Multiple level-sets for elliptic Cauchy problems in three dimensional domains

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Abstract: We analyze a multiple level-set method for solving elliptic Cauchy problems with piecewise constant solutions. This method corresponds to an iterated Tikhonov method for a particular Tikhonov functional based on TV-H^1 penalization. Generalized minimizers for our Tikhonov functional are defined and an existence result is established. Moreover, convergence and stability results of the proposed Tikhonov method are derived. The proposed multiple level-set method is tested numerically, and our experiments demonstrate that the method is able to accurately recover multiple objects as well as multiple contrast levels.

Keywords. elliptic Cauchy problems, level-set methods, multiple levels, Tikhonov regularization

AMS subject classifications. 65J20, 35J60

1   Introduction

The model and the inverse problem

Let $\Omega \subset \mathbb{R}^3$, be an open bounded set with piecewise Lipschitz boundary $\partial\Omega$. Moreover, we assume that $\partial\Omega = \overline{\Gamma}_1 \cup \overline{\Gamma}_2$, where Γ_i are two open connected disjoint parts of $\partial\Omega$. We denote by $\mathcal{P}$ the elliptic operator defined in Ω by

$$\mathcal{P}(u) \;:=\; -\sum_{i,j=1}^{d} D_i(a_{i,j}D_j u)\,, \tag{1}$$

where the real functions $a_{i,j} \in L^\infty(\Omega)$ are such that the matrix $A(x) := (a_{i,j})_{i,j=1}^d$ satisfies $\xi^t A(x)\,\xi > \alpha||\xi||^2$, for all $\xi \in \mathbb{R}^d$ and for a.e. $x \in \Omega$, where $\alpha > 0$.

We denote by *elliptic Cauchy problem* the boundary value problem (BVP)

$$(CP) \qquad \begin{cases} \mathcal{P}u = f\,, \text{ in } \Omega \\ \quad u = g_1\,, \text{ at } \Gamma_1 \\ u_\nu = g_2\,, \text{ at } \Gamma_1 \end{cases},$$

where the functions $(g_1, g_2) \in H^{1/2}(\Gamma_1) \times H_{00}^{1/2}(\Gamma_1)'$ correspond to the problem data, *Cauchy data*, and the real function $f \in L^2(\Omega)$ is a given source term in the model.

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If there exists a distribution $u \in H^1(\Omega)$, which solves the weak formulation of the elliptic equation $\mathcal{P}u = f$ in Ω, and also satisfies the boundary conditions above (at Γ_1) in the sense of the trace operator, we say that u is a (variational) solution of (CP).

Let u be a solution of (CP). It is worth noticing that, if the Neumann trace at Γ_2 of u is known, then u can be computed as a solution of the mixed boundary value problem

$$\mathcal{P} = f \,, \text{ in } \Omega \qquad u = g_1 \,, \text{ at } \Gamma_1 \qquad u_\nu = g_2 \,, \text{ at } \Gamma_1 \,,$$

which is a well posed problem in the sense of Hadamard [25]. Therefore, it is enough to consider the task of determining the Neumann trace of u at Γ_2.

Brief overview on elliptic Cauchy problems

Elliptic Cauchy problems are not well posed in the sense of Hadamard. A tutorial example given by Hadamard almost 90 years ago during his seminars shows that the solution of (CP) does not depend continuously on the Cauchy data [25, 32]. For a recent analytical investigation of the degree of ill-posedness of elliptic Cauchy problems in two-dimensional bounded Lipschitz domains, we refer the reader to [5].

Existence of solutions for (CP) for arbitrary Cauchy data (g_1, g_2) do not hold. For details we refer the reader to [32, 20]. Actually, one cannot prove existence even in the case of analytical Cauchy data (g_1, g_2) [24].

A given pair of Cauchy data (g_1, g_2) is called *consistent* if the corresponding problem (CP) admits an H^1-solution. It was proved in [3] that the set $M := \{(g_1, g_2) \in H^{1/2}(\Gamma_1) \times [H_{00}^{1/2}(\Gamma_1)]'; (g_1, g_2) \text{ consistent Cauchy data}\}$ is a dense subset of $H^{1/2}(\Gamma_1) \times [H_{00}^{1/2}(\Gamma_1)]'$.

What concerns the issue of uniqueness of solutions for (CP), it has been proven that elliptic Cauchy problems admit a unique weak solution in $H^1(\Omega)$ (see, e.g., [20]). A classical uniqueness result for solutions in $C^2(\Omega)$ can be found in [10]. Moreover, a uniqueness result for a class of nonlinear elliptic Cauchy problems can be found in [30].

What concerns numerical investigations of (CP), a large variety of methods can be found in the literature:

    M.1) Optimization approach [22, 3];

    M.2) Iterative methods [29, 32, 26, 20, 30];

    M.3) Backus-Gilbert method [31, 27];

    M.4) Optimal control [12];

    M.5) Quasi-reversibility [6, 7];

    M.6) Level set method [33, 18].

The reason for this strong interest resides on the fact that elliptic Cauchy problems arise in many industrial, engineering and biomedical applications including:

    A.1) Expansion of measured surface fields inside a body from partial boundary measurements [3];

    A.2) A classical thermostatics problem, which consists in recovering the temperature in a given domain when its distribution and the heat flux are known over the accessible region of the boundary [20];

    A.3) The analogous electrostatics case encountered in electric impedance tomography [3];

    A.4) Inverse problems related to corrosion detection [2, 33].

Multiple level-set methods for elliptic Cauchy problems

Our main goal in this work is to study multiple level-sets methods [13, 15] for obtaining regularized solutions of (CP). Multiple level-set approaches for elliptic Cauchy problems are manageable whenever the unknown solution is a simple function defined on Γ_2 assuming at most N different values, i.e. there exists disjoint measurable subsets $D_j \subset \Gamma_2$ and constants c_j, $j = 1 : N$, such that $|\Gamma_2| = \sum_j |D_j|$ and u at Γ_2 is known *a-priori* to satisfy $u_\nu|_{\Gamma_2} = \sum_j c_j \chi_{D_j}$, almost everywhere.

