function on K. If $\int \phi d\mu < \lambda, \forall\, \mu \in \mathcal{M}_1(g)$, then there exists $N \in \mathbb{N}$ such that $\forall n \geq N$ we have*

$$\frac{1}{n} \sum_{i=0}^{n-1} \phi(g^i(x)) < \lambda, \forall x \in K.$$

Notice that if $g \in NEnd^1(M)$ and the condition in hypothesis of the lemma holds for $\phi = \log \|Dg^{-1}\|$ then g is expanding. Analogously, if $f \in Diff^1(M)$, Λ is a compact invariant set with the continuous splitting $T_\Lambda M = E^{cs} \oplus E^{cu}$, and the functions $\phi_1 = \log \|Df|_{E^{cs}}\|$, $\phi_2 = \log \|Df^{-1}|_{E^{cu}}\|$ satisfy the condition of the lemma then Λ is a hyperbolic set for f. Indeed, this is an immediate consequence of the following simple proposition:

Proposition 3.5. *Let $g : K \to K$ be a continuous map defined in a compact metric space K. Let $\phi_n : K \to \mathbb{R}$ be a g-subadditive sequence of continuous functions (that is, $\phi_{n_1+n_2}(x) \leq \phi_{n_1}(x) + \phi_{n_2} \circ g^{n_1}(x), \forall n_1, n_2 \in \mathbb{N}$). Suppose that $\int_M \phi_1 d\mu < \lambda, \forall \mu \in \mathcal{M}_1(g)$. Then there exists N such that for all $n \geq N$,*

$$\phi_n(x) \leq n \cdot \lambda, \forall x \in K.$$

Proof. By lemma 3.4 applied to $\phi = \phi_1$, there is $N \in \mathbb{N}$ such that

$$\frac{1}{n} \sum_{i=0}^{n-1} \phi_1(g^i(x)) < \lambda, \forall x \in K.$$

The subadditivity of ϕ_n then implies that

$$\phi_n \leq \sum_{i=0}^{n-1} \phi_1(g^i(x)) < n \cdot \lambda, \forall n \geq N, \forall x \in K.$$

$\square$

Taking into account these preliminar results we proceed to prove the main results stated in the previous section.

Proof of Theorem B. Let $\tilde{\mathcal{R}}$ and $\mathcal{R}$ be the residual sets given by Pugh's and Mañé's Closing Lemmas, respectively, and let $f : M \to M$ be a diffeomorphism in the residual set $\mathcal{A}$ given by the intersection of $\mathcal{S}$, $\tilde{\mathcal{R}}$ and $\mathcal{R}$. Therefore, not only $\Omega(f) = \overline{Per(f)}$ as the space of invariant probabilities for f is the closed convex hull of ergodic probabilities supported in periodic orbits. Take $\phi_s := \log(\|Df|_{E^{cs}}\|)$, $\phi_n = \log(\|Df^n|_{E^{cs}}\|)$, $\phi_u := \log(\|[Df|_{E^{cu}}]^{-1}\|)$, $\psi_n = \log(\|[Df^n|_{E^{cu}}]^{-1}\|)$ for $n \in \mathbb{N}$ and $K = \Omega(f)$ in Proposition 3.5. Observe that such functions are continuous since the subbundles E^{cs} and E^{cu} vary continuously.

Since $f \in \mathcal{S}$, we can use Remark 3.1 to deduce that there exists $\lambda < 0$ such that for every periodic point p with period $t(p)$ we have

$$\int_{\Omega(f)} \phi_s d\big(\tfrac{1}{t(p)} \sum_{j=0}^{t(p)-1} \delta_{f^j(p)}\big) = \frac{1}{t(p)} \sum_{j=0}^{t(p)-1} \phi_s(f^j(p)) \leq \lambda$$

and

$$\int_{\Omega(f)} \phi_u d\big(\tfrac{1}{t(p)} \sum_{j=0}^{t(p)-1} \delta_{f^j(p)}\big) = \frac{1}{t(p)} \sum_{j=0}^{t(p)-1} \phi_u(f^j(p)) \leq \lambda,$$

where $\frac{1}{t(p)} \sum_{j=0}^{t(p)-1} \delta_{f^j(p)}$ is the ergodic measure supported in the periodic orbit of p. Using the Ergodic Closing Lemma for f, all f-invariant probability μ is the limit of a convex combination of such measures supported in periodic orbits and so we conclude that

$$\int_{\Omega(f)} \phi_s d\mu \leq \lambda \quad \text{and} \quad \int_{\Omega(f)} \phi_u d\mu \leq \lambda, \forall \mu \in \mathcal{M}_1(f).$$

Note that $\mathcal{M}_1(f|_{\Omega(f)}) \simeq \mathcal{M}_1(f)$. We are under the hypothesis of Proposition 3.5 (just exchange λ by $0 > \lambda' > \lambda$ to guarantee the strict inequality in the statement of lemma 3.4). Hence, we conclude that $\overline{Per(f)} = \Omega(f)$ is a hyperbolic set for f.

Now, let us see that (residually) $\Omega(f)$ has no cycles, and therefore, by [19], f is Ω−stable. In particular, there is a neighborhood of f whose elements are Axiom A.

Given a set $S \subset M$ let $S(\epsilon)$ be the ϵ−neighborhood of S in M.

Consider $\mathcal{C}$ the collection of compact subsets of M endowed with Hausdorff distance given by

$$d_H(K_1, K_2) := \inf\{\epsilon > 0, K_1 \subset K_2(\epsilon) \text{ and } K_2 \subset K_1(\epsilon)\}, \forall K_1, K_2 \in \mathcal{C},$$

Let $\mathcal{K}$ be the set of Kupka-Smale diffeomorphisms, which is residual. Since hyperbolic periodic points are robust, the map $\Psi : \mathcal{K} \to \mathcal{C}$ that assigns each f to $\overline{Per(f)}$ is lower semicontinuous. Hence, there exists a residual subset $\mathcal{R}' \subset \mathcal{U}$ of continuity points for Ψ.

Note that if $g\, Diff^1(M)$ is an Axiom A with cycles, then g is a discontinuity point for Ψ. In fact, by [18], there exists an wandering point y out from a neighborhood of $\Omega(g)$ which by an arbitrarily small perturbation can be changed into a point of transversal intersection between Stable and Unstable Manifolds of some basic set of g. Such point y then becomes a nonwandering point for g_n in a sequence tending to g as $n \to +\infty$, and is contained in $\overline{Per(g_n)}$. Therefore, $\overline{Per(g_n)} \not\to \overline{Per(g)}$ as $n \to +\infty$. Since $\mathcal{K}$ is a residual subset of $Diff^1(M)$, and transversal intersection between submanifolds is an open property, we can take g_n to be Kupka-Smale diffeomorphisms. This implies that g is not a point of continuity for Ψ.

