TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 00, Number 0, Pages 000–000
S 0002-9947(XX)0000-0

HYPERBOLCITY IN THE VOLUME PRESERVING AND SYMPLECTIC SCENARIO

ALEXANDER ARBIETO AND THIAGO CATALAN

ABSTRACT. Hayashi has extended a result of Mañé, proving that every element in $\mathcal{F}^1(M)$ satisfies Axioma A, i.e., every diffeomorphism f with a neighborhood $\mathcal{U}$, where all periodic points of any $g \in \mathcal{U}$ are hyperbolic, it is an Axioma A diffeomorphism. Here, we prove an analogue result in the volume preserving and symplectic scenario, and using this we give a proof of the analogous version of Palis conjecture in the volume preserving scenario.

1. INTRODUCTION AND STATEMENT OF THE RESULTS

Let M be a C^∞ Riemannian manifold without boundary and let $\text{Diff}^1_m(M)$ denote the set of diffeomorphisms which preserves the Lebesgue measure m induced by the Riemannian metric. Also, by $\text{Diff}^1_\omega(M)$ we mean the set of diffeomorphisms which are symplectic, those are the diffeomorphisms that preserves a symplectic form, i.e. $f^*\omega = \omega$, for a closed and non-degenerated 2-form ω. Both sets are endowed with the C^1-topology.

In the theory of dynamical systems, one important question is to know whether robust dynamical properties in the phase space leads to differentiable properties of the system. For instance, one of the most important properties that a system can have is stability. This says that any system close enough to the initial have the same orbit structure of the initial. In other terms, this says that there is a topological conjugacy between this system and the initial one.

In a striking article [12] Mañé proves that any C^1-Ω-stable diffeomorphism is in fact an Axiom A diffeomorphism. Actually, Mañé believe that a weaker property than Ω-stability should be enough to guarantee the Axiom A property. Let us elaborate on this property.

Given a diffeomorphism f over M, a periodic point p of f is *hyperbolic* if $Df^{\tau(p)}$ has eigenvalues with absolute values different of one, where $\tau(p)$ is the period of p. In the space of C^1 diffeomorphisms over M, $\text{Diff}^1(M)$, we can define the set $\mathcal{F}^1(M)$ as the set of diffeomorphisms $f \in \text{Diff}^1(M)$ which have a C^1-neighborhood $\mathcal{U} \subset \text{Diff}^1(M)$ such that if $g \in \mathcal{U}$ then any periodic point of g is hyperbolic. In [9], Hayashi proved that any diffeomorphism in $\mathcal{F}^1(M)$ is Axiom A, which means that the periodic points are dense in the nonwandering set $\Omega(f)$ and the last one is a hyperbolic set. We recall that in dimension two, this was proved by Mané

Received by the editors April 9, 2010.
2010 *Mathematics Subject Classification.* Primary 37D20; Secondary 37C20.
Key words and phrases. Hyperbolic orbits, Volume-preserving, symplectic, Palis conjecture.
A.A. was partially supported by CNPq Grant and Faperj.
T.C. was partially supported by Fapesp.

[11]. Observe that in the volume preserving and symplectic scenario, the Axiom A condition is equivalent to the diffeomorphism be Anosov, since $\Omega(f) = M$ by Poincaré Recurrence Theorem. Hence, it is a natural question if Hayashi's and Mané's results still holds in the volume preserving and symplectic scenario, this is the purppose of this article.

We define the set $\mathcal{F}^1_m(M)$ as the set of diffeomorphisms $f \in \mathrm{Diff}^1_m(M)$ which have a C^1-neighborhood $\mathcal{U} \subset \mathrm{Diff}^1_m(M)$ such that if $g \in \mathcal{U}$ then any periodic point of g is hyperbolic. Analogously, we define the set $\mathcal{F}^1_\omega(M)$ using $\mathrm{Diff}^1_\omega(M)$ instead $\mathrm{Diff}^1_m(M)$.

If $f \in \mathrm{Diff}^1(M)$ then an $f-$invariant compact set Λ of M is called a *hyperbolic set* if there is a continuous and Df-invariant splitting $T_\Lambda M = E^s \oplus E^u$ such that there are constants $0 < \lambda < 1$ and $C > 0$, satisfying

$$\|Df^k_x|E^s(x)\| \leq C\lambda^k \quad \text{and} \quad \|Df^{-k}_x|E^u(x)\| \leq C\lambda^k,$$

for every $x \in \Lambda$ and $n > 0$. We say that f is an *Anosov diffeomorphism*, if M is a hyperbolic set for f. The main result of this article is the following,

Theorem 1.1. *Any diffeomorphism in $\mathcal{F}^1_m(M)$ or $\mathcal{F}^1_\omega(M)$ is Anosov.*

We would like to observe that the symplectic case was already solved by Newhouse in [13]. But, since our proof in the symplectic case is different of him (he uses the unfolding of tangencies far from Anosov diffeomorphisms) and since it rises naturally of the methods for the proof in the volume preserving case we decided to write it also.

Note, since the neighborhoods of the diffeomorphisms are taken in the respectively spaces $\mathrm{Diff}^1_m(M)$, $\mathrm{Diff}^1_\omega(M)$, or $\mathrm{Diff}^1(M)$ we could take no relation between $\mathcal{F}^1_m(M)$, $\mathcal{F}^1_\omega(M)$ and $\mathcal{G}^1(M)$ direct from definition. But, as a corollary of the previous theorem we obtain,

Corollary 1.2. $\mathcal{F}^1_\omega(M) \subset \mathcal{F}^1_m(M) \subset \mathcal{F}^1(M)$.

We define the *index* of a hyperbolic periodic point as the dimension of its stable manifold. We say that a diffeomorphism f exhibit a *heterodimensional cycle* if there are hyperbolic periodic saddles p and q of f with different indices such that the stable manifold of p intersects the unstable one of q and vice-versa, one of them is required to be transversal and the other to be quasi-transversal, see [5] for instance.

The following result was announced by Crovisier in [7], but his explanation was unclear for us without the use of the previous theorem and some bifurcations. Actually, the next result is a version of Palis conjecture [14] in the volume preserving scenario.

Corollary 1.3. If $f \in \mathrm{Diff}^1_m(M)$ is not an Anosov diffeomorphism, then it can be approximated by one exhibiting an heterodimensional cycle.

This paper is organized as follows. In section 2 we announce and prove the Franks lemma in conservative and symplectic case, the main tools used to perturbations. In section 3, we prove the index for hyperbolic periodic is constant in neighborhoods of any $f \in \mathcal{F}^1_m(M)$. Finally, in section 4 we give a proof for the Theorem A.

2. Franks-type Lemmas

One of the most useful and basic perturbation lemmas is the Franks lemma [8]. This lemma enable us to perform non-linear perturbations along a finite piece of an orbit simply performing arguments from Linear Algebra. However, in the volume-preserving or symplectic case, more arguments are needed, since the perturbation must preserve also the underlying structure. In the volume-preserving scenario one helpful tool is the Arbieto-Matheus Pasting Lemma [2] and in the symplectic scenario the theory of generating functions will do the job, as we shall see.