The manuscript is outlined as follows: In Section 2 we write the elliptic Cauchy problem in the functional analytical framework of an (ill-posed) operator equation. This is the starting point for the level set approach derived in the sequel. In Section 3 we investigate a multiple level-set approach for (CP) based on the ideas presented in [15]. First we define a Tikhonov functional related to (CP). This functional is based in the introduction of TV-H^1 penalization. Moreover, we define the concept of generalized minimizers for this functional. Existence of (generalized) minimizers for this Tikhonov functional is proven. Relevant properties of the generalized minimizers as well as properties of the penalization term are investigated. In the sequel we prove convergence and stability results for this Tikhonov regularization method. In Section 4 we introduce a stabilized (smooth) Tikhonov functional. In the main result of this section we prove that the minimizers of the stabilized functional asymptotically approximate the minimizers of the original Tikhonov functional, as the stabilization parameter goes to zero. The corresponding multiple level-set method is derived from an explicit Euler method for solving the evolution equation related to the first order optimality condition of the stabilized functional. Section 5 is devoted to numerics. An efficient implementation of the multiple level-set method is investigated. We are able to improve the performance of the method in [15] by using a specially suited pre-conditioning strategy. Several experiments are provided, in order to illustrate the effectiveness of the multiple level-set method considered in Section 4.

2   Formulation of the inverse problem

We begin by defining the auxiliary problem:

$$\begin{cases} \mathcal{P}v = f\,, \text{ in } \Omega \\ \quad v = g_1\,, \text{ at } \Gamma_1 \\ \quad v_\nu = \varphi\,, \text{ at } \Gamma_2 \end{cases}. \tag{2}$$

This mixed BVP defines the operator $T : \varphi \mapsto v_\nu|_{\Gamma_1}$. Notice that, if $\varphi = u_\nu|_{\Gamma_2}$, where u is the solution of (CP), then it would follow $T(\varphi) = g_2$. A first least square approach [22] consists in solving the optimization problem

$$\|T(\varphi) - g_2\|^2 \to \min.$$

Due to the superposition principle for linear elliptic BVPs [24], one can split the solution of (2) in $v = v_a + v_b$, where

$$\mathcal{P}v_a = 0\,, \text{ in } \Omega \qquad v_a = 0\,, \text{ at } \Gamma_1 \qquad (v_a)_\nu = \varphi\,, \text{ at } \Gamma_2\,; \tag{3}$$

$$\mathcal{P}v_b = f\,, \text{ in } \Omega \qquad v_b = g_1\,, \text{ at } \Gamma_1 \qquad (v_b)_\nu = 0\,, \text{ at } \Gamma_2\,. \tag{4}$$

Now, we define from (3) the linear operator

$$L : \varphi \mapsto (v_a)_\nu|_{\Gamma_1}\,, \tag{5}$$

and from (4) we define the function $z := (v_b)_\nu|_{\Gamma_1}$. Since $T(\varphi) = L\,\varphi + z$, the Cauchy problem (CP) can be written in the form of the operator equation

$$L\,\varphi \;=\; g_2 - z\,, \tag{6}$$

where the constant term z depends only on the Dirichlet data g_1, on the source term f and on the operator $\mathcal{P}$. Therefore, it can be computed *a-priori*.

In the sequel we shall assume $\Omega \subset \mathbb{R}^3$ and define a functional analytical framework to analyze (6). The Cauchy data is assumed to satisfy

$$(g_1, g_2) \;\in\; H^{1/2}(\Gamma_1) \times H_{00}^{1/2}(\Gamma_1)' \tag{7}$$

and the source term f to be a $L^2(\Omega)$-distribution.

From this choice of g_1 and f, the elliptic theory allow us to conclude that the mixed BVP in (4) has a unique solution $v_b \in H^1(\Omega)$ [14, 24]. Therefore, $z := (v_b)_\nu|_{\Gamma_1} \in H_{00}^{1/2}(\Gamma_1)'$ and the term $g_2 - z$ on the right hand side of (6) is now a distribution in $H_{00}^{1/2}(\Gamma_1)'$.

The next result [18, Proposition 2.1] shows that the linear operator L in (5) is well defined and continuous from $L^{3/2}(\Gamma_2)$ to $H_{00}^{1/2}(\Gamma_1)'$.

Proposition 2.1. *Let $\Omega \subset \mathbb{R}^3$ be defined as in Section 1 and assume the Cauchy data (g_1, g_2) to be given in $H^{1/2}(\Gamma_1) \times H_{00}^{1/2}(\Gamma_1)'$. Then, the operator defined in (5) is an injective bounded linear map $L : L^{3/2}(\Gamma_2) \to H_{00}^{1/2}(\Gamma_1)'$.*

To conclude this section we address an issue related to noisy Cauchy data. If only corrupted noisy data (g_1^δ, g_2^δ) are available for problem (CP), we assume the existence of a consistent Cauchy data (g_1, g_2) satisfying (7) such that

$$\|g_1 - g_1^\delta\|_{L^2(\Gamma_1)} + \|g_2 - g_2^\delta\|_{L^2(\Gamma_1)} \le \delta. \tag{8}$$

Since z in (6) depends continuously on g_1 in the $H^{1/2}(\Gamma_1)$ topology, it is natural to ask whether it is possible to obtain from measured data (g_1^δ, g_2^δ) satisfying (8), a corresponding $z^\delta \in [H_{00}^{1/2}(\Gamma_1)]'$ such that $\|z - z^\delta\|_{[H_{00}^{1/2}(\Gamma_1)]'} \le \delta$. The next result [18, Lemma 2.3] gives a positive answer to this question

Lemma 2.2. *Let the noise Cauchy data be given as in (8), where $g_1 \in H^s(\Gamma_1)$ for some $s > 1/2$. Then (CP) reduces to the operator equation $L\varphi = g_2^\delta - z^\delta$, where the right hand side satisfies*

$$\|(g_2 - z) - (g_2^\delta - z^\delta)\|_{[H_{00}^{1/2}(\Gamma_1)]'} \le h(\delta). \tag{9}$$

Here $h : \mathbb{R}^+ \to \mathbb{R}^+$ is a function satisfying $\lim_{\delta \to 0} h(\delta) = 0$.