So we conclude that for f in a residual subset of $\mathcal{U}$, f is Axiom A and has no cycles. Therefore, we obtained an open and dense subset $\mathcal{A} \subset \mathcal{U}$, whose elements are Axiom A (with no cycles).

$\square$

Remark 3.6. Under the conditions of either Theorem A or Theorem B, if it occurs that for $f \in \mathcal{A}$ is such that $\Omega(f)$ has the no cycle condition (see [18]) then we conclude that $\Omega(\hat{f})$ is hyperbolic for all $\hat{f}$ belonging in an open and dense subset of $\mathcal{U}$.

As we discussed in the previous section the results in Theorem A follow from Theorem B by proving that the existence of a dominated splitting over the periodic set is continuous. More precisely, it is enough to recall the following lemma.

Lemma 3.7. [7, Lemma 14] *Let $f : M \to M$ be a diffeomorphism on a compact manifold M. Let $X \subset M$ be some f−invariant set. Suppose there exists some invariant dominated splitting $T_X M = E \oplus \hat{E}$. Then, such splitting is continuous in $T_X M$, and unique once we fix the dimensions of $E, \hat{E}$. Moreover, it extends uniquely and continuously to a splitting of $T_{\overline{X}} M$.*

Note that Corollary 2.1 is an immediate consequence of the proofs of Theorems B and A above, as in such corollary we are assuming that the thesis in Ergodic Closing Lemma holds for $f \in Diff^1(M)$.

Remark 3.8. Just as in the case of Remark 3.1, $Per(g)$ is NUE if and only if there exists $\lambda < 0$ so that $\sum_{j=0}^{t(p)-1} \log(\|[Dg(g^j(p))]^{-1}\|) \leq \lambda \cdot t(p)$ for each periodic point p with period $t(p)$.

The remainder of this section is now devoted to the proof of Theorem C and D in the non-invertible context. As the reader may guess the proof of these theorems goes along the same lines and ideas used in the proofs of Theorems A and B, using the Ergodic Closing Lemma version for Nonsingular Endomorphisms (Th. E). For completeness we write down the proofs anyway.

Proof of Theorem C. Let $g : M \to M$, $g \in \mathcal{S} \subset NEnd^1(M)$ belonging in the residual set $\mathcal{R}$ given by Th. E. Therefore, in particular, we have that the space of invariant probabilities for g is the closed convex hull of ergodic probabilities supported in periodic orbits. Put $\phi := \log(\|[Dg]^{-1}\|)$. Since $g \in \mathcal{S}$, there exists $\lambda < 0$ such that, for any periodic point p with period $t(p)$ we have (see Remark 3.8):

$$\int_M \phi d\Big(\frac{1}{t(p)} \sum_{j=0}^{t(p)-1} \delta_{g^j(p)}\Big) = \sum_{j=0}^{t(p)-1} \phi(g^j(p)) \leq \lambda,$$

where $\frac{1}{t(p)} \sum_{j=0}^{t(p)-1} \delta_{g^j(p)}$ is the ergodic measure supported in the periodic orbit of p. As the thesis of Ergodic Closing Lemma holds for g, all g-invariant probability μ is the limit of a convex combination of such measures supported in periodic orbits and so we conclude that

$$\int_M \phi d\mu \leq \lambda, \forall \mu \in \mathcal{M}_1(g).$$

We are again under the hypothesis of the fundamental proposition 3.5, which implies that g is an expanding map.

$\square$

Corollary 2.2 is a straightforward consequence of the proof above.

Proof of Theorem D. Let $g : M \to M$ be a nonsingular endomorphism belonging to the residual set $\mathcal{A}$ given by the intersection of $\mathcal{S}$, $\tilde{\mathcal{R}}$ and $\mathcal{R}$, the last ones given by the Closing Lemmas for Endomorphisms. Therefore, in particular, we have $\Omega(g) = \overline{Per(g)}$ and the space of invariant measures for g is the closed convex hull of ergodic measures supported in periodic orbits. Put $\phi_s := \log(\|Dg|_{E^{cs}}\|)$, $\phi_u := \log(\|[Dg|_{C^{cu}}]^{-1}\|)$. Such functions are continuous, since the subbundle E^{cs} and the cone field C^{cu} are assumed to be continuous. ϕ_s and ϕ_u are defined in $\Omega(g)$ and $X = \overline{\cup_{n \geq 0} g^{-n}(Per(g))}$, respectively. Since $g \in \mathcal{S}$, there exists $\lambda < 0$ such that, for any periodic point p with period $t(p)$ we have (just as in Remark 3.1):

$$\int_{\Omega(g)} \phi_s d\Big(\frac{1}{t(p)} \sum_{j=0}^{t(p)-1} \delta_{g^j(p)}\Big) = \frac{1}{t(p)} \sum_{j=0}^{t(p)-1} \phi_s(g^j(p)) \leq \lambda$$

and

$$\int_X \phi_u d\big(\tfrac{1}{t(p)} \sum_{j=0}^{t(p)-1} \delta_{g^j(p)}\big) = \frac{1}{t(p)} \sum_{j=0}^{t(p)-1} \phi_u(g^j(p)) \leq \lambda,$$

where $\frac{1}{t(p)} \sum_{j=0}^{t(p)-1} \delta_{g^j(p)}$ is the ergodic measure supported in the periodic orbit of p. The Ergodic Closing Lemma for Nonsingular Endomorphisms implies that

$$\int_{\Omega(g)} \phi_s d\mu \leq \lambda \quad \text{and} \quad \int_X \phi_u d\mu \leq \lambda, \forall \mu \in \mathcal{M}_1(g).$$

Just as in the previous theorems, we apply Proposition 3.5 to conclude that $\overline{Per(g)} = \Omega(g)$ is a hyperbolic set for g.

$\square$

4. Proof of the Ergodic Closing Lemma for nonsingular endomorphisms

For the proof of Theorem E, let us start by fixing some notation. Given $x \in M$, we define $B_\epsilon(f, x)$ as an $\epsilon-$neighborhood of the orbit of x. Define $\Sigma(f)$ as the set of points $x \in M$ such that for every neighborhood $\mathcal{U}$ of f and every $\epsilon > 0$, there exist $g \in \mathcal{U}$ and $y \in M$ such that $y \in Per(g)$, $g = f$ on $M \setminus B_\epsilon(f, x)$ and $d(f^j(x), g^j(y)) \leq \epsilon$, $\forall 0 \leq j \leq m$, where m is the $g-$period of y.

