PREVALENT DYNAMICS AT THE FIRST BIFURCATION OF HÉNON-LIKE FAMILIES

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Abstract. We study the dynamics of strongly dissipative Hénon-like maps, around the first bifurcation parameter a^* at which the uniform hyperbolicity is destroyed by the formation of tangencies inside the limit set. We prove that a^* is a full Lebesgue density point of the set of parameters for which Lebesgue almost every initial point diverges to infinity under positive iteration. A key ingredient is that a^* corresponds to "non-recurrence of every critical point", reminiscent of Misiurewicz parameters in one-dimensional dynamics. Adapting on the one hand Benedicks & Carleson's parameter exclusion argument, we construct a set of "good parameters" having a^* as a full density point. Adapting Benedicks & Viana's volume control argument on the other, we analyze Lebesgue typical dynamics corresponding to these good parameters.

1. Introduction

One important problem in dynamics is to describe transitions from structurally stable to unstable regimes. Equally important is to describe how strange attractors are created. A prototypical example intimately connected to these problems is given by the Hénon family

$$H_a\colon (x,y)\mapsto (1-ax^2+\sqrt{b}y,\pm\sqrt{b}x),\quad 0<b\ll 1.$$

For all large a, one gets a uniformly hyperbolic horseshoe [12], a paradigmatic example of structurally stable chaotic systems. As one decreases a, the horseshoe loses its stability at a bifurcation parameter, and then a nonuniformly hyperbolic strange attractor is created, with positive probability in parameter space [5]. The aim of this paper is to shed some light on the process of this sort of transition from horseshoes to strange attractors.

We work within a framework set up by Palis for studying bifurcations of diffeomorphisms: consider arcs of diffeomorphisms losing their stability through generic bifurcations, and analyze which dynamical phenomena are more frequently displayed (in the sense of Lebesgue measure in parameter space) in the sequel of the bifurcation. More precisely, let (φ_a) be a parametrized family of diffeomorphisms which undergoes a first bifurcation at $a=a^*$, i.e., φ_a is structurally stable for $a>a^*$ and φ_{a^*} has a cycle. We assume (φ_a) unfolds the bifurcation generically. A dynamical phenomenon $\mathcal{P}$ is *prevalent* at a^* if

$$\liminf_{\varepsilon\to+0}\varepsilon^{-1}\mathrm{Leb}(\{a\in[a^*-\varepsilon,a^*]\colon \varphi_a \text{ displays } \mathcal{P}\})>0.$$

This framework originates in the work of Newhouse and Palis [19], on the frequency of bifurcation sets in the unfoldings of homoclinic tangencies. In that paper, diffeomorphisms before the first bifurcation are Morse-Smale. Palis and Takens [20, 21, 22], inspired by works of Newhouse, studied the prevalence of uniform hyperbolicity in arcs of diffeomorphisms for

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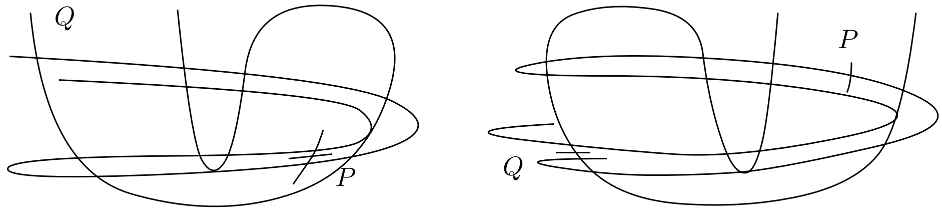

FIGURE 1. The case $a = a^*$. There exist two hyperbolic fixed saddles P, Q near $(1/2, 0)$, $(-1, 0)$ correspondingly. In the orientation preserving case (left), $W^u(Q)$ meets $W^s(Q)$ tangencially. In the orientation reversing case (right), $W^u(P)$ meets $W^s(Q)$ tangencially.

which the non-wandering set of the diffeomorphism at the bifurcation is a union of a non-trivial basic set of saddle type and an orbit of tangency. In opposite direction, the frequency of non-hyperbolicity was studied by Palis and Yoccoz [23, 24, 25].

For the Hénon family, the first bifurcation where the horseshoe ceases to be stable corresponds to the formation of homoclinic or heteroclinic tangencies [2]. This tangency is quadratic, and $(H_a)_a$ unfolds the tangency generically [3]. The orbit of the tangency is accumulated by transverse homoclinic points, and hence contained in the limit set. In [10], all these statements are extended to Hénon-like families, a perturbation of the Hénon family (see Section 2 for a precise definition).

This sort of bifurcation is completely different from the one treated in [20, 21, 22, 23, 24, 25]. A key aspect of models treated in these papers is that the orbit of tangency at the first bifurcation is not contained in the limit set. This implies a global control on new orbits added to the underlying basic set, and moreover allows one to use its invariant foliations to translate dynamical problems to the problem on how two Cantor sets intersect each other. This argument is not viable, if the orbit of tangency, responsible for the loss of the stability of the system, is contained in the limit set, as in the case of Hénon-like families. Let us call such a bifurcation an *internal tangency bifurcation*.

For an Hénon-like family (f_a), we aim to describe changes in the set

$$K_a = \left\{ z \in \mathbb{R}^2 \colon \{f_a^n(z)\}_{n \in \mathbb{Z}} \text{ is bounded} \right\}.$$

By a result of [10], there is a parameter a^* such that K_a is a hyperbolic set for $a > a^*$, and $(f_a)_a$ unfolds a quadratic tangency at $a = a^*$ generically. This suggests that the structure of K_a depends in a very discontinuous way upon a. For instance, a^* is accumulated from left by: a-intervals for which f_a has sinks [1, 15]; sets with positive Lebesgue measure for which f_a has nonuniformly hyperbolic attractors [18], etc. A consequence of our theorem is that the frequency of such parameters tends to zero as $a \to a^*$. Let

$$K_a^+ = \left\{ z \in \mathbb{R}^2 \colon \{f_a^n(z)\}_{n \geq 0} \text{ is bounded} \right\}.$$

Theorem. *For an Hénon-like family (f_a) there exists a set Δ of a-values such that:*

(a) $\lim_{\varepsilon \to +0} \varepsilon^{-1} \mathrm{Leb}(\Delta \cap [a^* - \varepsilon, a^*]) = 1$*;*

(b) *if $a \in \Delta$, then K_a^+ has zero Lebesgue measure.*

(c) *if $a \in \Delta$, then f_a is transitive on K_a.*

To grasp the meanings of the theorem, it is worthwhile to recall Jakobson's theorem for the quadratic family $x \to 1 - ax^2$, which states that $a = 2$ is a (one-sided) full Lebesgue density

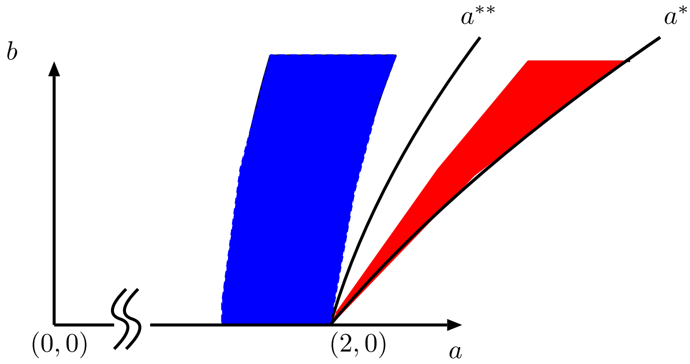

FIGURE 2

point of the set of parameters corresponding to absolutely continuous invariant probability measures. These measures allow one to statistically predict the asymptotic "fate" of Lebesgue almost every initial conditions. For $a > 2$, the orbit of the critical point $x = 0$ is escaping, and thus the invariant set is uniformly hyperbolic. In other words, $a = 2$ is a first bifurcation parameter of the quadratic family. Immediately right after the bifurcation one mainly gets "observable chaos". Our theorem asserts a sharp contrast to this sort of transition. For $a \in \Delta$, K_a behaves like a basic set of saddle type, in that Lebesgue typical points escape from any neighborhood of it. This means that, physically observable complicated behaviors are chaotic transient around K_a, not sustained in time.

This striking difference at the first bifurcation stems from a simple fact intrinsic to two-dimension: at the parameter a^*, the unstable manifold of the saddle fixed point(s) is not confined in any bounded region. Indeed, one key step in the proof of the theorem is to show that, for carefully chosen parameters, the unstable manifold intersects K_a^+ in a set with zero Lebesgue measure on the manifold.

Figure 2 indicates a landscape in the (a, b)-plane (as usual, b controls the closeness to the quadratic family, see (1)). The a^*-line consists of the parameters of the first bifurcation. The a^{**}-line consists of parameters corresponding to the manifold organization indicated in Figure 3. The parameter set Δ is in the red region bounded by these two lines. For $b > 0$ small, Benedicks and Carleson [5], Mora and Viana [18] constructed a set of a-values near 2, corresponding to maps which exhibit nonuniformly hyperbolic strange attractors. These parameter sets are in the blue region at the left of the a^{**}-line.

In view of the theorem, one might speculate that maps in $\{f_a : a \in \Delta\}$ would retain some weak form of hyperbolicity, as a memory of the uniform hyperbolicity before the bifurcation. For the moment, we do not know if the uniform hyperbolicity is prevalent at a^*. See Remark 5.1 for a further discussion. To our knowledge, the only presently known result on the prevalence of hyperbolicity in internal tangency bifurcations is due to Rios [26], on arcs of surface diffeomorphisms destroying type 3 horseshoes (horseshoes with three symbols [21]).

To prove the theorem, we build on and develop the machinery for the analysis of strongly dissipative Hénon maps [5, 6, 8, 18, 30]. Excluding undesirable parameters inductively, we construct the parameter set Δ having a^* as a full density point. We then investigate the dynamics of $f \in \{f_a : a \in \Delta\}$.

A parameter exclusion argument in the spirit of Jakobson [16], Benedicks and Carleson [4, 5] was first brought into the study of homoclinic bifurcations by Palis and Yoccoz [24, 25]. As we mentioned in the beginning, the underlying basic set at the bifurcation is used in a

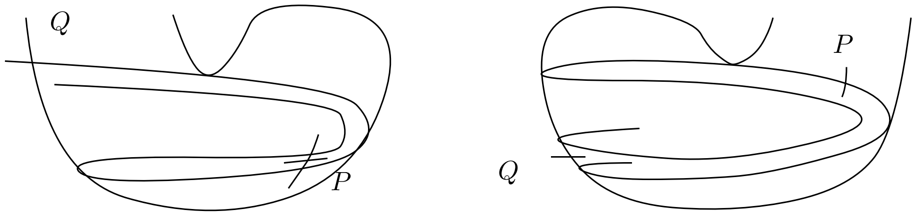

FIGURE 3. The case $a = a^{**}$: orientation preserving (left); orientation reversing (right).

crucial way there, and the same approach does not work in our context of internal tangency bifurcation. In order to prove that K_a^+ has zero Lebesgue measure, we develop the volume control argument of Benedicks and Viana [6].

The rest of this paper consists of six sections and one appendix. In Section 2 we analyze one fixed map, collecting results from [5, 6, 18, 30] and [27] as far as we need them. In Section 3 we recall the procedure in [27] for finding suitable critical approximations, used as guides for orbits falling in critical regions.

The parameter set Δ is constructed in Section 4. This part closely follows the previous construction of the parameter set in [27], modulo the assertion that a^* is a full density point of Δ. It is at this point where the characteristic of the first bifurcation is crucial. We show that the map f_{a^*} behaves as if it is a "two-dimensional Misiurewicz map", in the sense that *every* critical approximation of it is non-recurrent. Then it is possible, as in the one-dimensional case [4, 16], to arrange the induction construction in such a way that less and less proportions of parameters in $[a^* - \varepsilon, a^*]$ are excluded as $\varepsilon \to +0$, and the total fractions of Δ in the intervals get closer to one. Consequently, Δ must have a^* as a full density point.

For the remaining three sections we consider the dynamics of one fixed map $f \in \{f_a \colon a \in \Delta\}$. In Section 5 we identify an well-organized geometric structure of the unstable manifold, close to the one identified by Wang and Young [30]. Using this structure, in Section 6 we analyze the dynamics on the unstable manifold. Combining a classical large deviation argument [5, 8, 9] with a continuity argument from the first bifurcation, we prove that K^+ intersects the unstable manifold in a set with zero Lebesgue measure. In Section 7 we study the dynamics on K^+. A careful adaptation of the volume control argument [6] together with the conclusion of Section 6 shows that K^+ cannot have positive two-dimensional Lebesgue measure.

2. Preliminaries

In this section we analyze one fixed map f, collecting results from [5, 6, 18, 30] and [27] as far as we need them.

2.1. **Hénon-like families.** We deal with a parameterized family (f_a) of diffeomorphisms on $\mathbb{R}^2$ such that $f = f_a$ has the form

$$f_a \colon (x, y) \mapsto (1 - ax^2, 0) + b \cdot \Phi(a, b, x, y), \tag{1}$$

where (a, b) is close to $(2, 0)$ and Φ is bounded, continuous, C^4 in (a, x, y).

Although f is globally defined on $\mathbb{R}^2$, it is possible to localize our consideration to a compact domain defined as follows. If f preserves orientation, let $W^u = W^u(Q)$. Otherwise, let $W^u = W^u(P)$. Let R_0 denote the compact domain bounded by W^u and $W^s(Q)$, as indicated in Figure 4. By a result of [10], points outside of R_0 escape to infinity either by positive or

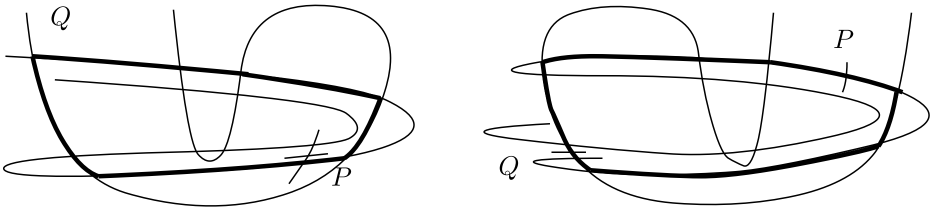

FIGURE 4. The region R_0

negative iterations. Hence $K \subset R_0$ holds. Let $D_0 = \{(x, y) \notin R_0 \colon x \geq \sqrt{2}\}$. It can be read out from [10] that $K^+ \subset D_0 \cup R_0$ holds. By the obvious uniform hyperbolicity on D_0, $K^+ \cap D_0$ has zero Lebesgue measure. Therefore, for the proof of the theorem, it suffices to show that $K^+ \cap R_0$ has zero Lebesgue measure. To this end, the next lemma allows us to focus on the dynamics inside R_0.

Claim 2.1. $K^+ \cap R_0 = \bigcap_{n \geq 0} f^{-n}(R_0)$.

Proof. Let $z \in K^+ \cap R_0$. Suppose that $z \notin f^{-n}(R_0)$ holds for some $n > 0$. Let n_0 denote the smallest integer with this property. Then $f^{n_0+1}(z) \in D_1$, where D_1 is the set of points (x, y) which is at the left of $W^s_{\rm loc}(Q)$ and $|y| \leq \sqrt{b}$. As $D_1 \cap K^+ = \emptyset$, $z \notin K^+$ holds, which is a contradiction. Consequently, $K^+ \cap R_0 \subset \bigcap_{n \geq 0} f^{-n}(R_0)$ holds. The reverse inclusion is obvious. □

To structure the dynamics inside R_0, we construct *critical points* and use them as *guides*. Unlike the attractor context [5, 18, 30], the construction of critical points has to take into consideration possible leaks out of R_0 under iteration, and unbounded derivatives at infinity is a bit problematic. To bypass this problem, we work with a new family $(\tilde{f}_{a,b})$ which is obtained by modifying the quadratic map $x \to 1 - ax^2$, and Φ in (1) so that the following holds:

(M1) $f = \tilde{f}$ on R_0 and $\tilde{f}(D_1) \subset D_1$;
(M2) if $z \in R_0$ and $\tilde{f}(z) \notin R_0$, then for any $n \geq 1$ and a nonzero tangent vector v at $\tilde{f}^n(z)$ with ${\rm slope}(v) \leq \sqrt{b}$, ${\rm slope}(D\tilde{f}(\tilde{f}^n(z))v) \leq \sqrt{b}$ and $\|D\tilde{f}(\tilde{f}^n(z))v\| \geq 2\|v\|$;
(M3) there exists a constant $C_0 > 0$ such that $\|\partial^i f\| \leq C_0$ and $|\det D\tilde{f}| \leq C_0 b$ on $D_1 \cup R_0 \cup f(R_0)$ $(i = 1, 2, 3, 4)$, where ∂^i denotes any partial derivative in a, x, y of order i.

2.2. **Hyperbolic behavior.** Constructive constants are α, M, δ, chosen in this order. The α, δ are small, and M is a large integer. Having chosen all of them, we choose sufficiently small b.

From this point on, let us denote $\tilde{f}$ by f. We start with basic properties of f. For $\delta > 0$, define $I(\delta) = \{(x, y) \in R_0 \colon |x| < \delta\}$. The next lemma establishes a uniform hyperbolicity outside of $I(\delta)$. Not only for orbits staying inside R_0, the hyperbolicity estimates hold for orbits which leak out of R_0.

Lemma 2.1. *For any $\lambda_0 \in (0, \log 2)$ and $\delta > 0$, the following holds for (a, b) close to $(2, 0)$. Let $z \in R_0$ be such that z, $f(z), \cdots, f^{n-1}(z) \notin I(\delta)$, and let v be a tangent vector at z with* ${\rm slope}(v) \leq \sqrt{b}$. *Then:*

(a) ${\rm slope}(Df^n(z)v) \leq \sqrt{b}$ *and* $\|Df^n(z)v\| \geq \delta e^{\lambda_0 n}\|v\|$;

(b) *if, in addition, $f^n(z) \in I(\delta)$, then $\|Df^n(z)v\| \geq e^{\lambda_0 n}\|v\|$.*

Proof. If $z, f(z), \cdots, f^{n-1}(z) \in R_0$, then (a) (b) follow from the closeness of f to the top quadratic map. Otherwise, the orbit splits into the part $z, f(z), \cdots, f^{k-1}(z)$ $(k < n)$ in R_0, and the rest out of R_0. (b) is vacuous because of $f^n(z) \notin I(\delta)$. We have $\text{slope}(Df^k(z)v) \leq \sqrt{b}$ and $\|Df^k(z)v\| \geq \delta e^{\lambda_0 k}\|v\|$. Combining these with (M2) we obtain (a). $\square$

2.3. **Quadratic behavior.** The letter C denotes any generic constants which depend only on (f_a) restricted to $[-2,2]^2$. Let us agree that $a \approx b$ indicates that $C^{-1} \leq a/b \leq C$ holds for some $C \geq 1$.

In the next lemma we assume γ is a *horizontal curve*, that is, a C^2-curve such that the slopes of its tangent directions are $\leq 1/10$ and the curvature is everywhere $\leq 1/10$. For $z \in \gamma$, let $t(z)$ denote any unit vector tangent to γ at z. In addition, we assume there exists $\zeta \in \gamma$ such that $\text{slope}(Df(\zeta)t(\zeta)) \geq C\sqrt{b}$. Let e denote any unit vector tangent to $f(\gamma)$ at $f(\zeta)$. Split $Df(z)t(z) = A(z)\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) + B(z)e$.

Lemma 2.2. ([27] Lemma 2.2.) *There exists* C *such that for all* $z \in \gamma \cap I(\delta)$, $|z-\zeta| \approx |A(z)|$ *and* $|B(z)| \leq C\sqrt{b}$.

Remark 2.1. Let Γ be a C^1 curve located near $f(\zeta)$ of the form $\Gamma = \{(x(y), y) \colon |x'| \leq C\sqrt{b}, |y| \leq \sqrt{b}\}$. By Lemma 2.2, either: Γ is tangent to $f(\gamma)$ and the tangency is quadratic; or Γ intersects $f(\gamma)$ exactly at two points.

2.4. **Most contracting directions.** Some versions of results in this section were obtained in [5, 18]. Our presentation follows [30]. Let M be a 2×2 matrix. Denote by e the unit vector (up to sign) such that $\|Me\| \leq \|Mu\|$ holds for any unit vector u. We call e, when it exists, the *most contracting direction* of M.

For a sequence of matrices M_1, $M_2 \cdots$, we use $M^{(i)}$ to denote the matrix product $M_i \cdots M_2 M_1$, and e_i to denote the mostly contracting direction of $M^{(i)}$.

Hypothesis for Sect.2.2. The matrices M_i satisfy $|\det M_i| \leq Cb$ and $\|M_i\| \leq C_0$.

Lemma 2.3. ([30] Lemma 2.1.) Let $i \geq 2$, and suppose that $\|M^{(i)}\| \geq \kappa^i$ and $\|M^{(i-1)}\| \geq \kappa^{i-1}$ for some $\kappa \geq b^{1/10}$. Then e_i and e_{i-1} are well-defined, and satisfy

$$\|e_i \times e_{i-1}\| \leq \left(\frac{Cb}{\kappa^2}\right)^{i-1}.$$

Corollary 2.1. ([30] Corollary 2.1.) *If* $\|M^{(i)}\| \geq \kappa^i$ *for* $1 \leq i \leq n$*, then:*

(a) $\|e_n - e_1\| \leq \frac{Cb}{\kappa^2}$;

(b) $\|M^{(i)}e_n\| \leq \left(\frac{Cb}{\kappa^2}\right)^i$ *holds for* $1 \leq i \leq n$.

Next we consider for each i a parametrized family of matrices $M_i(s_1, s_2, s_3)$ such that $\|\partial^j \det M_i(s_1, s_2, s_3)\| \leq C_0^i b$, and $|\partial^j M_i(s_1, s_2, s_3)| \leq C_0^i$ for each $0 \leq j \leq 3$. Here, ∂^j represents any one of the partial derivatives of order j with respect to s_1, s_2, or s_3.

Corollary 2.2. ([30] Corollary 2.2.) Suppose that $\|M^{(i)}(s_1, s_2, s_3)\| \geq \kappa^i$ for $1 \leq i \leq n$. Then for $j = 1, 2, 3$ and $2 \leq i \leq n$,

$$|\partial^j(e_i \times e_{i-1})| \leq \left(\frac{Cb}{\kappa^{2+j}}\right)^{i-1}, \tag{2}$$

$$\|\partial^j(M^{(i)}e_i)\| \leq \left(\frac{Cb}{\kappa^{2+j}}\right)^i. \tag{3}$$

Let $e_1(z)$ denote the most contracting direction of $Df(z)$ when it makes sense. From the form of our map (1), $e_1(z)$ is defined for all $z \notin I(\sqrt{b})$. In view of [[18] pp. 21], we have

$$\text{slope}(e_1) \geq C/\sqrt{b} \quad \text{and} \quad \|\partial e_1\| \leq C\sqrt{b}. \tag{4}$$

Definition 2.1. We say z is *κ-expanding up to time n*, or simply *expanding*, if there exists a tangent vector v at z and $\kappa \geq b^{1/10}$ such that for every $1 \leq i \leq n$,

$$\|Df^i(v)\| \geq \kappa^i\|v\|.$$

For $n \geq 1$, let $e_n(z)$ denote the most contracting direction of $Df(z)$ when it makes sense. From Corollaries 2.1, 2.2 and (4) we get

Corollary 2.3. *If z is κ-expanding up to time n, then* $\text{slope}(e_n) \geq C/\sqrt{b}$ *and* $\|\partial e_n\| \leq \frac{Cb}{\kappa^3}$.

2.5. **Long stable leaves.** A C^2-curve Γ of the form

$$\Gamma = \{(x(y), y) \colon |y| \leq \sqrt{b}, |x'(y)| \leq C\sqrt{b}, |x''(y)| \leq C\sqrt{b}\}.$$

is called a *vertical curve.* A C^2-distance between two vertical curves is measured by regarding them as C^2-functions on $[-\sqrt{b}, \sqrt{b}]$.

Lemma 2.4. (cf.[18] Section 6.) *Let $\kappa \geq \delta^{15}$. If z is κ-expanding up to time n, then for every $1 \leq i \leq n$, the maximal integral curve Γ_i of e_i through z contains a vertical curve. In addition, for every $1 < i \leq n$, $d_{C^2}(\Gamma_i, \Gamma_{i-1}) \leq \left(\frac{Cb}{\kappa^4}\right)^{i-1}$.*

By a *long stable leaf of order i through z* we mean the curve Γ_i as in the statement.

Remark 2.2. In the construction of long stable leaves, the relation between the lengths of leaves and the value of κ is crucial [18]. In [6], long stable leaves of length $\approx 1/5$ are used. To this end, they require $\kappa \geq e^{-20}$. For our purpose, long stable leaves of length $\approx 2\sqrt{b}$ suffices. Hence, $\kappa \geq \delta^{15}$ suffices.

Lemma 2.5. (cf.[6] Proposition 2.4.) *Let $\kappa \geq \delta^{15}$. If z is κ-expanding, the stable set $W^s(z)$ contains a vertical curve $\Gamma_\infty(z)$ through z, and $|f^n(\xi) - f^n(\eta)| \leq \left(\frac{Cb}{\kappa}\right)^n$ holds for all ξ, $\eta \in \Gamma_\infty(z)$ and $n \geq 1$. Moreover, if z_1, z_2 are expanding, then*

$$\text{angle}(t_\Gamma(\xi_1), t_\Gamma(\xi_2)) \leq C\sqrt{b}|\xi_1 - \xi_2|,$$

where $t_\Gamma(\xi_i)$ denotes any unit vector tangent to $\Gamma_\infty(z_i)$ at ξ_i, $i = 1, 2$.

We call a *long stable leaf through z* the curve $\Gamma_\infty(z)$ as in the statement, and a *stable leaf* any compact curve having some iterate contained in a long stable leaf.

Let $\Gamma_\infty(z_1)$, $\Gamma_\infty(z_2)$ be as in Corollary 2.5 and $\xi_1, \eta_1 \in \Gamma(z_1)$. Let ξ_2, η_2 denote the points in $\Gamma_\infty(z_2)$ whose y-coordinate coincides with that of ξ_1 and η_1 correspondingly. Lemma 2.5 and the Gronwall inequality give

$$|\xi_1 - \xi_2| \leq e^{C\sqrt{b}}|\eta_1 - \eta_2|. \tag{5}$$

In particular, two distinct long stable leaves do not intersect each other.

Notation. Let $z \in I(\delta)$ and suppose that $f(z)$ is δ^{15}-expanding up to time n. The long stable leaf of order n through $f(z)$ is denoted by $\Gamma_n(z)$. If $f(z)$ is δ^{15}-expanding, then the long stable leaf through $f(z)$ is denoted by $\Gamma(z)$.

