Level-set approaches of L^2–type for recovering shape and contrast in ill-posed problems

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Abstract

We investigate level-set type approaches for solving ill-posed inverse problems, under the assumption that the solution is a piecewise constant function. Our goal is to identify the level sets as well as the level values of the unknown parameter function.

Two distinct level-set frameworks are proposed for solving the inverse problem. In both of them the level-set function is assumed to be in L^2. Corresponding Tikhonov regularization approaches are derived and analyzed. Existence of minimizers for the Tikhonov functionals is proven. Moreover, convergence and stability results of the variational approaches are established, characterizing the Tikhonov approaches as regularization methods.

1 Introduction

Several inverse problems of interest consist of identifying an unknown physical quantity $u \in X$, that can be represented by a piecewise constant real function over a bounded given domain Ω, from the set of data $y \in Y$, where X, Y are Hilbert spaces. The relation between the unknown parameter function and the problem data is described by the model

$$F(u) \; = \; y \,, \tag{1}$$

where $F : D(F) \subset X \to Y$, what corresponds to the fact that the set of data is obtained by indirect measurements of the parameter. Because of this, in practical applications the exact data $y \in Y$ is, in general, not known. Given is only approximate measured data $y^\delta \in Y$, corrupted by noise of level $\delta > 0$ and satisfying

$$\|y^\delta - y\|_Y \;\le\; \delta \,. \tag{2}$$

In the case where the unknown function u is a piecewise constant function distinguishing between two given values, level-set approaches were considered in [20, 17, 13, 4, 2, 3]. In this case, since the level values of u are known, one needs only to identify the level sets of u, i.e. the inverse problem reduces to a shape identification problem. In the case where the unknown

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function u is a piecewise constant function distinguishing between several given values, multiple level set approaches were considered in [3, 5, 8].

If the level values of u are also unknown the inverse problem becomes harder, since one has to identify both the level sets as well as the level values of the unknown parameter u. In this case, the dimension of the parameter space increases by the number of unknown level values.

Our starting point in this article is the assumption that the parameter function u in (1) is a piecewise constant function assuming two distinct unknown values, i.e. $u(x) \in \{c^1, c^2\}$ a.e. in $\Omega \subset \mathbb{R}^d$. In this case one can assume the existence of an open mensurable set $D \subset\subset \Omega$ s.t. $u(x) = c^1$, $x \in D =: D_1$ and $u(x) = c^2$, $x \in \Omega/D =: D_2$.

In this article we propose two level set approaches to represent the unknown parameter u:

1) Standard level set approach (sLS): This approach consists in introducing the level set function ϕ, in $L^2(\Omega)$, which acts as a regularization on the parameter space. We use the Heaviside projector H, to represent a solution of (1) in the form

$$u \ = \ c^2 H(\phi) + c^1(1 - H(\phi)) \ =: \ P_s(\phi, c^j)\,. \tag{3}$$

Notice that $u(x) = c^i$, $x \in D_i$, where the sets D_i are defined by $D_1 = \{x \in \Omega\,;\ \phi(x) > 0\}$ and $D_2 = \{x \in \Omega\,;\ \phi(x) < 0\}$. Thus, the operator P_s establishes a straightforward relation between the level sets of ϕ and the sets D_i representing our *a priori* knowledge about the solution u.

Within this sLS framewok, the inverse problem in (1), with data given as in (2), can be written in the form of the operator equation

$$F(P_s(\phi, c^j)) \ = \ y^\delta\,. \tag{4}$$

In order to obtain approximate solutions to (4), we propose the minimization of the Tikhonov functional

$$\mathcal{G}_{\alpha,s}(\phi, c^j) \ := \ \|F(P_s(\phi, c^j)) - y^\delta\|_Y^2 + \alpha\Big\{\beta_1 |H(\phi)|_{\mathtt{BV}} + \beta_2\|\phi\|^2_{L^2(\Omega)} + \beta_3\|c^j\|^2_{\mathbb{R}^2}\Big\} \tag{5}$$

based on TV–L^2 penalization. Concurrent approaches were proposed in [2, 3, 21] (using TV penalization) and in [13, 8] (using TV-H^1 penalization).

2) Piecewise constant level set approaches (pcLS): In the sequel we introduce the *piecewise constant* level set function $\phi \in L^2(\Omega)$ such that $\phi(x) = i$, $x \in D_i$, $i = 1, 2$. Then, defining the auxiliary functions $\psi_1(t) := 2-t$ and $\psi_2(t) := t-1$, we represent the characteristic functions of the subdomains D_i in the form $\chi_{D_i}(x) = \psi_i(\phi(x))$. Consequently, a solution of (1) can be written in the form

$$u \ = \ c^1\psi_1(\phi) + c^2\psi_2(\phi) \ =: \ P_{pc}(\phi, c^j)\,. \tag{6}$$

Notice that the piecewise constant assumption on ϕ corresponds to the constraint $\mathcal{K}(\phi) = 0$, where $\mathcal{K}(\phi) := (\phi - 1)(\phi - 2)$ is a smooth nonlinear functional.