At the end of this section, we shall obtain the **residual** version of the Ergodic Closing Lemma (our Theorem E) as consequence of the following result:

Theorem 4.1. *For any nonsingular endomorphism f, $\Sigma(f)$ is a total probability set, that is, $\Sigma(f)$ is a full probability set for any $f-$invariant probability.*

Remark 4.2. Theorem 4.1 above, with $f \in Diff^1(M)$ instead of $f \in NEnd^1(M)$ in its statement, was the former Ergodic Closing Lemma proved by Mañé. In fact, Mañé, in [14], did not explicitly give the proof of his corresponding classical residual version, although he stated such version in [16]. This gap was filled out very recently by Abdenur et al in [1].

Definition 4.3. ($\epsilon-$shadowing by a periodic point.) Let f and g maps on a compact metric space Λ. Given $\epsilon > 0$ and $x \in \Lambda$, we say that a $g-$periodic point p with period n $\epsilon-shadows$ x iff $d(g^j(p), f^j(x)) < \epsilon, \forall 0 \leq j \leq n$.

Let $\epsilon > 0$ and a neighborhood $\mathcal{U} \ni f$, $\mathcal{U} \subset NEnd^1(M)$ be given. We define $\Sigma(f, \mathcal{U}, \epsilon)$ as the set of points $x \in M$ such that there exist $g \in \mathcal{U}$ and $y \in M$ such that $y \in Per(g)$, $g = f$ on $M \setminus B_\epsilon(f, x)$ and $d(f^j(x), g^j(y)) \leq \epsilon, \forall 0 \leq j \leq m$, where m is the $g-$period of y. That is, $\Sigma(f, \mathcal{U}, \epsilon)$ is the set of points $x \in M$ which are $\epsilon-$shadowed by a periodic point $y \in Per(g)$, for some $g \in \mathcal{U}$. Everytime there is no chance of misunderstanding, we will just write $\Sigma(\mathcal{U}, \epsilon)$ instead of $\Sigma(f, \mathcal{U}, \epsilon)$. If we take a nested neighborhood basis $\mathcal{U}_n$ of f in $NEnd^1(M)$ then

$$\Sigma(f) = \cap_{n \in \mathbb{N}} \Sigma(f, \mathcal{U}_n, 1/n).$$

Therefore, Theorem 4.1 is an immediate consequence of

Proposition 4.4. *For any nonsingular endomorphism f, any neighborhood $\mathcal{U}$ of f and $\epsilon > 0$, $\Sigma(\mathcal{U}, \epsilon) = \Sigma(f, \mathcal{U}, \epsilon)$ is a total probability set for f.*

Since the proof of Proposition 4.4 is long and involving we divide it in two main parts that we shall describe next for the reader's convenience. Part 1 consists in an improvement of Closing Lemma (see [25], [26], [27], [32]), roughly stating in particular that given a nonsingular endomorphism f, for any point $x \in M$ there are $\rho > 1$ and $r > 0$ such that if x returns to a ball $B(x, r')$, $r' < r$, then there is an intermediate iterate $y = f^{m(x)}(x)$ which returns to $B(x, \rho r')$ that is shadowed by a g-periodic point, where g is some endomorphism close to f. The precise statement corresponds to Lemma 4.10, whose proof we write down further in this paper. Let us point out that this first part of the argument is entirely topological.

Part 2 uses ergodicity and Birkhoff's Theorem to show that the set of points $x \in M$ that are shadowed by periodic points of some endomorphism close to f has total probability with respect to $f-$invariant measures. In fact, recall first that typical points for an ergodic invariant measure are recurrent. The arguments in Part 1 yield that at every time a recurrent point x returns to a ball there is an intermediate iterate y with the shadowing property we look for, that also returns to a ball with the same center and whose radius is ρ times bigger. Notice that if we could take ρ equal to one above there would be nothing to do: the set of recurrent points and of the $\epsilon-$shadowable points $\Sigma(\mathcal{U}, \epsilon)$ would visit equally the same balls which imply that they have the same measure. More generally, using standard Radon-Nikodym derivative calculations, the same conclusion would be obtained if the $\mu-$measure of a ball and its $\rho-$homothetic image were proportional, at least for balls with sufficiently small radius. In fact, given a set S it is well known that $\mu-$a.e. point $x \in M$ we have

$$\chi_S(x) = \lim_{r \to 0} \frac{\mu(B(x, r) \cap S)}{\mu(B(x, r))}.$$

Applying Part 1 when $S = \Sigma(\mathcal{U}, \epsilon)$ one has that for some $\rho = \rho(x)$

$$\liminf_{r \to 0} \frac{\mu(B(x, \rho r) \cap S)}{\mu(B(x, r))} \geq 1$$

holds $\mu-$a.e. $x \in M$, due to ergodicity of μ and the fact that asymptotic frequency of visits of a typical point to $B(x, \rho r) \cap S$ is greater or equal than to $B(x, r)$. If the measures of $B(x, r)$ and $B(x, \rho r)$ were proportional, say, by a factor of $\zeta > 0$, we could obtain

$$\chi_S(x) \geq \liminf_{r \to 0} \frac{\mu(B(x, \rho r) \cap S)}{\mu(B(x, \rho r))} \cdot \frac{\mu(B(x, r))}{\mu(B(x, r))} \geq \zeta > 0,$$

which implies that the indicator function χ_S of S is equal to 1 a.e., and so S has total probability. Even though such proportionality on balls with same center and different radius may not hold for general measures, rough estimates in such direction do hold, which are sufficient for our proof. Following the ideas above, we use a Vitali's covering like argument to prove that the set of points that are shadowed by a periodic point of some nearby endomorphism g has total probability for f. The core of these arguments is contained in Lemma 4.13 and Theorem 4.15 below.

We proceed to prove Part 1 and its main technical Lemma 4.10. For that purpose we shall first introduce some notations and proceed to some perturbation lemmas. As the manifold M is compact, there is δ such that $\{\exp_p, p \in M\}$ is an equilipschitz family of diffeomorphisms, such that each exponential map $\exp_p$ embeds $B(0, \delta)$ in a neighborhood B_p of p. Given $p \in M$, we define a metric $d' = d'_p : B_p \times B_p \to [0, +\infty)$ given by

$$d'(x,y) := |\exp_p^{-1}(x) - \exp_p^{-1}(y)|.$$

Obviously, d' is Lipschitz-equivalent to the manifold usual metric restricted to B_p. Setting d' as the metric in B_p, then $\exp_p$ isometrically maps $B(0, \delta)$ on $B_p = B'(p, \delta)$, where the quote ' signs the ball in the metric d'.