2.6. **Recovering expansion.** Let γ be a horizontal curve and $n \geq M$. We say $\zeta \in \gamma$ is a *critical approximation of order* n on γ if:
(i) $\|Df^i(f(\zeta))\| \geq 1/10$ for $1 \leq i \leq n$;
(ii) $e_n(f(\zeta))$ is tangent to $Df(\zeta)t(\zeta)$, where $t(\zeta)$ is any unit vector tangent to γ at ζ.

Notation. For $z \in I(\delta)$ and $i \geq 1$, let $w_i(z) = Df^{i-1}(f(z))\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$.

We now introduce three conditions, which are taken as inductive assumptions in the construction of the parameter set Δ. Let $\lambda := \lambda_0/2$, where λ_0 is the one in Lemma 2.1. A critical approximation ζ of order n on γ *has a good critical behavior up to time* $k \geq M$ if:

(G1) $\|w_i(\zeta)\| \geq e^{\lambda(i-1)}$ for $1 \leq i \leq k$;
(G2) $\|w_j(\zeta)\| \geq e^{-2\alpha i}\|w_i(\zeta)\|$ for $1 \leq i < j \leq k$;
(G3) there exists a monotone increasing function $\chi\colon [M,k]\cap\mathbb{N} \circlearrowleft$ such that for each $j \in [M,k]$ there exists $\chi(j) \in [(1-\sqrt{\alpha})j, j]$ such that $\|w_{\chi(j)}(\zeta)\| \geq \delta\|w_i(\zeta)\|$ holds for $0 \leq i < \chi(j)$.

Hypothesis for the rest of Sect.2.6: ζ is a critical approximation of order n on γ, and has a good critical behavior up to time $20n$.

For $M \leq k \leq 20n-1$, let

$$D_k(\zeta) = e^{-3\alpha k} \cdot \min_{1\leq i\leq k} \min_{i\leq j\leq k+1} \frac{\|w_j(\zeta)\|^2}{\|w_i(\zeta)\|^3}.$$

Represent the long stable leaf of order n through $f(\zeta)$ as a graph of a function $\Gamma_n(f(\zeta)) = \{(x_n(y), y)\colon |y|\sqrt{b}\}$. Let

$$V_k = \{(x,y)\colon |x - x_n(y)| \leq D_k(\zeta)/2, |y| \leq \sqrt{b}\}.$$

Take a monotone increasing function χ satisfying condition (G3). Let v denote any nonzero vector tangent to γ at z. If $f(z) \in V_k \setminus V_{k+1}$, then we say v is in *admissible position relative to* ζ. Define a *bound period* $p = p(\zeta, z)$ by

$$p = \chi(k),$$

and a *fold period* $q = q(\zeta, z)$ by

$$q = \min\left\{i \in [1,p)\colon |\zeta - z|^{\tilde{\alpha}} \cdot \|w_{j+1}(\zeta)\| \geq 1 \text{ for every } i \leq j < p\right\},$$

where

$$\tilde{\alpha} = \frac{2\log C_0}{\log 1/b}. \tag{6}$$

It is easy to check that (G1-3) and the assumption on z give $|\zeta - z|^{\tilde{\alpha}} \cdot \|w_p(\zeta)\| \geq 1$. Hence q is well-defined. If $f(z) \in V_{20n-1}$, then we say v is in *critical position relative to* ζ.

Proposition 2.1. ([27] Proposition 2.2.) *Let γ, ζ, z, v be as above.*
(i) *If v is in admissible position relative to ζ and $f(z) \in V_k \setminus V_{k+1}$, then:*
 (a) $\log|\zeta - z|^{-\frac{3}{\log C_0}} \leq p \leq \log|\zeta - z|^{-\frac{3}{\lambda}}$;
 (b) $q \leq C\tilde{\alpha}p$;
 (c) $|f^i(\zeta) - f^i(z)| \leq e^{-2\alpha p}$ *for* $1 \leq i \leq p$;
 (d) $|\zeta - z|\|v\| \leq \|Df^q(z)v\| \leq |\zeta - z|^{1-\tilde{\alpha}}\|v\|$;
 (e) $\|Df^p(z)v\| \geq \|v\| \cdot |\zeta - z|^{-1+\frac{\alpha}{\log C_0}} \geq e^{\frac{\lambda p}{3}}\|v\|$;
 (f) $\|Df^p(z)v\| \geq (\delta/10)\|Df^i(z)v\|$ *for* $0 \leq i < p$;

(ii) *If v is in critical position relative to ζ, then* $\|Df^n(z)v\| \leq e^{-8\lambda n}\|v\|$.

A proof of this proposition follows the line that is now well understood [5, 18, 30]. We split $Df(z)v$ into the direction of $\binom{1}{0}$ and that of $e_n(f(z))$, iterate them separately, and put them together at the expiration of the fold period.

Proposition 2.1 indicates the usefulness of the following terminology. Let ζ be a critical approximation on a horizontal curve γ. Let $t(\zeta)$ denote any unit vector tangent to γ at ζ. A nonzero vector v is in *tangential position relative to* ζ if there exists a horizontal curve which is tangent to both v and $t(\zeta)$.

3. Existence of binding points

To deal with returns to the region $I(\delta)$, we look for suitable critical approximations and use them as guides to keep further evolution in track. Such critical approximations, if exists, are called *binding points*. In this section we recall the procedure in [27] for finding binding points.

3.1. **Creation of new critical approximations.** By a $C^2(b)$-*curve* we mean a C^2-curve such that the slopes of all its tangent vectors are $\leq \sqrt{b}$ and the curvature is everywhere $\leq \sqrt{b}$. The next two lemmas are used to create new critical approximations around the existing ones. For corresponding versions, see: [5] p.113, Lemma 6.1; [18] Sect.7A, 7B; [30] Lemma 2.10, 2.11.

Lemma 3.1. *Let γ be a $C^2(b)$-curve in $I(\delta)$ parameterized by arc length and such that $\gamma(0)$ is a critical approximation of order n. Suppose that:*
(i) *$\gamma(s)$ is defined for $s \in [-b^{\frac{n}{4}}, b^{\frac{n}{4}}]$;*
(ii) *there exists $m \in [n/3, 20n]$ such that $\|Df^i(f\gamma(0))\| \geq 1$ for $1 \leq i \leq m$.*
There exists $s_0 \in [-b^{\frac{n}{4}}, b^{\frac{n}{4}}]$ such that $\gamma(s_0)$ is a critical approximation of order m on γ.

Next we consider two $C^2(b)$-curves γ_1, γ_2 in $I(\delta)$ parametrized by arc length, in a way that the x-coordinate of $\gamma_1(0)$ coincides with that of $\gamma_2(0)$. Let $t_\sigma(s)$ denote any unit vector tangent to γ_σ at $\gamma_\sigma(s)$, $\sigma = 1, 2$.

Lemma 3.2. *Let γ_1, γ_2 be as above and suppose that:*
(i) *$\gamma_1(s)$, $\gamma_2(s)$ are defined for $s \in [-\varepsilon^{\frac{n}{2}}, \varepsilon^{\frac{n}{2}}]$, $\varepsilon \leq C_0^{-5}$;*
(ii) *$\gamma_1(0)$ is a critical approximation of order n on γ_1 and $\|Df^i(f\gamma_1(0))\| \geq 1$ for $1 \leq i \leq n$;*
(iii) *$|\gamma_1(0) - \gamma_2(0)| \leq \varepsilon^n$ and $\mathrm{angle}(t_1(0), t_2(0)) \leq \varepsilon^n$.*
There exists $s_0 \in [-\varepsilon^{\frac{n}{2}}, \varepsilon^{\frac{n}{2}}]$ such that $\gamma_2(s_0)$ is a critical approximation of order n on γ_2.

3.2. **Hyperbolic times.** Let

$$\theta = \alpha^3, \quad \kappa_0 = C_0^{-10}. \tag{7}$$

Let v be a tangent vector at z and let $m \geq 1$. We say v is r-*regular up to time* m if for $0 \leq i < m$,

$$\|Df^m(z)v\| \geq r\delta\|Df^i(z)v\|.$$

We say $\mu \in [0, m]$ is an m-*hyperbolic time* of v if $Df^\mu(z)v$ is $\kappa_0^{\frac{1}{2}}$-expanding up tp time $m - \mu$. Results related to the next lemma can be found in [[5] Lemma 6.6], [[18] Lemma 9.1], [[30] Claim 5.1].

Lemma 3.3. ([27] Lemma 2.12; Abundance of well-distributed hyperbolic times) *Let $m \geq \log(1/\delta)$ and suppose that a tangent vector v at z is $1/10$-regular up to time m. There exist $s \geq 2$ and a sequence $\mu_1 < \mu_2 < \cdots < \mu_s$ of m-hyperbolic times of v such that:*

(a) $\|Df^{\mu_j}(z)v\|$ *is* $\kappa_0^{\frac{1}{4}}$*-expanding up to time* $m - \mu_j$*;*
(b) $1/16 \leq (m - \mu_{j+1})/(m - \mu_j) \leq 1/4$ *for* $1 \leq j \leq s - 1$*;*
(c) $0 \leq \mu_1 < m/2$ *and* $m - \log(1/\delta) \leq \mu_s \leq m - \log(1/\delta)/2$.

3.3. **Nice critical approximations.** Let ζ be a critical approximation of order n on a horizontal curve γ. We say ζ is *nice* if:
(C1) $\|Df^i(f\zeta)\| \geq 1$ for $1 \leq i \leq n$;
(C2) $f^{-i}(\zeta) \in [-2, 2] \times [-\sqrt{b}, \sqrt{b}]$ for $1 \leq i \leq [\theta n]$;
(C3) let $t(\zeta)$ denote any unit vector tangent to γ at ζ. Then $Df^{-[\theta n]}(\zeta)t(\zeta)$ is $\kappa_0^{\frac{1}{3}}$-expanding and $\delta/10$-regular, both up to time $[\theta n]$.

Here, the square bracket denotes the integer part.

Hypothesis for the rest of Sect.3: m, n are integers with $m \geq \log(1/\delta)$, $n \geq \log(1/\delta)$, and:

• each nice critical approximation ζ has a good critical behavior up to time $20 \min(n, \xi)$, where ξ is the order of ζ;
• a tangent vector v at z is r_0-regular up to time m, and $f^m(z) \in I(\delta)$.

3.4. **Binding procedure.** We describe how to choose a binding point relative to which $Df^m(z)v$ is in horizontal position. Fix once and for all a sequence $\mu_1 < \mu_2 < \cdots < \mu_s$ of m-hyperbolic times of v satisfying

$$m - \mu_1 \leq \theta n, \quad \frac{1}{2}\log(1/\delta) \leq m - \mu_s \leq \log(1/\delta), \quad \frac{1}{16} \leq \frac{m - \mu_{i+1}}{m - \mu_i} \text{ for } 1 \leq i < s. \tag{8}$$

The existence of such a sequence is guaranteed by Lemma 3.3. Any sequence does the job. Correspondingly, fix once and for all a sequence $n \geq n_1 > \cdots > n_s > n_{s+1} > \cdots > n_{s_0} := M$ of integers such that

$$m - \mu_i = [\theta n_i] \text{ for } 1 \leq i \leq s, \quad n_i = n_{i+1} + 1 \quad \text{for } s \leq i < s_0. \tag{9}$$

Lemma 3.4. ([27] Proposition 3.1.) *There exist $i \in [1, s]$ and a critical approximation ζ_i of order n_i such that $Df^m(z)v$ is in tangential position relative to ζ_i.*

Sketch of the proof. One way to find such n_i and ζ_i are described as follows. Let l_i denote the straight segment of length $\kappa_0^{3\theta n_i}$ centered at $f^{\mu_i}(z)$ and tangent to $Df^{\mu_i}(z)v$. Then $\gamma_i := f^{\mu_i}(l_i)$ is a $C^2(b)$-curve extending to both sides around $f^m(z)$ to length $\geq \kappa_0^{4\theta n_i}$. Lemma 3.1, Lemma 3.2 and the hypothesis of f allow us to show the following: if $Df^m(z)v$ is in critical position relative to a critical approximation of order n_i on γ_i, then there exists a critical approximation of order n_{i-1} on γ_{i-1} relative to which $Df^m(z)v$ is in horizontal position. A recursive use of this argument yields the conclusion. □

Definition 3.1. Let $i_0 \in [1, s]$ denote the largest integer such that there exists a critical approximation of order n_{i_0} relative to which $Df^m(z)v$ is in tangential position. We call any such critical approximation a *binding point for* $Df^m(z)v$.

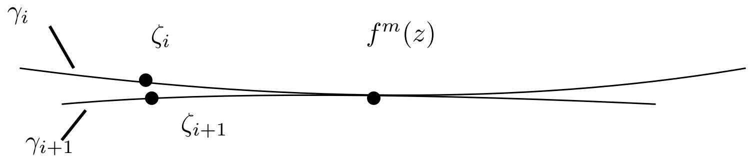

FIGURE 5. critical approximations on $C^2(b)$-curves

Remark 3.1. Obviously, critical approximations eligible as binding points of $Df^m(z)v$ are not unique. This does not matter.

Let ζ denote any binding point for $Df^m(z)v$. By the definitions in Sect.2.6, there are two mutually exclusive cases:

(a) $i_0 = 1$, and $f^m(z)$ is in critical position relative to ζ;

(b) $Df^m(z)(v)$ is in admissible position relative to ζ.

Remark 3.2. Let ζ, ζ' denote two different binding points for $Df^m(z)v$. If (a) occurs for ζ, then (a) has to occur for ζ' as well, for otherwise one can find a horizontal curve on which ζ and ζ' lie, a contradiction.

In case (a), the contraction estimate (ii) in Proposition 2.1 is in place. In case (b), all the estimates in (i) in Proposition 2.1 are in place: the loss of expansion and regularity suffered from the return are recovered at the end of the bound period.

In addition, in case (b), one can repeat the binding procedure in the following manner. Write $m = m_1$. Let p_1 denote the bound period. (e,f) Proposition 2.1 implies that v is 1/10-regular up to time $m_1 + p_1$. Let $m_2 \geq m_1 + p_1$ denote the smallest such that $f^{m_2}(z) \in I(\delta)$. By Lemma 2.1, v is 1/10-regular up to time m_2. Subsequently one may repeat the binding procedure once again, replacing $m \to m_2$, $f^m(z) \to f^{m_2}(z)$, $Df^m(z)v \to Df^{m_2}(z)v$.

In this way, one can (if (a) does not occur) define integers

$$m_1 < m_1 + p_1 \leq m_2 < m_2 + p_2 \leq m_3 < \cdots$$

inductively as follows: for $k \geq 1$, let p_k be the bound period of $f^{m_k}(z)$, and let m_{k+1} be the smallest $j \geq m_k + p_k$ such that $f^j(z) \in I(\delta)$. (Note that an orbit may return to $I(\delta)$ during its bound periods, i.e. (m_k) are not the only return times to $I(\delta)$.) This decomposes the orbit of z into segments corresponding to time intervals $(m_k, m_k + p_k)$ and $[m_k + p_k, m_{k+1}]$, during which we describe the orbit of z as being"bound" and "free" states respectively; m_k are called times of *free returns*.

Remark 3.3. Let us consider the case where the above hypothesis is satisfied for every $n \geq \log(1/\delta)$. Then, the binding procedure allows us to keep in track the evolution of any individual complete orbit in W^u, decomposing it into bound and free segments. However, this procedure is not well-adapted to our phase-space construction in later sections, because:

- the choice of binding points relies only on the individual orbit under consideration and neglects a global information on W^u;
- there are ambiguities in the choice of binding points.

These issues will be resolved in Section 5, for parameters in Δ.

4. Parameter exclusion

In this section we construct the parameter set Δ in the theorem, having a^* as a full density point. The construction is done by induction: $\Delta = \bigcap_{n\geq 0} \Delta_n$, where Δ_n is constructed at step n, excluding from Δ_{n-1} all those undesirable parameters for which some critical approximation may not have good critical behavior up to time $20n$.

4.1. **Critical approximations of f_{a^*} are non-recurrent.**

The construction of Δ and a measure estimate of it closely follow [27], in which a positive measure set of parameters was constructed corresponding to Hénon-like maps with nonuniformly hyperbolic behavior. One key difference from [27] is the assertion that a^* is a full density point of Δ. A key ingredient for this is the next proposition, which states that the orbit of every critical approximation of f_{a^*} is non-recurrent.

Proposition 4.1. *For every critical approximation ζ of f_{a^*} of order n, $f_{a^*}^i(\zeta) \in \{(x,y) \in \mathbb{R}^2 \colon |x| \geq 9/10\}$ holds for every $1 \leq i < 20n$.*

We postpone a proof of this proposition to Sect.4.10.

4.2. **Definition of parameter sets.**

Let

$$N = \left[\theta^{-1}\log(1/\delta)\right].$$

Choose sufficiently small ε_0 and b so that for any $f \in \{f_a \colon a \in [a^* - \varepsilon_0, a^*]\}$, any critical approximation has a good critical behavior up to time $20N$. This requirement is feasible by the elementary fact that all critical approximations are contained in $I(\sqrt{b})$. Set $\Delta_n = [a^* - \varepsilon_0, a^*]$ for $1 \leq n \leq N$.

Definition 4.1. Let $n > N$, $a \in \Delta_{n-1}$ and suppose that f_a has a good critical behavior up to time $20(n-1)$. Let $20(n-1) \leq m < 20n$. We say a nice critical approximation ζ of f_a of order $\geq n$ satisfies $(G)_m$ if:

(i) there is an well-defined decomposition of the orbit $f(\zeta), f^2(\zeta), \cdots, f^m(\zeta)$ into bound and free segments in the sense of Sect.3.4;

(ii) let $n_1 < n_2 < \cdots < n_s \leq m$ denote all the free return times of ζ, with $z_1, \cdots, z_s$ the corresponding binding points. They are of order $< n$ and

$$\sum_{i=1}^{s} \log|f^{n_i}(\zeta) - z_i| \geq -\alpha m. \tag{10}$$

For $n > N$, define Δ_n to be the set of all $a \in \Delta_{n-1}$ for which every nice critical approximation of order $\geq n$ satisfies $(G)_{20n-1}$. In other words,

$$\Delta_{n-1} \setminus \Delta_n = \left\{ \begin{array}{c} a \in \Delta_{n-1} \colon\ (G)_m \text{ fails for some } 20(n-1) \leq m < 20n \\ \text{and some nice critical approximation of order} \geq n \text{ of } f_a \end{array} \right\}.$$

The next proposition allows us to proceed to the definition of Δ_{n+1} with no vicious cycle.

Proposition 4.2. ([27] Proposition 5.1.) *Let $a \in \Delta_n$ and let ζ be a nice critical approximation of order $\geq n$ of f_a. Then ζ has a good critical behavior up to time $20n$.*

Sketch of the proof. We just sketch the proofs of (G1), (G2). By the definition of Δ_n, ζ satisfies $(G)_{20n-1}$. The length of the total bound states in the first $20n$ iterates of ζ is bounded by

(a) Proposition 2.1 and Condition (10). It follows that the total length of free states is proportional to $20n$. Lemma 2.1 gives (G1).

The function χ in $(G3)$ is defined as follows. Let $j \in [M, 20n]$ and $h_0 := j$. Define a finite sequence $h_1 > \cdots > h_{t(j)}$ of free return times of ζ inductively as follows. Let $\hat{h}_{k+1}$ denote the largest free return time before h_k, when it makes sense. Let p_{k+1} denote the corresponding bound period. If

$$h_k - \hat{h}_{k+1} - p_{k+1} \leq (1/\lambda_0) \log(10\delta), \tag{11}$$

then let $h_{k+1} = \hat{h}_{k+1}$. In all other cases, h_{k+1} is undefined, namely $k = t(j)$. Define $\chi(j) = h_{t(j)}$. Obviously, $\chi(j) \leq j$ holds. If $(1 - \sqrt{\alpha})j \leq \chi(j)$ did not hold, (11) would imply that the total number of bound iterates in the interval $[(1 - \sqrt{\alpha})j, j]$ were bigger than a constant multiple of $\sqrt{\alpha}j$. While by condition (G), the total number of bound states in the interval is smaller than a constant multiple of αj. If α is small, then these two estimates are not compatible. $\square$

4.3. **Combinatorics.** To estimate the measure of $\Delta_{n-1} \backslash \Delta_n$, we first decompose it into a finite number of subsets, based on certain combinatorics on itineraries of critical approximations. We then estimate the measure of each subset separately, and unify them at the end. In this subsection we introduce two integral components of the combinatorics.

Definition 4.2. Let $f \in \{f_a : a \in \Delta_{n-1} \setminus \Delta_n\}$. Let ζ be a critical approximation of f of order $\geq n$. Let $\nu < 20n$ be a free return time of ζ, with the binding point z. If ν is not the first return time to $I(\delta)$, then let $n_1 < \cdots < n_t$ denote all the free return times of ζ before ν, with $z_1, \cdots, z_t$ the corresponding binding points. Write $n_{t+1} = \nu$ and $z_{t+1} = z$. We say ν is a *deep return time*, if it is the first return time to $I(\delta)$, or else for $1 \leq s \leq t$,

$$\sum_{j=s+1}^{t+1} 2 \log |f^{n_j}(\zeta) - z_j| \leq \log |f^{n_s}(\zeta) - z_s|. \tag{12}$$

Remark 4.1. If ν is not a deep return time, by definition, the reverse inequality holds for some $i \in [1, t]$. This means that the effect of the free return at time ν is negligible, compared to that of the free return at time n_i

Let f, ζ be as above and ν a deep return time. Let p_s, q_s $(1 \leq s \leq t)$ denote the corresponding bound and fold periods. For each n_s, let

$$\sigma_{n_s}(\zeta) = \frac{\|w_{n_s+q_s}(\zeta)\|^\beta}{\|w_{n_s}(\zeta)\|^{1+\beta}},$$

where $\beta = \frac{10}{9}$. For each $i \in [1, \nu) \setminus \bigcup_{1 \leq s \leq t} [n_s, n_s + p_s - 1]$, let

$$\sigma_i(\zeta) = \frac{\|w_{i+1}(\zeta)\|}{\|w_i(\zeta)\|^2}.$$

Define

$$\Theta_\nu(\zeta) = \kappa_0 \cdot \left[\sum_{i=1}^{\nu-1} \sigma_i(\zeta)^{-1} \right]^{-1}.$$

It is understood that the sum runs over all i such that $f^i(\zeta)$ is free.

Lemma 4.1. ([27] Lemma 5.2.) *For the above f,ζ,ν,z, if ν is a deep return time of ζ, then*

$$\|w_\nu(\zeta)\||\Theta_\nu(\zeta)|\geq |f^\nu(\zeta)-z|^{\frac{1}{2}}.$$

Definition 4.3. (Addresses on the horizontal H) For each $\mu\geq\theta M$, we fix a subdivision of $\mathbb{R}\times\{\sqrt{b}\}$ into right-open horizontals of equal length $\kappa_0^{3\mu}$. We label all of them intersecting the horizontal $H:=[-2,2]\times\{\sqrt{b}\}$ with $l=1,2,3,\cdots$, from the left to the right. By a *μ-address* of a point x on H we mean the integer l which is a label of the horizontal containing x.

4.4. **Decomposition of the exclude parameter set at step n.** Fix positive integers $m\in[20(n-1),20n)$, s, t, R. Fix the following combinatorics:
- sequences $(\mu_1,\cdots,\mu_s),(x_1,\cdots,x_s)$ of s positive integers;
- sequences $(\nu_1,\cdots,\nu_t)$, $(n_1,\cdots,n_t)$, $(r_1,\cdots,r_t)$, $(y_1,\cdots,y_t)$ of t positive integers.

Let $E_n(*)=E_{m,s,t,R}(\cdots)$ denote the set of all $a\in\Delta_{n-1}\setminus\Delta_n$ for which there exists a nice critical approximation ζ of $f_a=f$ of order $n'\geq n$ such that the following holds:
(Z1) $(G)_{m-1}$ holds, and $(G)_m$ fails;
(Z2) $\mu_1<\cdots<\mu_s$ is a maximal sequence of $[\theta n']$-hyperbolic times of the tangent vector $Df^{-[\theta n']-1}(f(\zeta))e_{n'}(f(\zeta))$ satisfying

$$\frac{1}{2}\log(1/\delta)\leq n'-\mu_s\leq\log(1/\delta),\quad n'-\mu_1\leq\theta n,\quad \frac{1}{16}\leq\frac{n'-\mu_{i+1}}{n'-\mu_i}\leq\frac{1}{4}\text{ for }1\leq i<s. \tag{13}$$

Since $[\theta n']\geq\log(1/\delta)$, Lemma 3.3 ensures the existence of such a maximal sequence;
(Z3) the point of intersection between H and the long stable leaf of order $[\theta n']-\mu_i$ through $f^{-[\theta n']+\mu_i}(\zeta)$ has x_i as its $[\theta n']-\mu_i$-address;
(Z4) $\nu_1<\cdots<\nu_t=m$ are all the free return times in the first m iterates of ζ, with $\zeta_1,\cdots\zeta_t$ the corresponding binding points;
(Z5) for each $k\in[1,t]$, the order of ζ_k is $n_k<n$. If $\nu_k<m$, then r_k is such that $|\zeta_k-f^{\nu_k}(\zeta)|\in[e^{-r_k},e^{-r_k+1})$ holds. If $\nu_k=m$, which means $k=t$ and $\nu_t=m$, then r_t is defined as follows. If $|f^m(\zeta)-z_t|>e^{-\alpha m}$, then r_t is such that $|f^m(\zeta)-z_t|\in[e^{-r_t},e^{-r_t+1})$ holds. Otherwise, $r_t=\alpha m$;
(Z6) the point of intersection between H and the long stable leaf of order $[\theta n_k]$ through $f^{-[\theta n_k]}(\zeta_k)$ has y_k as its $[\theta n_k]$-address.

Definition 4.4. If $a\in E_n(*)$, then any nice critical approximation of f_a of order $\geq n$ for which (Z1-6) hold is called *responsible* for a. The parameter set $E_n(*)$ an *n-class*.

By definition, any parameter excluded from Δ_{n-1} belongs to some n-class. We estimate the measure of $\Delta_{n-1}\setminus\Delta_n$ by estimating a contribution from each n-class first, and then counting the total number of n-classes.

4.5. **Digestive remarks on the combinatorics.** Let us give some remarks on the meanings of the hypotheses in the definition of $E_n(*)$. (Z1) is a completely reasonable hypothesis. (Z4), (Z5) are also reasonable. The remaining three hypotheses will be used to deal with two problems intrinsic to two-dimension.