3 A Tikhonov regularization approach

3.1 Generalized minimizers

The starting point of our approach is the assumption that the solution $\overline{\varphi}$ of (6) is a simple function taking only a finite number of possible values. Moreover, we assume the existence

of disjoint subsets $D_j \subset \Gamma_2$ and constants c_j, $j = 1 : N$, such that $|\Gamma_2| = \sum_j |D_j|$ and

$$\overline{\varphi} = \sum_{j=1}^{N} c_j \chi_{D_j}, \text{ a.e.} \tag{10}$$

Next we introduce the H^1-functions $\{\phi^j\}_{j=1}^{p}$, where p is the smallest integer satisfying $2^p \geq N$ (for simplicity of the presentation we shall assume in this paper $N = 4$ and $p = 2$), such that

$$D_1 = \{x \,|\, \phi^1(x) > 0, \, \phi^2(x) > 0\}, \quad D_2 = \{x \,|\, \phi^1(x) > 0, \, \phi^2(x) < 0\},$$
$$D_3 = \{x \,|\, \phi^1(x) < 0, \, \phi^2(x) > 0\}, \quad D_4 = \{x \,|\, \phi^1(x) < 0, \, \phi^2(x) < 0\}.$$

Define $\mathcal{V} = \{u \in L^\infty(\Gamma_2) \,|\, u = \chi_D, \, D \subset \Gamma_2 \text{ measurable}, \, \mathcal{H}^{n-1}(\partial D) < \infty\}$, where $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff-measure, and $q : \mathcal{V} \times \mathcal{V} \to L^\infty(\Gamma_2)$ is given by

$$q(u_1, u_2) = c_1 u_1 u_2 + c_2(1 - u_1)u_2 + c_3 u_1(1 - u_2) + c_4(1 - u_1)(1 - u_2). \tag{11}$$

Set $\mathcal{U} = q(\mathcal{V}, \mathcal{V})$, i.e., $\mathcal{U} = \{u \in L^\infty(\Gamma_2) \,|\, u = q(u_1, u_2), \, u_1, u_2 \in \mathcal{V}\}$, and define the operator $P : H^1(\Gamma_2) \times H^1(\Gamma_2) \ni (\phi^1, \phi^2) \mapsto q(H(\phi^1), H(\phi^2)) \in \mathcal{U}$, where $H : H^1(\Gamma_2) \to \mathcal{V}$, is the Heavyside projector. Using (10) and the above definitions we can represent $\overline{\varphi}$ by

$$\overline{\varphi} = P(\phi^1, \phi^2),$$

and the inverse problem (6) can be written in the form of the operator equation

$$L(P(\phi^1, \phi^2)) = g_2^\delta - z^\delta, \tag{12}$$

with noise data satisfying (9). Once an approximate solution (ϕ^1, ϕ^2) of (12) is obtained, a corresponding solution of (6) is given by $\overline{\varphi} = P(\phi^1, \phi^2)$.

The multiple level-set method proposed in this paper corresponds to a continuous evolution of the functions ϕ^1 and ϕ^2 for an artificial time t. This evolution aims to minimize of the Tikhonov functional

$$\mathcal{G}_\alpha(\phi^1, \phi^2) = \|L\left(P(\phi^1, \phi^2)\right) - (g_2^\delta - z^\delta)\|_Y^2 + \alpha \sum_{j=1}^{2} \left\{\beta |H(\phi^j)|_{BV} + \|\phi^j - \phi_0^j\|_{H^1}^2\right\} \tag{13}$$

based on $TV\text{--}H^1$ penalization. Here $\alpha > 0$ plays the rule of a regularization parameter and $\beta > 0$ is a scaling factor. The BV-seminorm terms penalize the length of the Hausdorff measure of the boundary of the set $\{x \,|\, \phi^1(x) \geq 0 \text{ and } \phi^2(x) \geq 0\}$ and play an important role in the analysis of convergence of our regularization procedure.

Since P is discontinuous, one cannot prove that the Tikhonov functional (13) attains a minimizer. In order to guarantee existence of minimizers for $\mathcal{G}_\alpha$, it is necessary to use the concept of generalized minimizers introduced in [15].

Definition 3.1. *Let the boundary part $\Gamma_2 \subset \partial\Omega$ be defined as in Section 1.*

1. *The set Ad of admissible parameters consists of tuples $(z^1, z^2, \phi^1, \phi^2) \in (L^\infty(\Gamma_2))^2 \times (H^1(\Gamma_2))^2$ such that there exist sequences $\{\phi_k^1\}_{k \in \mathbb{N}}$, $\{\phi_k^2\}_{k \in \mathbb{N}}$ in $H^1(\Gamma_2)$ and a sequence $\{\varepsilon_k\}_{k \in \mathbb{N}}$ of positive numbers converging to zero, satisfying[1]*

$$\lim_{k \to \infty} \|\phi_k^j - \phi^j\|_{L^2(\Gamma_2)} = 0 \quad \text{and} \quad \lim_{k \to \infty} \|H_{\varepsilon_k}(\phi_k^j) - z^j\|_{L^{3/2}(\Gamma_2)} = 0, \, j = 1, 2.$$