Lemma 4.5. [27] *For any $\eta > 0$, for any $f \in \mathrm{NEnd}^1(M)$, there is an $\alpha > 0$ such that for any $f \in \mathrm{NEnd}^1(M)$, any $q \in M$, any two points $v_1, v_2 \in T_q M$ with $B(v_2, |v_1 - v_2|/\alpha) \subset B(0, \delta) \subset T_q M$, there is a diffeomorphism $h = h_{q,\alpha,v_1,v_2} : M \to M$, called an $\alpha-$lift, such that:*

(1) $h(\exp_q(v_2)) = \exp_q(v_1)$;
(2) *The closure of set of points where h differs to the identity is contained in $\exp_q(B(v_2, |v_1 - v_2|/\alpha)$;*
(3) $d_1(hf, f) < \eta$.

Definition 4.6. (Dynamical neighborhood.) We say that a neighborhood V of a point $p \in M$ is $N-$ dynamical for f if each connected component $\cup_{j=0}^N f^{-j}(V)$ contains exactly one point of $\cup_{j=0}^N f^{-j}(\{p\})$.

Lemma 4.7. [9], [32] *Let $f \in \mathrm{NEnd}^1(M)$, $p \in M$, $N \geq 1$ given such that all terms in $\cup_{j=0}^{N+1} f^{-j}(p)$ are distinct. Then, for any $\eta > 0$, there is a $\beta > 0$, and a map $f_1 \in \mathrm{NEnd}^1(M)$, called a local linearization of f with the following properties (1)-(5).*

(1) $B'(p, \beta)$ is $(N+1)$-dynamical *for both f and f_1, and $f^{-j}(B'(p, \beta)) = f_1^{-j}(B'(p, \beta))$ for $j = 1, \ldots, N+1$.*
(2) *For $q \in \cup_{j=1}^{N+1} f^{-j}(p)$, let $V(q)$ be the open connected component of $\cup_{j=1}^{N+1} f^{-j}(B'(p, \beta/4))$ containing q. Then, $f_1|_{V(q)} = \exp_{f(q)} \circ (T_q f) \circ \exp_q^{-1}$.*
(3) $f_1^{N+1}(x) = f^{N+1}(x), \forall x \in f^{-N-1}(B'(p, \beta))$.
(4) $f_1 = f$ *on* $M \setminus \cup_{j=1}^{N+1} f^{-j}(B'(p, \beta))$.
(5) $d_1(f_1, f) < \eta$.

Remark 4.8. For the sequel, we need to emphasize two aspects from the last lemma. On the one hand, we obtain as a direct consequence that if for some $k \in \mathbb{N}$ we have that $x, f^k(x)$ are both out of $\cup_{j=0}^N f^{-j}(B'(p, \beta))$, then $f^k(x) = f_1^k(x)$. On the other hand, notice that as η goes to 0, one can take β arbitrarily small in the last lemma.

Theorem 4.9. *(Theorem A in [32].) Let $(\mathcal{T}, T_q)$ a complete tree of isomorphisms associated to the pre-orbit of a point $q_0 \in M$, that is, a collection of $n-$dimensional inner product spaces E_q and isomorphisms $T_q : E_q \to E_{q_0}$ associated to each q in the pre-orbit of q_0, with T_{q_0} equal to identity. Given $\alpha > 0$, there are $\rho > 0$ and $N \geq 1$ such that: for any ordered*

set $X = \{x_0 \prec \cdots \prec x_t\} \subset E_{q_0}$, there is a point $y \in X \cap B(x_t, \rho|x_0 - x_t|)$ such that for any branch $\Gamma = \{q_0, q_1, \ldots, \}$ of $\mathcal{T}$, there is a point $w = w(\Gamma) \in X \cap B(x_t, \rho|x_0 - x_t|)$ which is before y in the order of X, together with $N+1$ points $c_0(\Gamma) = c_0, \ldots, c_N(\Gamma) = c_N \in B(x_t, \rho|x_0 - x_t|)$ satisfying the following two conditions:

- $c_0 = w, c_N = y$; and
- $|T_{q_n}^{-1}(c_j) - T_{q_n}^{-1}(c_{j+1})| \leq \alpha d(T_{q_n}^{-1}(c_{j+1}) - T_{q_n}^{-1}(A))$, where $A := \{x \in X, w \prec x \prec y\} \cup \partial B(x_t, \rho|x_0 - x_t|)$.

The next lemma is the main target in the first part of our Ergodic Closing Lemma, whose arguments are just topological. It implies in particular that, given $\epsilon > 0$ and *any* f−recurrent point x, then x has an iterate which is ϵ−shadowed by a periodic point of some g close to f.

Lemma 4.10. *Given $f \in \text{NEnd}^1(M)$, $p \in M$, $\epsilon > 0$ and a neighborhood $\mathcal{U}$ of f, there exist $r > 0$, $\rho' > 1$ such that if for some natural $t > 0$, we have $x, f^t(x) \in B'_{\overline{r}}(p)$, with $0 < \overline{r} \leq r$, then there exist $0 \leq t_1 < t_2 \leq t$ and $g \in \mathcal{U}$ such that:*

- $w = f^{t_1}(x), y = f^{t_2}(x) \in \overline{B'_{\rho'\overline{r}}(p)}$;
- $g^{t_2-t_1}(w) = w$;
- $g(z) = f(z)$ for $z \notin B_\epsilon(f, x)$ and $d(g^j(w), f^j(w)) \leq \epsilon, \forall 0 \leq j \leq t_2 - t_1$.

Proof. Take an $\eta > 0$ such that the η−ball with center f is contained in $\mathcal{U}$. Take $1 > \alpha > 0$ such that $d_1(h \circ f, f) < \eta/2$, for any α−lift h. Without loss of generality, we assume that $\epsilon < \alpha^2$. We also assume that $\epsilon < \delta$.

We assume that p is not periodic for f, otherwise, there is nothing to prove. This implies that all points in the pre-orbit of p are distinct. Let $\rho > 2$ and $N \geq 1$ be the numbers provided by Theorem 4.9, for $\alpha > 0$ taken as above, and for $q_0 = p$, each q_j to be some j−pre-image of p, $E_{q_j} = T_{q_j}M$ and $T_{q_j} = Df^j(q_j)$.

So, take $r > 0$ such that $r < \epsilon/(6\rho)$ and $\text{diam}(f^{-j}(B'(p, 3\rho r)) < \epsilon, \forall j = 0, \ldots, N+1$. We assume that each connected component of $\cup_{j=0}^{N+1} f^{-j}(B'(p, 3\rho r))$ contains exactly one point $q_j \in f^{-j}(p), j = 1, \ldots, N+1$. In particular, if $z, f^{\hat{t}}(z) \in B'(p, 3\rho r)$, then $\hat{t} > N+1$.