In what follows, we will give an statement in both scenarios, but we will explain how to prove it independently. In particular, by D we mean $\mathrm{Diff}^1_\omega(M)$ or $\mathrm{Diff}^1_m(M)$, and by E we mean linear maps which are symplectic or with determinant one, respectively.

Lemma 2.1. *Let $f \in D$ and $\mathcal{U}$ be a C^1-neighborhood of f in D. Then, there exist a neighborhood $\mathcal{U}_0 \subset \mathcal{U}$ of f and $\delta > 0$ such that if $g \in \mathcal{U}_0(f)$, $S = \{p_1, \dots, p_m\} \subset M$ and $\{L_i : T_{p_i}M \to T_{p_{i+1}}M\}_{i=1}^m$ are linear maps belonging to E satisfying $\|L_i - Dg(p_i)\| \leq \delta$ for $i = 1, \dots m$ then there exists $h \in \mathcal{U}(f)$ such that $h(p_i) = g(p_i)$ and $Dh(p_i) = L_i$.*

Remark 2.2. As we shall see in the proof, in the symplectic case, if U is a neighborhood of S then h can be taken such that $h(x) = g(x)$ for every $x \in S \cup (M - U)$.

2.0.1. *Proof in the volume-preserving case.* In fact, the proof in the volume-preserving scenario is contained in the paper by [10]. Where Arbieto-Matheus Pasting Lemma [2] is used to proof the following basic lemma.

Lemma 2.3. *For every $N \in \mathbb{N}$ and $\varepsilon > 0$ there is a neighborhood $\mathcal{G}$ of the identity in $SL(N, \mathbb{R})$, the special linear subgroup, such that for every $A \in \mathcal{G}$ there exists $h \in \mathrm{Diff}^1_m(\mathbb{R}^N)$ satisfying the following properties:*

1. *h coincides with the identity outside the unit ball at the origin;*
2. *$h(0) = 0$ and $Dh(0) = A$;*
3. *$\|Dh - Id\| < \varepsilon$.*

From this the proof of lemma 2.1 can be deduced easily (see [10]).

2.0.2. *Proof in the symplectic case.* All of our approximations are local and will be done in local coordinates using *generating functions.* Hence, let us define these functions. Let $(u, v) = (u_1, \dots, u_n, v_1, \dots, v_n)$ be a system of coordinates on $\mathbb{R}^{2n}$ and $\omega = \sum_{i=1}^n du_i \wedge dv_i$ be a 2-form. Let $f(u, v) = (\xi(u, v), \eta(u, v))$ be a C^1 symplectic diffeomorphism defined on a simply connected neighborhood V of the origin. Thus, $\sum_{i=1}^n du_i \wedge dv_i = \sum_{i=1}^n d\xi_i \wedge d\eta_i$.

We can assume that $f(0, 0) = (\xi^0, \eta^0)$ and that $\dfrac{\partial \eta(u, v)}{\partial v}$ is non-singular at each point of V. Then we can solve v as a C^1 function of u and η, this means, $v = v(x, \eta)$. So, $(u_1, \dots, u_n, \eta_1, \dots, \eta_n)$ defines new C^1 coordinates on a small neighborhood of $(0, \eta^0)$. Consider now $\alpha = \sum_{i=1}^n v_i(u, \eta) du_i + \xi(u, \eta) d\eta_i$ the 1-form on $\mathbb{R}^{2n}$, and since f is symplectic, we can see that $d\alpha = 0$. Thus there exists a real valued function $S = S(u, \eta)$, unique up to a constant, defined in a neighborhood of $(0, \eta^0)$, such that $dS = \alpha$. S is called a *generating function* for f and satisfies $\frac{\partial S}{\partial \eta_i} = \xi_i$, $\frac{\partial S}{\partial u_i} = v_i$ and $\frac{\partial^2 S}{\partial \eta_i \partial u_i}$ is non-singular for each (u, η) near $(0, \eta^0)$ in the domain of S.

Conversely, if $S(u, \eta)$ is a C^2 function defined in a neighborhood of $(0, \eta^0)$ such that $\dfrac{\partial^2 S}{\partial \eta_i \partial u_i}$ is non-singular for each point in the domain, then setting $\xi_i(u, \eta) = \frac{\partial S}{\partial \eta_i}$ and $v_i(u, \eta) = \frac{\partial S}{\partial u_i}$ we may solve $\eta = \eta(u, v)$ as a C^1 function of u and v, such that

$f(u,v) = (\xi(u,\eta(u,v)),\eta(u,v))$ is a C^1 symplectic diffeomorphism defined in a neighborhood of the origin. The amazing fact of the generating functions is that symplectic diffeomorphisms are C^1 near if, and only if, their generating functions are C^2 near.

Claim 1: Let $R : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be a symplectic linear transformation δ_0 near Id. Then, there is a diffeomorphism r which is $K_0\delta_0 - C^1$-close to Id, such that $r = Id$ outside an arbitrary small neighborhood of origin and $Dr(0) = R$.

Without loss of generality we may choose δ_0 such that $\dfrac{\partial R(u,v)}{\partial v}$ is non-singulare. We denote by $S_R(u,\eta)$ and $S_{Id}(u,\eta)$ the generating functions of R and Id, respectively.

Claim 1.1: Given $\gamma > 0$, $|S_R(u,\eta) - S_{Id}(u,\eta)| < \delta_0\gamma^2$ for $|(x,\eta)| < \gamma$.

In fact, let $A(x,\eta) = S_R(x,\eta) - S_{Id}(x,\eta)$. Fix a point (x,η) such that $|(x,\eta)| < \gamma$ and let α be a segment in $\mathbb{R}^{2n}$ such that $\alpha(0) = 0$ and $\alpha(1) = (x,\eta)$. By the Calculus Fundamental Theorem, we have $A(x,\eta) = \int_{(0,1)}(A(\alpha(t)))'dt$, since $A(0) = 0$ by the construction of the generating functions in case. Thus

$$
\begin{aligned}
|A(x,\eta)| &\le \int_0^1 \|DA\||(\alpha(t))|\,|\alpha'(t)|\,dt,\\
&< \delta_0\gamma\int_0^1 |\alpha'(t)|dt,\\
&< \delta_0\,\gamma^2,
\end{aligned}
$$

where we use the fact that $DA : \mathbb{R}^{2n} \to \mathbb{R}^{2n*}$ is linear in the first inequality. This proves claim 1.1.