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[1] $H_\varepsilon(\phi^j)(x) := \begin{cases} 0 & \text{for} \quad \phi^j(x) < -\varepsilon, \\ 1 + \frac{\phi^j(x)}{\varepsilon} & \text{for} \quad \phi^j(x) \in [-\varepsilon, 0], \\ 1 & \text{for} \quad \phi^j(x) > 0. \end{cases}$

2. A **generalized minimizer** of $\mathcal{G}_\alpha$ is an admissible paramter $(z^1, z^2, \phi^1, \phi^2)$ minimizing

$$\mathcal{G}_\alpha(z^1, z^2, \phi^1, \phi^2) := \|L(q(z^1, z^2)) - (g_2^\delta - z^\delta)\|_Y^2 + \alpha\,\rho(z^1, z^2, \phi^1, \phi^2) \qquad (14)$$

over Ad. Here the functional ρ is defined by

$$\rho(z^1, z^2, \phi^1, \phi^2) := \inf\Big\{ \liminf_{k\to\infty} \sum_{j=1}^{2} \big(\beta|H_{\varepsilon_k}(\phi_k^j)|_{BV} + \|\phi_k^j - \phi_0^j\|_{H^1}^2\big)\Big\},$$

where the infimum is taken with respect to all sequences $\{\varepsilon_k\}$ and $\{(\phi_k^1, \phi_k^2)\}$ as in 1.

The set Ad is non empty [15, Remark 2]. Moreover, since the solution of (6) is assumed to satisfy $\overline{\varphi} = P(\phi^1, \phi^2)$, if the functions $\phi^j \in H^1(\Gamma_2)$ are such that $|\nabla\phi^j| \neq 0$ in a neighborhood of the level set $\{\phi^j = 0\}$, we can define the constant sequences $\{\phi_k^j = \phi^j\}_k$ and $z^j = H(\phi^j)$, and estimate

$$\|H_{\varepsilon_k}(\phi_k^j) - z^j\|_{L^{3/2}(\Gamma_2)} \leq C \int_{-\varepsilon_k}^{0} 1\,dt \to 0, \quad \varepsilon_k \to 0,$$

where C is a positive number and $\{\varepsilon_k\}$ is any positive sequence converging to zero. Consequently $(z^1, z^2, \phi^1, \phi^2)$ is in Ad and satisfies

$$L(q(z^1, z^2)) = g_2 - z. \qquad (15)$$

3.2 Convergence analysis

In the next theorem we summarize results on coercivity and lower semicontinuity of the functional ρ and well-posedness of $\mathcal{G}_\alpha$ which are scaterred along [15].

Theorem 3.1. *Let the functionals $\rho, \mathcal{G}_\alpha$ be defined as in item 2 of Definition 3.1. The following assertions hold true:*

1. *The functional $\rho(z^1, z^2, \phi^1, \phi^2)$ is coercive and strong-weak lowersemicontinuous in the $BV(\Gamma_2)^2 \times H^1(\Gamma_2)^2$ and $L^{3/2}(\Gamma_2)^2 \times H^1(\Gamma_2)^2$ topology, respectively;*

2. *The functional $\mathcal{G}_\alpha$ in (13) attains minimizers on the set Ad.*

Proof. See Lemma 4, Lemma 5 and Theorem 6 in [15]. □

In the next theorem, we present the results of converge and stability of the approximations for the solution of (6). These results were originally proved for a nonlinear inverse problem in [15], where a notion of minimum-norm solution was introduced.

Theorem 3.2. *The following assertions hold true:*

1. *Assume that we have exact data, i.e. $\delta = 0$ and $\beta > 0$. For every $\alpha > 0$ let $(z_\alpha^1, z_\alpha^2, \phi_\alpha^1, \phi_\alpha^2)$ denote a minimizer of $\mathcal{G}_\alpha$ on the set Ad. Then, for every sequence of positive numbers $\{\alpha_k\}_{k\in\mathbb{N}}$ converging to zero, there exists a subsequence (denoted again by $\{\alpha_k\}_{k\in\mathbb{N}}$) such that $(z_{\alpha_k}^1, z_{\alpha_k}^2, \phi_{\alpha_k}^1, \phi_{\alpha_k}^2)$ is strongly convergent in $(L^{3/2}(\Gamma_2))^2 \times (L^2(\Gamma_2))^2$ to some limit point $(z^1, z^2, \phi^1, \phi^2)$. Moreover, the limit (z^1, z^2) is a solution of (6), i.e. $L(q(z^1, z^2)) = g_2 - z$.*

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2. *Let the Cauchy data (g_1, g_2) be consistent and $\alpha = \alpha(\delta)$ be a positive function satisfying $\lim_{\delta \to 0} \alpha(\delta) = 0$ and $\lim_{\delta \to 0} \delta^2 \alpha(\delta)^{-1} = 0$. Moreover, let $\{\delta_k\}_{k \in \mathbb{N}}$ be a sequence of positive numbers converging to zero and $\{(g_1^{\delta_k}, g_2^{\delta_k})\}_{k \in \mathbb{N}}$ be a corresponding noisy data satisfying (8). Then, there exist a subsequence (denoted again by $\{\delta_k\}$) and a sequence $\{\alpha_k := \alpha(\delta_k)\}_{k \in \mathbb{N}}$ such that $(z^1_{\alpha_k}, z^2_{\alpha_k}, \phi^1_{\alpha_k}, \phi^2_{\alpha_k})$ converges in $(L^{3/2}(\Gamma_2))^2 \times (L^2(\Gamma_2))^2$ to some limit point $(z^1, z^2, \phi^1, \phi^2)$. Moreover, (z^1, z^2) satisfy $L(q(z^1, z^2)) = g_2 - z$.*

Proof. The proof follows from the results in [15, Theorems 8 and 9] applied to the operator equation $L(q(z^1, z^2)) = g_2 - z$. □

4 Multiple level-set approximations

4.1 A Stabilized functional

From a numerical viewpoint it is important to define a functional which can be handled numerically. Our aim in this direction is to try to find minimizers which "approximate" the minimizers of $\mathcal{G}_\alpha$ by means of the stabilized functional defined by

$$\mathcal{G}_{\varepsilon,\alpha}(\phi^1, \phi^2) := \|L(P_\varepsilon(\phi^1, \phi^2)) - (g_2^\delta - z^\delta)\|_Y^2 + \alpha \sum_{j=1}^{2} \left\{ \beta |H_\varepsilon(\phi^j)|_{BV} + \|\phi^j - \phi_0^j\|_{H^1}^2 \right\}, \quad (16)$$

where ε, α and β are positive real numbers and $P_\varepsilon(\phi^1, \phi^2) = q(H_\varepsilon(\phi^1), H_\varepsilon(\phi^2))$ is a smooth approximation of the discontinuous operator P.