Now, assuming that $x, f^t(x) \in B'(p, \overline{r})$, for some $0 < \overline{r} < r$ we can apply Theorem 4.9 to the set $X = \{x, f(x), \ldots, f^t(x)\} \cap B'(p, 3\rho\overline{r})$ endowed with the order given by the iterate number: if $f^k(x), f^{\hat{k}}(x) \in X$, then $f^k(x) \prec f^{\hat{k}}(x) \Leftrightarrow k < \hat{k}$. Therefore, set $\rho' = 3\rho$. We then obtain $f^{t_2}(x) = y \in \{x, \ldots, f^t(x)\} \cap B'(f^t(x), \rho \cdot d'(f^t(x), x)) \subset B'(p, \rho'\overline{r})$ such that for any branch $\Gamma = \{p = p_0, p_1, \ldots, p_n, \ldots\}$ of the pre-orbit of p, there is $w = w(\Gamma) = f^{t_1}(x) \in \{x, \ldots, f^t(x)\} \cap B'(f^t(x), \rho \cdot d'(f^t(x), x))$, with $t_1 = t_1(\Gamma) < t_2$ together with points $c_0 = c_0(\Gamma), \ldots, c_N = c_N(\Gamma) \in B'(f^t(x), \rho \cdot d'(f^t(x), x))$ such that:

(a) $c_0 = w, c_N = y$; and

(b) $|T_{p_j}^{-1}(c_j) - T_{p_j}^{-1}(c_{j+1})| \leq \alpha d(T_{p_j}^{-1}(c_{j+1}), T_{p_j}^{-1}(A))$, where $A := \{f^j(x) \in X; t_1 < j < t_2\} \cup \partial B'(f^t(x), \rho \cdot d'(f^t(x), x))$.

As $w = w(\Gamma)$ and y are both in X, there is a natural number $k(\Gamma) \geq 1$ such that $f^{k(\Gamma)}(w(\Gamma)) = y$. Note that $k(\Gamma) > N+1$, as $\cup_{j=0}^{N+1} f^{-j}(\{y\}) \cap B'(p, 3\rho r) = y$, from our choice of r. Setting $z := f^{k(\Gamma)-N-1}(w(\Gamma))$, we see that z does not depend on the branch Γ of

p, since $w(\Gamma)$ and y are in X, $f^{k(\Gamma)}(w(\Gamma)) = y$ and y, N do not depend on Γ. By our choice of r, since $y \in B'(p, 3\rho r)$ there is a unique connected component $V_{N+1} \subset f^{-(N+1)}(B'(p, 3\rho r))$ such that $z \in V_{N+1}$. Also, there is a unique $p_{N+1} \in f^{-(N+1)}(p) \cap V_{N+1}$. From now on, we fix Γ as some branch of p containing p_{N+1} (That is, $\Gamma = (p = p_0, \dots, p_{N+1}, \dots)$), and we consider all constants $w, c_0, \dots, c_N$, k obtained by applying Theorem 4.9 with respect to such branch. For each $p_j, j = 0, \dots, N - 1$, let h_{p_j} be the $\alpha-$kernel lift obtained by treating in lemma 4.5 $q = p_j$, $v_1 = [Df^j(p_j)]^{-1}(c_j)$, $v_2 = [Df^j(p_j)]^{-1}(c_{j+1})$. Defining a map $g : M \to M$ by

$$g := \begin{cases} h_{p_j} \circ f_1 \text{ on } V(p_{j+1}); \\ f_1 \text{ on the rest of } M, \end{cases}$$

we have that $g \in NEnd^1(M)$ and $d_1(g, f) < \eta$. Thus $g \in \mathcal{U}$.

Due to condition (b) above, the $g-$orbit from w to z never touches the region in which $g \neq f_1$. Therefore, $g^{k-(N+1)}(w) = f_1^{k-(N+1)}(w)$. By remark 4.8, we also have that $f^{k-(N+1)}(w) = f_1^{k-(N+1)}(w)$, and thus

$$g^{k-(N+1)}(w) = f^{k-(N+1)}(w) = z.$$

Now, it is easy to see that $g^{N+1}(z) = w$ and then $g^k(w) = w$. In fact, $f_1^{N+1}(z) = y$, and the lifts $h_{p_{N-1}}, \dots h_0$ gradually and slightly modifies f_1-orbit segment joining z and y, in such way that $g^{N+1}(z) = w$ and $d(g^j(z), f^j(z)) \leq d(g^j(z), f_1^j(z)) + d(f_1^j(z), f^j(z)) < \epsilon$, $\forall j = 1, \dots, N + 1$.

$\square$

Now we proceed to prove the second part of the proof of Proposition 4.4. Although the main idea of this part is borrowed from [14], the proofs we have written are presented in an abstract setting for future use and bookkeeping purposes. This will also clarify the sort of arguments which are used.

We start by introducing some notation. We say that a subset C of the torus T^s is a cube if it can be written as $A = I_1 \times \dots \times I_s$, where the sets I_i are intervals of same length in S^1 (containing both, none, or one of its boundary points). If p_i is the middle point of I_i, we say that the point $(p_1, \dots, p_s)$ is the center of A. The length of the intervals I_i is called the side of the cube. For each $k \in \mathbb{N}^+$, let $(\mathcal{P}_j^{(k)})_{j \in \mathbb{N}^+}$ be a sequence of partitions of T^s by cubes whose side is $2\pi/k^j$. For every atom P of a partition $\mathcal{P}_j^{(k)}$, we can associate cubes $\hat{P}$ and $\tilde{P}$ having the same center of P, but with sides $2\pi/k^{j-1}$ and $6\pi/k^{j-1}$, respectively. If $x \in T^s$, denote by $P_j^{(k)}(x)$ the atom of $\mathcal{P}_j^{(k)}$ containing x. Suppose that M is isometrically embedded in T^s. We recall the following useful fact on such kind of partitions.

Lemma 4.11. [14, Lemma I.5] *For every probability measure μ on the Borel sets of T^s, every $\delta > 0$ and for all odd natural k, the following inequalities holds for any $j \geq 1$:*

$$\mu(\{x; \mu(P_j^{(k)}(x)) \geq \delta\mu(\hat{P}_j^{(k)}(x))\}) \geq 1 - \delta k^s$$

and

$$\mu(\{x; \mu(P_j^{(k)}(x)) \geq \delta\mu(\tilde{P}_j^{(k)}(x))\}) \geq 1 - \delta 3^s k^s.$$

Let $f \in NEnd^1(M)$, $\epsilon > 0$, a neighborhood $\mathcal{U}$ of f and an ergodic $\mu \in \mathcal{M}_1(f)$ be given. Extend μ to a measure on T^s by $\mu(A) := \mu(A \cap M)$, for all Borel set $A \subset T^s$. Let $\mho \subset M$ be some Borelian set and suppose that $\mho(r, \rho)$, where $r > 0$, $\rho > 1$, is some Borelian set whose elements are points $x \in M$ with the following property: if $y \in B'_{r'}(x)$ for some $0 < r' \leq r$ and $f^t(y) \in B'_{r'}(x)$, for some $t > 0$ then there exist $0 \leq t_1 < t$, such that $f^{t_1}(y) \in \overline{B'_{\rho r'}(x)} \cap \mho$. Take $r_i > 0$, $\rho_l > 1$ two monotone sequences converging respectively to 0 and $+\infty$.