Now, let β be a C^∞ bump function, vanishing for $|t| > 1$ and constant equal to one for $|t| < 1/2$. We define $K_1 = \sup\{|\beta'|, |\beta''|\}$. For some $\gamma > 0$ let us define

$$
S(x,\eta) = \beta\left(\frac{|(x,\eta)|}{\gamma}\right) S_R(x,\eta) + \left(1 - \beta\left(\frac{|(x,\eta)|}{\gamma}\right)\right) S_{Id}(x,\eta).
$$

Using the proximity of S_R and S_{Id}, and the claim 1.1, it is easy to see that S and S_{Id} are $K_1K_2\delta_0 - C^2$-close, where K_2 is some constant depending of n. Thus the symplectic diffeomorphism r generated by S is $K_1K_2\delta_0 - C^1$-close to Id. Moreover $r = Id$ outside $B(0, K_3\gamma)$ and $Dh(0) = R$, where K_3 depends on R. And so, taking $K_0 = K_1K_2$ we complete the proof of claim 1.

Now, let $\varepsilon > 0$ such that every $g \in \mathrm{Diff}^1_\omega(M)$ which is $2\varepsilon - C^1$-close to f is in $\mathcal{U}$. Thus, we choose $\mathcal{U}_0 \subset \mathcal{U}$ as the $\varepsilon-$neighborhood of f in $\mathrm{Diff}^1_\omega(M)$. Let us also consider two finite open cover (ψ_i, U_i) and (ϕ_i, V_i) of M by symplectic coordinates such that $f(\overline{U_i}) \subset V_i$. We recall that symplectic coordinates are also so called Darboux coordinates, and they are coordinates (ϕ, U) such that $\phi^*\omega$ is the standard 2-form in $\mathbb{R}^{2n}$. Note the neighborhood $\mathcal{U}_0$ can be taken to guarantee $g(\overline{U_i}) \subset V_i$ for all $g \in \mathcal{U}_0$ and all i. Without loss of generality we can suppose that every ball of radius δ_0 is contained in some U_i, for some i. We denote $K_4 = \max\{Lip(\phi_i^{-1})\}$, where $Lip(\phi)$ is the Lipchitz constant of ϕ.

Let $\gamma_0 > 0$ such that the balls $B(x_j, \gamma_0)$, are pairwise disjoints and $f(B(x_j, \gamma_0)) \subset B(f(x_j), \delta_0)$. Finally, we take $\gamma_1 > 0$ such that $f^{-1}(\phi^{-1}(B(\phi_i(x_j), \gamma_1))) \subset B(x_j, \gamma_0)$, where i depends on x_j. Taking a smaller neighborhood $\mathcal{U}_0$ of f, if necessary, we can suppose that this holds for every $g \in \mathcal{U}_0$.

Fixe now some j and for simplicity denote $x = x_j$, $(\psi, U) \in \{(\psi_i, U_i)\}$ and $(\phi, V) \in \{(\phi_i, V_i)\}$ the associated symplectic coordinates. Now, let $g \in \mathcal{U}_0$. We can assume $\phi(g(x)) = 0$, modulo a translation of ϕ which is symplectic. Now, considering $R = D\phi(g(x)) \circ L \circ Dg^{-1}(x) \circ D\phi^{-1}(0) : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ a symplectic linear application. We have that R is sufficiently C^1-close to identity if L is sufficiently C^1-close to $Dg(x)$.

Using claim 1 for R, we can find $\overline{r}$ $\delta_0 - C^1$-close to identity such that $\overline{r} = Id$ outside $B(0, \gamma_1)$ and $D\overline{r}(0) = R$. Then, taking $r = \phi^{-1} \circ \overline{r} \circ \phi : V \to M$ and $r = Id$ outside V, this is a well defined symplectic diffeomorphism $K_0\, K_4\delta_0 - C^1$ near Id. Hence, $\tilde{h} = r \circ g$ is a symplectic diffeomorphism $K_0\, K_4\delta_0 - C^1$-close to g. Moreover $\tilde{h} = g$ outside $B(x, \gamma_0)$ and $D\tilde{h}(x) = L$.

Finally, taking $\delta = \min\{K_0\, K_4\delta_0,\ \varepsilon\}$ and using the above calculations, we can build symplectic difeomorphisms $h_j \in \mathcal{U}$ such that $h_j = g$ outside $B(x_j, \gamma_0)$ and $Dh_j(x_j) = L_j$. Thus, the symplectic diffeomorphism $h = h_1 \circ \ldots \circ h_m$ belongs to $\mathcal{U}$, $h = g$ outside a small neighborhood of $\{x_1, \ldots, x_m\}$ and $Dh(x_j) = L_j$.

The proof is now complete.

3. Index of Periodic Orbits

Let $f \in \mathcal{F}^1_\omega(M)$, since all periodic orbits are hyperbolic and we are in the symplectic scenario the index is always half of the dimension of the ambient manifold.

In the volume preserving case, the property of have two periodic hyperbolic saddles with different indices is forbidden by the creation of heteroclinic cycles.

Proposition 3.1. Let $f \in \mathcal{F}^1_m(M)$ then there exists a neighborhood $\mathcal{U}$ of f in $\mathrm{Diff}^1_m(M)$ and an integer i such that for every diffeomorphism $g \in \mathcal{U}$ and every hyperbolic periodic orbit p of g, the index of p with respect to g is i.

Before we prove this let us recall some good properties of the periodic set, that will be very useful.

We say that a f-invariant compact set Λ has *dominated splitting* if there are a continuous splitting $T_\Lambda M = E \oplus F$ and constants $m \in \mathbb{N}$, $0 < \lambda < 1$ such that

$$\|Df^m_x|E(x)\|\ \|Df^{-m}_{f^m(x)}|F(f^m(x))\| \leq \lambda.$$

Now, let $\Lambda_i(f)$ denote the close of the set formed by hyperbolic periodic points of f with index i. The following proposition is essential and it can be deduced from the Franks Lemma 2.1 and a linear algebra result of Mañé (Lemma II.3 of [11]) in the same way as Mañé proves Proposition II.1 of [11]. From now $\mathcal{F}$ may mean $\mathcal{F}^1_\omega(M)$ or $\mathcal{F}^1_m(M)$. We also recall that D means $\mathrm{Diff}^1_m(M)$ or $\mathrm{Diff}^1_\omega(M)$ depending of the context.