By Lemma 10 of [15] we have that $\mathcal{G}_{\varepsilon,\alpha}$ is well-posed, i.e., attains minimizers on $(H^1(\Gamma_2))^2$.

The next theorem shows that, for $\varepsilon \to 0$, the minimizers of $\mathcal{G}_{\varepsilon,\alpha}$ approximate a (generalized) minimizer of $\mathcal{G}_\alpha$.

Theorem 4.1. *Let $\alpha, \beta > 0$ be given. For each $\varepsilon > 0$ denote by $(\phi^1_{\varepsilon,\alpha}, \phi^2_{\varepsilon,\alpha})$ a minimizer of $\mathcal{G}_{\varepsilon,\alpha}$. There exists a sequence of positive numbers $\{\varepsilon_k\}$ converging to zero such that $(H_{\varepsilon_k}(\phi^1_{\varepsilon_k,\alpha}), H_{\varepsilon_k}(\phi^2_{\varepsilon_k,\alpha}), \phi^1_{\varepsilon_k,\alpha}, \phi^2_{\varepsilon_k,\alpha})$ converges strongly in $(L^{3/2}(\Gamma_2))^2 \times (L^2(\Gamma_2))^2$ and the limit is a generalized minimizer of $\mathcal{G}_\alpha$.*

Proof. Let $(z^1_\alpha, z^2_\alpha, \phi^1_\alpha, \phi^2_\alpha)$ be a minimizer of $\mathcal{G}_\alpha$ on the set *Ad*. From Definition (3.1), there exists a sequence $\{\varepsilon_k\}_{k \in \mathbb{N}}$ of positive numbers converging to zero and a corresponding sequence $\{(\phi^1_k, \phi^2_k)\}_{k \in \mathbb{N}}$ in $H^1(\Gamma_2)^2$ satisfying

$$\phi^j_k \to \phi^j_\alpha \quad \text{and} \quad H_{\varepsilon_k}(\phi^j_k) \to z^j_\alpha \quad \text{in} \quad L^{3/2}(\Gamma_2), \, j = 1, 2.$$

From Lemma 3 of [15] we can assume that

$$\rho(z^1_\alpha, z^2_\alpha, \phi^1_\alpha, \phi^2_\alpha) = \lim_{k \to \infty} \sum_{j=1}^{2} \left(\beta |H_{\varepsilon_k}(\phi^j_k)|_{BV} + \|\phi^j_k - \phi_0^j\|_{H^1(\Gamma_2)}^2 \right).$$

Let $(\phi^1_{\varepsilon_k}, \phi^2_{\varepsilon_k})$ be a minimizer of $\mathcal{G}_{\varepsilon_k,\alpha}$. The sequences $\{\phi^j_{\varepsilon_k}\}$, $j = 1, 2$ are uniformly bounded in $H^1(\Gamma_2)$. Thus, there are weakly convergent subsequences (denoted again by the same indices) and the weak limits are denoted by $\tilde{\phi}^j$, $j = 1, 2$. Similarly, the sequences $\{H_{\varepsilon_k}(\phi^j_{\varepsilon_k})\}_{k \in \mathbb{N}}$, $j = 1, 2$ are uniformly bounded in $BV(\Gamma_2)$. By the compact Sobolev embedding theorem [1, 21]

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there exist convergent subsequences (denoted with the same indices) and limits are denoted by $\tilde{z}^j$, $j = 1, 2$. Summarizing, we have

$$\phi^j_{\varepsilon_k} \to \tilde{\phi}^j \quad \text{in} \quad L^2(\Gamma_2) \quad \text{and} \quad H_{\varepsilon_k}(\phi^j_{\varepsilon_k}) \to \tilde{z}^j \quad \text{in} \quad L^{3/2}(\Gamma_2), \quad j = 1, 2,$$

as $k \to \infty$. Thus $(\tilde{z}^1, \tilde{z}^2, \tilde{\phi}^1, \tilde{\phi}^2) \in L^{3/2}(\Gamma_2)^2 \times H^1(\Gamma_2)^2$ is admissible.

Arguing with definition of ρ, as in Lemma 1 of [15] and with the continuity of L, we conclude that

$$\|L(q(\tilde{z}^1, \tilde{z}^2)) - (g_2^\delta - z^\delta)\|_Y^2 = \lim_{k\to\infty} \|L(P_{\varepsilon_k}(\phi^1_{\varepsilon_k}, \phi^2_{\varepsilon_k})) - (g_2^\delta - z^\delta)\|_Y^2,$$

$$\rho(\tilde{z}^1, \tilde{z}^2, \tilde{\phi}^1, \tilde{\phi}^2) \le \liminf_{k\to\infty} \sum_{j=1}^{2} \left(\beta |H_{\varepsilon_k}(\phi^j_k)|_{BV} + \|\phi^j_k - \phi^j_0\|^2_{H^1(\Gamma_2)} \right).$$