Our first target in this second part is to obtain an abstract result (Theorem 4.15) which will be essential in both proofs of Proposition 4.4 and Theorem E. Such result says that, if $\cup_{i,l}\mho(r_i, \rho_l) = M$, then $\mho$ has total probability for f.

Remark 4.12. All results from this point of the paper up to Theorem 4.15 do not use much regularity of f. In fact, specifically for the statements from Lemma 4.13 up to Theorem 4.15, we only request $f : M \to M$ to be a Borelian map such that $\mathcal{M}_1(f) \neq \emptyset$. This occurs, for instance, if f is a continuous map.

For each pair (i, l), we can find and odd natural $k = k(i, l)$ and $j(i, l)$ such that $\forall j \geq j(i, l)$ and $x \in T^s$ there exists $0 \leq r \leq r_i$ satisfying

$$P_j^{(k)}(x) \subset B_r(x)$$

and

$$\hat{P}_j^{(k)}(x) \supset \overline{B_{\rho_l r}(x)},$$

the balls here are taken in the torus. The next lemma is where the μ-ergodicity is necessary for the proof of Proposition 4.4.

Lemma 4.13. If $x \in \mho(r_i, \rho_l)$, $j \geq j(i, l)$, $k = k(i, l)$ and $\mu(P_j^{(k)}(x)) \geq \delta\mu(\hat{P}_j^{(k)}(x))$, we have:

$$\mu(\hat{P}_j^{(k)}(x) \cap \mho) \geq \delta\mu(\hat{P}_j^{(k)}(x)).$$

Proof. As μ is ergodic, for $\mu-$typical $y \in M$, we have that

$$\mu(\hat{P}_j^{(k)}(x) \cap \mho) = \lim_{n \to +\infty} \frac{1}{n}\#\{1 \leq t \leq n; f^t(y) \in \hat{P}_j^{(k)}(x) \cap \mho\},$$

and

$$\mu(P_j^{(k)}(x)) = \lim_{n \to +\infty} \frac{1}{n}\#\{1 \leq t \leq n; f^t(y) \in P_j^{(k)}(x)\}.$$

By the definition of $\mho(r_i, \rho_l)$, between any pair of natural numbers n_1 and n_2 such that $f^{n_1}(y), f^{n_2}(y) \in P_j^{(k)}(x) \subset B'_r(x)$, there exists $n_1 \leq t_1 < n_2$, such that $f^{t_1}(y) \in (\overline{B'_{\rho_l r}(x)} \cap \mho) \subset (\hat{P}_j^{(k)}(x) \cap \mho)$. This implies that

$$\#\{1 \leq t \leq n; f^t(y) \in \hat{P}_j^{(k)}(x) \cap \mho\} \geq$$
$$\#\{1 \leq t \leq n; f^t(y) \in P_j^{(k)}(x)\} - 1.$$

Hence

$$\mu(\hat{P}_j^{(k)}(x) \cap \mho) \geq \mu(P_j^{(k)}(x)) \geq \delta\mu(\hat{P}_j^{(k)}(x)).$$

$\square$

Now define $\Lambda_\delta^0(i,l)$, for $\delta > 0$, as the set of points $x \in T^s$ such that for $k = k(i,l)$, we have

$$\mu(P_j^{(k)}(x)) \geq \delta\mu(\hat{P}_j^{(k)}(x)) \quad \text{and} \quad \mu(P_j^{(k)}(x)) \geq \delta\mu(\tilde{P}_j^{(k)}(x)),$$

for an infinite sequence $\varsigma(x)$ of values of j, and set $\Lambda_\delta(i,l) := \Lambda_\delta^0(i,l) \cap \mho(r_i, \rho_l)$. The next lemma, a kind of Vitali's covering lemma, will be useful to estimate the measure of $\mho^c$.

Lemma 4.14. *Given a neighborhood V of $\mho^c \cap \Lambda_\delta(i,l)$, there exist sequences $x_q \in \mho^c \cap \Lambda_\delta(i,l)$, $(j_q), j_q \in \varsigma(x_q) \subset \mathbb{N}$, $q = 1, 2, \ldots$, such that*

(1) *The sets $\hat{P}_{j_q}^{(k)}(x_q)$, $q \in \mathbb{N}$ are disjoint and contained in V;*

(2) $\mu\big((\mho^c \cap \Lambda_\delta(i,l)) \setminus \cup_{q\in\mathbb{N}}\hat{P}_{j_q}^{(k)}(x_q)\big) = 0.$

Proof. By standard measure theoretical arguments, a translation $\tau : T^s \to T^s$ can be found in such way that

$$\mu(\tau(\cup\{\partial\hat{A}; A \in \mathcal{P}_j^{(k)}, k \geq 1, j \geq 1\})) = 0;$$

where $\partial\hat{A}$ is the boundary of $\hat{A} \in \hat{\mathcal{P}}_j^{(k)}$. Denoting by $\mathcal{F}$ the family of sets $P_j^{(k)}(x)$ with $x \in \Lambda_\delta(i,l) \cap \mho^c$ and $j \in \varsigma(x)$. Take a sequence $A_u \in \mathcal{F}$ satisfying:

(1) $\hat{A}_u \subset V$, $\forall u \in \mathbb{N}$, and $\mu(\hat{A}_u \cap \hat{A}_e) = 0$, $\forall 1 \leq e < u$.

(2) $\text{diam}(A_u) = \max\{\text{diam}(A); \hat{A} \subset V \text{ and } \mu(\hat{A} \cap \hat{A}_e) = 0, \forall 1 \leq e < u\}.$

Such properties imply that $\lim_{u\to+\infty} \text{diam}(A_u) = 0$ and

$$\sum_u \mu(A_u) = \mu(\cup_u A_u) \leq 1. \tag{1}$$

We claim that for $N \geq 1$

$$\big(\Lambda_\delta(i,l) \cap \mho^c\big) \setminus \cup_{u=1}^N \overline{\hat{A}_u} \subset \cup_{u>N}\tilde{A}_u. \tag{2}$$