Proposition 3.2. If $f \in \mathcal{F}$, there exists a neighborhood $\mathcal{U}$ of f in D, and constants $K > 0$, $m \in \mathbb{N}$ and $0 < \lambda < 1$ such that

a) For every $g \in \mathcal{U}$ and $p \in Per(g)$ with minimum period $\tau(p) \geq m$

$$\prod_{i=0}^{k-1} \|Dg^m(g^{mi}(p))|E^s_{g^{mi}(p)}(g)\| \leq K\lambda^k$$

and

$$\prod_{i=0}^{k-1} \|Dg^{-m}(g^{-mi}(p))|E^u_{g^{-mi}(p)}(g)\| \leq K\lambda^k,$$

where $k = [\tau(p)/m]$.

b) For all $0 < i < dimM$ there exists a continuous splitting $T_{\Lambda_i(g)}M = E_i \oplus F_i$ such that

$$\|Dg^m(x)|E_i(x)\| \; \|Dg^{-m}(g^m(x))|F_i(g^m(x))\| \leq \lambda.$$

for all $x \in \Lambda_i(g)$ and $E_i(p) = E^s_p(g)$, $F_i(p) = E^u_p(g)$ when $p \in Per(g)$ and $dimE^s_p(g) = i$.

c) For all $p \in Per(g)$

$$\limsup_{n\to+\infty} \frac{1}{n}\sum_{i=0}^{n-1} \log \|Dg^m(g^{mi}(p))|E^s_{g^{mi}(p)}(g)\| < 0$$

and

$$\limsup_{n\to+\infty} \frac{1}{n}\sum_{i=0}^{n-1} \log \|Dg^{-m}(g^{-mi}(p))|E^u_{g^{-mi}(p)}(g)\| < 0.$$

If proposition 4.1 is not true, then there exists two hyperbolic periodic orbits p and q of f with respectively indices i and $i + j$, for some $j > 0$.

Now, the following result can be proved by the same methods of Abdenur-Bonatti-Crovisier-Diaz-Wen in [1].

Proposition 3.3. For any neighborhood $\mathcal{U}$ of $f \in D$, if there exists $p, q \in Per(f)$ with indices i and $i + j$, respectively, then for any positive integer α between i and $i + j$ there exists $g \in \mathcal{U}$ and a hyperbolic periodic point of g with index α.

Before the proof of this Proposition, let us first to show how we can use it to prove Proposition 3.1.

Hence, by Proposition 3.3 we can find hyperbolic periodic points p and q of f, by some perturbation of f, with indices i and $i + 1$, respectively.

In the sequence, we will see how to perturb f in order to get a heterodimensional cycle between these two hyperbolic periodic points. First, we remember a result by Bonatti-Crovisier [4]:

Lemma 3.4. *There exists a residual subset $\mathcal{R}$ of $\mathrm{Diff}^1_m(M)$ such that if $g \in \mathcal{R}$ then $M = H(p, g)$, where $H(p, g)$ is the homoclinic class for a hyperbolic periodic point p of g. In particular, g is transitive.*

Hence, perturbing and using the hyperbolicity of p and q, we can suppose that our diffeomorphism $f \in \mathcal{R}$ and so it is transitive. Then using the connecting lemma of Hayashi for conservative diffeomorphisms, see [4], we can create an intersection between $W^u(p)$ and $W^s(q)$, also perturbing if necessary, we can assume that this intersection is transversal. Hence, this intersection is robust, and we can suppose that this new diffeomorphism also belongs to $\mathcal{R}$. Using the connecting lemma once more, we can create an intersection between $W^s(p)$ and $W^u(q)$. Thus we create a heterodimensional cycle.

Continuing the proof of Proposition 3.1, what we want to do now it's to find a periodic point with at least one lyapunov exponent as near as we want of zero.

We suppose first that p and q are fixed points.

Now, let $\mathcal{R}_1$ be the set of volume preserving diffeomorphisms where the homoclinic class are disjoint or coincide. Using a result by Carballo, Morales and Pacifico in [6] we know that $\mathcal{R}_1$ is a residual subset in $\mathrm{Diff}^1_m(M)$. So, before the perturbation for constructing the heterodimensional cycle we could take $f \in \mathcal{R} \cap \mathcal{R}_1$, instead of just in $\mathcal{R}$. Hence, $M = \Lambda_i(f) = \Lambda_{i+1}(f)$, i.e., hyperbolic periodic points with indices i and $i+1$ are dense in M. Therefore, using Proposition 3.2 we have a dominated splitting for f, $TM = E \oplus C \oplus F$, such that the dimension of E and C are equal to i and one, respectively. And thus, by the continuation of the dominated splitting we still have this one for the perturbation of f that exhibit a heterodimensional cycle between p and q as before.

Let $\mathcal{U}$ be some neighborhood of f in $\mathcal{F}^1_m(M)$ and $\mathcal{U}_1 \subset \mathcal{U}$ such that we still have the previous dominated splitting for every $g \in \mathcal{U}_1$.

We remember now a perturbation result of Xia in [16].

Lemma 3.5. *Fixed $\phi \in D$, there exists constants $\varepsilon_0 > 0$ and $c > 0$, depending on ϕ, such that for any $x \in M$, and any $\psi \in D$ $\varepsilon_0 - C^1$ near ϕ, and any positive numbers $0 < \delta < \varepsilon_0$, $0 < \varepsilon < \varepsilon_0$, the following facts hold:*

if $d(y,x) < c\delta\varepsilon$, then there is a $\psi_1 \in D$ $\varepsilon - C^1$ near ψ such that $\psi_1(\psi^{-1}(x)) = y$, $\psi_1(z) = z$ for all $z \notin \psi^{-1}(B_\delta(x))$.

Then, we fixe ε_0 and $c > 0$ for f according the previous lemma. Now, let $0 < \varepsilon < \varepsilon_0$ be such that if $f_1 \in D$ is $\varepsilon - C^1$ near of f then $f \in \mathcal{U}_1$.

Continuing, let $x \in W^s(p) \cap W^u(q)$ and $y \in W^u(p) \cap W^s(q)$. Now, let B_p and B_q small balls in M centered at p and q, respectively. Moreover, given any $\gamma > 0$ we may choose B_p such that $\|Df(z) - Df(p)\| \leq \gamma$, for every $z \in B_p$. By the choice of x and y we can choose m_1, m_2, m_3 and m_4 positive integers such that $f^{m_1}(x), f^{-m_3}(y) \in B_p$ and $f^{-m_2}(x), f^{m_4}(y) \in B_q$. Now, let $0 < \delta < \varepsilon_0$ such that

$$f^{-1}(B_\delta(f^{m_1}(x))) \cap B_\delta(f^{m_1}(x)) = \varnothing,\ f^{-1}(B_\delta(f^{-m_2+1}(x))) \cap B_\delta(f^{-m_2+1}(x)) = \varnothing,$$

and

$$f^{-1}(B_\delta(f^{m_4}(y))) \cap B_\delta(f^{m_4}(y)) = \varnothing,\ f^{-1}(B_\delta(f^{-m_3+1}(y))) \cap B_\delta(f^{-m_3+1}(y)) = \varnothing.$$

Using the λ-Lemma we can find z_m $c\delta\varepsilon-$near $f^{m_1}(x)$ such that $f^m(z_m)$ is also $c\delta\varepsilon-$near $f^{-m_3}(y)$ and $f^r(z_m) \in B_p$, $0 \leq r \leq m$, for every $m > 0$ big enough. Analogously, we can find $\overline{z}_n$ satisfying similar conditions changing p for q and the respectively iterates of x and y.