• *Infinitely many responsible critical approximations.* The first problem is that critical approximations responsible for a single parameter are far from unique, and even infinite. Of course, all of them have to be taken into consideration in the measure estimate of $E_n(*)$. Conditions

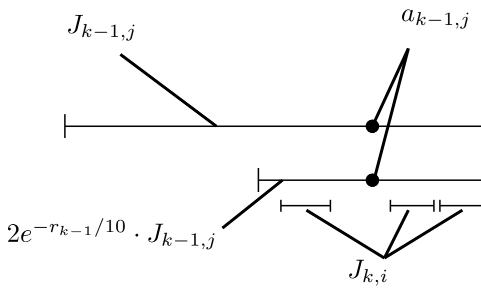

FIGURE 6. Organization of $J_{k,i}$-intervals

(Z2), (Z3) are used to deal with this problem. They allow us to reduce our consideration to a finite number of parameter-dependent orbits, called *deformations*, introduced in Sect.4.7.

• *Infinitely many binding points.* Binding points are far from unique, because of the way of our definition of binding points in Sect.3.4. (Z6) allows us to deal with this problem, with the help of deformations.

4.6. **Full Lebesgue density at the first bifurcation parameter.** We conclude that Δ has a^* as a full Lebesgue density point. Let $|\cdot|$ denote the one-dimensional Lebesgue measure. For a compact interval I centered at x and $r > 0$, let $r \cdot I$ denote the interval of length $r|I|$ centered at x. One main step is a proof of the next

Proposition 4.3. (Covering by intervals) *Let $m \in [20(n-1), 20n)$, s, t, R be positive integers. For any n-class $E_n(m, s, t, R, \cdots,) = E_n(*)$, for any $\varepsilon \in (0, \varepsilon_0)$, $k \in [1, t]$, there exist a finite number of pairwise disjoint intervals $\{J_{k,i}\}_i$ with the following properties:*
(a) $E_n(*) \cap [a^* - \varepsilon, a^*] \subset \bigcup_i e^{-r_k/3} \cdot J_{k,i}$;
(b) *if $t > 1$, then for each $k \in [2, t]$ and $J_{k,i}$ there exists $J_{k-1,j}$ such that $J_{k,i} \subset 2e^{-r_{k-1}/3} \cdot J_{k-1,j}$;*
(c) $\sum_i |J_{1,i}| \leq 3\varepsilon$.

This sort of covering originates in the works Tsujii [28, 29], and has been used in [27] for the construction of positive measure set of parameters corresponding to maps with nonuniformly hyperbolic behavior. For our purpose we need to develop it further.

Proposition 4.3 gives $|E_n(*) \cap [a^* - \varepsilon, a^*]| \leq 3\varepsilon e^{-\frac{1}{3}R}$, where $R = r_1 + r_2 \cdots + r_t$. To conclude, we need to count the number of all feasible n-classes. The counting argument in [27] shows

$$\sharp((\mu_1, x_1), \cdots, (\mu_s, x_s)) \leq \kappa_0^{-7\theta n}$$

and

$$\sharp(\nu_1, \cdots, \nu_t)\sharp(r_1, \cdots, r_t)\sharp(n_1, \cdots, n_t)\sharp(y_1, \cdots, y_t) \leq e^{\tau(\delta)n + \frac{\theta R}{5\lambda}},$$

where $\tau(\delta) \to 0$ as $\delta \to 0$. [[27] Lemma 5.3] gives $r_1 + \cdots + r_t \geq \alpha m/2$. Taking contributions from all n-classes into consideration,

$$\begin{aligned}|(\Delta_{n-1} \setminus \Delta_n) \cap [a^* - \varepsilon, a^*]| &\leq \varepsilon \sum_{m,s,t} \sum_{R \geq \alpha m/2} \sum_{r_1 + \cdots + r_t = R} |E_n(m, s, t, R, *) \cap [a^* - \varepsilon, a^*]| \\ &\leq \varepsilon e^{\tau(\delta)n} \sum_{R \geq \alpha n} \exp\left(-\frac{R}{6}\right) \leq \varepsilon e^{-\alpha n/8}.\end{aligned}$$

Let

$$n_0(\varepsilon) = \frac{1}{2\log C_0}\log\left(\frac{2\varepsilon}{\kappa_0\delta}\right). \tag{14}$$

The next lemma indicates that no parameter is deleted from $[a^*-\varepsilon,a^*]$ up to step $[n_0(\varepsilon)/20]$, namely $[a^*-\varepsilon,a^*]\subset\Delta_n$ holds for every $0\leq n\leq[n_0(\varepsilon)/20]$.

Lemma 4.2. *Let $a_0\in[a^*-\varepsilon,a^*]$, and let ζ_0 be a critical approximation of f_{a_0}. Then $f_{a_0}^n(\zeta_0)\notin I(\delta)$ holds for every $1\leq n<\min(n_0(\varepsilon),20\xi)$, where ξ is the order of ζ_0.*

Therefore

$$\begin{aligned}|\Delta\cap[a^*-\varepsilon,a^*]| &= |\Delta_0\cap[a^*-\varepsilon,a^*]|-\sum_{n=1}^{\infty}|\,(\Delta_{n-1}\setminus\Delta_n)\cap[a^*-\varepsilon,a^*]|\\ &=\varepsilon-\sum_{n>[n_0(\varepsilon)/20]}|\,(\Delta_{n-1}\setminus\Delta_n)\cap[a^*-\varepsilon,a^*]|\geq\varepsilon\left(1-\sum_{n>[n_0(\varepsilon)/20]}e^{-\alpha n}\right).\end{aligned}$$

Since $n_0(\varepsilon)\to\infty$ as $\varepsilon\to0$, we obtain $\lim_{\varepsilon\to0}\varepsilon^{-1}|\Delta\cap[a^*-\varepsilon,a^*]|=1$ as desired. □

4.7. **Parameter dependence of nice critical approximations.** The rest of this section is entirely devoted to the proof of Proposition 4.3 and Lemma 4.2. A key ingredient is a *deformation of a quasi critical approximation*, developed in [[27] Section 4,5] for dealing with the parameter dependence of critical approximations.

Definition 4.5. Let ζ be a critical approximation of order n on a $C^2(b)$-curve γ. We say ζ is a *quasi critical approximation of order n on γ* if $Df^{-[\theta n]}(\zeta)t(\zeta)$ is $\kappa_0^{\frac12}$-expanding up to time $[\theta n]$, where $t(\zeta)$ denotes any unit vector tangent to γ at ζ.

For the purpose of introducing deformations, we make the following assumption. Let $\hat a\in[a^*-\varepsilon_0,a^*]$. Write f for $f_{\hat a}$. Let γ be a $C^2(b)$-curve in $I(\delta)$. Let ζ be a quasi critical approximation of order n on γ. Assume:

- $\|Df^i(f(\zeta))\|\geq1$ for $1\leq i\leq n$;
- $Df^{-[\theta n]}(\zeta)t(\zeta)$ is $\kappa_0^{\frac13}$-expanding and $\delta/160$-regular, both up to time $[\theta n]$.

Let r denote the point of intersection between H and the long stable leaf of order $[\theta n]$ through ξ. Let $l\subset H$ denote the horizontal of length $2\kappa_0^{3\theta n}$ centered at r. By [[27] Lemma 4.1], $f_{\hat a}^{[\theta n]}(l)$ is a $C^2(b)$-curve, and there exists a quasi critical approximation of order n on it, denoted by $\zeta(\hat a)$ for which $|\zeta-\zeta(\hat a)|\leq(Cb)^{\frac{\theta n}{4}}$ holds.

It turns out that this picture persists, for a small variation of parameters. Let

$$I_n(\hat a)=[\hat a-\kappa_0^n,\hat a+\kappa_0^n]. \tag{15}$$

By [[27] Lemma 4.2], for all $a\in I_n(\hat a)$, $f_a^{[\theta n]}(l)$ is a $C^2(b)$-curve. By [[27] Proposition 4.1], there exists a quasi critical approximation of order n of f_a on it, which we denote by $\zeta(a)$.

Definition 4.6. The map $a\in I_n(\hat a)\to\zeta(a)$ is called a *deformation* of ζ.

The next lemma states that the "speeds" of deformations are uniformly bounded. Let " $\cdot$ " denote the differentiation with respect to a.

Lemma 4.3. ([27] Proposition 4.2.) *The $a \in I_n(\hat{a}) \to \zeta(a)$ is C^3 and for all $a \in I_n(\hat{a})$,*

$$\max\left(\|\dot{\zeta}(a)\|, \|\ddot{\zeta}(a)\|\right) \leq \kappa_0^{3\log\delta}.$$

To introduce a main proposition we need some notation. Let ζ be a nice critical approximation of $f_{\hat{a}}$ of order n. For the deformation $a \in I_n(\hat{a}) \to \zeta(a)$ of ζ and $i > 0$, let $f_a^i(\zeta(a)) = \zeta_i(a)$. If $\nu < 20n$, $f^\nu(\zeta)$ is free and ζ has good critical behavior up to time ν, then define

$$J_\nu(a, \zeta) = [a - \Theta_\nu(\zeta), a + \Theta_\nu(\zeta)].$$

Proposition 4.4. [[27] Section 5] *Let $a_0 \in \Delta_{n-1}$ and let ζ_0 be a nice critical approximation of f_{a_0}, with a good critical behavior time ν and $f^\nu(\zeta_0)$ is free. There exist an integer m and a quasi critical approximation ζ of order m such that:*
(i) $|f_{a_0}^\nu(\zeta_0) - f_{a_0}^\nu(\zeta)| \leq (Cb)^{\frac{1}{2}\theta\nu}$;
(ii) *the deformation $a \in I_m(a_0) \to \zeta(a)$ satisfies:*
- (a) $J_\nu(\zeta_0, a_0) \subset I_m(a_0)$*:*
- (b) *the set $\{\zeta_\nu(a)\colon a \in J_\nu(a_0, \zeta_0)\}$ is a horizontal curve;*
- (c) $\|\zeta_\nu(a) - \zeta_\nu(b)\| \approx \|w_\nu(\zeta_0)\||a - b| \ll 1$ *for all $a, b \in J_\nu(\zeta_0, a_0)$.*

4.8. **Proof of Proposition 4.3.** We choose each $J_{k,i}$ so that it has the form $J_{k,i} = J_{\nu_k}(a_{k,i}, \zeta_{k,i})$, where $a_{k,i} \in E_n(*) \cap [a^* - \varepsilon, a^*]$ and $\zeta_{k,i}$ is some responsible critical approximation of $f_{a_{k,i}}$. In what follows we describe how to choose $(a_{k,i}, \zeta_{k,i})_i$.

Start with $k = 1$. We describe how to choose $(a_{1,i}, \zeta_{1,i})_i$ such that (a) holds with $k = 1$. First, choose arbitrary $a_{1,1} \in E_n(*) \cap [a^* - \varepsilon, a^*]$. Let $\zeta_{1,1}$ denote any responsible critical approximation of $f_{a_{1,1}}$. We show

$$E_n(*) \cap (J_{1,1} \setminus e^{-r_1/3} J_{1,1}) = \emptyset. \tag{16}$$

If $J_{1,1}$ covers $E_n(*)$, then the desired inclusion follows. Otherwise, choose $a_{1,2} \in E_n(*) - J_{1,1}$. We claim that

$$J_{1,1} \cap J_{1,2} = \emptyset. \tag{17}$$

If $J_{1,1} \cap J_{1,2}$ covers $E_n(*)$, then the desired inclusion follows. Otherwise, choose $a_{1,3} \in E_n(*) - J_{1,1} \cup J_{1,2}$. Repeat this. As the length of these intervals are uniformly bounded from below, there must come a point when the inclusion is fulfilled.

Below we sketch the proofs of (16) and (17). To ease notation, write $a_i := a_{1,i}$, $\zeta_i := \zeta_{1,i}$ and $J_i = J_{1,i}$, $i = 1, 2$.

Sketch of the proof of (16). Choose an inter m, a quasi critical approximation ζ of f_{a_1} of order m, and its deformation $a \in I_m(a_1) \to \zeta(a)$ for which the conclusions of Proposition 4.4 hold up to time ν_1. In fact, (Z2), (Z3) allow us to choose such a deformation so that the following holds:
- $|f_{a_1}^{\nu_1}(\zeta_1) - f_{a_1}^{\nu_1}(\zeta(a_1))| \leq e^{-r_1}$;
- if $a \in J_1 \cap E_n(*)$ and x is *any* responsible critical approximation of f_a, then $|f_a^{\nu_1}(x) - f_a^{\nu_1}(\zeta(a))| \ll e^{-r_1}$.

The second item states that, although responsible critical approximations for a single parameter a are not unique, all of their positions at time ν_k are well-approximated by that of $f_a^{\nu_1}(\zeta(a))$.

Now, let z_1 denote the binding point of order n_1 for $f_{a_1}^{\nu_1}(\zeta_1)$ and let $a \in I_{n_1}(a_1) \to z_1(a)$ denote its deformation. (Z6) allows us to show that this deformation satisfies:
• $|z_1 - z_1(a_1)| \leq e^{-r_1}$;
• if $a \in J_1 \cap E_n(*)$ and x is *any* responsible critical approximation of f_a, with y a binding point for $f_a^{\nu_1}(x)$, then $|y - z_1(a)| \ll e^{-r_1}$.
The second item states that, although binding points are not unique, they are well approximated by $z_1(a)$.

These four conditions altogether prohibit any parameter in $a \in J_{1,1} \setminus e^{-r_1/3} \cdot J_{1,1}$ from belonging to $E_n(*)$, and consequently (16) holds. To see this, suppose that this is not the case and $a \in J_{1,1} \setminus e^{-r_1/3} \cdot J_{1,1}$, $a \in E_n(*)$. Let x denote *any* critical approximation responsible for a. Let y denote *any* binding point for $f_a^{\nu_1}(x)$. The triangle inequality gives

$$\begin{aligned}|f_a^{\nu_1}(x) - y| \geq &|f_a^{\nu_1}(\zeta(a)) - f_{a_1}^{\nu_1}(\zeta(a_1))| - |f_a^{\nu_1}(\zeta(a)) - f_a^{\nu_1}(x)| - |f_{a_1}^{\nu_1}(\zeta(a_1)) - f_{a_1}^{\nu_1}(\zeta_1)| \\ &- |f_{a_1}^{\nu_1}(\zeta_1) - z_1| - |z_1 - z_1(a)| - |z_1(a) - y|,\end{aligned}$$

where, for the last term, $z_1(a)$ makes sense, because of $J_1 \subset I_{n_1}(a_1)$. On the first term, Proposition 4.4 and Lemma 4.1 give

$$|f_{a_1}^{\nu_1}(\zeta(a_1)) - f_a^{\nu_1}(\zeta(a))| \approx \|w_{\nu_1}(\zeta_1)\| \cdot |a_1 - a| \gg e^{-r_1}.$$

The remaining four terms are $\leq e^{-r_1}$. It follows that $|f_a^{\nu_1}(x) - y| \gg e^{-r_1}$. This yields a contradiction to the assumption that x is responsible for a. Hence $a \notin E_n(*)$ holds.

Sketch of the proof of (17). For the discussions to follow, we need to introduce *critical parameters* [27]. For the purpose of this we make the following assumption and observation. Let $\hat{a} \in E_n(*)$ and let $\hat{\zeta}$ denote any critical approximation responsible for a. Let z denote any binding point for $f_{\hat{a}}^{\nu_k}(\hat{\zeta})$, and let $a \in I_{n_k}(\hat{a}) \to z(a)$ denote its deformation. Take an integer m, a quasi critical approximation ζ of $f_{\hat{a}}$ of order m, and its deformation $a \in I_m(\hat{a}) \to \zeta(a)$ for which the conclusions of Proposition 4.4 hold up to time ν_k. The "speed" of $z(a)$ as a sweeps in the interval $I_{n_k}(\hat{a})$ is bounded from above by in Lemma 4.3. On the other hand, the "speed" of $\zeta_{\nu_k}(a)$ as a sweeps in the interval $J_{\nu_k}(\hat{a}, \hat{\zeta})$ is much faster, by Proposition 4.4. From the proposition, $J_{\nu_k}(\hat{a}, \hat{\zeta}) \subset I_{n_k}(\hat{a})$ holds. Hence, the comparison of the speeds and Lemma 4.1 together imply that there exists a unique parameter $c_0 \in e^{-r_k/3} \cdot J_{\nu_k}(\hat{a}, \hat{\zeta})$ such that the x-coordinate of $\zeta_{\nu_k}(c_0)$ coincides with that of $z(c_0)$.

Definition 4.7. The c_0 is called a *critical parameter in* $J_{\nu_1}(\hat{a}, \hat{\zeta})$.

A proof of (17) is outlined as follows. Let c_0, c_0' denote the critical parameters in $J_{1,1}$, $J_{1,2}$ respectively. Suppose that (17) does not hold. Then, from a distortion argument, $|J_{1,1}| \approx |J_{1,2}|$ follows. As $a_{1,2} \notin J_{1,1}$, this implies $c_0 \neq c_0'$. In addition, it is possible to extend the domain of definition of the deformation of $\zeta_{1,1}$ to the larger interval $J_{1,1} \cup J_{1,2}$, so that all the above properties of the deformation continue to hold. As $a_{1,2} \notin J_{1,1}$, the argument used in the proof of (16) gives $a_{1,2} \notin E_n(*)$. This is a contradiction. Hence (17) holds.

Having chosen $(a_{k-1,i}, \zeta_{k-1,i})_i$ and the corresponding intervals $(J_{k-1,i})_i$, we choose $(a_{k,j}, \zeta_{k,j})_j$ as follows. For each $J_{k-1,i}$, in the same way as the proof of (16) it is possible to choose a finite number of parameters $a_{k,1}, a_{k,2}, \cdots$ in $E_n(*) \cap [a^* - \varepsilon, a^*] \cap e^{-r_{k-1}/3} \cdot J_{k-1,i}$ such that the corresponding intervals $J_{k,1}, J_{k,2}, \cdots$ are pairwise disjoint and altogether cover $E_n(*) \cap e^{-r_{k-1}/3} \cdot J_{k-1,i}$. Now the issue is to show the inclusion $\bigcup_j J_{k,j} \subset 2e^{-r_{k-1}/3} \cdot J_{k-1,i}$. This is a consequence of the fact that the center $a_{k,j}$ of each $J_{k,j}$ belongs to $e^{-r_{k-1}/3} \cdot J_{k-1,i}$, and any $J_{k,j}$ does not contain the critical parameter in $J_{k-1,i}$. $\square$

Lemma 4.4. *For every i, $\Theta_{\nu_1}(\zeta_{1,i}) \leq 2\varepsilon$.*

As the intervals $(J_{1,i})_i$ are pairwise disjoint and intersect $[a^* - \varepsilon, a^*]$, Lemma 4.4 gives $\sum_i |J_{1,i}| \leq 3\varepsilon$. This proves (c).

It is left to prove Lemma 4.4. We use the following which can be proved by slightly extending the arguments in Sect.4.1 and using the definition of quasi critical approximations.

Claim 4.1. *Let ζ be a quasi critical approximation of order n of f_{a^*}. There exists a critical point z of f_{a^*} such that $|\zeta - z| \leq (Cb)^{\frac{1}{2}\theta n}$.*

Proof of Lemma 4.4. Take an integer m, a quasi critical approximation ζ of $f_{a_{1,i}}$ of order m, and its deformation $a \in I_m(a_{1,i}) \to \zeta(a)$ for which the conclusions of Proposition 4.4 hold up to time ν_1. If $|J_{1,i}| > 2\varepsilon$, then $a^* \in J_{1,i}$ holds, because of $a_{1,i} \in [a^* - \varepsilon, a^*]$. Then $\zeta(a^*)$ makes sense and we have $|f_{a_{1,i}}^{\nu_1}(\zeta_{1,i}) - \zeta_{\nu_1}(a^*)| \leq |f_{a_{1,i}}^{\nu_1}(\zeta_{1,i}) - \zeta_{\nu_1}(a_{1,i})| + |\zeta_{\nu_1}(a_{1,i}) - \zeta_{\nu_1}(a^*)| \ll 1$. As ν_1 is a return time, $f_{a_{1,i}}^{\nu_1}(\zeta_{1,i}) \in I(\delta)$ holds. It follows that $\zeta_{\nu_1}(a^*)$ is near $I(\delta)$. On the other hand, Proposition 4.1 and Claim 4.1 together imply $\zeta_{\nu_1}(a^*) \in \{(x, y) \colon |x| \geq 4/5\}$. A contradiction arises. $\square$

4.9. **Proof of Lemma 4.2.** A proof of this lemma also relies on deformations. We argue by induction. Let ζ_0 be an arbitrary critical approximation of f_{a_0} of order ξ. Let $20N \leq k \leq \min(n_0(\varepsilon), 20\xi)$ and suppose that $f_{a_0}^i(\zeta_0) \notin I(\delta)$ holds for $1 \leq i \leq k-1$. Note that this assumption for $k = 20N$ is satisfied. We show $f_{a_0}^k(\zeta) \notin I(\delta)$. As ζ_0 is arbitrary, the claim follows.

By the assumption of induction, $f_{a_0}^k(\zeta_0)$ is free. Take an integer m, a quasi critical approximation ζ of f_{a_0} of order m, and its deformation $a \in I_m(\hat{a}) \to \zeta(a)$ for which the conclusions of Proposition 4.4 hold up to time k. The definition of the interval $J_k(a_0, \zeta_0)$ and (14) give

$$|J_k(a_0, \zeta_0)| \geq \kappa_0 \delta C_0^{-2k} \geq 2\varepsilon.$$

As $a_0 \in [a^* - \varepsilon, a^*]$, $a^* \in J_k(a_0, \zeta_0)$ holds. Proposition 4.4, $\zeta(a^*)$ makes sense and we have $|f_{a_0}^k(\zeta_0) - \zeta_k(a^*)| \leq |f_{a_0}^k(\zeta_0) - \zeta_k(a_0)| + |\zeta_k(a_0) - \zeta_k(a^*)| \ll 1$. Proposition 4.1 and Claim 4.1 give $\zeta_k(a^*) \notin \{(x, y) \colon |x| \leq 4/5\}$. Hence $f_{a_0}^k(\zeta_0) \notin I(\delta)$ follows. $\square$

4.10. **Proof of Proposition 4.1.** In this subsection, all dynamical objects pertain to f_{a^*}, and f indicates f_{a^*}. Let r denote the point of the quadratic tangency near $(0, 0)$. Let S denote the lenticular compact domain in $I(\delta)$ bounded by the segment in W^u and the parabola in $W^s(Q)$ containing r (cf. Figure 1). By (M1), all points in $f(S)$ do not return to R_0 under positive iteration, and thus they are expanding. By Proposition 2.3, $f(S)$ is foliated by long stable leaves. Note that the leaf through $f(r)$ contains the boundary of R_0.

Definition 4.8. Let γ be a $C^2(b)$-curve in $W^u(Q)$ stretching across $I(\delta)$. We say $\zeta \in \gamma$ is a *critical point on* γ if $z \in S$, and the long stable leaf through $f(z)$ is tangent to $W^u(Q)$ at $f(z)$.

A proof of Proposition 4.1 is briefly outlined as follows. We approximate any critical approximation by a critical point. By definition, the orbit of every critical point do not return to R_0. Hence the claim follows.

Lemma 4.5. *Let γ be a $C^2(b)$-curve in $W^u(Q)$ stretching across $I(\delta)$. There exists a unique critical point on γ. In addition, For every $n \geq M$ there exists a critical approximation of order n on γ within the distance $(Cb)^{\frac{n}{4}}$ from the critical point.*

Proof. Let Γ denote any long stable leaf which is at the right of $\Gamma(r)$. By Remark 2.1, Γ intersects $f(\gamma)$ at two points, or else Γ is tangent to $f(\gamma)$ and the point of tangency is quadratic. There exists only one leaf for which the latter holds, for otherwise two distinct leaves intersect each other, a contradiction to the remark below Lemma 2.5. The pull-back of the point of tangency is a critical point on γ, denoted by ζ. Hence, the first statement holds.

Take $z \in \gamma$ with $|\zeta - z| = b^{\frac{n}{4}}$, and write $f(z) = (x_0, y_0)$. Represent the two long stable leaves as graphs of functions on $[-\sqrt{b}, \sqrt{b}]$: $\Gamma_n(z) = \{(x(y), y)\}$ and $\Gamma_n(\zeta) = \{(\tilde{x}(y), y)\}$. Since the Hausdorff distance between $\Gamma_n(\zeta)$ and $\Gamma(\zeta)$ is $\leq (Cb)^n$, Lemma 2.2 gives $|x(y_0) - \tilde{x}(y_0)| = |x_0 - \tilde{x}(y_0)| \approx b^{\frac{n}{2}}$. Since e_n is Lipschitz, it follows that $|x(y) - \tilde{x}(y)| \approx Cb^{\frac{n}{2}}$ for all $y \in [-\sqrt{b}, \sqrt{b}]$. Hence $f^{-1}(\Gamma_n(z))$ intersects γ at two points within $(Cb)^n$ from ζ. This and Remark 2.1 together imply the second statement. □

Let ζ_0 denote the critical point which is closest to Q in the Riemannian distance in $W^u(Q)$. Let G denote the segment in $W^u(Q)$ with endpoints Q, $f(\zeta_0)$. A proof of the next lemma is given in Appendix.

Lemma 4.6. *For every $n \geq 0$, any component of $f^n(G) \cap I(\delta)$ is a $C^2(b)$-curve.*

Proof Proposition 4.1. If $|f^{-[\theta n]}(\zeta) - f(r)| \leq 1/10$, then let $m = [\theta n] - 1$. Otherwise, let $m = [\theta n]$. Then $f^{-m}(\zeta)$ is expanding. Let z denote the point of intersection between the long stable leaf of order m through $f^{-m}(\zeta)$ and G. It is possible to take a curve γ in G extending both sides around z to length $b^{\frac{m}{3}}$. For otherwise the contraction along the long stable leaf gives $f^m(Q) \in I(\delta)$, a contradiction because Q is a fixed point and $Q \notin I(\delta)$. By the definition of m, γ avoids the turn near $f(\zeta_0)$, and hence is $C^2(b)$. Then $f^m(\gamma)$ is a $C^2(b)$-curve extending both sides around $f^m(z)$ to length $\geq b^{\frac{m}{2}}$. By Lemma 3.2, there exists a critical approximation $\bar{z}$ of order n on $f^m(\gamma)$ such that $|\zeta - \bar{z}| \leq (Cb)^{\frac{\theta n}{4}}$ holds. By Lemma 4.5and Lemma 4.6, there exist a $C^2(b)$-curve γ' in W^u containing $f^m(\gamma)$ and stretching across $I(\delta)$, and a critical point ζ'' on γ' such that $|\bar{z} - \zeta''| \leq (Cb)^{\frac{\theta n}{4}}$. It follows that $|f^i(\zeta) - f^i(\zeta'')| \leq (Cb)^{\frac{\theta n}{5}}$ for $1 \leq i < 20n$. As the orbit of ζ'' is out of R_0, the claim holds. □

Standing hypothesis for the rest of the paper: $f \in \{f_a \colon a \in \Delta \cap (a^{**}, a^*]\}$. Recall that a^{**} is the parameter corresponding to the manifold organization as in Figure 3. The positive measure sets of parameters constructed in [5, 18, 30] corresponding to Hénon-like strange attractors are not concerned, because they are at the left of a^{**}.