In fact, if $x \in \big(\Lambda_\delta(i,l)\cap\mho^c\big)\setminus\cup_{u=1}^N\overline{\hat{A}_u}$, there exist $A \in \mathcal{F}$ with $x \in A$ and $\hat{A}\cap(\cup_{u=1}^N\overline{\hat{A}_u}) = \emptyset$. Take $N_1 > N$ such that $\hat{A}\cap\hat{A}_u = \emptyset$, $\forall 1 \leq u < N_1$ and $\hat{A}\cap\hat{A}_{N_1} \neq \emptyset$. By item (2) above, it follows that $\text{diam}(\hat{A}) \leq \text{diam}(\hat{A}_{N_1})$. This implies that $\hat{A} \subset \tilde{A}_{N_1}$ and then

$$x \in A \subset \tilde{A}_{N_1} \subset \cup_{u>N}\tilde{A}_u,$$

which concludes the proof of equation 2. By such equation and our assumption that partition elements borders have zero measure, we obtain that

$$\mu\Big(\big(\Lambda_\delta(i,l) \cap \mho^c\big) \setminus \cup_{u=1}^N \hat{A}_u\Big) = \mu\Big(\big(\Lambda_\delta(i,l) \cap \mho^c\big) \setminus \cup_{u=1}^N \overline{\hat{A}_u}\Big) \leq$$

$$\mu\big(\cup_{u>N}\tilde{A}_u\big) \leq \sum_{u>N}\mu(\tilde{A}_u) \leq \delta^{-1}\sum_{u>N}\mu(A_u).$$

Due to eq. (1) the tail sum above goes to zero as $N \to +\infty$, which implies the lemma. $\square$

Lemmas 4.13 and 4.14 are the key ingredients in the

Theorem 4.15. *Let M be a compact Riemannian manifold and let $f : M \to M$ be a measurable Borelian map such that $\mathcal{M}_1(f) \neq \emptyset$. Let $\mho \subset M$ and $\mho(r, \rho)$ be Borelian subsets of M, where $r > 0$, $\rho > 1$.*

Suppose that the points $x \in \mho(r, \rho)$ have the following property: if $y \in B_{r'}(x)$ for some $0 < r' \leq r$ and $f^t(y) \in B_{r'}(x)$, for some $t > 0$ then there exist $0 \leq t_1 \leq t$, such that $f^{t_1}(y) \in \overline{B_{\rho r'}(x)} \cap \mho$. Suppose also that $r_i > 0$, $\rho_l > 1$ are two monotone sequences converging respectively to 0 and $+\infty$, such that $\cup_{i,l} \mho(r_i, \rho_l) = M$. Then, $\mho$ has total probability with respect to the map f.

Proof. Consider $\Lambda^0_\delta(i, l)$ and $\Lambda_\delta(i, l) = \Lambda^0_\delta(i, l) \cap \mho(r_i, \rho_l)$ the same sets defined above in our text. By Lemma 4.11, this implies that

$$\mu(\Lambda^0_\delta(i, l)) \geq 1 - \delta(k^s + 3^s k^s).$$

Since the last inequality implies that $\cup_{n=1}^{+\infty} \Lambda_{1/n}(i, l) = \mho(r_i, \rho_l)(\mod 0)$ it is enough to prove that

$$\mu(\mho^c \cap \Lambda_\delta(i, l)) = 0, \forall 0 < \delta < 1.$$

In fact this implies that $\mho \supset \mho(r_i, \rho_l) \mod (0)$ and, consequently,

$$\mho \supset \left(\cup_{i,l} \mho(r_i, \rho_l) \right) = M \quad \mod (0).$$

We will then have $\mu(\mho) = \mu(M) = 1$, and the proof of Theorem 4.15 will be completed.

Fix (i, l) and $\delta > 0$. Let V a neighborhood of $\Lambda_\delta(i, l) \cap \mho^c$. By lemmas 4.13 and 4.14 it follows that

$$\mu(V) \geq \sum_q \mu\big(\hat{P}^{(k)}_{j_q}(x_q)\big) \geq \frac{1}{1 - \delta} \sum_q \mu\big(\hat{P}^{(k)}_{j_q}(x_q) \cap \mho^c\big) =$$

$$\frac{1}{1 - \delta} \mu\Big(\big(\cup_q \hat{P}^{(k)}_{j_q}(x_q) \big) \cap \mho^c \Big) \geq \frac{1}{1 - \delta} \mu\big(\Lambda_\delta(i, l) \cap \mho^c\big).$$

But if $\mu\big(\Lambda_\delta(i, l) \cap \mho^c\big) > 0$, one can take V satisfying

$$\mu(V) < \frac{1}{1 - \delta} \mu\big(\Lambda_\delta(i, l) \cap \mho^c\big),$$

contradicting the last inequality. Hence $\mu\big(\Lambda_\delta(n, m) \cap \mho^c\big) = 0$. $\qquad\square$

Now, let us finish the proof of Proposition 4.4 (which implies Theorem 4.1).

Proof of Proposition 4.4. Define $\Sigma(\mathcal{U}, \epsilon, r, \rho)$, where $r > 0$, $\rho > 1$, as the set of points $x \in M$ such that if $y \in B_{r'}(x)$ for some $0 < r' \leq r$ and $f^t(y) \in B_{r'}(t)$, for some $t > 0$ then there exist $0 \leq t_1 < t_2 \leq t$, $g \in \mathcal{U}$ and $z \in M$ such that $g = f$ on $M \setminus B_\epsilon(f, x)$,

$$g^{t_2 - t_1}(z) = z, d(g^j(z), f^j(f^{t_1}(y)) \leq \epsilon, \forall 0 \leq j \leq t_2 - t_1,$$

and

$$f^{t_1}(y) \in \overline{B_{\rho r'}(x)}.$$

In particular, $f^{t_1}(y) \in \Sigma(\mathcal{U}, \epsilon)$. It is easy to see that $\Sigma(\mathcal{U}, \epsilon, r, \rho)$ is a Borelian set. Again, let $r_i > 0$ and $\rho_l > 1$ to be two monotone sequences converging respectively to 0 and $+\infty$. We note that Lemma 4.10 implies that

$$M = \cup_{i \geq 1} \cup_{l \geq 1} \Sigma(\mathcal{U}, \epsilon, r_i, \rho_l),$$

for every neighborhood $\mathcal{U}$ of f and every $\epsilon > 0$. So, taking $\mho = \Sigma(\mathcal{U}, \epsilon)$ and $\mho(r_i, \rho_l) = \Sigma(\mathcal{U}, \epsilon, r_i, \rho_l)$ in Theorem 4.15, we conclude that $\mu(\mho) = \mu(\Sigma(\mathcal{U}, \epsilon)) = 1$ for all f−ergodic probability. By Ergodic Decomposition Theorem, this implies that $\Sigma(\mathcal{U}, \epsilon)$ has total probability.

$\square$

So far, we have proven the raw version of Ergodic Closing Lemma for Endomorphisms (Theorem 4.1). The next lemma will be used in the proof of the residual version of Ergodic Closing Lemma. We denote by $\mathcal{M}(M)$ the set of probabilities on M endowed with the weak-* topology.