Hence, the set

$$O_{mn} = \{z_m, ..., f^m(z_m), f^{-m_3}(y), ..., f^{m_4}(y), \overline{z}_n, ..., f^n(\overline{z}_n), f^{-m_2}(x), ..., f^{m_1}(x)\}$$

is a pseudo periodic orbit. And using the Lemma 3.5, we can perturb f in order to find a periodic orbit p_{mn} that shadows O_{mn}. Moreover, note that $\{z_m, ..., f^{m-1}(z_m), f^{-m_3}(y), ..., f^{m_4-1}(y), \overline{z}_n, ..., f^{n-1}(\overline{z}_n), f^{-m_2}(x), ..., f^{m_1-1}(x)\}$ is the orbit of the periodic point p_{mn}, indeed. Observe also that p_{mn} pass m and n times in B_p and B_q, respectively. Then, by the dominated splitting of f, m and n could be chosen such that the index of p_{mn} is either $i+1$ or i.

Now, fixed some big n, we choose $m = m(n)$ the biggest one such that, as defined before, p_{mn} and $p_{m-1\,n}$ are hyperbolic periodic points of different perturbations of f with indices i and $i+1$, respectively. We will call these perturbations of f by g and h, i.e., $p_{mn} \in Per(g)$ and $p_{m-1\,n} \in Per(h)$. We would like to remark that the way to perturb f in order to construct these points, it gives us $g = h$ outside B_p. Finally, taking n bigger if necessary we have $g, h \in \mathcal{U}_1$.

By the previous process we have that the orbit of the hyperbolic periodic points p_{kn}, $k = m$, $m-1$, is

$$\{z_k, ..., f^{k-1}(z_k), f^{-m_3}(y), ..., f^{m_4-1}(y), \overline{z}_n, ..., f^{n-1}(\overline{z}_n), f^{-m_2}(x), ..., f^{m_1-1}(x)\},$$

for $k = m$, $m-1$, where z_k and $\overline{z}_n$ can be found by $\lambda-$lemma, depending of k, as before.

Now, denoting by τ de period of $p_{mn} = f^{-m_3}(y)$ and taking $K = \|Df(p)|C(f)\|$, we have

$$\begin{aligned}
0 &< \frac{1}{\tau}\log\|Dg^{\tau}(p_{mn})|_{C(g)}\| \\
&= \frac{1}{\tau}\sum_{t=0}^{\tau-1}\log\|Dg(g^t(p_{mn}))|_{C(g)}\| \\
&< \frac{1}{\tau}(\sum_{t=0}^{\tau-1}\log\|Df(g^t(p_{mn}))|_{C(f)}\| + \gamma) \\
&\le \frac{1}{\tau}(\sum_{t=0}^{\tau-m-1}\log\|Df(g^t(p_{mn}))|_{C(f)}\| + m(\log\|Df(p)|_{C(f)}\| + \gamma) + \gamma\tau)
\end{aligned}$$

(3.1)

$$< \frac{1}{\tau-1}(\sum_{t=0}^{\tau-m-1}\log\|Df(g^t(p_{mn}))|_{C(f)}\| + m-1(\log\|Df(p)|_{C(f)}\|)) + 2\gamma + \frac{K}{\tau},$$

where we use that the central direction C is one-dimensional in the first equality, the continuity of the dominated splitting in the second equality and the choice of B_p in the third one. On the other hand, using the hyperbolic periodic point $p_{m-1\,n} = f^{-m_3}(y)$ of h and similarly arguments we have the following:

$$\begin{aligned}
0 &> \frac{1}{\tau-1}\log\|Dh^{\tau-1}(p_{m-1\,n})|_{C(h)}\| \\
&= \frac{1}{\tau-1}\sum_{t=0}^{\tau-1}\log\|Dh(h^t(p_{m-1\,n})|_{C(h)}\| \\
&> \frac{1}{\tau-1}(\sum_{t=0}^{\tau-1}\log\|Df(h^t(p_{m-1\,n}))|_{C(f)}\| - \gamma)
\end{aligned}$$

(3.2)

$$\ge \frac{1}{\tau-1}(\sum_{t=0}^{\tau-m-1}\log\|Df(h^t(p_{m-1\,n}))|_{C(f)}\| + m-1(\log\|Df(p)|_{C(f)}\|)) - 2\gamma.$$

Now, since $g = h$ outside B_p and the orbit of p_{mn} and $p_{m-1\,n}$ also coincides outside B_p we have $g^t(p_{mn}) = h^t(p_{m-1\,n})$ for $0 \le t \le \tau-m-1$. Hence, using this we can substitute the inequality (3.2) in (3.1) and to obtain the following

$$0 < \frac{1}{\tau}\log\|Dg^{\tau}(p_{mn})|C(g)\| < 4\gamma + \frac{K}{\tau}. \tag{3.3}$$

Therefore, since the the period τ goes to infinity when n goes to infinity, and $\gamma > 0$ is arbitrary, it's possible to find a hyperbolic periodic point p_{mm} with a Lyapunov exponent as near as we want of zero.

In the general case, when p and q are hyperbolic periodic points, the difference is that the neighborhoods B_p and B_q must be neighborhoods of the orbits of p and q, respectively, and then the numbers m and n will be multiples of the periods of p and q, respectively. Hence, by the same arguments as before we can find the periodic point p_{mn} with at least one lyapunov exponent near of zero.

Finally, using Franks lemma 2.1 again we can perturb once more such that we reduce a little bit the force of the eigenvalue associated to the Lyapunov exponent near zero in each point of the orbit of p_{mn} such that we can get indeed a zero Lyapunov exponent for the periodic point p_{mn}. This means, we have an eigenvalue with absolute value one, and then p_{mn} is not a hyperbolic periodic point. Since all of these perturbations can be done inside $\mathcal{U} \subset \mathcal{F}$ we have a contradiction. Therefore we proved the Proposition 3.1.

Now, we will prove the Proposition 3.3.

First, we will prove a conservative version of the Proposition 2.3 in [1].

Proposition 3.6. There is a residual subset $\mathcal{R}_2$ of $\mathrm{Diff}^1_m(M)$ consisting of diffeomorphisms f such that $Per_{\mathbb{R}}(H(p,f))$, the set of hyperbolic periodic points of f with the same index of p and with only real eigenvalues of multiplicity one, is dense in $H(p,f)$ for every non-trivial homoclinic class $H(p,f)$ of f.

We recall that a *periodic linear systems (cocycles)* is a 4-tuple $\mathcal{P} = (\Sigma, f, \mathcal{E}, A)$, where f is a diffeomorphism, Σ is an infinite set of periodic points of f, $\mathcal{E}$ an Euclidian vector bundle defined over Σ, and $A \in GL(\Sigma, f, \mathcal{E})$ is such that $A(x) : \mathcal{E}_x \to \mathcal{E}_{f(x)}$ is a linear isomorphism for each x ($\mathcal{E}_x$ is the fiber of $\mathcal{E}$ at x). For the precise definition we refer the reader to the work of Bonatti-Diaz-Pujals [5].