5. Dynamics on the unstable manifold

In this section we develop a one-dimensional analysis on the unstable manifold W^u. In Sect.5.1, we define a *critical set* $\mathcal{C}$ in W^u, as a set of accumulation points of critical approximations. Each element of $\mathcal{C}$ is called a *critical point.* In Sect.5.2, 5.3 we prove some key estimates on critical points. In Sect.5.4 we identify a geometric structure of W^u near the critical set.

Notation. For $z \in W^u$, let $t(z)$ denote any unit vector tangent to the unstable manifold at z. The boundaries of R_0 in W^u is called *unstable sides*, and denoted by ∂R_0. Let $\partial R_n := f^n(\partial R_0)$.

5.1. **The critical set.** In the case $W^u = W^u(Q)$, fix a fundamental domain $\mathcal{F}$ in $W^u_{\rm loc}(Q)$. For $z \in \mathcal{F}$, define a sequence $n_1 < n_1+p_1 \leq n_2 < n_2+p_2 \leq n_3 < \cdots$ inductively as follows: n_1 is the smallest such that $f^{n_1}(z) \in I(\delta)$ and p_1 is the bound period of $f^{n_1}(z)$; $n_k \geq n_{k-1}+p_{k-1}$ is the smallest such that $f^{n_k}(z) \in I(\delta)$, and p_k is the bound period of $f^{n_k}(z)$. From the fact that Q is a fixed saddle, it follows that this sequence is defined indefinitely, or else there exists an integer m such that $Df^m(z)t(z)$ is in critical position relative to critical approximations of arbitrarily high order. If the latter case occurs, we let $f^m(z) \in \mathcal{C}$. Since each such point is isolated in W^u, $\mathcal{C}$ is a countable set. In the case $W^u = W^u(P)$, $\mathcal{C}$ is constructed in the same way, with Q replaced by P.

Proposition 5.1. *The following holds for each $\zeta \in \mathcal{C}$:*

(a) $\|w_n(\zeta)\| \geq e^{\lambda(n-1)}$ *for* $n \geq 1$*;*
(b) $\|w_j(\zeta)\| \geq e^{-2\alpha i}\|w_i(\zeta)\|$ *for* $1 \leq i < j$*;*
(c) *there exists a monotone increasing function* $\chi\colon \mathbb{N} \circlearrowleft$ *such that for each* n*,* $\chi(n) \in [(1-\sqrt{\alpha})n, n]$ *and* $\|w_{\chi(n)}(\zeta)\| \geq \delta\|w_k(\zeta)\|$ *for* $1 \leq k < \chi(n)$*;*
(d) *the long stable leaf* $\Gamma(f\zeta)$ *is tangent to* W^u *at* $f(\zeta)$ *and the tangency is quadratic.*

Proof. By definition, for each $\zeta \in \mathcal{C}$ there exists a strictly increasing sequence $m_1 < m_2 < \cdots$ of integers and a sequence $\zeta_{m_1}, \zeta_{m_2}, \cdots$ of critical approximations with good critical behavior, such that ζ_{m_ℓ} is of order m_ℓ, and $\zeta_{m_\ell} \to \zeta$ as $\ell \to \infty$. (a) (b) (c) are direct consequences of this convergence. By the definition of $\mathcal{C}$ and (ii) in Proposition 2.1, $t(\zeta)$ is contracted exponentially by positive iterations. Thus $t(f(\zeta))$ is tangent to $\Gamma(f\zeta)$. By Remark 2.1, this tangency is quadratic, and (d) holds. □

Remark 5.1. In the creation of the theory of nonuniformly hyperbolic strange attractors [5, 8, 9, 18, 30], a key point of the arguments is to consider *critical sets*, which play the role of critical points in one-dimensional dynamics. Our set $\mathcal{C}$ is designed to play the same role in our context.

In the attractor context, critical sets have dynamically intrinsic characterizations [30], as unique sources of non-hyperbolicity. On the other hand, our $\mathcal{C}$ may not have an intrinsic meaning, because of possible escapes from R_0. If this happens for orbits of $\mathcal{C}$, that means that $\mathcal{C}$ depends on the initial modification of the maps on D_1, as in Section 2.2.

Nevertheless, the set $\mathcal{C}$ has to do with the hyperbolicity of the system in the following sense. For each $\zeta \in \mathcal{C}$, let $n(\zeta) > 0$ denote the smallest such that $f^{n(\zeta)}(\zeta)$ is out of R_0. If no such integer exists, let $n(\zeta) = \infty$. The system is uniformly hyperbolic if and only if $\sup_{\zeta\in\mathcal{C}}\{n(\zeta)\} < \infty$. This criterion is reminiscent of the classical fact in one-dimensional dynamics [17], which states that an invariant set is uniformly hyperbolic if and only if it does not contain critical points.

5.2. **Recovering expansion.** In this and the next subsection we assume that ζ is a critical point on a horizontal curve γ in $I(\delta)$. By this we mean $\Gamma(f\zeta)$ is tangent to $f(\gamma)$ at $f(\zeta)$. We state a version of Proposition 2.1. The difference is that ζ is no longer an approximation and a "genuine" critical point, and thus the estimates are available entirely on γ.

Write $\Gamma(f\zeta) = \{(x(y), y)\colon |y| \leq \sqrt{b}\}$. For each $k \geq M$, let $V_k = \{(x,y)\colon |x - x(y)| \leq D_k(\zeta)/2, |y| \leq \sqrt{b}\}$. If $f(z) \in V_k \setminus V_{k+1}$, define a *bound period* $p = p(\zeta, z)$ by

$$p = \chi(k),$$

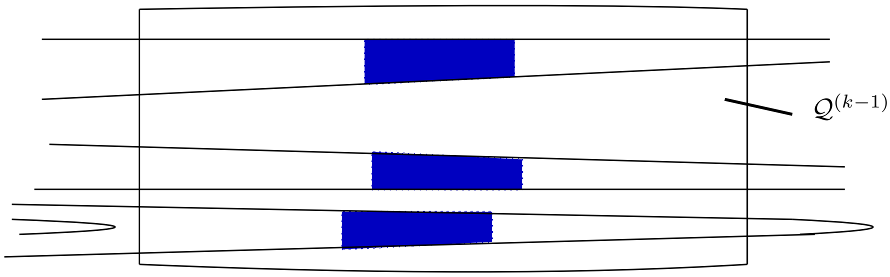

FIGURE 7. The relation between $\mathcal{C}^{(k-1)}$ and $\mathcal{C}^{(k)}$. The shaded regions are components of $\mathcal{C}^{(k)}$.

and a *fold period* $q = q(\zeta, z)$ by

$$q = \min \left\{ i \in [1, p) \colon |\zeta - z|^{\tilde{\alpha}} \cdot \|w_{j+1}(\zeta)\| \geq 1 \text{ for } i \leq j < p \right\}.$$

The next proposition is proved similarly to the proof of Proposition 2.1.

Proposition 5.2. *Let $z \in \gamma \setminus \{\zeta\}$ and let $t(z)$ denote any unit vector tangent to γ at z. Then:*
(a) $p \leq \log |\zeta - z|^{-\frac{3}{\lambda}}$;
(b) $q \leq C\tilde{\alpha}p$;
(c) $|f^i(\zeta) - f^i(z)| \leq e^{-2\alpha p}$ *for* $1 \leq i \leq p$;
(d) $|\zeta - z| \leq \|Df^q(z)t(z)\| \leq |\zeta - z|^{1-\tilde{\alpha}}$;
(e) $\|Df^p(z)t(z)\| \geq |\zeta - z|^{-1+\frac{\alpha}{\log C_0}} \geq e^{\frac{\lambda p}{3}}$;
(f) $\|Df^p(z)t(z)\| \geq (\delta/10)\|Df^i(z)t(z)\|$ *for* $0 \leq i < p$;
(g) $\|Df^i(z)t(z)\| \approx |\zeta - z|\|w_i(\zeta)\|$ *for* $q \leq i \leq p$;
(h) $\|Df^i(z)t(z)\| < 1$ *for* $1 \leq i \leq q$.

5.3. **Critical partitions.** Using the family (V_k) of vertical strips, we construct a *critical partition* of γ as follows. By Remark 2.1, $\gamma \cap f^{-1}(V_k \setminus V_{k+1})$ consists of two components, one at the right ζ and the other at the left. For simplicity, let us denote both by γ_k. If $f(\gamma_k)$ does not intersect the vertical boundary of V_k, then we take γ_k together with the adjacent γ_{k+1}. We cut each γ_k into $[e^{3\alpha k}]$-number of curves of equal length, and denote them by $\gamma_{k,s}$ $(s = 1, 2, \cdots)$.

Let $\mathrm{dist}(\gamma_{k,s}, \zeta)$ denote the distance between $\gamma_{k,s}$ and ζ. Let

$$\alpha_0 := \frac{1}{\frac{6 \log C_0}{\alpha} + 2}. \tag{18}$$

A proof of the next lemma is given in Appendix.

Lemma 5.1. *For each $\gamma_{k,s}$ we have:*
(a) $\mathrm{length}(\gamma_{k,s}) \leq \mathrm{dist}(\gamma_{k,s}, \zeta)^{1+\frac{\alpha}{3 \log C_0}}$;
(b) $f^{\chi(k)}(\gamma_{k,s})$ *is a $C^2(b)$-curve of length* $\geq e^{-4\alpha k}$;
(c) *For all $\xi, \eta \in \gamma_{k,s}$,*

$$\left| \frac{\|Df^{\chi(k)}(\xi)t(\xi)\|}{\|Df^{\chi(k)}(\eta)t(\eta)\|} - 1 \right| \leq |f^{\chi(k)}(\xi) - f^{\chi(k)}(\eta)|^{\alpha_0}.$$

5.4. **Geometry of critical regions.** We identify a geometric structure of critical regions, close the one depicted in ([30] Sect.1.2). Let $\mathcal{C}^{(0)} = \{(x,y) \in R_0 \colon |x| \leq \delta\}$.

Proposition 5.3. *There exists a nested sequence $\mathcal{C}^{(0)} \supset \mathcal{C}^{(1)} \supset \mathcal{C}^{(2)} \supset \cdots$ such that the following holds for $k = 0,1,2,\cdots$:*

(S1) *$\mathcal{C}^{(k)}$ has a finite number of components called $\mathcal{Q}^{(k)}$ each one of which is diffeomorphic to a rectangle. The boundary of $\mathcal{Q}^{(k)}$ is made up of two $C^2(b)$-curves of ∂R_k connected by two vertical lines: the horizontal boundaries are $\approx \min(2\delta, \kappa_0^k)$ in length, and the Hausdorff distance between them is $\mathcal{O}(b^{\frac{k}{2}})$;*

(S2) *On each horizontal boundary γ of each component $\mathcal{Q}^{(k)}$ of $\mathcal{C}^{(k)}$, there is a critical point located within $\mathcal{O}(b^{\frac{k}{4}})$ of the midpoint of γ.*

(S3) *$\mathcal{C}^{(k)}$ is related to $\mathcal{C}^{(k-1)}$ as follows: $\mathcal{Q}^{(k-1)} \cap R_k$ has at most finitely many components, each of which lies between two $C^2(b)$ subsegments of ∂R_k that stretch across $\mathcal{Q}^{(k-1)}$ as shown in FIGURE 7. Each component of $\mathcal{Q}^{(k-1)} \cap R_k$ contains exactly one component of $\mathcal{C}^{(k)}$.*

(S4) *Let $\Xi^{(k)}$ denote the set of critical points on the horizontal boundaries of $\bigcup_{j=0}^{k} \mathcal{C}^{(j)}$. Then $\mathcal{C} = \bigcup_{k\geq 0} \Xi^{(k)}$.*

The rest of this section is entirely devoted to an inductive proof of (S1), (S2), (S3). (S4) is a direct consequence of this. In Section 5.5, we first describe a structure of the induction, to make clear how to proceed from one to the next step. In Section 5.6 we treat an initial step of the induction. In Section 5.7 we treat a generic step.

This induction includes a selection of binding points for orbits in W^u. In view of our construction in later sections, and to resolve the problems mentioned in Remark 3.3, these binding points are selected from $\mathcal{C}$ according to a definite rule, and fixed once and for all.

5.5. **Structure of induction.** (S1), (S2) for $k = 0$ are trivial. (S3) for $k = 0$ is an empty condition. Let us say that ∂R_0 *is controlled up to time* 0 *by* $\Xi^{(0)}$. Using the critical partition in Sect.5.3, we assign to all points in $\partial R_0 \cap I(\delta)$ their binding points in $\Xi^{(0)}$ and bound periods. This makes sense to refer to points in ∂R_1 as being free or bound.

Definition 5.1. Let $j \geq 1$ and assume:

$(I)_{j-1}$: (S1-3) hold for $0 \leq k \leq j-1$, and ∂R_0 is controlled up to time $j-1$ by $\Xi^{(j-1)}$.

Under this assumption, we say:

• a segment in ∂R_j is a *free segment* if all points on it are free;

• a maximal free segment in ∂R_j is a free segment in ∂R_j which is not contained in any other free segment in ∂R_j;

• a bound segment in ∂R_j is any connected component of $\partial R_j \setminus \{$maximal free segment in $\partial R_j\}$.

In the sequel we need two curvature-related estimates.

Lemma 5.2. *Any free segment in ∂R_j is a $C^2(b)$-curve.*

Proof. Let γ be a free segment in ∂R_j. Then $1 \geq C\delta\|Df^{-n}(z)t(z)\|$ holds for all $z \in \gamma$ and $n > 0$. Hence, the curvature of γ is $\leq \sqrt{b}$, by the curvature estimate in [[27] Lemma 2.4] and the boundedness of the curvature of $W^u_{\rm loc}$. The inequality for $n = -1$ implies that the slopes of the tangent directions of γ are $\leq \sqrt{b}$. □

Lemma 5.3. *For any free segment γ and $n \geq 0$, the curvature of $f^{-n}(\gamma)$ is everywhere $\leq 4^{3n}$.*

Proof. For $z \in \gamma$, let $\kappa_{-n}(z)$ denote the curvature of $f^{-n}(\gamma)$ at $f^{-n}(z)$. If $f^{-n}(z)$ is free, then $\kappa_{-n}(z) \leq \sqrt{b}$, by Lemma 5.2. Otherwise, let $m < -n$ denote the largest integer such that $f^m(z)$ is a free return. [[27] Lemma 2.4] and $\kappa_m(z) \leq \sqrt{b}$ give

$$\kappa_{-n}(z) \leq \sqrt{b}(Cb)^{-n-m}\frac{\|Df^m(z)t(z)\|^3}{\|Df^{-n}(z)t(z)\|^3} + \sum_{i=1}^{-n-m}(Cb)^i\frac{\|Df^{-n-i}(z)t(z)\|^3}{\|Df^{-n}(z)t(z)\|^3}.$$

Since z is free, $\|Df^{-n-i}(z)t(z)\| \leq 1/(r_0\delta)$, and thus for $1 \leq i \leq -n-m$,

$$\frac{\|Df^{-n-i}(z)t(z)\|}{\|Df^{-n}(z)t(z)\|} \leq (r_0\delta)^{-1}4^n.$$

Replacing all these in the above inequality, we obtain $\kappa_{-n}(z) \leq Cb\delta^{-3}4^{3n} \leq 4^{3n}$. □

Definition 5.2. Suppose that (S1-3) hold for every $0 \leq k \leq j$. We say ∂R_0 *is controlled up to time* j *by* $\Xi^{(j)}$, if for any maximal free segment γ in ∂R_j there exist a horizontal curve $\tilde{\gamma}$ which contains γ and a critical point $\zeta \in \Xi^{(j)}$ on $\tilde{\gamma}$.

At step $j-1$ of the induction, we show the implication $(I)_{j-1} \Longrightarrow (I)_j$. Then, for all points in $\partial R_j \cap I(\delta)$ which are free, we assign their binding points as follows. For a maximal free segment γ in ∂R_j, take $(\tilde{\gamma}, \zeta)$ as in Definition 5.2. We use ζ as a common binding point for points in $\gamma \cap I(\delta)$. Their bound periods are given by considering the critical partition of $\tilde{\gamma}$. This makes sense to refer to points in ∂R_{j+1} as being free or bound.

5.6. **From step** 0 **to step** N. Let $1 \leq j \leq N$ and suppose $(I)_{j-1}$. The bound parts of ∂R_j do not come back to $\mathcal{C}^{(0)}$, and $\partial R_j \cap I(\delta)$ consists of $C^2(b)$ curves, each of which admits a critical point. Define $\mathcal{C}^{(j)} = R_j \cap \mathcal{C}^{(0)}$. $(I)_j$ obviously holds.

5.7. **From step** 2^mN **to** $2^{m+1}N$. The same argument cannot be continued indefinitely, because bound segments return to $I(\delta)$. To deal with these returns, we need the help of critical points.

Lemma 5.4. *For each* $\zeta \in \mathcal{C}$ *there exist positive integers* $n_1 < n_1 + p_1 \leq n_2 < n_2 + p_2 \leq n_3 < \cdots$ *such that, for each* n_l, $f^{n_l}(\zeta) \in I(\delta)$, *and there exists a critical approximation* $\hat{z}_l$ *relative to which* $w_{n_l}(\zeta)$ *is in admissible position, with* $|f^{n_l}(\zeta) - \hat{z}_l| \geq e^{-\alpha n_l}$.

The integers $n_1, n_2, \cdots$ are called *free return times* of ζ.

Proof. We argue by induction. First, let $n_1 = \min\{n > 0 \colon f^n(\zeta) \in I(\delta)\}$. As $I(\delta)$ is open, $n_1 = \min\{n > 0 \colon f^n(\zeta_{m_\ell}) \in I(\delta)\}$ holds for all sufficiently large ℓ. Let z_{m_ℓ} denote the binding point for $f^{n_1}(\zeta_{m_\ell})$, with a bound period p_{m_ℓ}. Passing to subsequences, we may assume that both converge as $\ell \to \infty$. Define $\hat{z}_1$, p_1 to be the corresponding limits.

Given $(n_k, \hat{z}_k, p_k)$, define $n_{k+1} = \min\{n \geq n_k + p_k \colon f^n(\zeta) \in I(\delta)\}$. Passing to subsequences again, we may assume that $f^{n_{k+1}}(\zeta_{m_{\ell'}})$ is a free return to $I(\delta)$, with a binding point $z_{m_{\ell'}}$ and a bound period $p_{m_{\ell'}}$, both converging as $\ell \to \infty$. Define $\hat{z}_{k+1}$, p_{k+1} to be the corresponding limits. □

Definition 5.3. Let $\zeta \in \mathcal{C}$, with $n_1, n_2, \cdots$ and $\hat{z}_1, \hat{z}_2, \cdots$ as in Lemma 5.4. We say ζ is *controlled up to time* n *by* $\Xi^{(k)}$ if, for each $n_l \leq n$ there exists $z_l \in \Xi^{(k)}$ such that $|z_l - \hat{z}_l| = \mathcal{O}(b^{\frac{\theta\xi}{5}})$, where ξ is the order of $\hat{z}_l$. Such z_l is called a *binding point* for ζ.

Clearly, every $\zeta \in \mathcal{C}$ is controlled up to time $2N$ by $\Xi^{([\theta N])}$. To proceed from step $2^m N$ to step $2^{m+1}N$, it suffices to show

Lemma 5.5. *Let $m \geq 0$. Suppose that $(I)_{2^m N}$ holds, and that every $\zeta \in \mathcal{C}$ is controlled up to time $2^{m+1}N$ by $\Xi^{([2^m\theta N])}$. Then:*
(a) $(\mathrm{I})_k$ *holds for $2^m N < k \leq 2^{m+1}N$;*
(b) *every $\zeta \in \mathcal{C}$ is controlled up to time $2^{m+2}N$ by $\Gamma^{([2^{m+1}\theta N])}$.*

Proof of (a). Assume $(I)_{j-1}$ for some $2^m N < j \leq 2^{m+1}N$. Then $\Xi^{(j-1)}$ makes sense. We prove $(I)_j$ in three steps.

Step 1: Treatment of bound segments in ∂R_j.

Lemma 5.6. *Let B be a bound segment in ∂R_j. There exist $N < l < j$ and $\zeta \in \Xi^{(j-1)}$ such that $f^l(\zeta)$ is free and $d(f^l(\zeta), B) \leq e^{-2\alpha l}$.*

Proof. We define a sequence $z_0, \cdots, z_s$ in $\Xi^{(j-1)}$ and a sequence $n_0, \cdots, n_s$ of positive integers inductively as follows. By the definition of bound segments, there exists $0 < n_0 \leq k$ such that $f^{-n_0}(B)$ contains a critical point in $\Xi^{(j-n_0)}$, denoted by z_0. If $f^{n_0}(z_0)$ is bound, let $n_1 < n_0$ denote the free return time of z_0 with bound period p_1, such that $n_1 < n_0 < n_1 + p_1$. Let z_1 denote the corresponding binding point, which is in $\Xi^{([\theta n_1])} \subset \Xi^{(j-1)}$ by the assumption of induction. If $f^{n_0-n_1}(z_1)$ is bound, then let $n_2 < n_0 - n_1$ denote the free return time of z_1 with bound period p_2, such that $0 < n_2 < n_0 - n_1 < n_2 + p_2$. Let z_2 denote the binding point, which is in $\Xi^{([\theta n_2])} \subset \Xi^{(j-1)}$, and so on.

We must reach some n_s and z_s such that $f^{n_0-n_1-\cdots-n_s}(z_s)$ is free. By the inductive assumption, each z_i is controlled up to time $k-1$. Hence, for each $i = 1, \cdots, s$ we have $p_i < \frac{4\alpha}{\lambda}p_{i-1}$. Let d denote the Hausdorff distance. We have

$$\begin{aligned} d(B, f^{n_0-n_1-\cdots-n_s}(z_s)) \leq & d(B, f^{n_0}(z_0)) + |f^{n_0}(z_0) - f^{n_0-n_1}(z_1)| + |f^{n_0-n_1}(z_1) - f^{n_0-n_1-n_2}(z_2)| \\ & + \cdots + |f^{n_0-n_1-\cdots-n_{s-1}}(z_{s-1}) - f^{n_0-n_1-\cdots-n_s}(z_s)| \\ & \leq \sum_{k=0}^{s} 2e^{-2\alpha p_k} \leq 3e^{-2\alpha p_s} \leq 3e^{-2\alpha(n_0-n_1-\cdots-n_s)}, \end{aligned}$$

where we have used Proposition 2.1 for the second inequality. As z_{s-1} is bound at time $n_0 - n_1 - \cdots - n_{s-1}$, $n_0 - n_1 - \cdots - n_{s-1} < p_s$ holds. Hence $n_0 - n_1 - \cdots - n_s < p_s$ and the last inequality holds. Take $l = n_0 - n_1 - \cdots - n_s$ and $\zeta = z_s$. The argument shows $N < l$. □

Corollary 5.1. *For any bound segment B in ∂R_j and $\alpha j \leq i < j$, $B \cap \mathcal{C}^{(i)} = \emptyset$.*

Proof. Take $l < j$ and $\zeta \in \Xi^{(j-1)}$ such that the conclusion of Lemma 5.6 holds. If $f^l(\zeta) \in I(\delta)$, then let $z \in \Xi^{([\theta l])}$ denote the binding point. We have $d(B, z) \geq |f^l(\zeta) - z| - \mathrm{diam}(B) \geq e^{-\alpha l} - 6e^{-2\alpha l} \geq e^{-2\alpha l}$. This implies $B \cap \mathcal{C}^{([\alpha l])} = \emptyset$, and the claim holds. If $f^l(\zeta) \notin I(\delta)$, then let $O = (0,0)$. If l is large so that $d(B, O) \geq |f^l(\zeta) - O| - \mathrm{diam}(B) \geq \delta - 2e^{-2\alpha l} \geq \delta/2$ holds, then the claim follows, because $j > 0$. If l is so small that the last inequality does not hold, then $f^l(\zeta)$ is near $f(I(\delta))$, which is away from $I(\delta)$. □

Step 2: Construction of $\mathcal{C}^{(j)}$. Let $\mathcal{Q}^{(j-1)}$ denote any component of $\mathcal{C}^{(j-1)}$ which intersects ∂R_j. By Corollary 5.1, bound segments in ∂R_j do not intersect $\mathcal{C}^{(j-1)}$. Hence, each component of $\mathcal{Q}^{(j-1)} \cap R_j$ is bounded by two free segments stretching across $\mathcal{Q}^{(j-1)}$.

Lemma 5.7. *For any free segment γ in ∂R_j stretching across $\mathcal{Q}^{(j-1)}$, there exists a critical point on γ within $\mathcal{O}(b^{\frac{j}{4}})$ of the midpoint of γ.*

This lemma allows us to construct $\mathcal{C}^{(j)}$ so that (S2) (S3) hold.

Proof. By the closeness and the disjointness of the boundaries of $\mathcal{Q}^{(j-1)}$, their tangent directions are close enough, for Lemma 3.2 to yield a critical approximation ζ_0 of order $m_0 := j$ on γ, within $\mathcal{O}(b^{\frac{j}{3}})$ of the midpoint of γ.

We inductively construct a sequence $\zeta_0, \zeta_1, \cdots$, of nice critical approximations on γ, of order $m_0 < m_1 < \cdots$, such that: (a) $m_{i+1} \in [5m_i/4, 20m_i)$; (b) $|\zeta_i - \zeta_{i+1}| \leq (Cb)^{\frac{m_i}{2}}$. The limit of the sequence $(\zeta_i)_i$ is a critical point with the desired property.

Given ζ_i of order m_i for some $i \geq 0$, ζ_{i+1} is constructed as follows. Let $\mu_1 < \mu_2 < \cdots$ denote any infinite sequence of integers such that $\frac{1}{16} \leq \frac{\mu_j}{\mu_{j+1}} \leq \frac{1}{4}$ for $j = 1, 2, \cdots$, and $\|Df^{k-\mu_j}(\zeta_i)(t(\zeta_i))\| \geq \kappa_0^{\frac{\mu_j - k}{4}}$ for $0 \leq k \leq \mu_j$. Lemma 3.3 ensures the existence of such a sequence. Let $\mu_{j(i)}$ be such that $\mu_{j(i)} \leq 20\theta m_i < \mu_{j(i)+1}$. Define m_{i+1} to be the smallest integer such that $[\theta m_{i+1}] = \mu_{j(i)}$ holds. We have $\theta m_{i+1} \geq \mu_{j(i)+1}/16 \geq 5\theta m_i/4$. (a) allows us to use Lemma 3.1, to create a critical approximation of order m_{i+1}, denoted by ζ_{i+1}. (b) is a consequence of Lemma 3.1.