Therefore,

$$\begin{aligned}
\mathcal{G}_\alpha(\tilde{z}^1, \tilde{z}^2, \tilde{\phi}^1, \tilde{\phi}^2) &= \|L(q(\tilde{z}^1, \tilde{z}^2)) - (g_2^\delta - z^\delta)\|_Y^2 + \alpha\rho(\tilde{z}^1, \tilde{z}^2, \tilde{\phi}^1, \tilde{\phi}^2) \\
&\le \liminf_{k\to\infty} \mathcal{G}_{\varepsilon_k,\alpha}(\phi^1_{\varepsilon_k}, \phi^2_{\varepsilon_k}) \\
&\le \liminf_{k\to\infty} \mathcal{G}_{\varepsilon_k,\alpha}(\phi^1_k, \phi^2_k) \le \limsup_{k\to\infty} \mathcal{G}_{\varepsilon_k,\alpha}(\phi^1_k, \phi^2_k) \\
&\le \limsup_{k\to\infty} \|L(P_{\varepsilon_k}(\phi^1_k, \phi^2_k)) - (g_2^\delta - z^\delta)\|_Y^2 \\
&\quad + \alpha \limsup_{k\to\infty} \sum_{j=1}^{2} \left(\beta |H_{\varepsilon_k}(\phi^j_k)|_{BV} + \|\phi^j_k - \phi^j_0\|^2_{H^1(\Gamma_2)} \right) \\
&= \|L(q(z_\alpha^1, z_\alpha^2)) - (g_2^\delta - z^\delta)\|_Y^2 + \alpha\rho(z_\alpha^1, z_\alpha^2, \phi_\alpha^1, \phi_\alpha^2) \\
&= \mathcal{G}_\alpha(z_\alpha^1, z_\alpha^2, \phi_\alpha^1, \phi_\alpha^2) = \inf \mathcal{G}_\alpha,
\end{aligned}$$

characterizing $(\tilde{z}^1, \tilde{z}^2, \tilde{\phi}^1, \tilde{\phi}^2)$ as a minimizer of $\mathcal{G}_\alpha$. □

4.2 Optimility conditions for the stabilized functional

Numerical algorithms for minimizing the stabilized functional (16) are typically based on attempts to satisfy the first order optimality conditions. To this end we consider $\mathcal{G}_{\varepsilon,\alpha}$ with $Y = L^2(\Gamma_2)$ and derive the directional derivatives with respect to ϕ^j for $j = 1, 2$, which read

$$R^j_{\varepsilon,\alpha,\beta} = 0, \qquad\qquad j = 1, 2 \qquad\qquad (17)$$

with

$$\begin{aligned}
R^j_{\varepsilon,\alpha,\beta}(\phi^1, \phi^2) :=\ & \Psi^j_\varepsilon H'_\varepsilon(\phi^j) L^*(L(P_\varepsilon(\phi^j, \phi^j)) - (g_2^\delta - z^\delta)) \\
&- \alpha \left[\frac{\beta}{2} \nabla \cdot \left(H'_\varepsilon(\phi^j) \nabla H_\varepsilon(\phi^j)/|\nabla H_\varepsilon(\phi^j)| \right) - (I - \Delta)(\phi^j - \phi^j_0) \right]
\end{aligned}$$

and

$$\Psi^1_\varepsilon(\phi^1, \phi^2) = (c_1 - c_2 - c_3 + c_4) H_\varepsilon(\phi^2) + (c_3 - c_4), \qquad (18)$$

$$\Psi^2_\varepsilon(\phi^1, \phi^2) = (c_1 - c_2 - c_3 + c_4) H_\varepsilon(\phi^1) + (c_3 - c_4). \qquad (19)$$

Below we consider certain fixed point iterations for soving the system of first order optimality conditions. Let us first note that (17) is equivalent to

$$M\begin{pmatrix} \phi^1 - \phi_0^1 \\ \phi^2 - \phi_0^2 \end{pmatrix} = M\begin{pmatrix} \phi^1 - \phi_0^1 \\ \phi^2 - \phi_0^2 \end{pmatrix} - \begin{pmatrix} R_{\varepsilon,\alpha,\beta}^1 \\ R_{\varepsilon,\alpha,\beta}^2 \end{pmatrix}. \tag{20}$$

If we let M vary in each iteration, we obtain algorithms of the following form

$$M_k\begin{pmatrix} \phi_{k+1}^1 - \phi_k^1 \\ \phi_{k+1}^2 - \phi_k^1 \end{pmatrix} = M_k\begin{pmatrix} \phi_k^1 - \phi_0^1 \\ \phi_k^2 - \phi_0^2 \end{pmatrix} - \begin{pmatrix} R_{\varepsilon,\alpha,\beta}^1(\phi_k^1, \phi_k^2) \\ R_{\varepsilon,\alpha,\beta}^2(\phi_k^1, \phi_k^2) \end{pmatrix}. \tag{21}$$

These iteration can be considered as preconditioned fixed-point iterations for (17).

4.3   Multiple level-set algorithms

In the following, we will consider three different choices for M_k.

Algorithm 1 (Simple iteration). As a first method, we consider the fixed point iteration discussed in [15], which has the form (21) with

$$M_k := \begin{pmatrix} \alpha(I - \Delta) & 0 \\ 0 & \alpha(I - \Delta) \end{pmatrix}.$$

In this case, the two equations in (21) decouple, and the iteration can be written as

$$\alpha(I - \Delta)(\phi_{k+1}^j - \phi_k^j) = \Psi_\varepsilon^j H_\varepsilon'(\phi^j) L^*(L(P_\varepsilon(\phi^j, \phi^j)) - (g_2^\delta - z^\delta))$$
$$- \alpha\frac{\beta}{2}\nabla \cdot \left(H_\varepsilon'(\phi^j)\nabla H_\varepsilon(\phi^j)/|\nabla H_\varepsilon(\phi^j)|\right).$$