Lemma 3.7 (Lemma 1.9 in [5])**.** *Let $H(p,f)$ be a non-trivial homoclinic class. Then the derivative Df of f induces a periodic linear system with transitions over $Per_h(H(p,f))$, the set of hyperbolic periodic points homoclinically related with p.*

By the previous lemma and Franks lemma 2.1, where the last one permits us to perform dynamically the perturbations of a cocycle, our problem in Proposition 3.6 becomes just a linear problem. We say that a periodic linear system with transitions $\mathcal{P} = (\Sigma,\, f,\, \mathcal{E},\, A)$ is *diagonalizable* at the point $x \in \Sigma$ if the linear map

$$M_A(x) : \mathcal{E}_x \to \mathcal{E}_x, \quad M_A(x) = A(f^{\tau(x)-1}(x)) \circ \ldots \circ A(f^2(x)) \circ A(x),$$

only has positive real eigenvalues of multiplicity one.

Lemma 3.8. *For every periodic linear system with transitions $\mathcal{P} = (\Sigma,\, f,\, \mathcal{E},\, A)$ and every $\varepsilon > 0$ there is a dense subset Σ' of Σ and an $\varepsilon-$perturbation A' of A defined on Σ' which is diagonalizable, that is, $M_{A'}(x)$ has positive real eigenvalues of multiplicity one for every $x \in \Sigma'$.*

By the Remark 7.2 in [5] we can consider the perturbation A' such that $detA'(x) = 1$ for every $x \in \Sigma'$. Then, as we have noted before, we can use Franks Lemma 2.1 and the previous lemmas to show the Proposition 3.6. In fact, after we have done all these observations the proof is identically the proof of Proposition 2.3 in [1].

Hence, we can suppose $f \in \mathcal{R}_2 \cap \mathcal{R}$, where the residual set $\mathcal{R}$ is given by Proposition 3.4, and then we may assume that p and q have only real eigenvalues of multiplicity one for $Df^{\tau(p)}(p)$ and $Df^{\tau(q)}(q)$, respectively.

As we did before we can suppose, unless some perturbation, that f exhibit a heterodimensional cycle between p and q. Hence the proof of the Proposition 3.3 follows direct from the next result stated in [1].

Proposition 3.9 (Theorem 3.2 in [1])**.** Let f be a diffeomorphism having a heterodimensional cycle associated to periodic saddles p and q, of indices i and $i+j$ with $j>0$, with real eigenvalues. Then, for any C^1-neighborhood $\mathcal{U}$ of f and for any integer α with $i \leq \alpha \leq i+j$, there exists $g \in \mathcal{U}$ having a periodic point with index α.

Although this proposition has been stated there for dissipative diffeomorphisms, all the perturbations of f did there is using franks lemma and Hayashi's connecting lemma. Then, this proposition is still true in the conservative case since we have these two perturbation tools in the conservative world.

4. Proof of Theorem A

In the sequence we shall prove the hyperbolicity of $\overline{Per(f)}$ for every $f \in \mathcal{F}^1_m(M)$ or $f \in \mathcal{F}^1_\omega(M)$, since we already know the index of the hyperbolic periodic points is constant.

Let us fixe $f \in \mathcal{G}$ and a continuous dominated splitting $T_{\overline{Per(f)}}M = E \oplus F$ given by the Proposition 3.2. From now, we also consider $m \in \mathbb{N}$, $0<\lambda<1$ and $K>0$ as in the Proposition 3.2. We will prove that this splitting gives us a hyperbolic structure on $\overline{Per(f)}$, indeed. We will follow here similar arguments as in the proof of Mane of Theorem B in [11].

To show this we need prove that we have contraction and expansion in the sub-bundles E and F, respectively, unless a certain finite time iterate. Hence, by compactness of $\overline{Per(f)}$, we just need to show the following

$$\liminf_{n\to+\infty} \|Df^n(x)|E(x)\| = 0 \tag{4.1}$$

and

$$\liminf_{n\to+\infty} \|Df^{-n}(x)|F(x)\| = 0,$$

for all $x \in \overline{Per(f)}$.

Note it's enough to prove the first case since the second one can be deduced from the first one changing f by f^{-1}.

Suppose now (4.1) isn't true. Then there exists $x \in M$ such that

$$\|Df^{jm}(x)|F(x)\| \geq c > 0, \text{ for all } j > 0.$$

Defining the following probability measure

$$\mu_j = \frac{1}{j}\sum_{i=0}^{j-1} \delta_{f^{mi}(x)},$$

where δ is the dirac measure, we can find a subsequence $j_n \to \infty$ such that μ_{j_n} converge to an f^m-invariant probability measure μ in the weak* topology and

$$\lim_{n\to+\infty} \frac{1}{j_n} \log \|Df^{mj_n}(x)|E(x)\| \geq 0. \tag{4.2}$$

Hence, taking the continuous functional $\phi(x) = \log \|Df^m(x)|E(x)\|$ over $\overline{Per(f)}$, we obtain:

$$\int_{\overline{Per(f)}} \phi \, d\mu = \lim_{n\to+\infty} \frac{1}{j_n} \sum_{i=0}^{j_n-1} \log \|Df^m(f^{mi}(x))|E(f^{mi}(x))\|$$
$$\geq \lim_{n\to+\infty} \frac{1}{j_n} \log \|Df^{mj_n}(x)|E(x)\| \geq 0.$$

And so, using Ergodic Birkhoff's Theorem

(4.3)
$$0 \leq \int_{\overline{Per(f)}} \phi \, d\mu = \int_{\overline{Per(f)}} \lim_{n\to+\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log \|Df^m(f^{mi}(y))|E(f^{mi}(y))\| \, d\mu(y).$$

Now let $\Sigma(f) \subset M$ a total probability set given by the Ergodic closing lemma in the conservative (symplectic) case. See [3]. Hence, denote by $\nu = \frac{1}{m}\sum_{i=0}^{m-1} {f^i}^* \mu$ the $f-$invariant probability measure induced by μ, we have $\nu(\Sigma(f) \cap \overline{Per(f)}) = 1$ since ν is supported on $\overline{Per(f)}$. But now, by the invariance of $\Sigma(f) \cap \overline{Per(f)}$ for f, it's easy to see that this is also a total probability set for μ. And so, this together with (4.3) implies the existence of a point $y \in \Sigma(f) \cap \overline{Per(f)}$ such that:

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log \|Df^m(f^{mi}(y))|E(f^{mi}(y))\| \geq 0. \tag{4.4}$$

Observe that part (c) of Proposition 3.2 is an obstruction for y be periodic. Hence $y \notin Per(f)$.

By (4.4), we can take $\lambda < \lambda_0 < 1$ and $n_0 > 0$ such that:

$$\frac{1}{n} \sum_{i=0}^{n-1} \log \|Df^m(f^{mi}(y))|E(f^{mi}(y))\| \geq \log \lambda_0, \tag{4.5}$$

when $n \geq n_0$.

The next step will consist in resume to find a hyperbolic periodic point $p \in Per(g)$ with orbit "near" the orbit of y, for g near f, and to use the Lemma 2.1 for changing the derivative at the orbit of p such that the inequality (4.5) gives us some contradiction with part (a) of Proposition 3.2.