We show that ζ_{i+1} is a nice critical approximation of order m_{i+1} on γ. To this end, it suffices to show the two conditions in (C3) in Sect.8. The second one is straightforward, because γ is a free segment. The first one is checked as follows. Since γ is a free segment,

$$|f^{-[\theta m_{i+1}]}(\zeta_i) - f^{-[\theta m_{i+1}]}(\zeta_{i+1})| \leq (r_0\delta)^{-1}(Cb)^{\frac{m_i}{2}}.$$

Let γ' denote the curve in $f^{-n}(\gamma)$ connecting these two points. Lemma 5.3 implies, for $1 \leq j \leq [\theta m_{i+1}]$,

$$\|Df^j(z)t(z)\| \geq \frac{1}{2}\kappa^{\frac{j}{4}} \geq \kappa^{\frac{j}{3}}.$$

This completes the construction of $(\zeta_i)_i$ and also the proof of Lemma 5.7. □

Step 3: Verification of $(I)_j$. To show the assertion on the Hausdorff distance in (S1), we regard the horizontal boundaries of the component of $\mathcal{C}^{(j-1)} \cap R_j$ containing $\mathcal{Q}^{(j)}$ as graphs of functions γ_1, γ_2 defined on an interval I of length $2\kappa_0^{j-1}$. Let $L(x) = |\gamma_1(x) - \gamma_2(x)|$. (S1) gives $L^{\frac{1}{2}}(x) \leq (Cb)^{\frac{j-1}{4}} <$ length(I). Moreover $|\gamma_1'(x) - \gamma_2'(x)| \leq L^{\frac{1}{2}}(x)$ holds, for otherwise γ_1 intersects γ_2. By this and the $C^2(b)$-property, $L(y) \geq L(x) - (L^{\frac{1}{2}}(x) - C\sqrt{b}|x-y|)|x-y|$ holds for $x, y \in I$, which is $\geq L(x)/2$ provided $|x-y| \leq L^{\frac{2}{3}}(x)$. Hence, area$(\mathcal{Q}^{(j)}) \geq L^{\frac{5}{3}}(x)/2$ holds. If $L(x) \geq b^{\frac{j}{2}}$, then area$(\mathcal{Q}^{(j)}) \geq b^{\frac{5j}{6}}/2$, which yields a contradiction to area$(\mathcal{Q}^{(j)}) <$ area$(R_j) \leq (Cb)^j$.

We show that ∂R_0 is controlled up to time j. Let γ denote any maximal free segment in ∂R_j intersecting $I(\delta)$. We indicate how to choose the horizontal curve $\tilde{\gamma}$.

If $\gamma \cap \mathcal{Q}^{(j-1)} \neq \emptyset$, then γ stretches across a component $\mathcal{Q}^{(j-1)}$, and there exists a critical point on γ, by Lemma 5.7. In this case, we take $\tilde{\gamma} = \gamma$. If $\gamma \cap \mathcal{Q}^{(k-1)} = \emptyset$, let $k_0 < k-1$ denote the largest such that $\mathcal{C}^{(k_0)} \cap \gamma \neq \emptyset$. Let $\mathcal{Q}^{(k_0)}$ denote the component intersecting γ. Let $\mathcal{Q}^{(k_0+1)}$ denote any component of $\mathcal{C}^{(k_0+1)}$ in $\mathcal{Q}^{(k_0)}$. Since the bound segments are small, there exists a horizontal curve $\tilde{\gamma}$ which contains γ and a critical point on $\tilde{\gamma}$.

(Proof of (b)). Let $\zeta \in \mathcal{C}$, $2^{m+1}N < n_l \leq 2^{m+2}N$ and suppose that n_l is a free return time of ζ. Let $\hat{z}_l$ denote the binding point of order ξ, as in Lemma 5.4. If $f^{-[\theta\xi]}(\hat{z}_l) \notin f(I(\delta))$, then the long stable leaf of order $[\theta\xi]$ through $f^{-[\theta\xi]}(\hat{z}_l)$ intersects ∂R_0 at one point, which we denote by x. Otherwise, the long stable leaf of order $[\theta\xi]-1$ through $f^{-[\theta\xi]+1}(\hat{z}_l)$ intersects ∂R_0 at one point, which we denote by x. In either of the two cases, $|f^{[\theta\xi]}(x)-\hat{z}_l| \leq (Cb)^{\theta\xi}$, and

$$\xi \leq C\alpha n_l < 2^{m+1}N.$$

Claim 5.1. *$f^{[\theta\xi]}(x)$ is free.*

Proof. Suppose the contrary. Let B denote the bound segment containing $f^{[\theta\xi]}(x)$, which is in $\partial R_{[\theta\xi]}$. By Lemma 5.6, $B \subset I(\delta)$ and there exists $l < [\theta\xi]$, $z \in \Xi^{([\theta\xi]-1)}$ such that $f^l(z)$ is free and $d(f^l(z), B) \leq e^{-2\alpha l}$. Let z' denote the binding point for $f^l(z)$. It follows that ζ and z' lie on the same horizontal curve, a contradiction. $\square$

Let γ denote the maximal free segment containing $f^{[\theta\xi]}(x)$. Lemma 5.6 implies that γ stretches across $\mathcal{Q}^{([\theta\xi]-1)}$. By the assumption of induction, there exists $z_l \in \Xi^{([\theta\xi])} \subset \Xi^{([2^{m+1}\theta N])}$, located within $\mathcal{O}(b^{\frac{[\theta\xi]}{4}})$ of the midpoint of $\gamma \cap \mathcal{Q}^{([\theta\xi]-1)}$. By Lemma 3.2, there exists a critical approximation z of order ξ on γ such that $|f^{[\theta\xi]}(x)-z| = \mathcal{O}(b^{\frac{\theta\xi}{2}})$. Lemma 3.1 implies $|z_l - z| \leq (Cb)^{\frac{\theta\xi}{5}}$. Hence

$$|\hat{z}_l - z_l| \leq |\hat{z}_l - f^{[\theta\xi]}(x)| + |f^{[\theta\xi]}(x) - z| + |z - z_l| = \mathcal{O}(b^{\frac{\theta\xi}{5}}),$$

which means that ζ is controlled up to time n_l by $\Xi^{([2^{m+1}\theta N])}$. This completes the proof of Proposition 5.3.

6. The measure of $W^u \cap K^+$

Let $|\cdot|$ denote the arc length measure on W^u (we will also denote by $|\cdot|$ the two-dimensional Lebesgue measure, but never for both things simultaneously). The aim of this section is to prove

Proposition 6.1. $|W^u \cap K^+| = 0$.

The main step in the proof of this proposition is the next

Lemma 6.1. (Abundance of stopping times) *Suppose that ω is an element of some critical partition constructed in Section 5.2. If $\omega \cap K^+$ has positive Lebesgue measure, there exist a sequence $\mathcal{Q}^{(1)}, \mathcal{Q}^{(2)}, \cdots$ of collections of pairwise interior-disjoint curves in W^u, and a sequence of stopping time functions $S_1, S_2 \cdots, S_k \colon \mathcal{Q}^{(k)} \to \mathbb{N}$ such that:*

(a) *for a.e. $z \in \omega \cap K^+$ there exists a sequence $\omega^{(1)} \supset \omega^{(2)} \supset \cdots$ of curves such that $\omega^{(k)} \in \mathcal{Q}^{(k)}$ for each $k \geq 1$ and $\{z\} = \bigcap_{k\geq 1} \omega^{(k)}(z)$;*

(b) $0 < S_1(\omega^{(1)}) < S_2(\omega^{(2)}) < \cdots$, *and* $\frac{\|Df^{S_k(\omega^{(k)})}(\xi)t(\xi)\|}{\|Df^{S_k(\omega^{(k)})}(\eta)t(\eta)\|} \leq e^{\frac{C}{\delta^{1+\alpha_0}}}$ *for all $\xi, \eta \in \omega^{(k)}$;*

(c) *$f^{S_k}(\omega^{(k)})$ is a $C^2(b)$-curve, stretching across one of the components of $I(11\delta/10) \setminus I(\delta)$.*

This is a direct consequence of the next large deviation estimate.

Lemma 6.2. (Growth to a fixed size) *Let ω_0 be an element of a critical partition, or a free segment not intersecting $I(\delta)$ and stretching across one of the components of $I(11\delta/10) \setminus I(\delta)$. If $\omega_0 \cap K^+$ has positive Lebesgue measure, there exist a collection $\mathcal{Q}$ of pairwise interior-disjoint curves in W^u and a stopping time function $S \colon \mathcal{Q} \to \mathbb{N}$ such that:*

(a) *for a.e.* $z \in \omega_0 \cap K^+$, *there exists* $\omega \in \mathcal{Q}$ *containing* z;
(b) *for each* $\omega \in \mathcal{Q}$, $f^{S(\omega)}(\omega)$ *is a free segment not intersecting* $I(\delta)$ *and stretching across one of the components of* $I(11\delta/10) \setminus I(\delta)$. *The distortion of* $f^{S(\omega)}|\omega$ *is uniformly bounded;*
(c) *there exists a constant* C *depending only on the length of* ω_0 *such that for every* $n \geq 0$,

$$|\{S > n\}| \leq Ce^{-\frac{\lambda n}{2}}. \tag{19}$$

Here, $\{S > n\}$ *denotes the union of all* $\omega \in \mathcal{Q}$ *such that* $S(\omega) > n$.

A large part of this section is devoted to the proof of Lemma 6.2. In Section 6.1 we define and describe the combinatorics of the partition $\mathcal{Q}$ and the stopping time S. In Section 6.2 we estimate the size of a curve with a given combinatorics, and combine it with a counting argument, and prove Lemma 6.2. Lemma 6.1 follows from this. In Section 6.3 we show that stable manifolds with "good shapes" are more or less dense. Combining this topological result with Lemma 6.1 we complete the proof of Proposition 6.1.

6.1. **Combinatorial structure.** Let ω_0 be a free segment in W^u as in Lemma 6.2. For each $n \geq 0$, considering n-iterates we construct a partition $\mathcal{P}_n$ of ω_0, and its subset $\mathcal{Q}_n$. Each element of $\mathcal{P}_n$ is a countable union of elements of $\mathcal{P}_{n+1}$. Each element of $\mathcal{Q}$ is an element of some $\mathcal{Q}_n$. If $\omega \in \mathcal{Q} \cap \mathcal{Q}_n$, then $S(\omega) = n$ holds.

If ω_0 is an element of a critical partition, let p_0 denote the bound period. Otherwise, that is, if $\omega_0 \cap I(\delta) = \emptyset$, let $p_0 = 0$. Let $n_1 = \min\{n \geq p_0 \colon f^n(\omega) \cap I(\delta) \neq \emptyset\}$. For every $0 \leq n < n_1$, set $\mathcal{P}_n = \{\omega_0\}$, the trivial partition of ω_0.

Let $n \geq n_1$. Given $\omega \in \mathcal{P}_{n-1}$, $\mathcal{P}_n|\omega$ is defined as follows. The n is either *cutting time* or *non-cutting time* of ω. If n is a cutting time of ω, $f^n(\omega)$ is cut into pieces. A pull-back of this partition defines $\mathcal{P}_n|\omega$. If n is a *non-cutting time of* ω, let $\mathcal{P}_n|\omega = \{\omega\}$.

We describe when n is a cutting or non-cutting time of ω. If $f^n(\omega) \cap I(\delta) = \emptyset$, or $f^n(\omega)$ is bound, then n is a *non-cutting time of* ω. If $f^n(\omega) \cap I(\delta) \neq \emptyset$ and $f^n(\omega)$ is free, Proposition 5.5 ensures the existence of a $C^2(b)$-curve γ containing $f^n(\omega)$, and a critical point on γ (any $C^2(b)$-curve with this property does the job). There are two mutually exclusive cases:

- ω_0 contains at least one element of the critical partition $\{\gamma_{k,s}\}$ of γ. In this case, n is a *cutting time* of ω_0. We cut $\omega \cap I(\delta)$ into pieces, by intersecting it with the elements of $\{\gamma_{k,s}\}$. The partition elements containing the boundary of $\omega \cap I(\delta)$ are taken together with the adjacent ones, so that all the resultant elements contains exactly one element of $\{\gamma_{k,s}\}$. If the component of $\omega \setminus I(\delta)$ is $\geq \delta/10$ in length, then we treat it as an element of our partition of ω. Otherwise, we take it together with the adjacent $\gamma_{k,s}$.
- ω_0 contains no element of $\{\gamma_{k,s}\}$. In this case, n is a non-cutting time of ω.

Lemma 6.3. *For each* $\omega \in \mathcal{P}_{n-1}$ *and all* $\xi, \eta \in \omega$ *and every* $k \in [0, n]$ *such that* $f^n(\omega)$ *is free,*

$$\log \frac{\|Df^n(\xi)t(\xi)\|}{\|Df^n(\eta)t(\eta)\|} \leq \frac{C}{\delta^{1+\alpha_0}} |f^n(\xi) - f^n(\eta)|^{\alpha_0}.$$

If $f^n(\xi), f^n(\eta) \in I(11\delta/10)$, *then the factor* $\delta^{1+\alpha_0}$ *can be dropped.*

Proof. Let $k < n$ and suppose that $f^k(\omega)$ is free. The time interval $[k, n]$ is decomposed into bound and free segments. Applying Proposition 5.2 to each bound segment and Lemma 2.1 to each free segment, we have $\|Df^{n-k}(z)t(z)\| \geq c\delta e^{\frac{\lambda}{3}(n-k)}$ for all $z \in f^k(\omega)$. Since $f^k(\omega)$ and $f^n(\omega)$ are $C^2(b)$, it then follows that

$$|f^k(\xi) - f^k(\eta)| \leq (c\delta)^{-1} e^{-\frac{\lambda}{3}(n-k)} |f^n(\xi) - f^n(\eta)|. \tag{20}$$

If $f^n(\xi), f^n(\eta) \in I(11\delta/10)$, then the factor δ can be dropped, by Lemma 2.1.

Let $n_1 < \cdots < n_s < n_{s+1} := n$ denote all the free returns in the first n-iterates of ω, with p_j the corresponding bound period. Let

$$S_j = \log \frac{\|Df^{p_j}(f^{n_j}(\xi))t(f^{n_j}(\xi))\|}{\|Df^{p_j}(f^{n_j}(\eta))t(f^{n_j}(\eta))\|} \text{ and } S_j' = \log \frac{\|Df^{n_{j+1}-n_j-p_j}(f^{n_j+p_j}(\xi))t(f^{n_j+p_j}(\xi))\|}{\|Df^{n_{j+1}-n_j-p_j}(f^{n_j+p_j}(\eta))t(f^{n_j+p_j}(\eta))\|}.$$

By Lemma 5.1 and (20),

$$\sum_{j=0}^{s} S_j \leq \sum_{j=0}^{s} |f^{n_j+p_j}(\xi) - f^{n_j+p_j}(\eta)|^{\alpha_0} \leq \frac{C}{\delta^{\alpha_0}} |f^n(\xi) - f^n(\eta)|.$$

By Lemma 2.1, $f^i(\omega)$ is a $C^2(b)$-curve outside of $I(\delta)$, for $n_j + p_j \leq i < n_{j+1}$. Hence

$$\sum_{j=0}^{s} S_j' = \sum_{j=0}^{s} \sum_{i=n_j+p_j}^{n_{j+1}-1} \log \frac{\|Df(f^i(\xi))t(f^i(\xi))\|}{\|Df(f^i(\eta))t(f^i(\eta))\|} \leq \frac{C}{\delta} \sum_{j=0}^{s-1} \sum_{i=n_j+p_j}^{n_{j+1}-1} |f^i(\xi) - f^i(\eta)| \leq \frac{C}{\delta} |f^n(\xi) - f^n(\eta)|.$$

In addition, if $f^n(\xi), f^n(\eta) \in I(11\delta/10)$, then the factor δ can be dropped. □

Definition 6.1. We say $\omega \in \mathcal{P}_n$ is an *escaping element* if (i) $\omega \cap K^+ \neq \emptyset$; (ii) n is a cutting time of the element of $\mathcal{P}_{n-1}$ containing ω, and $f^n(\omega) \cap I(\delta) = \emptyset$ holds.

Remark 6.1. By construction, if $\omega \in \mathcal{Q}_n$ is an escaping element, then $f^n(\omega)$ is a free segment, not intersecting $I(\delta)$ and stretching across one of the components of $I(\frac{11}{10}\delta) \setminus I(\delta)$.

Let $\mathcal{Q}_n$ denote the collection of all escaping elements of $\mathcal{Q}_n'$ which is not contained in escaping elements in $\bigcup_{0 \leq k < n} \mathcal{Q}_k'$. Define $\mathcal{Q} = \bigcup_n \mathcal{Q}_n$. Define a stopping time function $S \colon \mathcal{Q} \to \mathbb{N}$ by $S(\omega) = n$ for each $\omega \in \mathcal{Q}_n''$.

6.2. **Large deviations.** We show (19). By construction, this would imply that the elements of $\mathcal{P}$ altogether cover $\omega_0 \cap K^+$.

Let $\mathcal{Q}_n'$ denote the collection of all $\omega \in \mathcal{Q}_n$ such that there exists no element of $\bigcup_{0 \leq k < n} \mathcal{Q}_k$ containing ω, and let $|\mathcal{Q}_n'| = \sum_{\omega \in \mathcal{Q}_n'} |\omega|$. Clearly, $|\mathcal{Q}_n' \setminus \mathcal{Q}_n''| = |\{\omega \in \mathcal{Q} \colon S(\omega) > n\}|$ holds.

Let $n_1 > 0$ denote the cutting time of ω_0. It is finite, and depends only on the length of ω_0. This implies that, for $n \geq n_1$, any element of $\mathcal{Q}_n'$ has an well-defined itinerary that is described as follows. For each $\omega_n \in \mathcal{Q}_n' \setminus \mathcal{Q}_n''$ there exist a sequence of integers $0 < n_1 < \cdots < n_s \leq n$ called *essential free returns*, and an associated sequence $\omega_1 \supset \cdots \supset \omega_s \supset \omega$ such that ω_i is the element of $\mathcal{Q}_{n_i}$ containing ω, and n_i is a cutting time of ω_{i-1}, with $f^{n_i}(\omega_i) \subset I(2\delta)$. Let $\zeta_i \in \mathcal{C}$ denote the binding point for $f^{n_i}(\omega_{i-1})$. Let p_i denote the bound period. By an *itinerary* of ω we mean the sequence $(n_1, \pm p_1, \zeta_1), (n_2, \pm p_2, \zeta_2), \cdots, (n_s, \pm p_s, \zeta_s)$, where $+, -$ indicates whether $f^{n_i}(\omega_i)$ is at the right or left of ζ_i.

From this point on we assume

$$n \geq 2n_1. \tag{21}$$

By construction, $f^{n_i}(\omega_i)$ and $f^{n_i+p_i}(\omega_{n_i})$ are free segments. The following estimates are used in the proof:

$$|f^{n_i}(\omega_i)| \leq e^{-\lambda p_i} \text{ and } |f^{n_i+p_i}(\omega_{n_i})| \geq e^{-4\alpha p_i}.$$

The first one follows from the definition of the critical partition. The second one is from Lemma 5.1.

Let n_{s+1} denote the cutting time of ω_s, which is well-defined because ω_s intersects K^+.

Claim 6.1. *$n_{i+1}-n_i-p_i\leq \frac{20p_i}{\lambda}$ for every $1\leq i\leq s$.*

Proof. Since $f^{n_{i+1}}(\omega_i)$ is also a free segment, in view of Proposition 5.2 and Lemma 2.1 we have $3\geq |f^{n_{i+1}}(\omega_i)|\geq c\delta e^{\frac{\lambda}{3}(n_{i+1}-n_i-p_i)}e^{-4\alpha p_i}$. Rearranging gives $n_{i+1}-n_i-p_i\leq \frac{3}{\lambda}\left(\log(1/\delta)+5\alpha p_i\right)\leq \frac{20}{\lambda}p_i$, where the last inequality follows from $p_i\geq \frac{\log 1/\delta}{2\log 2}$. □

By definition, $n_{s+1}>n$ holds. Summing the above inequality over all $1\leq k\leq s$ and then using (21), we have

$$n\leq \frac{40}{\lambda}\sum_{i=1}^{s}p_i. \tag{22}$$

Write $\omega=\omega_{s+1}$. By $f^{n_{i+1}}(\omega_{i+1})\subset I(11\delta/10)$, the better version of the bounded distortion estimate in Lemma 6.3 is available and we have

$$|\omega|\leq|\omega_s|=|\omega_1|\frac{|\omega_2|}{|\omega_1|}\cdots\frac{|\omega_s|}{|\omega_{s-1}|}\leq 2^{s-1}\prod_{i=1}^{s-1}\frac{|f^{n_{i+1}}(\omega_{i+1})|}{|f^{n_{i+1}}(\omega_i)|}\leq e^{-(\lambda-3\alpha)(p_1+\cdots+p_{s-1})}e^{-\lambda p_s}\leq e^{-(\lambda-3\alpha)R},$$

where $R=\sum_{i=1}^{s}p_i$, which is $\geq\frac{\lambda n}{40}$ by (22). Hence

$$|\mathcal{Q}_n'\setminus\mathcal{Q}_n''|=\sum_R\sum_{\substack{\omega\\ p_1+\cdots+p_s=R}}|\omega|\leq\sum_R\sum_s 2^s\binom{R}{s}e^{-(\lambda-3\alpha)R}\leq\sum_{R\geq\lambda n/40}e^{-(\lambda-4\alpha)R}\leq e^{-(\lambda-5\alpha)\lambda n/40}.$$

For the last inequality we have used $s/R\leq 2\log 2/\log(1/\delta)$ and $\binom{R}{s}\leq e^{\beta(\delta)R}$, where $\beta(\delta)\to 0$ as $\delta\to 0$, which follows from Stirling's formula for factorials.

Proof of Lemma 6.1. It is now straightforward to define the sequence $\mathcal{Q}_1,\mathcal{Q}_2,\cdots$ of collections of pairwise interior-disjoint curves in ω_0 and an associated sequence of stopping time functions $S_1,S_2,\cdots$ in Lemma 6.1. Let $\mathcal{Q}_1=\mathcal{Q}$ and $S_1=S$. Given $\mathcal{Q}_k$ and S_k, for each $\omega\in\mathcal{Q}_k$ define a partition $\mathcal{Q}'$ of $f^{S_k(\omega)}(\omega)$ and a stopping time function $S'\colon\mathcal{Q}'\to\mathbb{N}$, replacing ω_0 in Lemma 6.2 by $f^{S_k(\omega)}(\omega)$. This defines $\mathcal{Q}_{k+1}$ in the obvious way. For $\omega'\in\mathcal{Q}_{k+1}$ we define $S_{k+1}(\omega')=S_k(\omega)+S'(f^{S_k(\omega)}(\omega'))$, and so on. The bounded distortion follows from Lemma 6.3.

6.3. **Proof of Proposition 6.1.** The next lemma relies on a continuity argument within a small parameter range containing the first bifurcation parameter a^*, and is not valid for the parameter ranges treated in [5, 18, 30]. Recall that a^{**} denotes the parameter corresponding to the manifold organization indicated in Figure 3.

Lemma 6.4. *There exist $\varepsilon_1\in(0,a^*-a^{**})$ and $\sigma\in(0,1)$ such that for any $a\in[a^*-\varepsilon_1,a^*]$ and any $C^2(b)$-curve γ in W^u stretching across one component of $I(\frac{11}{10}\delta)\setminus I(\delta)$, $|\gamma\cap K^+|\leq\sigma\cdot|\gamma|$.*

We finish the proof of Proposition 6.1 assuming the conclusion of the lemma. Assume $|W^u\cap K^+|>0$. Then one can choose an element ω of some critical partition for which $|\omega\cap K^+|>0$ holds. By Lemma 6.1 and Lemma 6.4, for a.e. $z\in\omega\cap K^+$ there exists an arbitrarily small neighborhood of z in W^u in which the set of points which eventually escape from R_0 has a definite proportion. It follows that z is not a Lebesgue density point of $\omega\cap K^+$. This yields a contradiction to the Lebesgue density theorem.

It is left to prove Lemma 6.4. The following elementary observation is used, on the top quadratic map $g_2\colon [-1,1] \circlearrowleft$, $g_2(x) = 1-2x^2$: $1/2$ is a repelling fixed point, and the set of preimages $\bigcup_{n\geq 0} g_2^{-n}(1/2)$ is dense in $[-1,1]$, not containing 0.

By a vertical curve we mean a curve such that the slopes of its tangent directions are $\gg 1$. Let $l_0 \subset W^s(Q)$ denote the segment in $W^u(P)$ which contains P and stretches across R_0. Clearly, l_0 is a vertical curve. Iterating l_0 backward, it is possible to choose an integer N_0 independent of b, and to define a sequence $l_0, l_1, \cdots, l_{N_0}$ of vertical curves in $W^s(P)$ which stretch across R_0, and with the property that any $C^2(b)$-curve as in the statement of the lemma intersects one of them in its middle third. This picture persists for all $a \in (a^{**}, a^*)$ sufficiently close to a^*. By the definition of a^{**}, $W^u(P)$ is not contained in $[-2,2]^2$. By Inclination lemma, the conclusion holds. $\square$

7. Dynamics of Lebesgue typical points

In this last section we show $\bigcap_{n\geq 0} f^{-n}(R_0)$ has zero Lebesgue measure, and completes the proof of the theorem. The main step is a statistical argument, which enables us to show that *the occurrence of infinitely many close returns is improbable.* This sort of argument has been successfully undertaken by Benedicks and Viana [6] in the attractor context. We adapt it to our non-attracting context, with the help of the geometric structure of critical regions in Proposition 5.3. In addition, we dispense with any assumption on the Jacobian, which was assumed in [6, 30].

As a preliminary step, in Sect.7.1 we construct a family long stable leaves near each critical point. In Sect.7.2, using these leaves we define *critical rectangles.* In Sect.7.3 we introduce *close return times* as particular return times to critical rectangles, and show that the theorem follows from Proposition 7.2, which states that the occurrence of infinitely many close return times is improbable.

For the proof of Proposition 7.2, based on preliminary geometric constructions in Sect.7.4, 7.5, we construct in Sect.7.6 an infinite nested sequence $\Omega_0 \supset \Omega_1 \supset \cdots$. Each Ω_k is decomposed into rectangles, bordered by stable leaves and pieces of W^u and denoted by $R_{i_0\cdots i_k}$. The sequence $(i_0, \cdots, i_k)$ records the recurrent behavior of the iterates of the rectangle to the critical set. Combining these geometric ingredients with key analytic estimates in Sect.7.7,7.8, we complete the proof of Proposition 7.2 in Sect.7.9.