2a) Penalized pcLS approach: Within the pcLS framework, the inverse problem in (1), with data given as in (2), can be written in the form of the abstract operator equation

$$\begin{cases} F(P_{pc}(\phi, c^j)) \ = \ y^\delta\,, \\ \text{s.t.} \quad \phi \in \{L^2(\Omega);\ \mathcal{K}(\phi) = 0\}\,. \end{cases} \tag{7}$$

Approximate solutions to (7) can be obtained by minimizing the Tikhonov functional

$$\mathcal{G}_{\alpha,ppc}(\phi, c^j) \ := \ \|F(P_{pc}(\phi, c^j)) - y^\delta\|_Y^2 + \alpha\Big\{\beta_1 |P_{pc}(\phi, c^j)|_{\mathtt{BV}} + \beta_2\|K(\phi)\|^2_{L^2} + \beta_3\|c^j\|^2_{\mathbb{R}^2}\Big\}\,, \tag{8}$$

where the constraint $\mathcal{K}(\phi) = 0$ in (7) is enforced by the penalty term $\|K(\phi)\|^2_{L^2(\Omega)}$, with $K(t) := [\mathcal{K}(t) + 2t^2]^{1/2}$. The other penalization terms correspond to a $TV - L^2$ regularization strategy (for details see Section 3.1); here $\beta_j > 0$ are scaling factors.

2b) Strict pcLS approach: In the sequel we introduce yet another Tikhonov functional, based on the pcLS framework established above. We propose to obtain approximate solutions to the operator equation (7) by minimizing

$$\mathcal{G}_{\alpha,spc}(\phi, c^j) \;:=\; \|F(P_{pc}(\phi, c^j)) - y^\delta\|^2_Y + \|\mathcal{K}(\phi)\|_{L^1} + \alpha\Big\{\beta_1 |P_{pc}(\phi, c^j)|_{\mathtt{BV}} + \beta_2 \|c^j\|^2_{\mathbb{R}^2}\Big\}. \quad (9)$$

Notice that the minimization of the functional $\mathcal{G}_{\alpha,spc}$ furnishes a regularized solution to the system of operator equations:

$$\begin{bmatrix} F(P_{pc}(\phi, c^j)) \\ \mathcal{K}(\phi) \end{bmatrix} = \begin{bmatrix} y^\delta \\ 0 \end{bmatrix}.$$

The penalization term in (9) corresponds simply to a TV regularization strategy. There is actually no need to add an L^2 penalization term to the Tikhonov functional $\mathcal{G}_{\alpha,spc}$ (the reason will become clear in Section 3).

The main difference between the penalized-pcLS and strict-pcLS approaches resides on the fact that, in the limit case $\alpha \to 0$,[1] the limit $(\overline{\phi}, \overline{c}^j)$ of the minimizers $(\phi_\alpha, c^j_\alpha)$ of $\mathcal{G}_{\alpha,ppc}$ does not necessarily satisfy $\mathcal{K}(\overline{\phi}) = 0$. Therefore, the limit levelset function $\overline{\phi}$ does not have to be piecewise constant. On the other hand, the minimizers $(\psi_\alpha, c^j_\alpha)$ of $\mathcal{G}_{\alpha,spc}$ converge (as $\alpha \to 0$) to some limit $(\overline{\psi}, \overline{c}^j)$ satisfying $F(P_{pc}(\overline{\psi}, \overline{c})) = y$ and $\mathcal{K}(\overline{\psi}) = 0$. Thus, the limit levelset function $\overline{\psi}$ is indeed piecewise constant (as suggested by the name of the approach).

This article is outlined as follows: In Section 2 we introduce the concept of generalized minimizers for the functional $\mathcal{G}_{\alpha,s}$ in (5). Basic properties of the generalized minimizers are verified, as well as regularity properties of the penalization term of $\mathcal{G}_{\alpha,s}$. Moreover, we derive a convergence analysis for the Tikhonov method related to the sLS approach. We prove a well-posedness result, and also convergence results for exact and noisy data. In Section 3 we derive for the pcLS approaches a convergence analysis analog to the classical one presented for the sLS approach. Section 4 is devoted to numerical experiments. Level set type methods based on the sLS and pcLS approaches are implemented for solving a two-dimensional inverse potential problem.

2 The sLS approach

We shall consider the model problem described as in the introduction under the following general assumptions:

(A1) $\Omega \subseteq \mathbb{R}^d$, $d = 2$, is bounded with piecewise C^1 boundary $\partial\Omega$.

(A2) The operator $F : \mathbb{D} \subset L^p(\Omega) \to Y$ is continuous and Fréchet-differentiable on $\mathbb{D}$ with respect to the L^p-topology, where $1 \le p < d/(d-1) = 2$.

(A3) ε, α and β_j , $j = 1, 2, 3$ denote positive parameters.