Using that $y \in \Sigma(f)$ we can approximate f by diffeomorphisms g such that there exist $p \in Per(g)$ and the distance between $f^j(p)$ and $f^j(y)$ is arbitrary small, for $0 \leq j \leq n$, where n is the minimum period of p_g. Since y is not periodic the period n must goes to infinity when g approaches f. Hence we may choose g and p such that:

$$n \geq m, \tag{4.6}$$

$$k \geq n_0, \tag{4.7}$$

$$K\lambda^k < \lambda_0^k \tag{4.8}$$

and

$$\left(\frac{\lambda}{\lambda_0}\right)^k C^m \leq \frac{1}{2}, \tag{4.9}$$

where $k = [n/m]$ and $C = \sup_{x\in M} \|Df^{-1}(x)\|$.

These choices together with (4.5) and the dominated splitting of $f|\overline{Per(f)}$ give us the following

$$\begin{aligned} \|Df^{-n}_{f^n(y)}|F(f^n(y))\| &\leq \prod_{i=0}^{k-1} \|Df^{-m}_{f^{n-mi}(y)}|F(f^{n-mi}(y))\| \, \|Df^{-(n-mk)}_{f^{n-mk}(y)}|F(f^{n-mk}(y))\| \\ &\leq \lambda^k \, C^m \, \lambda_0^{-k} \leq \frac{1}{2}. \end{aligned} \tag{4.10}$$

Let U be a neighborhood of $\overline{Per(f)}$ small enough such that the maximal set in U

$$\Lambda_U(f) = \bigcap_{n \in \mathbb{Z}} f^n(U)$$

has dominated splitting and satisfying the thesis of the Proposition 3.2. Hence, we can chose $\mathcal{U} \subset \mathcal{G}$ a neighborhood of f such that every $h \in \mathcal{U}$ has a dominated splitting in $\Lambda_U(h)$ near of the dominated splitting in $\Lambda_U(f)$. Note now, that $E_g(p) = E^s_g(p)$ and $F_g(p) = E^u_g(p)$ since dominated splitting is unique if we fixe the dimensions, and the index of periodic points is constant for $g \in \mathcal{U}$. Hence, taking a smaller neighborhood $\mathcal{U}$ if necessary we can suppose $E^s_g(g^i(p))$ and $E^u_g(g^i(p))$ as near as we want of $E(f^i(y))$ and $F(f^i(y))$, $0 \leq i \leq n$, respectively.

In the sequence we will build some symplectic (conservative) isomorphisms $A_i : T_{g^i(p)}M \to T_{f^i(y)}M$, $0 \leq i \leq n$, near of identity in local coordinates. Moreover, for future convenience, we will have $A_i(E^s_g(g^i(p))) = E(f^i(x))$ isometrically and $A_i(E^u_g(g^i(p))) = F(f^i(x))$, $0 \leq i \leq n$.

Because of geometry restrictions the symplectic case is more complicated, hence we will show the cares to take in this case. The conservative case is similar and easier.

Before start the construction, it's easy to check that $E(x)$ and $F(x)$ are transversal lagrangian subspaces of T_xM for every $x \in \overline{Per(f)}$, since $E(q) = E^s(q)$ and $F(q) = E^u(q)$ for all $q \in Per(f)$ by Proposition 3.2, the two form ω is continue and f preserves ω.

We show how to construct A_0, the other cases are analogous. Using symplectic properties we can choose

$$\{e_1(i), \ldots, e_n(i), r_1(i), \ldots, r_n(i)\} \text{ basis for } T_iM,\ i = x, p,$$

such that $\{e_j(i),\ 1 \leq j \leq n\}$ is a orthonormal basis for $E(x)$ if $i = x$ or for $E^s_g(p)$ if $i = p$, and $\{r_j(i),\ 1 \leq j \leq n\}$ is a basis for $F(x)$ if $i = x$ or for $E^u_g(p)$ if $i = p$, and $\omega(i)(e_j(i), r_j(i)) = 1$, for every $1 \leq j \leq n$ and $i = x, p$.

By continuity of the two form ω these basis are as near as f and g are, unless a constant that depends only of ω. Then, let $A_0 : T_pM \to T_xM$ a linear map satisfying $A_0(e_j(p)) = e_j(x)$ and $A_0(r_j(p)) = r_j(x)$, $1 \leq j \leq n$. Therefore, by construction, A_0 is a symplectic linear map, and by the choice of the basis above, A_0 is near of the identity in local symplectic coordinates.

Now, let's back to the proof. Let $L_i : T_{g^i(p)}M \to T_{g^{i+1}(p)}M$ be symplectic (conservative) maps defined in the following way

$$L_i = A^{-1}_{i+1} \, Df(f^i(x)) \, A_i, \ \text{ for } 0 \leq i \leq n-1.$$

Hence, taking n bigger if necessary, L_i is as near of $Dg(g^i(p))$ as we want, for all $0 \leq i \leq n-1$. Then, using Lema 2.1, we can find $h \in \mathcal{U}$ such that $p \in Per(h)$ and $Dh(h^i(p)) = L_i$, $0 \leq i \leq n-1$. Observe that $E^s_g(p)$ and $E^u_g(p)$ still are invariants by $Dh^n(p)$, by construction of L_i's. This together with (4.10), the proximity of f and g, and the dimension of the subspaces give us that $E_h^u(p) = E_g^u(p)$. And so, $E_h^s(p) = E_g^s(p)$ too.

Finally, since $A_i|E_g^s(g^i(p))$ is isometry, we have the following

$$\|Dh^m(h^{im}(p))|E_h^s(h^{im}(p))\| = \|Df^m(f^{im}(x))|E(f^{im}(x))\|, \text{ for all } i \in \mathbb{N}.$$

Therefore,

$$\prod_{i=0}^{k-1} \|Dh^m(h^{im}(p))|E_h^s(h^{im}(p))\| = \prod_{i=0}^{k-1} \|Df^m(f^{im}(x))|E(f^{im}(x))\| \geq \lambda_0^k,$$

what contradicts Proposition 3.2. Therefore we showed that $\overline{Per(f)}$ is hyperbolic if $f \in \mathcal{G}$.

Finally to conclude that if $f \in \mathcal{G}$ then f is Anosov, we just need to show that $\Omega(g) = \overline{Per(f)}$ since $\Omega(f) = M$, by recurrence Poincare Theorem. To show this we will use basically Pugh's closing lemma.

Let $f \in \mathcal{G}$. We can chose some neighborhood $\mathcal{U}$ of f in D such that $\sharp H_n(g)$, number of hyperbolic periodic points of g with period smaller or equal than n, is finite and equal for every $g \in \mathcal{U}$, since all diffeomorphisms in $\mathcal{U}$ has only hyperbolic periodic points.