7.1. Construction of long stable leaves.

For the purpose of stating the next proposition, we introduce a *distance $d_{\mathcal{C}}(\cdot)$ to $\mathcal{C}$* as follows. Let $z \in W^u \setminus \bigcup_{n>0} f^n(\mathcal{C})$ and suppose that z is free. If $z \notin I(\delta)$, then let $d_{\mathcal{C}}(z) = |x|$, where $z = (x,y)$. Otherwise, let $\zeta \in \mathcal{C}$ denote the binding point for z and let $d_{\mathcal{C}}(z) = |z-\zeta|$. If z is bound, then $d_{\mathcal{C}}(z)$ is undefined. For a free segment ω, let $d_{\mathcal{C}}(\omega) = \min_{z\in\omega} d_{\mathcal{C}}(z)$.

The next proposition indicates the existence of a family of long stable leaves near each critical value. In addition, these leaves have a slow recurrence property to $\mathcal{C}$.

Proposition 7.1. (Long stable leaves through slowly recurrent points) *Let ζ be a critical point on a free segment γ. For each element ω_0 of the critical partition of γ there exists $z \in \omega_0$ such that $d_{\mathcal{C}}(f^n(z)) \geq \delta e^{-5\alpha n}$ holds for every $n > 0$ such that $f^n(z)$ is free. In addition, the long stable leaf through $f(z)$ exists.*

Proof. We divide the proof into three steps. First, we prove the existence of $z \in \omega_0$ with the property as in the first half of the statements. Next, we give an angle estimate. Finally, we show the existence of long stable leaves through $f(z)$.

Step1. Construction of slowly recurrent points. Let $n_0 = 0$. Let p_0 denote the bound period of ω_0. Let $\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2, \cdots$ denote the sequence of partitions of ω_0 constructed in the same way as in Sect.6.1. We construct a (possibly finite) sequence $p_0 \leq n_1 < n_2 < \cdots$ and a nested sequence $\omega_0 \supset \omega_1 \supset \omega_2 \supset \cdots$ of curves for which the following holds for every $k \geq 0$. Obviously, any point in the intersection $\bigcap_{k\geq 0} \omega_k$ satisfies the desired property:

- $\omega_k \in \mathcal{P}_{n_k}$, and for every $0 \leq n \leq n_k$ such that $f^n(\omega_k)$ is free, $d_{\mathcal{C}}(f^n\omega_k) \geq \delta e^{-5\alpha n}$;
- n_{k+1} is a cutting time of ω_k. If there exists no cutting time of ω_k, then n_{k+1} is undefined.

The construction of the sequence is by induction that is described as follows. Given n_k, $\omega_k \in \mathcal{P}_{n_k}$ such that $f^{n_k}(\omega_k) \subset I(\frac{11}{10}\delta)$, with a bound period p_k, define $n_{k+1} \geq n_k + p_k$ to be the cutting time of ω_k. We claim that $f^{n_{k+1}}(\omega_k)$ is a free segment of length $\geq \delta e^{-5\alpha n_{k+1}}$. Indeed, by Lemma 5.1, $f^{n_k+p_k}(\omega_k)$ is a free segment of length $\geq e^{-4\alpha p_k}$. Using Lemma 2.1 from time n_k+p_k to n_{k+1}, $|f^{n_{k+1}}(\omega_k)| \geq \delta |f^{n_k+p_k}(\omega_k)| \geq \delta e^{-4\alpha n_{k+1}}$. Hence, it is possible to take an element $\omega_{k+1} \in \mathcal{P}_{n_{k+1}}$ such that $\omega_{k+1} \subset \omega_k$, $f^{n_{k+1}}(\omega_{k+1}) \subset I(\frac{11}{10}\delta)$ and $d_{\mathcal{C}}(f^{n_{k+1}}\omega_{k+1}) \geq \delta e^{-5\alpha n_{k+1}}$. To recover the assumption of the induction, it suffices to show $d_{\mathcal{C}}(f^n\omega_k) \geq \delta e^{-5\alpha n}$ for every $n_k + p_k \leq n < n_{k+1}$ such that $f^n(\omega_k)$ is free. If $f^n(\omega_k) \cap I(\delta) = \emptyset$, then $d_{\mathcal{C}}(f^n\omega_k) \geq \delta \geq \delta e^{-5\alpha n}$ holds. To treat the case where n is a free return time, we need

Sublemma 7.1. *Let $\tilde{n}_1 < \cdots < \tilde{n}_s$ denote all the free return times of ω_k in $[n_k + p_k, n_{k+1})$, with $\tilde{p}_1, \cdots, \tilde{p}_s$ the corresponding bound periods. Then*

$$\tilde{p}_1 + \cdots + \tilde{p}_s \leq \frac{13\alpha p_k}{\lambda}.$$

Proof. Splitting the time interval $[n_k + p_k, n_{k+1})$ into bound and free segments, for all $z \in f^{n_k+p_k}(\omega_k)$ we have $\|Df^{n_{k+1}-n_k-p_k}(z)t(z)\| \geq e^{\frac{\lambda}{3}(\tilde{p}_1+\cdots+\tilde{p}_s)}$. Combining this with $|f^{n_k+p_k}(\omega_k)| \geq e^{-4\alpha p_k}$ from Lemma 5.1, $3 > |f^{n_{k+1}}(\omega_k)| \geq e^{\frac{\lambda}{3}(\tilde{p}_1+\cdots+\tilde{p}_s)-4\alpha p_k}$. The first inequality is due to the elementary fact that the forward iterates of ω_k cannot grow to a free segment of length > 3 without intersecting $I(\delta)$. Taking logs we obtain the desired inequality. $\square$

For each $\tilde{n}_i$ we have $d_{\mathcal{C}}(f^{\tilde{n}_i}(\omega)) \geq e^{-\frac{\log C_0}{3}\tilde{p}_i} \geq e^{-\frac{5\alpha \log C_0}{\lambda}p_k} \geq \delta e^{-5\alpha n}$. The last inequality follows from $p_k \leq -\frac{3}{\lambda}\log(\delta e^{-5\alpha n_k})$.

Step2. Angle estimates. We introduce a useful language along the way.

Definition 7.1. Let $z \in I(\delta) \setminus \mathcal{C}$. A tangent vector v at z is in *tangential position* relative to $\zeta \in \mathcal{C}$ if there exists a horizontal curve γ which is tangent to both v and $t(\zeta)$.

Let $z \in \omega_0$ have the property in Lemma 7.1. Let $\theta_n = \text{angle}(Df^n(z)t(z), w_n(z))$. Let $0 =: n_0 < n_1 < n_2 < \cdots$ denote all the free return times of z, with $\zeta_0, \zeta_1, \zeta_2, \cdots$ the corresponding binding points. The next lemma allows us to use ζ_k as a binding point for $w_{n_k}(\zeta)$.

Lemma 7.1. *For every free return time $n_k > 0$ of z, $\theta_{n_k} \leq (Cb)^{\frac{n_k}{3}}$ holds. In addition, $w_{n_k}(z)$ is in tangential position relative to ζ_k.*

Proof. Let p_k denote the binding period for n_k. The next three angle estimates follow from [[27] Sublemma 3.2.]:

$$\theta_{p_0} \leq \theta_1 (Cb)^{(p_0-1)/2} \frac{\|Df(z)t(z)\|}{\|Df^{p_0}(z)t(z)\|} \frac{\|w_1(z)\|}{\|w_{p_0}(z)\|} \leq (Cb)^{p_0/3}; \tag{23}$$

$$\theta_{n_{k+1}} \leq \theta_{n_k+p_k} (Cb)^{(n_{k+1}-n_k-p_k)/2} \frac{\|Df^{n_k+p_k}(z)t(z)\|}{\|Df^{n_{k+1}}(z)t(z)\|} \frac{\|w_{n_k+p_k}(z)\|}{\|w_{n_{k+1}}(z)\|} \quad \text{for } k \geq 0; \tag{24}$$

$$\theta_{n_k+p_k} \leq \theta_{n_k} (Cb)^{p_k/2} \frac{\|Df^{n_k}(z)t(z)\|}{\|Df^{n_k+p_k}(z)t(z)\|} \frac{\|w_{n_k}(z)\|}{\|w_{n_k+p_k}(z)\|} \quad \text{for } k \geq 1. \tag{25}$$

Using these, we prove the statement by induction on k. Take $k = 0$ in (24). By (23) and Lemma 2.1, the two fractions of the right-hand side are $\leq 1/\delta$ and $\theta_{n_1} \leq (Cb)^{n_1/3}$ holds. This estimate and the distance bound in Lemma 7.1 implies that $w_{n_1}(z)$ is in tangential position relative to ζ_1. Then, taking $k = 1$ in (25) we get $\theta_{n_1+p_1} \leq (Cb)^{(n_1+p_1)/3}$. Taking $k = 2$ in (24) we get $\theta_{n_2} \leq (Cb)^{n_2/3}$, and that $w_{n_2}(z)$ is in tangential position relative to ζ_2, and so on. □

Step 3. The existence of long stable leaves. In view of Lemma 2.5, it suffices to show that $f(z)$ is expanding.

Lemma 7.2. *For $n \geq 1$, $\|w_n(z)\| \geq \min(\delta e^{\frac{\lambda n}{3}}, 4^{-\frac{3\alpha n}{\lambda}})$.*

Proof. The inequality clearly holds for every $1 \leq n \leq p_0$. Suppose that $f^n(z)$ is free. Applying Lemma 2.1 to each free segment and Proposition 2.1 to each bound segment, we have $\|w_n(z)\| \geq \delta e^{\frac{\lambda n-1}{3}}$. Suppose that $f^n(z)$ is bound, that is, $n_k < n < n_k + p_k$ holds for some n_k. A better estimate $\|w_{n_k+p_k}(z)\| \geq e^{\frac{\lambda}{3}(n_k+p_k-1)} \geq 1$ dropping the factor δ and $\|Df\| \leq 4$ give $\|w_n(z)\| \geq 4^{-(n_k+p_k-n)}\|w_{n_k+p_k}(z)\| \geq 4^{-p_k}$. Substituting $p_k \leq \frac{3\alpha n_k}{\lambda} \leq \frac{3\alpha n}{\lambda}$ into the exponent yields the desired estimate. This completes the proof of Lemma 7.2 and hence that of Proposition 7.1. □

7.2. **Critical rectangles.** Let $\mathcal{Q}^{(k)}$ denote any component of $\mathcal{C}^{(k)}$. Let ζ_0, ζ_1 denote the critical points on the horizontal boundaries of $\mathcal{Q}^{(k)}$. Take curves γ_0, γ_1 of length $\delta^{\frac{k}{10}}$ in the horizontal boundaries of $\mathcal{Q}^{(k)}$ so that: (i) γ_0 (resp. γ_1) contains ζ_0 (resp. ζ_1) within $\mathcal{O}(b^{\frac{k}{4}})$ of the midpoint of it; (ii) γ_0, γ_1 are connected by two vertical lines. Let $\mathcal{B}^{(k)} \subset \mathcal{Q}^{(k)}$ denote the region bordered by γ_0 is connected to γ_1 by the two vertical lines through their endpoints.

We construct a region $\mathcal{B}_0^{(k)} \subset \mathcal{B}^{(k)}$ as follows. Assume $\Gamma(\zeta_0)$ is at the right of $\Gamma(\zeta_1)$. Choose a point $z \in \gamma_1$ for which $\delta^k \leq |z - \zeta_1| \leq \delta^{\frac{k}{2}}$, and $d_{\mathcal{C}}(f^n z) \geq \delta e^{-5\alpha n}$ holds for every $n \geq 1$. Proposition 7.1 ensures the existence of such a point. For the reason explained in Sect.7.4, $\Gamma(z)$ intersects $f(\gamma_1)$ exactly at two points.

By (5), the Hausdorff distance between $\Gamma(z)$ and $\Gamma(\zeta_0)$ is $\leq C|f(z) - f(\zeta_1)| + C|f(\zeta_1) - f(\zeta_0)| \leq C\delta^{\frac{k}{2}}$. Hence, $\Gamma(z)$ intersects $f(\gamma_0)$ at one point. For the reason explained in Sect.7.4, $\Gamma(z)$ intersects $f(\gamma_0)$ exactly at two points. Define $\mathcal{B}_0^{(k)}$ to be the region bordered by γ_0, γ_1 and the parabola $f^{-1}(\Gamma(z))$. By construction, the horizontal boundaries of $\mathcal{B}_0^{(k)}$ extend both sides around ζ_0, ζ_1 to length from $\approx \delta^k$ to $\approx \delta^{\frac{k}{2}}$. We call $\mathcal{B}_0^{(k)}$ a *critical rectangle* of order k. Let $\mathcal{A}^{(k)}$ denote the collection of all $\mathcal{B}_0^{(k)}$.

Definition 7.2. We say $z \in I(\delta)$ is *controlled up to time* $\nu > 0$ if $f^n(z) \notin \mathcal{A}^{(n)}$ holds for every $1 \leq n < \nu$. We say z is *controlled* if it is controlled up to any time.

The next lemma indicates that, if z is controlled, then there exists a long stable leaf through $f(z)$.

Lemma 7.3. *If $z \in I(\delta)$ is controlled up to time ν, then $\|w_n(z)\| \geq \delta^{\frac{12n\log 2}{\lambda}}$ holds for $1 \leq n < \nu$.*

Proof. We inductively define a sequence $0 < n_1 < n_1+p_1 \leq n_2 < n_2+p_2 \leq \cdots \leq n_s < n_s+p_s \leq \nu$ of integers and critical points $\zeta_1, \zeta_2, \cdots, \zeta_s$ such that: (i) $f^{n_l}(z) \in I(\delta)$ for each n_l, and $w_{n_l}(z)$ is in tangential position relative to ζ_l, with p_l the bound period and $|f^{n_l}(z) - \zeta_{n_l}| \geq \delta^{2n_l}$; (ii) n_{l+1} is the next time of returns to $I(\delta)$ after $n_l + p_l$.

Given n_l, ζ_l and p_l, let $n_{l+1} \geq n_l + p_l$ denote the smallest such that $f^{n_{l+1}}(z) \in I(\delta)$. By the assumption, $f^{n_{l+1}}(z) \notin \mathcal{A}^{(n_{l+1})}$ holds. Let k denote the largest integer such that $f^{n_{l+1}}(z) \in \mathcal{C}^{(k)}$, and let $\mathcal{Q}^{(k)}$ denote the component of $\mathcal{C}^{(k)}$ containing $f^{n_{l+1}}(z)$. By (S3), $f^{n_{l+1}}(z)$ is in tangential position relative to critical points on the horizontal boundaries of the component of $\mathcal{C}^{(k-1)}$ containing $\mathcal{Q}^{(k)}$. Choose one of them as ζ_{l+1}.

Suppose that $n_l < n < n_l+p_l$ holds. In the same way as in the proof of Lemma 7.2, we have $\|w_n(z)\| \geq 4^{-p_l}$. Substituting $p_l \leq \frac{6n_l}{\lambda}\log(1/\delta)$ into the exponent yields the desired inequality. For all other n it is immediate to show the desired inequality, in the same way as in the proof of Lemma 7.2. $\square$

7.3. **Infinitely many close returns are improbable.**

Definition 7.3. We say $z \in I(\delta)$ makes a *close return* at time ν if z is controlled up to time ν and $f^\nu(z) \in \mathcal{A}^{(\nu)}$. We say ν is a close return time of z.

Let $z \in I(\delta)$. Let $\nu_1, \nu_2, \cdots$ be defined inductively as follows: ν_1 is a close return time of z; given $\nu_1, \cdots, \nu_k$, let ν_{k+1} be the close return time of $f^{\nu_1+\nu_2+\cdots+\nu_k}(z) \in I(\delta)$. If $\nu_1, \cdots, \nu_k$ are defined in this way, we say z *has k close return times.* If the sequence is defined is indefinitely, we say z has *infinitely close return times.* Otherwise, we say z *only finitely many close return times.*

Let k_0 be a large integer. In what follows, we request that k_0 is sufficiently large, only finitely many times. Let Ω_∞ denote the set of all $z \in \mathcal{A}^{(k_0)}$ which has infinitely many close return times. We have $\Omega_\infty = \bigcap_{k\geq 1}\Omega_k$, where Ω_k denotes the set of all $z \in \mathcal{A}^{(k_0)}$ which has k close return times. The inclusion $\Omega_k \subset \Omega_{k-1}$ is obvious from the definition.

Proposition 7.2. $|\Omega_k|/|\Omega_{k-1}| \to 0$ *exponentially fast, as $k \to \infty$. In particular, Ω_∞ has zero Lebesgue measure.*

Let $\Lambda = \bigcap_{n\geq 0} f^{-n}(R_0)$. We show how $|\Lambda| = 0$ follows from this proposition. To see this, suppose $|\Lambda| > 0$. Lemma 2.1 indicates that Λ intersects $\bigcup_{n\geq 0} f^{-n}(I(\delta))$ in a set with positive Lebesgue measure. For almost every $z \in \Lambda \cap \bigcup_{n\geq 0} f^{-n}(I(\delta))$, define $m(z) > 0$ to be the smallest such that $f^{m(z)}(z)$ is controlled. Let us see $m(z)$ is well-defined. This is clear in the case $z \notin \bigcup_{n\geq 0} f^{-n}(\mathcal{A}^{(k_0)})$. Otherwise, take $i_0(z) \geq 0$ such that $f^{i_0(z)}(z) \in \mathcal{A}^{(k_0)}$. By Proposition 7.2, one of the following holds almost surely: either (i) $f^{i_0(z)}(z)$ is controlled, or else (ii) $f^{i_0(z)}(z)$ has only finitely many close return times, denoted by $\nu_1, \cdots, \nu_k$. By definition, $f^{i_0+\nu_1+\cdots+\nu_k}(z)$ is controlled.

Let $V_j = \{z \in \Lambda \cap \bigcup_{n\geq 0} f^{-n}(I(\delta))\colon m(z) = j\}$. Take j such that $|V_j| > 0$. By definition, any point in $f^j(V_j)$ is controlled, and thus $f^{j+1}(V_j)$ is foliated by long stable leaves. Consider the projection $\pi\colon f^{j+1}(V_j) \to \partial R_0$ along the long stable leaves. (5) says that π is Lipschitz continuous. In particular, $\pi(f^{j+1}(V_j))$ has positive one-dimensional Lebesgue measure in W^u. By the contraction along the leaves, $\pi(f^{j+1}(V_j)) \subset K^+$ holds. This yields a contradiction to Proposition 6.1.

The rest of this paper is devoted to the proof of Proposition 7.2. Before proceeding, let us give some estimates on close return times which will be used in the sequel. Let $z \in \Omega_k$, and let $\nu_1, \cdots, \nu_k$ denote the corresponding sequence of k close return times of z. By definition, for every $1 \leq l \leq k-1$, $f^{\nu_l}(z) \in \mathcal{A}^{(\nu_l)}$ holds. Let ζ denote any critical point on the horizontal boundary of the component of $\mathcal{A}^{(\nu_l)}$ containing $f^{\nu_l}(z)$. By definition, $|f^{\nu_l}(z) - \zeta| \leq C\delta^{\frac{\nu_l}{2}}$ holds. Then

$$|f^{\nu_l+i}(z) - f^i(\zeta)| \leq C\delta^{\frac{\nu_l}{2}} 4^i \ll e^{-\alpha i} \quad \text{for } 1 \leq i \leq 4\nu_l.$$

This implies

$$\nu_{l+1} \geq 4\nu_l \quad \text{for } 1 \leq l < k. \tag{26}$$

The same reasoning gives $\nu_1 \geq 4k_0$, and thus

$$\nu_l \geq 4^l k_0 \quad \text{for } 1 \leq l \leq k. \tag{27}$$

7.4. **Partitions of critical rectangles.** By a *rectangle* R we mean a compact region bounded by two disjoint curves in W^u and two disjoint stable leaves. The boundaries of R in W^u are called *unstable sides*. The boundaries in the stable leaves are called *stable sides*.

We define partitions of rectangles, using the families of long stable leaves constructed in Section 7.1. To this end, let us fix once and for all an enumeration $\mathcal{C} = \{\zeta_m\}_{m=1}^{\infty}$ of all the critical points and let γ_m denote the maximal free segment containing ζ_m. We deal with a rectangle R in $I(\delta)$ such that:

(R1) the unstable sides of R are made up of two free segments, each contained in γ_{m_0} and γ_{m_1}. In addition, $|\zeta_{m_0} - \zeta_{m_1}| \leq (Cb)^{\frac{k}{2}}$ holds for some $k \geq 1$;
(R2) the unstable sides of R extend to both sides around ζ_{m_0}, ζ_{m_1} to length $\approx \delta^k$;
(R3) $\Gamma(\zeta_{m_0})$ is at the right of $\Gamma(\zeta_{m_1})$;
(R4) there exists a long stable leaf Γ_∞ such that $f^{-1}(\Gamma_\infty)$ contains the stable sides of R.

One situation we have in mind is that two maximal free segments in ∂R_ν stretch across $\mathcal{B}_0^{(k)}$, where $k < \nu$. If this happens, then the region bounded by the two maximal free segments and the stable sides of $\mathcal{B}_0^{(k)}$ is a rectangle satisfying all the requirements.

By Lemma 7.1, in each element of the critical partition of γ_{m_1} there exists a point z such that the long stable leaf through $f(z)$ exists. Take just one such point from each element of the partition and denote the associated countable number of long stable leaves by Γ_Δ, $\Delta = -1, -2, -3, \cdots$ from the left to the right. We repeat essentially the same construction for γ_{m_0}. The difference is that, only those of the elements of the critical partition of γ_{m_0} come into play whose f-image is at the right of $\Gamma(\zeta_{m_1})$. We denote by Γ_Δ the associated countable number of long stable leaves at the right of $\Gamma(\zeta_{m_1})$, where $\Delta = 1, 2, 3, \cdots$ from the left to the right.

We claim that, if $\Delta > 0$, then $f^{-1}(\Gamma_\Delta)$ intersects the unstable side of R containing ζ_{m_0} exactly at two points, one on the right of ζ_{m_0} and the other on the left. To see this, Let

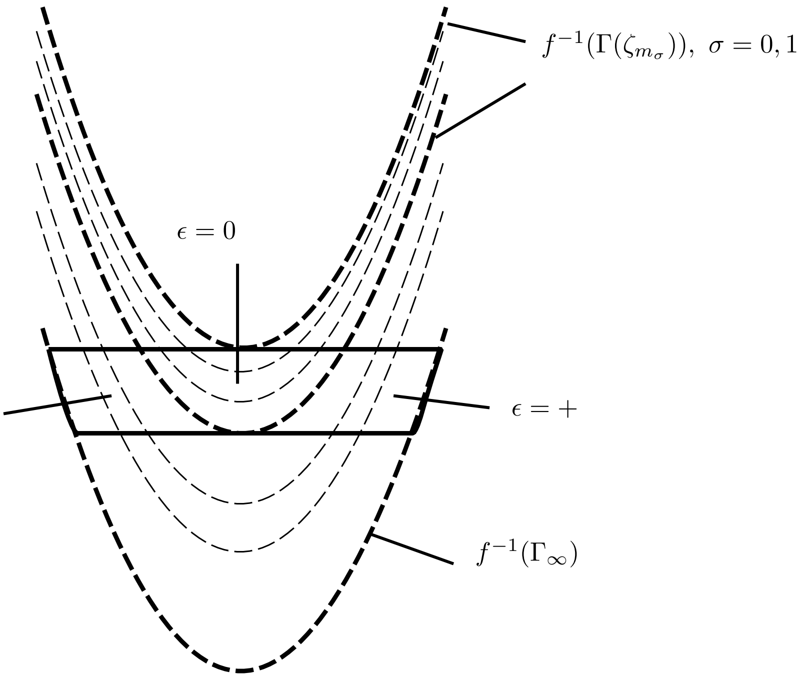

FIGURE 8. critical rectangle and its partition with long stable leaves

$z \in f^{-1}(\Gamma_\Delta)$ be on the unstable side and split $Df(z)t(z) = A(z)\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) + B(z)t_{\Gamma_\Delta}(z)$, where the first component of $t(z)$ is positive and $t_{\Gamma_\Delta}(z)$ denotes any unit vector tangent at $f(z)$ to Γ_Δ. The proof of Lemma 2.2 and (5) show $|A(z)| > 0$ and $A(z_1)A(z_2) > 0$ if and only if z_1, z_2 are on the same side of ζ_{m_0}. Hence the claim follows.

If $\Delta < 0$, then $f^{-1}(\Gamma_\Delta)$ intersects the stable side of R containing ζ_{m_0}. By the above claim, $f^{-1}(\Gamma_\Delta)$ intersects each of the unstable sides of R exactly at two points. These observations and the Lipschitz continuity of the tangent directions of long stable leaves as in (5) altogether indicate that the family of long stable leaves induces a partition of R. Each element of the partition is a rectangle, bounded by the unstable sides of R and two neighboring parabolas $f^{-1}(\Gamma_\Delta)$, $f^{-1}(\Gamma_{\Delta+1})$.

7.5. **Symbolic coding.** Each rectangle in the partition of R constructed in Section 7.4 is denoted by $R(\rho, \epsilon, \Delta, p)$. Here, the meanings of $(\rho, \epsilon, \Delta, p)$ are as follows:

- if the unstable sides of $R(\rho, \epsilon, \Delta, p)$ intersect both γ_{m_0} and γ_{m_1}, then $\rho = m_1$. Otherwise, $\rho = m_0$;
- if $\rho = m_0$, then $\epsilon = 0$. If $\rho = m_1$, then $\epsilon = +$ or $\epsilon = -$, depending on whether the unstable sides of $R(\rho, \epsilon, \Delta, p)$ is at the "right" or the "left" of ζ_{m_0} and ζ_{m_1};
- the stable sides of $f(R(\rho, \epsilon, \Delta, p))$ are contained in $\Gamma_\Delta \cup \Gamma_{\Delta+1}$.
- $p = \max\{p(\zeta_\rho, z) \colon z \in \gamma_\rho \cap R(\rho, \epsilon, \Delta, p)\}$.

The integer p is called a *bound period* of $R(\rho, \epsilon, \Delta, p)$. By the monotonicity of the function $p(\zeta_\rho, z)$, the maximum is attained at one of the edges of $R(\rho, \epsilon, \Delta, p)$. It is immediate to see the following:

(i) all points in $f(R(\rho, \epsilon, \Delta, p))$ are expanding up to time $p_{i_0} - 1$;

(ii) for all $\xi, \eta \in R(\rho, \epsilon, \Delta, p)$ and every $1 \le i \le p$, $\|w_i(\xi)\|/\|w_i(\eta)\| \le 2$.