Identifying $\alpha = 1/\Delta t$, $t_n = n\Delta t$, and $\phi_n^j = \phi^j(t_n)$, $n = 1, 2, \ldots$, we find that

$$(\Delta - I)\left(\frac{\phi^j(t_n) - \phi^j(t_{n-1})}{\Delta t}\right) = R_{\varepsilon,1/\Delta t,\beta}^{j,*}(\phi^1(t_{n-1}), \phi^2(t_{n-1})). \tag{22}$$

If we consider Δt as step length in a time discretization, we find that in a formal sence the iterative regularized solution ϕ_n^j, $j = 1, 2$, is an approximate solution of the dynamical system

$$(\Delta - I)\left(\frac{\partial\phi^j(t)}{\partial t}\right) = R_{\varepsilon,1/\Delta t,\beta}^j(\phi^1(t), \qquad \phi^2(t)), j = 1, 2. \tag{23}$$

Algorithm 2 (A Gauss-Newton algorithm). Our second choice for M_k is motivated by Gauß-Newton methods for solving (17), i.e., we set

$$M_k := \begin{pmatrix} \psi_k^1 H_\varepsilon'(\phi_k^1)L^*LH_\varepsilon'(\phi_k^1)\psi_k^1 + \alpha(I - \Delta) & \psi_k^1 H_\varepsilon'(\phi_k^1)L^*LH_\varepsilon'(\phi_k^2)\psi_k^2 \\ \psi_k^2 H_\varepsilon'(\phi_k^2)L^*LH_\varepsilon'(\phi_k^1)\psi_k^1 & \psi_k^2 H_\varepsilon'(\phi_k^2)L^*LH_\varepsilon'(\phi_k^2)\psi_k^2 + \alpha(I - \Delta) \end{pmatrix}.$$

Here, M_k is just the Gauß-Newton approximation for the second derivative of the functional (16) with $\beta = 0$. While we expect faster convergence of the Newton-type method in comparison to the simple iteration, now a coupled linear system has to be solved in each step of iteration (21). Since M_k is positive definite, these linear systems can be solved iteratively, e.g., by a conjugate gradient method.

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Algorithm 3 (Preconditined iterations)**.** As a third alternative we consider the choice

$$M_k = \begin{pmatrix} (I - \Delta)^{-1} & 0 \\ 0 & (I - \Delta)^{-1} \end{pmatrix} M_k^*,$$

where M_k^* is either the matrix of the simple iteration or of the Gauß-Newton method above. For a detailed analysis of similar preconditioned iterative methods, we refer to [16, 17].

5 Numerial exeriments

In this section we illustrate the advantages of the multiple levelset methods in reconstructing piecewise constant parameters over standard methods based on the regularized solution of the quadratice least-squares problem. Moreover, we compare the three different numerical methods for minimizing the minimizing the stabilized functional (16) outlined in the previous section.

5.1 The model problem and its discretization

For our numerical experiments we consider the following three dimensional Cauchy problem: Let $\alpha > 0$ be given, and consider the domain $\Omega := (0,1) \times (0,1) \times (0,a)$ with boundary $\partial\Omega = \overline{\Gamma_0} \cup \Gamma_a \cup \Gamma_L$ consisting of three parts

$$\Gamma_0 := (0,1)^2 \times \{0\}, \qquad \Gamma_a := (0,1)^2 \times \{a\}, \qquad \text{and} \qquad \Gamma_L := \partial\Omega \setminus \overline{\Gamma_0 \cup \Gamma_a}.$$

We consider the boundary value problem

$$-\Delta u = 0 \qquad \qquad \text{in } \Omega, \tag{24}$$
$$u = 0 \qquad \qquad \text{on } \Gamma_0 \cup \Gamma_L \tag{25}$$
$$u_\nu = \varphi \qquad \qquad \text{on } \Gamma_a, \tag{26}$$

and the corresponding Cauchy problem of determining the function ϕ from additional observations

$$u_\nu = g \qquad \qquad \text{on } \Gamma_0.$$

For the numerical solution of (24)–(26) we use a discretization method based on Fourier series. Let $\varphi_{n,m}$ denote the Fourier coefficients of ϕ, i.e.,

$$\varphi(x,y) = \sum_{m,n} \varphi_{m,n} \sin(m\pi x) \sin(n\pi y),$$

then the solution of the boundary value problem (24)–(26) is given by

$$u(x,y,z) = \sum_{m,n} u_{m,n} \sin(m\pi x) \sin(n\pi y) \sinh(\omega_{m,n}\pi a), \qquad \omega_{m,n} := \sqrt{m^2 + n^2}$$

with coefficient $u_{m,n}$ defined by

$$u_{m,n} = \frac{\varphi_{m,n}}{\omega_{m,n}\pi \cosh(\omega_{m,n}\pi a)}.$$

The corresponding forward operator L in (6) is then given by

$$(L\varphi)(x,y) = \sum_{m,n} u_{m,n} \sin(m\pi x) \sin(n\pi y) \sinh(\omega_{m,n}\pi a).$$

5.2 Numerical tests

In our numerical experiments, we try to identify the piecewise constant function

$$\phi(x, y) = \begin{cases} 2, & (x - 0.75)^2 + (y - 0.75)^2 < 0.15 \\ 1, & (x - 0.75)^2 + (y - 0.25)^2 < 0.15 \text{ or } (x, y) \in (0.15, 0.35)^2 \\ 0, & \text{else} \end{cases}$$