Lemma 4.16. *Let $f : M \to M$ be an endomorphism. Suppose that, for x in a total probability set $S \subset M$, given $\epsilon > 0$ and a neighborhood $\mathcal{U}$ of f, there exists $g_{x,\epsilon} \in \mathcal{U}$ with a periodic point $p = p(x, \epsilon)$ which ϵ−shadows x. Then, given any ergodic measure $\mu \in \mathcal{M}_1(f)$, there are $g_k \to f$ and g_k−periodic points p_k such that μ is the limit of the sequence (μ_k) of g_k−ergodic measures respectivelly supported in the orbit of p_k. Moreover, each p_k can be taken to be a hyperbolic periodic point for g_k.*

Proof. Let us consider an f-ergodic probability μ. We suppose, without loss of generality, that μ is not supported in a periodic orbit, otherwise there is nothing to prove. For a μ−typical point $x \in M$, we can assume that x is recurrent (by Poincaré's Recurrence Theorem), has the shadowing property as in lemma's statement, and that

$$\frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j(x)} \to_{\text{weak}-*} \mu, \tag{3}$$

as $n \to +\infty$. In the last claim we made use of the Ergodic Decomposition Theorem.

Set $\epsilon_1 = 1$ and $n_k > 0$ as the first return time of the orbit of x to $B(x, \epsilon_k)$, where $\epsilon_{k+1} := d(f^{n_k}(x), x)/2, \forall k \geq 1$.

Therefore, $n_k \to +\infty$ as $k \to +\infty$. By hypothesis, one can take a sequence of $g_k := g_{x, \epsilon_k}$, with $g_k \to f$, exhibiting g_k−periodic points (p_k) such that each p_k $\epsilon_k/3$-shadows the orbit of x. In particular, the period t_{k+1} of p_{k+1} is, at least, n_k (otherwise, the orbit of x would return to $B(x, \epsilon_k)$ before n_k). So, $t_{k+1} \geq n_k$ implies that $t_k \to +\infty$ as $k \to +\infty$, and (up to take a subsequence) we can suppose that t_k are distinct. Note that slightly perturbing g_k in the neighborhood of p_k, we can suppose that p_k is hyperbolic. Set μ_k as the g_k-ergodic probability supported in the orbit of p_k. We will show that $\mu_k \to_{\text{weak}-*} \mu$ as $k \to +\infty$. From equation 3 we have that

$$\nu_k = \frac{1}{t_k} \sum_{j=0}^{t_k-1} \delta_{f^j(x)} \to_{\text{weak}-*} \mu,$$

as $k \to +\infty$. Let $\alpha > 0$ and $\{\varphi_1, \ldots, \varphi_s\} \subset C^0(M)$ be given. All we need to see is that there exists $k_0 \in \mathbb{N}$ such that μ_k belongs to the neighborhood

$$V_{\varphi_1,\ldots,\varphi_s;\alpha} := \{\nu \in \mathcal{M}(M); |\int \varphi_i d\nu - \int \varphi_i d\mu| < \alpha, \forall i = 1, \ldots, s\},$$

forall $k \geq k_0$. In fact, as φ_i, $i = 1, \ldots, s$ are uniformly continuous, take $\epsilon > 0$ such that $|\varphi_i(y) - \varphi_i(z)| < \alpha/2$, $\forall i = 1, \ldots, s$, $\forall y, z \in M$ such that $d(y, z) < \epsilon$. Then, take k_0 such that $\epsilon_k < \epsilon/2$, and $|\int \varphi_i d\nu_k - \int \varphi_i d\mu| < \alpha/2$, $\forall k \geq k_0$, $\forall i = 1, \ldots, s$. We conclude that

$$|\int \varphi_i d\mu_k - \int \varphi_i d\mu| \leq |\int \varphi_i d\mu_k - \int \varphi_i d\nu_k| + |\int \varphi_i d\nu_k - \int \varphi_i d\mu| <$$

$$\frac{1}{t_k} \sum_{j=0}^{t_k-1} |\varphi_i(f^j(x)) - \varphi_i(g_k^j(p_k))| + \alpha/2 \leq \alpha, \forall i = 1, \ldots, s;$$

which implies the lemma.

$\square$

Now, we proceed with the proof of Theorem E, by deriving it from Theorem 4.1 and Lemma 4.16 above. The arguments here are basically the same as in Theorem 4.2 in [1].

Proof of Theorem E. For $m \in \mathbb{N}$ fixed, by standard transversality arguments, the collection $\mathcal{K}_m$ of endomorphisms f such that all periodic points of f, with period up to m are hyperbolic is an open and dense subset $NEnd^1(M)$. So $\hat{\mathcal{R}} := \cap_{m=1}^{+\infty} \mathcal{K}_m$ is a residual set. Let the set of probabilities $\mathcal{M}(M)$ on M to be endowed with the weak-* topology and let κ be the collection of compact subsets of $\mathcal{M}(M)$ endowed with Hausdorff distance. Given $f \in \mathcal{R}$, denote by $\mathcal{M}_{per}(f)$ the set of $f-$ergodic measures supported in $f-$periodic orbits. Set $\Upsilon : \hat{\mathcal{R}} \to \kappa$ given by

$$\Upsilon(f) = \overline{\mathcal{M}_{per}(f)}$$

Due to the robustness of hyperbolic periodic points, such Υ is lower semicontinuous. This implies that there is a residual subset $\mathcal{R} \subset \hat{\mathcal{R}}$ whose elements are continuity points for Υ.

From now on, let $f \in \mathcal{R}$. Let us prove that $\mathcal{M}_1(f)$ is the closed convex hull of $f-$ergodic measures supported in $f-$periodic orbits. By Ergodic Decomposition Theorem, all we need to prove is that any $f-$ergodic measure μ is in $\overline{\mathcal{M}_{per}(f)}$. By Lemma 4.16, such measure μ is accumulated by $\mu_k \in \mathcal{M}_{per}(g_k)$, where $g_k \to f$ as $k \to +\infty$. As $\hat{\mathcal{R}}$ is residual, by means of a slight perturbation, we can suppose that $g_k \in \hat{\mathcal{R}}$ (as we construct p_k to be hyperbolic in the proof of that Lemma 4.16, such p_k persist under any sufficiently small perturbation). Since f is a continuity point for Υ, we have that $\overline{\mathcal{M}_{per}(g_k)} \to \overline{\mathcal{M}_{per}(f)}$ as $k \to +\infty$, and this implies that $\mu \in \overline{\mathcal{M}_{per}(f)}$.

Therefore, $\overline{\mathcal{M}_{per}(f)}$ contains all $f-$ergodic measures, and by Ergodic Decomposition Theorem, we conclude that $\mathcal{M}_1(f)$ is the closed convex hull of f-ergodic measures supported in periodic orbits.

$\square$

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