Lemma 7.4. (Geometry of rectangles at the end of bound periods) *For all z in the unstable sides of $R(\rho,\epsilon,\Delta,p)$, $\|Df^p(z)t(z)\|\geq C\delta\|Df^i(z)t(z)\|$ holds for every $0\leq i<p$. In particular, the unstable sides of $f^p(R(\rho,\epsilon,\Delta,p))$ are made up of two $C^2(b)$-curves.*

Let us say that $z,z'\in\partial R_{i_0}$ belong to different unstable sides of R_{i_0} if they belong to different components of ∂R_{i_0}. Otherwise we say they belong to the same unstable side.

Proof. Let ζ denote the critical point on the unstable side of R which contains z. Let $p(\zeta,z),q(\zeta,z)$ denote the bound and fold periods of z with respect to ζ, as defined in Sect.5.2. In view of (ii) as above and (g) Proposition 5.2,

$$\|Df^i(z)t(z)\|\approx|\zeta-z|\cdot\|w_i(\zeta)\|\quad\text{for}\quad q(\zeta,z)\leq i\leq\max(p(\zeta,z),p).\tag{28}$$

Let ξ_1,ξ_2,ξ_3,ξ_4 denote the *edges*, namely, the points which belong to both the stable and the unstable sides of $R(\rho,\epsilon,\Delta,p)$. In the discussion to follow, we assume that ξ_1,ξ_2 are on the same unstable side of R, and $f(\xi_i),f(\xi_{i+2})$ $(i=1,2)$ are connected by the long stable leaf which contains the stable side of $f(R(\rho,\epsilon,\Delta,p))$.

Case 1: $\epsilon=0$. In this case, ξ_1,ξ_2,ξ_3,ξ_4 are on the same unstable side of R. We suppose that ξ_1 is closest to ζ. Then $p=\max(p(\zeta,\xi_1),p(\zeta,\xi_3))$ holds. (5) and Lemma 2.2 give $|\zeta-\xi_1|\approx|\zeta-\xi_3|$. Hence, (a,b) Proposition 5.2 gives

$$q(\zeta,z)\leq C\tilde\alpha\max(\log|\zeta-\xi_1|^{-1},\log|\zeta-\xi_3|^{-1})<p.$$

This means that (28) holds for $q(\zeta,z)\leq i\leq p$ and therefore

$$\frac{\|Df^p(z)t(z)\|}{\|Df^i(z)t(z)\|}\geq C\frac{\|w_p(\zeta)\|}{\|w_i(\zeta)\|}\geq C\delta.$$

For $1\leq i\leq q(\zeta,z)$,

$$\frac{\|Df^p(z)t(z)\|}{\|Df^i(z)t(z)\|}\geq\|Df^p(z)t(z)\|\geq C\delta|\zeta-z|\|w_{p(\zeta,z)}(\zeta)\|\geq C\delta^{\frac{\alpha}{\log C_0}}>\delta.$$

The first inequality follows from (h) Proposition 5.2. The second inequality follows from $\|w_p(\zeta)\|\geq C\delta\|w_{p(\zeta,z)}(\zeta)\|$. For the third inequality we have used $|\zeta-z|\|w_{p(\zeta,z)}(\zeta)\|\geq|\zeta-z|^{-1+\frac{\alpha}{\log C_0}}\geq\delta^{-1+\frac{\alpha}{\log C_0}}$ which follows from (e) Proposition 5.2.

Since the unstable sides of $R(\rho,\epsilon,\Delta,p)$ are $C^2(b)$, these two inequalities and the curvature estimate in [[27] Lemma 2.3] together imply that the unstable sides of $f^p(R(\rho,\epsilon,\Delta,p))$ are $C^2(b)$.

Case 2: $\epsilon=+$ or $-$. ξ_1 and ξ_3 (resp. ξ_2 and ξ_4) belong to different unstable sides of R. We suppose that $\Gamma(f(\xi_1))$ is at the right of $\Gamma(f(\xi_2))$, and that ξ_1 and ζ belong to the same unstable side of R. Let ζ' denote the other critical point of R on the unstable side of R. Then $p=\max\{p(\zeta,\xi_1),p(\zeta',\xi_3)\}$ holds. By (5) and Lemma 2.2 again, $|\zeta-z|\geq C|\zeta-\xi_1|$ and $|\zeta-z|\geq C|\zeta'-\xi_3|$. Hence, $q(\zeta,z)\leq C\tilde\alpha\log|\zeta-z|^{-1}<p$. This means that (28) holds for $q(\zeta,z)\leq i\leq p$. The rest of the argument is analogous to that in Case 1. □

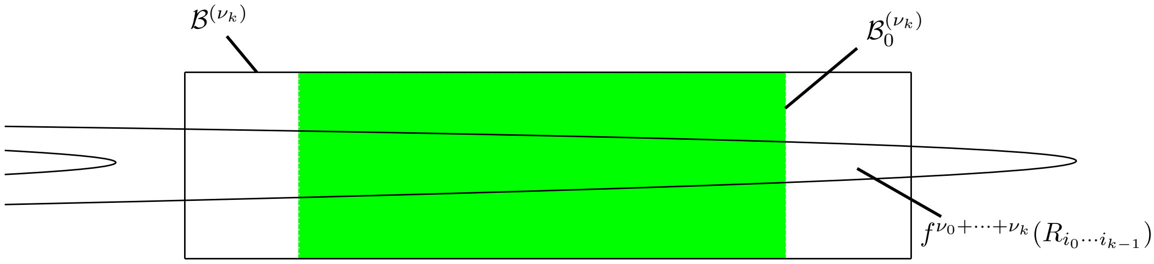

FIGURE 9. Situation in Proposition 7.3

7.6. **Construction of partitions.** Let $\Omega_0 = \mathcal{A}^{(k_0)}$. Putting the results in Sections 7.4, 7.5 together, for each $k \geq 0$ we inductively construct a partition of Ω_k into a countable number of rectangles. Each element of the partition of Ω_k will be denoted by $R_{i_0\cdots i_k}$, where $(i_0, \cdots, i_k)$ are itineraries which record the behavior of the rectangle under iteration, up to time $\nu_1 + \cdots + \nu_k$.

Construction of the partition of Ω_0. Take a component of $\mathcal{A}^{(k_0)}$ and denote it by R. Following the steps in Sect.7.4, define a partition of R with the family of long stable leaves. To each element of the partition, assign the set of symbols according to the rule described in Sect.7.5. Each element is denoted by R_{i_0}, where $i_0 = (\rho_0, \epsilon_0, \Delta_0, p_0)$ and $R_{i_0} = R(\rho_0, \epsilon_0, \Delta_0, p_0)$. We do the same construction for any component of $\mathcal{A}^{(k_0)}$.

Construction of the partition of Ω_k. Given the partition $\{R_{i_0\cdots i_{k-1}}\}_{i_0,\cdots,i_{k-1}}$ of Ω_{k-1} for some $k \geq 1$, define $R_{i_0\cdots i_{k-1}}(\nu_k) = \{z \in R_{i_0\cdots i_{k-1}} : \nu_k$ is a close return time of $f^{\nu_0+\cdots+\nu_{k-1}}(z)\}$. Here and for the rest of this section we adopt the next

Convention. $\nu_0 = 0$.

By definition, $\Omega_k = \bigcup_{(i_0,\cdots,i_{k-1})} \bigcup_{\nu_k} R_{i_0\cdots i_{k-1}}(\nu_k)$ holds.

Proposition 7.3. (Geometry of rectangles at close return times) *Let $z \in f^{\nu_0+\cdots+\nu_{k-1}}(R_{i_0\cdots i_{k-1}})$ and suppose $f^{\nu_k}(z) \in \mathcal{B}_0^{(\nu_k)} \subset \mathcal{B}^{(\nu_k)}$. Then the unstable sides of $f^{\nu_1+\cdots+\nu_k}(R_{i_0\cdots i_{k-1}}) \cap \mathcal{B}^{(\nu_k)}$ are $C^2(b)$-curves stretching across $\mathcal{B}^{(\nu_k)}$.*

We finish the construction of the partition of Ω_k assuming the conclusion of the proposition.

Take a component of $f^{\nu_1+\cdots+\nu_k}(R_{i_0\cdots i_{k-1}}(\nu_k))$ and denote it by R. By the proposition and the geometric structure of critical regions in Proposition 5.3, on each unstable side of R there exists a critical point, within $\mathcal{O}(b^{\frac{\nu_k}{8}})$ of its midpoint. In particular, R meets all the requirements (R1-4) in Sect.7.4. Following the steps in Sect.7.4, 7.5, define a partition of R with the family of long stable leaves and assign to each element the set of symbols. Let $R_{i_0\cdots i_{k-1}i_k} = f^{-(\nu_1+\cdots+\nu_k)}(R(\rho_k, \epsilon_k, \Delta_k, p_k))$, where $i_k = (\rho_k, \epsilon_k, \Delta_k, p_k, \nu_k)$. We do the same construction for any component of $f^{\nu_1+\cdots+\nu_k}(R_{i_0\cdots i_{k-1}}(\nu_k))$. This finishes the construction of the partition of Ω_k.

Proof of Proposition 7.3. Let $\Gamma_{\nu_k-1}(z) = \{(x(y), y) \colon |y| \leq \sqrt{b}\}$. Consider the vertical strip $V = \{(x, y) \colon |x - x(y)| \leq \delta^{\frac{\nu_k}{20}}, |y| \leq \sqrt{b}\}$.

Lemma 7.5. *V does not intersect the stable sides of $f^{\nu_0+\cdots+\nu_{k-1}+1}(R_{i_0\cdots i_{k-1}})$.*

Proof. Let σ denote any stable side of $f^{\nu_0+\cdots+\nu_{k-1}}(R_{i_0\cdots i_{k-1}})$. By construction, there exists $y\in W^u\cap\sigma$ such that $d_{\mathcal{C}}(f^n(y))\geq \delta e^{-5\alpha n}$ holds whenever $f^n(y)$ is free, and $f(\sigma)\subset\Gamma(y)$. Suppose $V\cap f(\sigma)\neq\emptyset$, and let $\xi\in V\cap f(\sigma)$. Let η denote the point of intersection between Γ and the horizontal through ξ. The definition of V gives $|\xi-\eta|\leq \delta^{\frac{\nu_k}{20}}$, and thus $|f^{\nu_k-1}(\eta)-f^{\nu_k-1}(\xi)|\leq \delta^{\frac{\nu_k}{21}}$. Since $\eta\in\Gamma$, $|f^{\nu_k}(z)-f^{\nu_k-1}(\eta)|\leq (Cb)^{\nu_k-1}$ holds. Hence $|f^{\nu_k}(z)-f^{\nu_k-1}(\xi)|\leq (Cb)^{\frac{\nu_k}{10}}$ follows. Meanwhile $|f^{\nu_k-1}(\xi)-f^{\nu_k-1}(y)|\leq (Cb)^{\nu_k-1}$ holds, and the assumption on z gives $|\zeta-f^{\nu_k}(z)|\leq C\delta^{\frac{\nu_k}{2}}$, where ζ is any critical point on the unstable sides of $\mathcal{B}_0^{(\nu_k)}$. Therefore

$$|\zeta-f^{\nu_k-1}(y)|\leq|\zeta-f^{\nu_k}(z)|+|f^{\nu_k}(z)-f^{\nu_k-1}(\xi)|+|f^{\nu_k-1}(\xi)-f^{\nu_k-1}(y)|\leq\delta^{\frac{\nu_k}{21}}.$$

This estimate and the proof of Corollary 5.1 together indicate that $f^{\nu_k-1}(y)$ is free. Hence, Proposition 7.1 gives a critical point ζ' relative to which $|\zeta'-f^{\nu_k-1}(y)|\geq\delta e^{-5\alpha\nu_k}$. Then it is possible to choose a horizontal curve γ such that both ζ and ζ' are on γ. This is a contradiction. □

By Lemma 7.5, V cuts a segment in each unstable side of $f^{\nu_0+\cdots+\nu_{k-1}+1}(R_{i_0\cdots i_{k-1}})$, denoted by γ. Let ζ' denote the critical point on the same unstable side of $f^{\nu_0+\cdots+\nu_{k-1}}(R_{i_0\cdots i_{k-1}})$ as that of $f^{-1}(\gamma)$. Let z' be an arbitrary point in γ. Let p_{k-1} denote the bound period of $f^{\nu_0+\cdots+\nu_{k-1}}(R_{i_0\cdots i_{k-1}})$. As z is controlled up to time ν_k, the distortion control gives $\|Df^j(f(z))\|\approx\|Df^j(z')\|\approx\|w_j(z)\|$ for $1\leq j<\nu_k$, and thus for $p_{k-1}-1\leq j<\nu_k$,

$$\|Df^jt(z')\|\approx|f^{-1}(z')-\zeta'|\cdot\|Df^j(z')\|\approx|f^{-1}(z')-\zeta'|\cdot\|w_j(z)\|. \tag{29}$$

By Lemma 7.4, $f^{p_{k-1}-1}(\gamma)$ is $C^2(b)$. Then, by [[27] Lemma 2.3.] and (29), the curvature of $f^{\nu_k-1}(\gamma)$ is everywhere bounded from above by

$$(Cb)^{\nu_k-p_{k-1}}\frac{\|w_{p_{k-1}}(z)\|^3}{\|w_{\nu_k}(z)\|^3}\sqrt{b}+\sum_{j=p_{k-1}}^{\nu_k-1}(Cb)^{\nu_k-j-1}\frac{\|w_{j+1}(z)\|^3}{\|w_{\nu_k}(z)\|^3}.$$

Since $\|w_{\nu_1}(z)\|\geq C\delta\|w_{j+1}(z)\|$ for $p_{k-1}\leq j<\nu_k$, it follows that the curvature of $f^{\nu_k-1}(\gamma)$ is everywhere $\leq\sqrt{b}$. (29) also implies that the slopes of the tangent directions of $f^{\nu_k-1}(\gamma)$ are $\leq\sqrt{b}$. Hence, $f^{\nu_k-1}(\gamma)$ is a $C^2(b)$-curve.

Parametrize γ by arc length s. Using $\|w_{\nu_k}(z)\|\geq1$, $|\zeta-f^{-1}(\gamma(s))|\geq e^{-\frac{1}{3}p_{k-1}\log C_0}$ for all s and the fact that the width of the strip V is $\delta^{\frac{\nu_k}{20}}$,

$$\int\|Df^{\nu_k-1}(\gamma(s))t(\gamma(s))\|ds\geq C\|w_{\nu_k}(z)\|\int|\zeta-f^{-1}(\gamma(s))|ds\geq Ce^{-\frac{1}{3}\nu_k\log C_0}\delta^{\frac{\nu_k}{20}}\gg\delta^{\frac{\nu_k}{10}}.$$

This implies that $f^{\nu_k-1}(\gamma)$ stretches across $\mathcal{B}^{(\nu_k)}$. This completes the proof of (a). □

7.7. **Unstable sides are roughly parallel.** Central to the proof of Proposition 7.2 is an estimate of the measure of the set

$$R_{i_0\cdots i_{k-1}}(\nu_k)=\{z\in R_{i_0\cdots i_{k-1}}\colon \nu_k \text{ is a close return time of } f^{\nu_0+\cdots+\nu_{k-1}}(z)\}.$$

This subsection and the next are devoted to obtaining this estimate. For the purpose of stating the next proposition we need some definitions.

- Choose C_1, C_2 as follows: $|\det Df|\geq C_1$ on R_0; for all ξ,η in the unstable sides of any component of $\mathcal{A}^{(k_0)}$, $\mathrm{angle}(u(\xi),u(\eta))\leq C_2|\xi-\eta|$. Let $C_3=C_0e^{\frac{6}{\log C_0}}$.

• We attach a *collar* to each rectangle in the following way. For each $R_{i_0} \subset \Omega_0$, let $Q(R_{i_0})$ denote the component of $\mathcal{A}^{(k_0)}$ containing R_{i_0}. Let $k \geq 1$. For each $R_{i_0\cdots i_k} \subset \Omega_k$, By Proposition 7.3, there exists exactly one component $\mathcal{B}^{(\nu_k)}$ of $\mathcal{A}^{(\nu_k)}$ containing $f^{\nu_1+\cdots+\nu_k}(R_{i_0\cdots i_k})$. Let $Q(R_{i_0\cdots i_k})$ denote the component of $f^{-(\nu_1+\cdots+\nu_k)}(\mathcal{B}^{(\nu_k)}) \bigcap R_{i_0\cdots i_{k-1}}$ containing $R_{i_0\cdots i_k}$.

• For any z in a free segment of W^u, let $u(z)$ denote the unit vector tangent to W^u at z such that the sign of the first component is positive.

Proposition 7.4. *For every $j \geq 0$ and any ξ, η in the unstable side of $f^{\nu_0+\cdots+\nu_j}Q(R_{i_0\cdots i_j})$,*

$$\text{angle}(u(\xi), u(\eta)) \leq C_2 C_3^{3\nu_j}|\xi - \eta|. \tag{30}$$

Proof of Proposition 7.4. We argue by induction on j. The choice of C_2 and the convention $\nu_0 = 0$ give (30) for $j = 0$. Let $k \geq 1$ and assume (30) for $j = k - 1$.

Lemma 7.6. *For any ξ, η in the unstable sides of $f^{\nu_0+\cdots+\nu_{k-1}+1}Q(R_{i_0\cdots i_{k-1}})$,*

$$\text{angle}(Df(\xi)u(\xi), Df(\eta)u(\eta)) \leq C_2 C_3^{\nu_k}|f(\xi) - f(\eta)|.$$

Proof. Let $\theta_i = \text{angle}(Df^i(\xi)u(\xi), Df^i(\eta)u(\eta))$, $i = 0, 1$. We have

$$\theta_1 \leq \frac{Cb\theta_0 + C|\xi - \eta|}{\|Df(\xi)u(\xi)\|\|Df(\eta)u(\eta)\|}.$$

Hence $\theta_1 \ll 1$, provided k_0 is sufficiently large. Hence

$$\theta_1 \leq CC_1^{-1}\left(|\xi - \eta| + \text{angle}(u(\xi), u(\eta))\right).$$

The inequality follows from the following elementary fact: for any nonzero vectors u, v such that $\text{angle}(u, v) \ll 1$, $\text{angle}(u, v) \leq 2|u - v|/\min\{\|u\|, \|v\|\}$. (30) with $j = k - 1$ and $|\xi - \eta| \leq C_1^{-1}|f(\xi) - f(\eta)|$ give

$$|\xi - \eta| + \text{angle}(u(\xi), u(\eta)) \leq 2C_1^{-1}C_2C_0^{3\nu_{k-1}}|f(\xi) - f(\eta)|.$$

Replacing this in the previous inequality,

$$\theta_1 \leq CC_1^{-2}C_2C_3^{3\nu_{k-1}}|f(\xi) - f(\eta)| \leq C_2C_3^{\nu_k}|f(\xi) - f(\eta)|.$$

The last inequality holds for sufficiently large k_0, because of $C_3^{4\nu_{k-1}} \leq C_3^{\nu_k}$ from (26). □

For any ξ on the unstable sides of $f^{\nu_0+\cdots+\nu_{k-1}+1}Q(R_{i_0\cdots i_k})$, let

$$v(\xi) = \rho \cdot Df(f^{-1}(\xi))u(f^{-1}(\xi)), \tag{31}$$

where $\rho > 0$ is the normalizing constant. If k_0 is sufficiently large, then $v(\xi)$ has a large slope. By the definition of $u(\cdot)$, the sign of the second component of $v(\xi)$ is constant for all ξ.

By Proposition 7.3 and the distortion control, the contractive field e_{ν_k-1} is well-defined on $f^{\nu_0+\cdots+\nu_{k-1}+1}Q(R_{i_0\cdots i_k})$. Fix once and for all the orientation of e_{ν_k-1} so that the second component of e_{ν_k-1} and that of $v(\xi)$ have the same sign. Let f_{ν_k-1} denote the unit vector field orthogonal to e_{ν_k-1}. Split $v(\xi) = A(\xi)e_{\nu_k-1}(\xi) + B(\xi)f_{\nu_k-1}(\xi)$.

Lemma 7.7. *For any ξ_1, ξ_2 on the unstable sides of $f^{\nu_0+\cdots+\nu_{k-1}+1}Q(R_{i_0\cdots i_k})$,*

$$\max\{|A(\xi_1) - A(\xi_2)|, |B(\xi_1) - B(\xi_2)|\} \leq 2C_2C_3^{\nu_k}|\xi_1 - \xi_2|.$$

Proof. The following elementary fact is used. For $u_i = \binom{\cos\theta_i}{\sin\theta_i}$, $0 \le \theta_i \le \pi$, $i = 1,2,3,4$,

$$|\mathrm{angle}(u_1,u_2) - \mathrm{angle}(u_3,u_4)| \le \mathrm{angle}(u_1,u_3) + \mathrm{angle}(u_2,u_4).$$

This can be checked using $\mathrm{angle}(u_i,u_j) = |\theta_i - \theta_j|$ and the triangle inequality.

We have $A(\xi_i) = \langle e_{\nu_k-1}(\xi_i), v(\xi_i)\rangle = \cos(\mathrm{angle}(e_{\nu_k-1}(\xi_i), v(\xi_i)))$, where the bracket denotes the scholar product. By the above definition of e_{ν_k-1}, $\mathrm{angle}(e_{\nu_k-1}(\xi_i), v(\xi_i)) \in [0,\pi]$ (in fact, it is ≈ 0). Considering $\arccos\colon [-1,1] \to [0,\pi]$ and $|\arccos'| \ge 1$ we have $|A(\xi_1) - A(\xi_2)| \le |\arccos(A(\xi_1)) - \arccos(A(\xi_2))|$, and

$$\begin{aligned}|\arccos(A(\xi_1)) - \arccos(A(\xi_2))| &= |\mathrm{angle}(e_{\nu_k-1}(\xi_1), v(\xi_1)) - \mathrm{angle}(e_{\nu_k-1}(\xi_2), v(\xi_2))\\ &\le \mathrm{angle}(v(\xi_1), v(\xi_2)) + \mathrm{angle}(e_{\nu_k-1}(\xi_1), e_{\nu_k-1}(\xi_2))\\ &\le 2C_2C_3^{\nu_k}|\xi_1 - \xi_2|.\end{aligned}$$

The first factor in the second line is bounded by Lemma 7.6. The second factor is bounded by Lemma 2.3. In the same way, we have $B_i = \langle f_{\nu_k-1}(\xi_i), v(\xi_i)\rangle = \cos(\mathrm{angle}(f_{\nu_k-1}(\xi_i), v(\xi_i)))$ and $\mathrm{angle}(f_{\nu_k-1}(\xi_i), v(\xi_i)) \in [0,\pi]$ (in fact, it is $\approx \pi/2$). Then

$$\begin{aligned}|\arccos(B(\xi_1)) - \arccos(B(\xi_2))| &= |\mathrm{angle}(f_{\nu_k-1}(\xi_1), v(\xi_1)) - \mathrm{angle}(f_{\nu_k-1}(\xi_2), v(\xi_2))|\\ &\le \mathrm{angle}(v(\xi_1), v(\xi_2)) + \mathrm{angle}(f_{\nu_k-1}(\xi_1), f_{\nu_k-1}(\xi_2))\\ &\le 2C_2C_3^{\nu_k}|\xi_1 - \xi_2|.\end{aligned}$$

For the last inequality we have used the orthogonality of f_{ν_k-1} to e_{ν_k-1}. □

Lemma 7.8. *There is a C^1 vector field ϕ_0 on $f^{\nu_0+\cdots+\nu_{k-1}+1}Q(R_{i_0\cdots i_k})$ such that $\|\phi_0\| \le 2$ and $\|D\phi_0\| \le 4C_2C_3^{2\nu_k}$.*

Proof. Recall the symbolic coding $i_{k-1} = (\rho_{k-1}, \epsilon_{k-1}, *, p_{k-1}, *)$. We deal with two cases separately.

Case1: $\epsilon_{k-1} \neq 0$. We introduce a C^1 coordinate $(\hat{x},\hat{y})$ on $f^{\nu_0+\cdots+\nu_{k-1}+1}Q(R_{i_0\cdots i_{k-1}})$ such that $9/10 \le \|\partial_{\hat{x}}\| \le 10/9$, $\|\partial_{\hat{y}}\| = 1$, $\langle\partial_{\hat{x}}, \partial_{\hat{y}}\rangle = 0$, $\langle\partial_{\hat{y}}, t(f\zeta_1)\rangle = 1$, $\Gamma(\zeta_1) = \{\hat{x} = 0\}$, $\Gamma(\zeta_2) = \{\hat{x} = c\}$, where $\zeta_1 = \zeta_{\rho_{k-1}}$ and ζ_2 is the critical point other than ζ_1 on the unstable side of $f^{\nu_0+\cdots+\nu_{k-1}}Q(R_{i_0\cdots i_{k-1}})$, and c is a constant. It is possible to choose such a coordinate, by the Lipschitz continuity of the tangent directions of $\Gamma(\zeta_1)$ and $\Gamma(\zeta_2)$ as in (5). Let $T\colon (x,y) \to (\hat{x},\hat{y})$ denote the coordinate transformation.

With respect to $(\hat{x},\hat{y})$-coordinate, we represent the unstable sides of $f^{\nu_0+\cdots+\nu_{k-1}+1}Q(R_{i_0\cdots i_k})$ by graphs of functions γ_1, γ_2, $\gamma_1(\hat{x}) < \gamma_2(\hat{x})$. For all ξ in the unstable sides the rectangle, let

$$(\gamma_2(\hat{x}) - \gamma_1(\hat{x}))\cdot v(\xi) = \tilde{A}(\xi)e_{\nu_k-1}(\xi) + \tilde{B}(\xi)f_{\nu_k-1}(\xi), \tag{32}$$

where $v(\xi)$ is the one in (31) and $T(\xi) = (\hat{x},\hat{y})$. We extend $\tilde{A}, \tilde{B}$ to C^1 functions on the entire $f^{\nu_0+\cdots+\nu_{k-1}+1}Q(R_{i_0\cdots i_k})$ so that $\max(\|\tilde{A}\|, \|\tilde{B}\|) \le 1$ and $\max(\|D\tilde{A}\|, \|D\tilde{B}\|) \le 3C_2C_3^{2\nu_k}$. For all z in the rectangle, define

$$\phi_0(z) = \tilde{A}(z)e_{\nu_k-1}(z) + \tilde{B}(z)f_{\nu_k-1}(z). \tag{33}$$

Of course, ϕ_0 is tangent to the unstable sides of the rectangle. Since $\|De_{\nu_{k-1}}\|$, $\|Df_{\nu_{k-1}}\|$ are bounded by Lemma 2.3, this yields the desired inequality.