from additional measurements of the Neumann trace $g := u_\nu$ at Γ_0. Throughout our numerical experiments we use synthetic data which are perturbed by random noise. The true solution and the corresponding data are displayed in Figure 1. In all numerical tests, we stop the iterations according to a discrepancy principle, i.e., we terminate the iterations when the first time the norm of the residual is less than $\tau\delta$, and we use $\tau = 1.5$ in our simulations. For the Gauß-Newton methods, we further utilize an adaptive strategy for choosing the regularization parameter α, i.e., we start with $\alpha = 1$ and decrease α in every Newton iteration by a factor 0.3. For the simple iterations, we choose $\alpha = \delta^2$. For the levelset methods, we utilize a
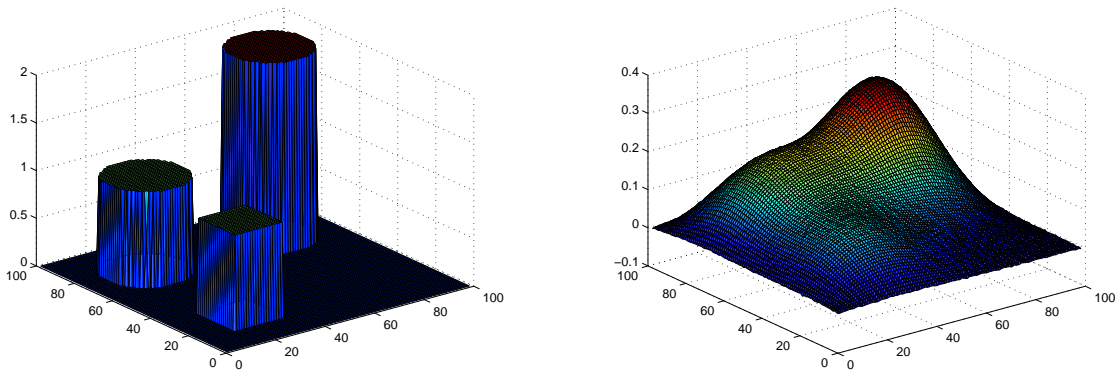

Figure 1: Exact solution (left) and data (right) perturbed with $\delta = 10^{-3}$ noise.

smoothed Heaviside projector

$$H_\varepsilon(z) := \frac{1}{2}(\text{erf}(z) + 1),$$

and we report only numerical experiments with $\beta = 0$, i.e., without bounded variation regularization term. Since the true solution only attains three different values, we choose $c_1 = 0$, $c_2 = 1$, $c_2 = 2$ and $c_4 = 0$ in the definition of the "projection" operator P, see (11).

In Figure 2, we display the solutions obtained with a standard conjugate gradient method applied to the linear Cauchy problem (6), and the one obtained with the preconditioned Gauß-Newton levelset method outlined in Algorithm 2 and 3. The reconstructions obtained with the different levelset algorithms are very similar, and we therefore only present the results for one of the methods. Also the reconstruction errors of the the methods are comparable, and we only list the error of conjugate gradient method applied to the solution of the linear Cauchy problem and the preconditioned Gauß-Newton method in Table 1. The rereconstructions of the levelset methods are clearly superior to the ones obtained with standard regularization methods, e.g., the L^2 reconstruction error of the levelset methods at noise level $\delta = 10^{-2}$ is comparable to the one of the CGNE method with noise level $\delta = 10^{-4}$. This shows that the
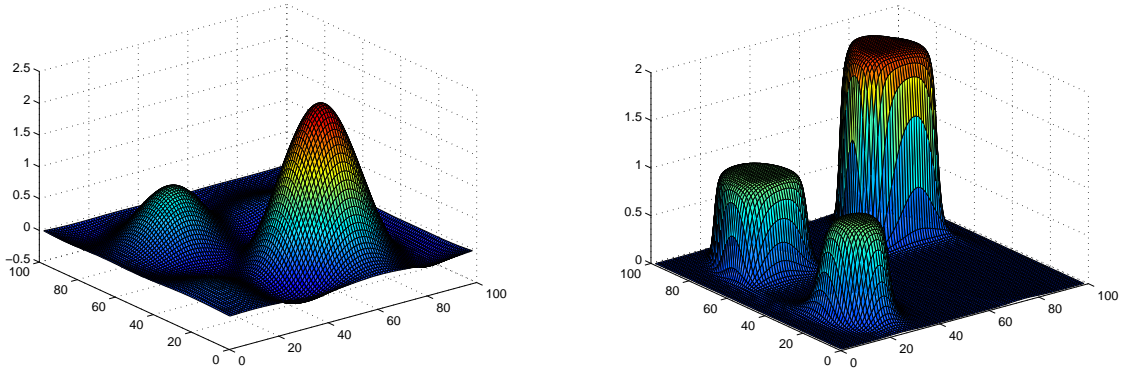
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Figure 2: Comparison of reconstructions obtained with conjugate gradients method applied to the linear Cauchy problem (left), and the preconditioned Gauß-Newton levelset method (right) for noise level $\delta = 10^{-3}$.

δ	0.01	0.001	0.0001	0.00001
CGNE	0.3940	0.2615	0.2356	0.2185
P-GN	0.2256	0.1804	0.1227	0.0763

Table 1: Reconstruction errors $\|\varphi^\dagger - \varphi_k\|_0$ of CGNE applied to the linear Cauchy problem (6), and the preconditioned Gauß-Newton levelset method (Algorithm 3) for varying noise level; see also Figure 2.

utilization of a-priori knowledge, i.e. the assumption of a piecewise constant solution, in the formulation of the algorithm can drastically improve the quality of reconstructions.

While the solutions obrtained with the different levelset algorithms are very similar, the computational cost of Algorithms 1–3 varies significantly. In Table 2, we compare the iteration numbers of the simple iteration and the Gauß-Newton algorithm, and their preconditioned variants.

δ	SIM	GN	P-SIM	P-GN
0.01	30	7 (10)	9	3 (10)
0.001	575	13 (37)	70	6 (28)
0.0001	>10000	20 (113)	4272	9 (62)

Table 2: Iteration numbers for the simple iteration (SIM) and the Gauß-Newton method (GN), as well as their preconditioned variants for various noise levels. For the Gauß-Newton iterations, the total number of inner CG iterations are listed in parenthesis.

Acknowledgments

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