Now, suppose $\overline{Per(f)} \subsetneq \Omega(f)$, and let $x \in \Omega(f)\backslash\overline{Per(f)}$. By Pugh's closing lemma we can fixe $k \in \mathbb{N}$ such that the perturbations of f in order to get a hyperbolic periodic point near x is done in a arbitrary small neighborhood of

$$\bigcup_{-k \leq j \leq k} f^j(x).$$

Hence, let U be a neighborhood of $\overline{Per(f)}$ such that $f^j(x) \notin \overline{U}$, $-k \leq j \leq k$. So, using the closing lemma we can get $g \in \mathcal{U}$ and $p \in Per(g)$ with $p \notin \overline{U}$. Nevertheless by choice of k and U, f is equal to g in U and since $p \notin \overline{U}$ we have $H_n(f) \neq H_n(g)$ for some $n \in \mathbb{N}$, what contradicts the fact of $g \in \mathcal{U}$.

Thefore, we have $\Omega(f) = \overline{Per(f)}$ as we wanted.

5. Palis Conjecture in the volume preserving scenario

Before we prove the Corollary 1.3 let us to show how we can perturb some diffeomorphism to create a hyperbolic periodic point.

Suppose $f \in \text{Diff}^1_m(M)$ and let p be a periodic point of f with prime period τ such that

$$Df^\tau(p) = \begin{bmatrix} a_1 & b_1 & & & \\ -b_1 & a_1 & * & & \\ & & \ddots & * & \\ & 0 & & a_t & b_t \\ & & & -b_t & a_t \end{bmatrix},$$

for some basis of T_pM, and some $t \leq n/2$ since some blocks on the diagonal can be 1×1. Hence, the eigenvalues are $a_j \pm i\, b_j$. Now, given two distinct blocks j and i, using Franks lemma 2.1 we can construct some perturbation of identity h such that $h = Id$ outside some small neighborhood U of p, $h(p) = p$ and $Dh(p) = (a_{kl})$, where $a_{jj} = c$, $a_{ii} = 1/c$, and the other $a_{kk} = 1$, $a_{kl} = 0$, since the matriz $Dh(p)$

has determinant 1, and it is as near to Id as $c > 0$ is near to one. So, considering $g = h \circ f$ we have g is near to f, p is still a periodic point of g and

$$Dg^{\tau}(p) = \begin{bmatrix} \begin{matrix} a_1 & b_1 \\ -b_1 & a_1 \end{matrix} & & & & & \\ & \ddots & & & & \\ 0 & & \begin{matrix} a_j & b_j \\ -c\,b_j & c\,a_j \end{matrix} & & & \\ & & & \cdot & & \\ & & & & \begin{matrix} a_i/c & b_i/c \\ -b_i & a_i \end{matrix} & & \\ & & & & & \ddots & \\ & & & & & & \begin{matrix} a_t & b_t \\ -b_t & a_t \end{matrix} \end{bmatrix}.$$

Then, it's easy to check that the modulus of the eigenvalues of Dg^{τ} are now

$$a_1^2 + b_1^2, \dots, c(a_j^2 + b_j^2), \dots, 1/c(a_i + b_i), \dots \ .$$

Using this we will prove the Corolary 1.3.

Suppose $f \in \mathrm{Diff}^1_m(M)$ is not Anosov. Then $f \notin \mathcal{F}^1_m(M)$, and therefore we can suppose there is a non hyperbolic periodic point p of f, unless some perturbation. Moreover, since $\omega(f) = M$ by the closing lemma we can also suppose there is a hyperbolic periodic point q of f. Now, using the previous perturbation in a neighborhood of p, we can choose $c > 0$ and appropriate blocks such that p and q are hyperbolic periodic points of this perturbation of f with different indices. And so, as we did before in the proof of Proposition 3.1, we can perturb once more and create a heterodimensional cycle between p and q.

References

1. Abdenur, F.; Bonatti, C.; Crovisier, S.; Diaz, L.; Wen; L. Periodic points and homoclinic classes, *Ergod. Th. and Dynam. Sys.* 27 (2007), pp. 1-22.
2. Arbieto A. and C. Matheus. On dominated splittings for conservative systems. Ergodic Theory and Dynamical Systems, 27, no. 5, (2007), pp. 1399-1417.
3. Arnaud, M-C. Le "Closing Lemma" en topologie C^1, *Supplment au Bull. Soc. Math. Fr.* 74(1998)
4. Bonatti, C. and Crovisier, S. Recurrence et generecite, *Inv. Math.* 158 (2004), no 1, pp. 33-104
5. Bonatti, C.; Diaz, L.; Pujals; H. A C^1-generic dichotomy for diffeomorphisms: Weak forms of hyperbolicity or infinitely many sinks or sources, *Annals of Math.* 158 (2003), pp. 355-418.
6. Carballo, C.; Moralles, C.; Pacifico, M. Homoclinic classes for generic C^1 vector fields, *Ergod. Th. and Dynam. Sys.* 23 (2003), pp. 403-415.
7. Crovisier, S. Perturbation de la dynamique de diffeomorphismes en topologie C^1. Preprint (2009).
8. Franks, J. Necessary conditions for stability of diffeomorphisms, *Trans. A.M.S.* 158 (1971), 301-308.
9. Hayashi, S. Diffeomorphisms in $\mathcal{F}^1(M)$ satisfy Axioma A, *Ergod. Th. and Dynamical Sys.* 12(1992), pp.233-253.
10. Liang, C. Liu, G. and Sun W. Equivalent Conditions of Dominated Splittings for Volume-Preserving Diffeomorphism, *Acta Math. Sinica* 23(2007), pp.1563-1576.
11. Mañé M. An Ergodic Closing Lemma. *The Annals of Mathematics* 2nd Ser., Vol 116, No. 3. (Nov., 1982), pp. 503-540.
12. Mañé M. A proof of the C^1 stability conjecture. *Publ. Math. de IHES*, Vol 66, (1987), pp. 161-210.

13. Newhouse, S.E. Quasi-eliptic periodic points in conservative dynamical systems, *American Journal of Mathematics*, Vol. 99, No. 5 (1975), 1061-1087 .
14. Palis, J. Global perspective for non-conservative dynamics. , *Annales I. H. Poincare - Analyse Non Lineaire*, v. 22, p. 485-507, 2005.
15. Pujals, E. and Sambarino, M. Homoclinic tangencies and hyperbolicity for surface diffeomorphisms, *Annals of Mathematics*, 151 (2000), 961-1023.
16. Xia, Z. Homoclinic points in symplectic and Volume-Preserving diffeomorphisms, *Communications in Mathematical Physics*, 177 (1996), 435-449.

Instituto de Matemática, Universidade Federal do Rio de Janeiro, P. O. Box 68530, 21945-970 Rio de Janeiro, Brazil.
E-mail address: arbieto@im.ufrj.bt

Instituto de Ciências, Matemática e Computação, Universidade de So Paulo, 16-33739153 São Carlos-SP, Brazil
E-mail address: catalan@icmc.usp.br