To simplify notation, write $A \circ T^{-1}$ for A, and the same for B, $\tilde{A}$, $\tilde{B}$. On the assumption that both $\gamma_1(\hat{x})$ and $\gamma_2(\hat{x})$ make sense, we extend $\tilde{A}$ affinely along the $\hat{y}$-direction. In other words, for $\hat{y} \in [\gamma_1(\hat{x}), \gamma_2(\hat{x})]$, define

$$\tilde{A}(\hat{x}, \hat{y}) = \tilde{A}(\hat{x}, \gamma_1(\hat{x})) + (\hat{y} - \gamma_1(\hat{x}))\,(A(\hat{x}, \gamma_2(\hat{x})) - A(\hat{x}, \gamma_1(\hat{x}))) . \tag{34}$$

In the same way, we extend $\tilde{B}$ affinely along the $\hat{y}$-direction. If, for instance, $\gamma_1(\hat{x})$ makes sense and $\gamma_2(\hat{x})$ does not, we enlarge the domain of definition of γ_2 so that $\gamma_2(\hat{x})$ makes sense. It is possible to show, using the long stable leaf of order $\nu_k - 1$ through $\gamma_1(\hat{x})$, that $\gamma_2(\hat{x})$ is sufficiently close to the unstable sides of the rectangle, so that all the preceding arguments go through.

The definition gives $\max(\|\tilde{A}\|, \|\tilde{B}\|) \leq \gamma_2(\hat{x}) - \gamma_1(\hat{x}) \ll 1$. Lemma 7.7 and the choice of $(\hat{x}, \hat{y})$-coordinate give $\max(\|\partial_{\hat{y}}\tilde{A}\|, \|\partial_{\hat{y}}\tilde{B}\|) \leq 3C_2C_3^{\nu_k}$. To evaluate the norms of $\hat{x}$-derivatives, we assume that ζ_σ and $f^{-1}(\gamma_\sigma(\hat{x}))$ belong to the same unstable side, $\sigma = 1, 2$. The choice of the coordinate implies

$$\left|\frac{d\gamma_\sigma}{d\hat{x}}(\hat{x})\right| \leq \frac{C}{|f^{-1}(\hat{x}, \gamma_\sigma(\hat{x})) - \zeta_\sigma|} \leq e^{\frac{3p_{k-1}}{\log C_0}} \leq e^{\frac{3\nu_k}{\log C_0}}. \tag{35}$$

Sublemma 7.7 gives

$$\left|\frac{dA}{d\hat{x}}(\hat{x}, \gamma_\sigma(\hat{x}))\right| \leq 3C_2C_3^{\nu_k}\left|\frac{d\gamma_\sigma}{d\hat{x}}(\hat{x})\right| \leq 3C_2C_3^{\nu_k}e^{\frac{3\nu_k}{\log C_0}}. \tag{36}$$

As $\tilde{A}(\hat{x}, \gamma_\sigma(\hat{x})) = (\gamma_2(\hat{x}) - \gamma_1(\hat{x}))A(\hat{x}, \gamma_\sigma(\hat{x}))$,

$$\left|\frac{d\tilde{A}}{d\hat{x}}(\hat{x}, \gamma_\sigma(\hat{x}))\right| \leq C_2C_3^{\nu_k}e^{\frac{3\nu_k}{\log C_0}}. \tag{37}$$

Differentiating (34) with $\hat{x}$ and then using (35) (36) (37), we obtain $\|\partial_{\hat{x}}\tilde{A}\| \leq C_2C_3^{2\nu_k}$. In the same way we obtain the desired upper estimate of $\|\partial_{\hat{x}}\tilde{B}\|$. Transforming all these derivative estimates back to the original (x, y)-coordinate, we obtain the desired estimates.

Case 2: $\epsilon_{k-1} = 0$. We introduce a C^1 coordinate $(\hat{x}, \hat{y})$ on $f^{\nu_0+\cdots+\nu_{k-1}+1}Q(R_{i_0\cdots i_k})$ such that $9/10 \leq \|\partial_{\hat{x}}\| \leq 10/9$, $\|\partial_{\hat{y}}\| = 1$, $\langle\partial\hat{x}, \partial\hat{y}\rangle = 0$, $\langle\partial\hat{y}, t(f\zeta_{\rho_{k-1}})\rangle = 1$, $\Gamma(\zeta_{\rho_{k-1}}) = \{\hat{x} = 0\}$. With respect to $(\hat{x}, \hat{y})$-coordinate, represent the unstable sides of $f^{\nu_0+\cdots+\nu_{k-1}+1}Q(R_{i_0\cdots i_k})$ by graphs of functions γ_1, γ_2, $\gamma_1(\hat{x}) < \gamma_2(\hat{x})$. Define functions $\tilde{A}$, $\tilde{B}$ on the unstable sides on the rectangle in the same say as (32), and extend it to the entire rectangle as in Case 1. Define ϕ_0 in the same way as (33). Derivative estimates of ϕ_0 are completely analogous to Case 1. □

We now introduce the projectivization f_* of Df, given by $f_*(\xi, v) = Df(\xi)v/\|Df(\xi)v\|$, and define vector fields ϕ_j on $f^{\nu_0+\cdots+\nu_{k-1}+j+1}Q(R_{i_0\cdots i_k})$ for $1 \leq j < \nu_k$, by push-forward under f_*:

$$\phi_j(z) = f_*(f^{-1}(z), \phi_{j-1}(f^{-1}(z))).$$

Lemma 7.9. *For all $z \in f^{\nu_0+\cdots+\nu_k}Q(R_{i_0\cdots i_k})$, $\|D\phi_{\nu_k-1}(z)\| \leq C_2C_3^{3\nu_k}$.*

(30) for $j = k$ is a direct consequence of this lemma. If $\epsilon_{k-1} \neq 0$, then for all ξ, η in the unstable side of $f^{\nu_0+\cdots+\nu_k}Q(R_{i_0\cdots i_k})$, $\mathrm{angle}(u(\xi), u(\eta)) = \mathrm{angle}(\phi_{\nu_k-1}(\xi), \phi_{\nu_k-1}(\eta))$ holds. If $\epsilon_{k-1} = 0$, then $\mathrm{angle}(u(\xi), u(\eta)) < \mathrm{angle}(\phi_{\nu_k-1}(\xi), \phi_{\nu_k-1}(\eta))$ holds.

It is left to prove Lemma 7.9. The following estimates, proved in Appendix, are used:

$$|\partial_v f_*(\xi, v)| \leq 2\frac{|\det Df(\xi)|}{\|Df(\xi)v\|^2}. \tag{38}$$

$$|\partial_\xi f_*(\xi,v)| \le \frac{\|D^2f(\xi)\|\|v\|}{\|Df(\xi)v\|}. \tag{39}$$

Differentiating the formula of ϕ_j and using the result recursively we get

$$\begin{aligned} D\phi_{\nu_k-1}(z) = &\sum_{i=1}^{\nu_k-1} \partial_v f_*^{i-1}(f^{-i+1}(z),\phi_{\nu_k-i})\partial_\xi f_*(f^{-i}(z),\phi_{\nu_k-1-i})Df^{-i}(z)\\ &+\partial_v f_*^{\nu_k-1}(f^{-\nu_k+1}(z),\phi_0)D\phi_0(f^{-\nu_k+1}(z))Df^{-\nu_k+1}(z), \end{aligned}$$

where ϕ_{ν_k-1-i} means $\phi_{\nu_k-1-i}(f^{-i}(z))$. In this computation we have used $Df^{-1}(f^{-1}z)Df^{-1}(z) = Df^{-2}(z)$, $\partial_v f_*(f^{-1}(z),\phi_{j-1})\partial_v f_*(f^{-2}(z),\phi_{j-2}) = \partial_v f_*^2(f^{-2}(z),\phi_{j-2}(f^{-2}(z)))$, and so on.

By (38), for every $1\le i<\nu_k$,

$$\|\partial_v f_*^{i-1}(f^{-i+1}(z),\phi_{\nu_k-i})\| \le \prod_{j=1}^{i-1} 2\frac{|\det Df(f^{-j}z)|}{\|Df(f^{-j}z)\phi_{\nu_k-1-j}\|} = 2^{i-1}\frac{|\det Df^{i-1}(f^{-i+1}z)|}{\|Df^{i-1}(f^{-i+1}z)\phi_{\nu_k-i}\|^2}. \tag{40}$$

We evaluate the fraction on the right-hand-side as follows. First,

$$|\det Df^{i-1}(f^{-i+1}z)| = \frac{\|Df^{i-1}(f^{-i+1}z)\|}{\|Df^{-i+1}(z)\|}.$$

(39) gives

$$\begin{aligned} \|\partial_\xi f_*(f^{-i}(z),\phi_{\nu_k-1-i})Df^{-i}(z)\| &\le \|\partial_\xi f_*(f^{-i}(z),\phi_{\nu_k-1-i})Df^{-1}(f^{-i+1}z)\|\|Df^{-i+1}(z)\|\\ &\le \frac{C\|Df^{-1}(f^{-i+1}z)\|\|Df^{-i+1}(z)\|}{\|Df(f^{-i}z)\|} \le CC_1^{-2}\|Df^{-i+1}(z)\|. \end{aligned}$$

Plugging these into (40),

$$\|\partial_v f_*^{i-1}(f^{-i+1}(z),\phi_{\nu_k-i})\partial_\xi f_*(f^{-i}(z),\phi_{\nu_k-1-i})Df^{-i}(z)\| \le 2^{i-1}CC_1^{-2}\frac{\|Df^{i-1}(f^{-i+1}z)\|}{\|Df^{i-1}(f^{-i+1}z)\phi_{\nu_{k+1}-i}\|^2}.$$

Replacing all these in the above equality,

$$\|D\phi_{\nu_k-1}(z)\| \le \sum_{i=1}^{\nu_k-1} 2^{i-1}CC_1^{-2}\frac{\|Df^{i-1}(f^{-i+1}z)\|}{\|Df^{i-1}(f^{-i+1}z)\phi_{\nu_{k+1}-i}\|^2} + \frac{\|Df^{\nu_k-1}(f^{-\nu_k+1}z)\|}{\|Df^{\nu_k-1}(f^{-\nu_k+1}z)\phi_0\|^2}\|D\phi_0(f^{-\nu_k+1}z)\|.$$

Lemma 7.10. *For all ξ in the unstable sides of $f^{\nu_0+\cdots+\nu_{k-1}+1}Q(R_{i_0\cdots i_k})$ and $0\le j<\nu_k$, $\|\phi_{\nu_k-1}(\xi)\|\ge C\delta\|\phi_j(\xi)\|$.*

Proof. Let ζ denote the critical point on the same unstable side of $f^{\nu_0+\cdots+\nu_{k-1}}Q(R_{i_0\cdots i_{k-1}})$ as that of $f^{-1}(\xi)$. Let $q(\zeta,\xi)$ denote the fold period. In view of Proposition 5.2 and the bounded distortion of f^{ν_k-1} on the rectangle, for $q(\zeta,\xi)\le j<\nu_k$ we have $\frac{\|\phi_{\nu_k-1}(\xi)\|}{\|\phi_j(\xi)\|}\ge C\frac{\|w_{\nu_k}(f^{-1}\xi)\|}{\|w_j(f^{-1}\xi)\|}\ge C\delta$, and for $0\le j<q(\zeta,\xi)$, $\|\phi_{q(\zeta,\xi)}\|>\|\phi_j(\xi)\|$. These two inequalities yield the desired one. □

Lemma 7.8 and Lemma 7.10 give

$$\|D\phi_{\nu_k-1}(z)\| \le \sum_{i=1}^{\nu_k-1} C\delta^{-2}8^i + C\delta^{-1}4^{\nu_k}C_2C_3^{2\nu_k} \le C_2C_3^{3\nu_k}.$$

The last inequality holds for sufficiently large k_0. This completes the proof of Lemma 7.9 and hence that of Proposition 7.4.

7.8. **Area distortion bounds.** Proposition 7.4 and the next area distortion bounds together allow us to estimate the Lebesgue measure of the set in question.

Proposition 7.5. *For every $k \geq 1$ and all $\xi_1, \xi_2 \in f^{\nu_0+\cdots+\nu_{k-1}}Q(R_{i_0\cdots i_k})$,*

$$\frac{|\det Df^{\nu_k}(\xi_1)|}{|\det Df^{\nu_k}(\xi_2)|} \leq e^{C_1^{-1}}.$$

Proof. Let γ denote one of the unstable sides of $f^{\nu_0+\cdots+\nu_{k-1}+1}Q(R_{i_0\cdots i_{k-1}})$. Let η_σ denote the point of intersection between $\Gamma_{\nu_k-1}(\xi_\sigma)$ and γ ($\sigma = 1, 2$). If η_1 and η_2 are on the unstable side of $f^{\nu_0+\cdots+\nu_{k-1}+1}Q(R_{i_0\cdots i_k})$, Lemma 7.10 implies for every $0 \leq i < \nu_k - 1$,

$$|f^i(\eta_1) - f^i(\eta_2)| \leq C\delta^{-1}|f^{\nu_k-1}(\eta_1) - f^{\nu_k-1}(\eta_2)| \leq C\delta^{-1}\delta^{\frac{\nu_k}{10}}.$$

By the contraction along the long stable leaves, $|f^i(f(\xi_\sigma)) - f^i(\eta_\sigma)| \leq (Cb)^{\frac{i}{2}}|f(\xi_\sigma) - \eta_\sigma| \leq (Cb)^{\frac{i+1}{2}}$ holds for every $1 \leq i < \nu_k - 1$. It follows that $|f^i(\xi_1) - f^i(\xi_2)| \leq C\delta^{-1}\delta^{\frac{\nu_k}{10}}$ for every $1 \leq i < \nu_k$. This yields

$$\sum_{i=0}^{\nu_k-1} |f^i(\xi_1) - f^i(\xi_2)| \leq C.$$

Combining this with $\|D \log |\det Df|\| \leq CbC_1^{-1}$ we obtain the desired inequality. Even if η_1 or η_2 is not on the unstable side of $f^{\nu_0+\cdots+\nu_{k-1}+1}Q(R_{i_0\cdots i_k})$, the constants in Lemma 7.10 are not significantly affected because $f(\xi_1), f(\xi_2) \in f^{\nu_0+\cdots+\nu_{k-1}+1}Q(R_{i_0\cdots i_k})$ holds. Hence we obtain the same conclusion. $\square$

7.9. **Proof of Proposition 7.2.** In what follows, we assume $k \geq k_0$ is large so that $C_2C_3^{3\nu_k} \leq C_3^{4\nu_k}$. Denote by γ_1 and γ_2 the two unstable sides of $f^{\nu_0+\cdots+\nu_k}Q(R_{i_0\cdots i_k})$, and consider their graph representations $\gamma_1 = \{(x, \gamma_1(x))\}$, $\gamma_2 = \{(x, \gamma_2(x))\}$. Let $L(x) = |\gamma_1(x) - \gamma_2(x)|$. Proposition 7.4 and the Gronwall inequality give $L(x)/L(y) \leq e^{C_3^{4\nu_k}|x-y|}$ for all x, y. As $|x - y| \leq C\delta^{\frac{\nu_k}{10}}$, $L(x)/L(y) \leq 2$ holds.

Let $S_{\nu_k,1}, S_{\nu_k,2}, \cdots$ denote the components of $R_{i_0\cdots i_{k-1}}(\nu_k)$, the total number of which is clearly $\leq 2^{\nu_k}$. For each $S_{\nu_k,m}$, the above estimate and Proposition 7.3 give

$$\frac{|\mathcal{B}_0^{(\nu_k)} \cap f^{\nu_0+\cdots+\nu_k}(S_{\nu_k,m})|}{|f^{\nu_0+\cdots+\nu_k}Q(R_{i_0\cdots i_k})|} \leq 2\delta^{\frac{\nu_k}{5}}.$$

Proposition 7.5 gives $\frac{|\det Df^{\nu_0+\cdots+\nu_k}(\xi_1)|}{|\det Df^{\nu_0+\cdots+\nu_k}(\xi_2)|} \leq e^{C_1^{-1}k}$ for all $\xi_1, \xi_2 \in Q(R_{i_0\cdots i_k})$. Hence

$$\frac{|f^{-(\nu_0+\cdots+\nu_k)}(\mathcal{B}_0^{(\nu_k)}) \cap S_{\nu_k,m}|}{|R_{i_0\cdots i_{k-1}}|} \leq \frac{|f^{-(\nu_0+\cdots+\nu_k)}(\mathcal{B}_0^{(\nu_k)}) \cap S_{\nu_k,m}|}{|Q(R_{i_0\cdots i_k})|} \leq 2e^{C_1^{-1}k}\delta^{\frac{\nu_k}{5}}.$$

The first inequality follows from the obvious inclusion $Q(R_{i_0\cdots,i_k}) \subset R_{i_0\cdots i_{k-1}}$. Summing this over all component, and then for all feasible ν_k,

$$\sum_{\nu_k}\sum_{m} \frac{|S_{\nu_k,m}|}{|R_{i_0\cdots i_{k-1}}|} \leq 2\sum_{\nu_k \geq 4^k k_0} 2^{\nu_k}e^{C_1^{-1}k}\delta^{\frac{\nu_k}{5}} \leq e^{C_1^{-1}k}\delta^{\frac{4^k k_0}{6}}.$$

Therefore

$$|\Omega_k| = \sum_{(i_0,\cdots,i_k)} |R_{i_0\cdots i_k}| = \sum_{(i_0,\cdots,i_{k-1})} |R_{i_0\cdots i_{k-1}}| \sum_{\nu_k,m} \frac{|S_{\nu_k,m}|}{|R_{i_0\cdots i_{k-1}}|} \leq e^{C_1^{-1}k}\delta^{\frac{4^k k_0}{6}}|\Omega_{k-1}|.$$

The multiplicative constant goes to 0 as $k \to \infty$. This completes the proof of Proposition 7.2.

7.10. **Transitivity on K.** Finally we show f is transitive on K. Let $H(Q)$ denote the closure of transverse homoclinic points of Q. Then $H(Q) \subset K$ holds. It suffices to show the reverse inclusion. Let $z \in K$, and let U be an open set containing z. Since the Lebesgue measure of $U \cap K^+$ is zero, U intersects $W^s(Q)$. It follows that $W^s(Q)$ is dense in K. Inclination lemma implies that z is accumulated by transverse homoclinic points of Q. Hence $K \subset H(Q)$ holds.

Appendix

A.1. Proof of Lemma 4.6. For $z \in W^u(Q)$, let $t(z)$ denote any unit vector tangent to $W^u(Q)$ at z.

Sublemma 7.2. *Let $n \geq 0$ and $z \in G$. Either $f^i(z) \notin I(\delta)$ for every $0 \leq i \leq n$, or else there exists a sequence $0 \leq n_1 < n_1 + p_1 \leq n_2 < n_2 + p_2 \leq n_3 < \cdots \leq n$ of integers such that:*
(a) $f^{n_i}(z) \in I(\delta)$;
(b) $f^j(z) \in \{(x, y) \in \mathbb{R}^2 \colon |x| \geq 9/10\}$ *for* $n_i + 1 \leq j \leq n_i + p_i$;
(c) $\|Df^{n_i}(z)t(z)\| \geq (\delta/10)\|Df^j(z)t(z)\|$ *for* $0 \leq j < n_i$.

Then it follows that $f^n(G) \cap I(\delta)$ is made up of $C^2(b)$-curves, and the conclusion of Lemma 4.6 holds.

It is left to prove Sublemma 7.2. The correct order for the reader is to go over Sect.5.2, 5.3 first before getting into the details of this proof.

The argument is an induction on n. For $n = 0$, the assertions are direct consequences of the definition of G. Suppose that they hold for $n = k$. From the fact that the orbits of all critical points on $W^u(Q)$ are out of R_0, all the estimates in Proposition 5.2 remain to hold for them. This allows us to decompose the orbit of z into bound and free segments as follows: $n_i \leq k$ is a return time to $I(\delta)$. By the assumption of the induction, there exists a $C^2(b)$-curve in $W^u(Q)$ tangent to $Df^{n_i}(z)t(z)$ stretching across $I(\delta)$. Let p_i denote the bound period, given by the critical point on the $C^2(b)$-curve and an associated critical partition in Sect.5.3. Let n_{i+1} denote the next return time to $I(\delta)$. By (c) in Proposition 5.2, bound parts of $f^{k+1}(G)$ do not return to $I(\delta)$. This recovers all the assertions for $n = k + 1$. □

A.2. Proof of Lemma 5.1. For $M \leq k < 20n - 1$, we show

$$e^{-3\alpha k} D_k(\zeta) \leq D_{k+1}(\zeta) \leq e^{-3\alpha} D_k(\zeta). \tag{41}$$

To this end, let $d_k(i) = \min_{j \in [i,k+1]} \frac{\|w_j(\zeta)\|^2}{\|w_i(\zeta)\|^3}$ and $d_{k+1}(i) = \min_{j \in [i,k+2]} \frac{\|w_j(\zeta)\|^2}{\|w_i(\zeta)\|^3}$. Then

$$\frac{D_{k+1}(\zeta)}{D_k(\zeta)} = e^{-3\alpha} \frac{\min_{i \in [1,k+1]} d_{k+1}(i)}{\min_{i \in [1,k]} d_k(i)} \leq e^{-3\alpha} \frac{\min_{i \in [1,k]} d_{k+1}(i)}{\min_{i \in [1,k]} d_k(i)} \leq e^{-3\alpha},$$

and the second inequality holds.

(G2) gives $\|w_{k+2}(\zeta)\| \geq e^{-2\alpha(k+1)}\|w_{k+1}(\zeta)\|$, and thus for $1 \leq i \leq k$,

$$d_{k+1}(i) = \min\left\{d_k(i), \frac{\|w_{k+2}(\zeta)\|^2}{\|w_i(\zeta)\|^3}\right\} \geq e^{-4\alpha k} d_k(i) \geq e^{-4\alpha(k+1)} D_k(\zeta).$$

Using $\|w_{k+1}(\zeta)\| \leq C_0\|w_k(\zeta)\|$ and $\|w_j(\zeta)\| \geq e^{-2\alpha k}\|w_{j-1}(\zeta)\|$ from (G2),

$$d_{k+1}(k+1) = \min_{j \in [k+1,k+2]} \frac{\|w_j(\zeta)\|^2}{\|w_{k+1}(\zeta)\|^2} \geq C_0^{-3} e^{-4\alpha(k+1)} \min_{j \in [k+1,k+2]} \frac{\|w_{j-1}(\zeta)\|^2}{\|w_k(\zeta)\|^3} = C_0^{-3} e^{-4\alpha(k+1)} d_k(k).$$

These two inequalities yield the first inequality in (41).

Using $D_{k+1}(\zeta) \geq C_0^{-3k}$ and $\text{length}(\gamma_k) \leq Ce^{2\alpha k}\sqrt{D_{k+1}(\zeta)}$ which follows from (41),

$$\text{length}(\gamma_{k,s}) \leq e^{-3\alpha k} \cdot \text{length}(\gamma_k) \leq D_{k+1}^{\frac{1}{2}+\frac{\alpha}{3\log C_0}}(\zeta).$$

This completes the proof (a).

From (f) Proposition 5.2, [[27] Lemma 2.3] and the fact that γ is $C^2(b)$, $f^{\chi(k)}(\gamma_{k,s})$ is a $C^2(b)$-curve. Using (41),

$$\begin{aligned}\text{length}(f^{\chi(k)}\gamma_{k,s}) &\geq Ce^{-3\alpha k}\|w_{\chi(k)}(\zeta)\|(D_k(\zeta) - D_{k+1}(\zeta)) \geq Ce^{-3\alpha k}\|w_{\chi(k)}(\zeta)\|D_k(\zeta)(1-e^{-3\alpha}) \\ &\geq Ce^{-3\alpha k}\|w_k(\zeta)\|D_k(\zeta)C_0^{-\sqrt{\alpha}k}(1-e^{-3\alpha}) \geq e^{-4\alpha k}.\end{aligned}$$

The third inequality follows from $\|w_{\chi(k)}(\zeta)\| \geq C_0^{-(k-\chi(k))}\|w_k(\zeta)\|$ and $k - \chi(k) \leq \sqrt{\alpha}k$ in (G2). This completes the proof of (b). For the proof of (c), see [[27] Lemma 5.11]. □

A.3. Derivative estimates of projectivization. We prove (38) (39). Let $v^\perp$ denotes any unit vector orthogonal to v. Then

$$\begin{aligned}|\partial_v f_*(\xi, v)| &= \lim_{\Delta\theta\to 0}\left\|\frac{1}{\Delta\theta}\left(\frac{Df(\xi)(v+\Delta\theta v^\perp)}{\|Df(\xi)(v+\Delta\theta v^\perp)\|} - \frac{Df(\xi)v}{\|Df(\xi)v\|}\right)\right\| \\ &\leq \frac{\|Df(\xi)v^\perp\|}{\|Df(\xi)v\|} + \lim_{\Delta\theta\to 0}\left\|\frac{1}{\Delta\theta}\frac{\|Df(\xi)v\| - \|Df(\xi)(v+\Delta\theta v^\perp)\|}{\|Df(\xi)v\|}\right\| \\ &\leq 2\frac{\|Df(\xi)v^\perp\|}{\|Df(\xi)v\|} = 2\frac{|\det Df(\xi)|}{\|Df(\xi)v\|^2}.\end{aligned}$$

Let $\xi = (x, y)$. Writing $\xi_x = \xi + (\Delta x, 0)$ we have

$$\begin{aligned}|\partial_x f_*(\xi, v)| &= \lim_{\Delta x\to 0}\left\|\frac{1}{\Delta x}\left(\frac{Df(\xi_x)v}{\|Df(\xi_x)v\|} - \frac{Df(\xi)v}{\|Df(\xi)v\|}\right)\right\| \\ &= \lim_{\Delta x\to 0}\left\|\frac{1}{\Delta x}\left(\frac{Df(\xi_x)v - Df(\xi)v}{\|Df(\xi)v\|} - \frac{\|Df(\xi_x)v\| - \|Df(\xi)v\|}{\|Df(\xi)v\|\|Df(\xi_x)v\|}Df(\xi_x)v\right)\right\| \\ &\leq 2\lim_{\Delta x\to 0}\left\|\frac{1}{\Delta x}\frac{(Df(\xi_x) - Df(\xi))v}{\|Df(\xi)v\|}\right\| = \frac{2}{\|Df(\xi)v\|}\left\|\left(\frac{\partial}{\partial x}Df(\xi)\right)v\right\|.\end{aligned}$$

In the same way we get

$$|\partial_y F(\xi, v)| \leq \frac{2}{\|Df(\xi)v\|}\left\|\left(\frac{\partial}{\partial y}Df(\xi)\right)v\right\|.$$

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