(A4) Equation (1) has a solution, i.e. there exists $u \in L^\infty(\Omega)$ satisfying $F(u) = y$; there exists a function $\phi \in L^2(\Omega)$ satisfying $|\nabla\phi| \ne 0$, in a neighborhood of $\{\phi = 0\}$ such that $H(\phi) = z$, for some $z \in L^\infty(\Omega)$; there exist constants values $c^j \in \mathbb{R}$ such that $P_s(z, c^j) = u$.

[1] Recall that in the presence of noise, $\delta > 0$, the regularization parameter α is a function of the noisy level, i.e., $\alpha = \alpha(\delta)$; see Theorem 8.

For each $\varepsilon > 0$, we define the operator

$$P_{s,\varepsilon}(\phi, c^j) \; := \; c^1 H_\varepsilon(\phi) + c^2(1 - H_\varepsilon(\phi)) \,, \tag{10}$$

where H_ε is the smooth approximation to H given by:

$$H_\varepsilon(t) := \begin{cases} 1 + t/\varepsilon & \text{for} \;\; t \in [-\varepsilon, 0] \\ H(t) & \text{for} \;\; t \in \mathbb{R}/[-\varepsilon, 0] \end{cases} .$$

2.1 The concept of generalized minimizers

In order to guarantee existence of a minimizer of $\mathcal{G}_{\alpha,s}$ in (5), we adapt to the level-set framework described above, the concept of generalized minimizers formulated in [13].

Definition 1. *Let the operators H, P_s, H_ε and $P_{s,\varepsilon}$ be defined as above.*

a) A **vector** $(z, \phi, c^j) \in L^\infty(\Omega) \times L^2(\Omega) \times \mathbb{R}^2$ *is called* **admissible** *when there exists a sequence $\{\phi_k\}$ of $L^2(\Omega)$-functions satisfying $\lim\limits_{k\to\infty} \|\phi_k - \phi\|_{H^{-1}(\Omega)} = 0$, and there exists a sequence $\{\varepsilon_k\} \in \mathbb{R}^+$ converging to zero such that $H_{\varepsilon_k}(\phi_k) \in L^p(\Omega)$ and $\lim\limits_{k\to\infty} \|H_{\varepsilon_k}(\phi_k) - z\|_{L^p(\Omega)} = 0$.*

b) A **generalized minimizer** *of $\mathcal{G}_{\alpha,s}$ is considered to be any admissible vector (z, ϕ, c^j) minimizing*

$$\mathcal{G}_\alpha(z, \phi, c^j) := \|F(q(z, c^j)) - y^\delta\|_Y^2 + \alpha R(z, \phi, c^j) \tag{11}$$

over the set of admissible vectors, where $q : L^\infty(\Omega) \times \mathbb{R}^2 \ni (z, c^j) \mapsto c^1 z + c^2(1 - z) \in L^\infty(\Omega)$, and the functional R is defined by

$$R(z, \phi, c^j) \; := \; \rho(z, \phi) + \beta_3 \|c^j\|_{\mathbb{R}^2}^2 \,, \tag{12}$$

with $\rho(z, \phi) := \inf \big\{ \liminf_{k\to\infty} (\beta_1 |H_{\varepsilon_k}(\phi_k)|_{\mathtt{BV}} + \beta_2 \|\phi_k\|_{L^2}^2) \big\}$. Here the infimum is taken over all sequences $\{\varepsilon_k\}$ and $\{\phi_k\}$ characterizing (z, ϕ, c^j) as an admissible vector.

2.2 Preliminary results

In the sequel we investigate relevant properties of the admissible vectors as well as properties of the penalization functional R in (12). We start by verifying some basic properties of the operators $P_{s,\varepsilon}$, H_ε and q that will be necessary in the subsequent analysis.

Lemma 1. *Let Ω and p be given as in (A1), (A2). The following assertions hold true.*

i) Let $\{z_k\}$ be a sequence in $L^\infty(\Omega)$ converging to some element $z \in L^\infty(\Omega)$ in the L^p-topology and $\{c^j_k\}$ be sequences of real numbers converging to c^j, $j = 1, 2$. Then $q(z_k, c^j_k)$ converges to $q(z, c^j)$ in the L^p-topology.

ii) Let $(z, \phi) \in L^\infty(\Omega) \times L^2(\Omega)$, be such that $H_\varepsilon(\phi) \to z$ in $L^p(\Omega)$ as $\varepsilon \to 0$ and let $c^j \in \mathbb{R}$. Then $P_{s,\varepsilon}(\phi, c^j) \to q(z, c^j)$ in $L^p(\Omega)$ as $\varepsilon \to 0$.

Proof. It is enough to prove assertion (i). Since Ω is bounded the constant functions are in $L^p(\Omega)$. Therefore,

$$\begin{aligned} ||q(z_k, c^j_k) - q(z, c^j)||_{L^p(\Omega)} &= \big\| c^1_k z_k + c^2_k(1 - z_k) - c^1 z - c^2(1 - z) \big\|_{L^p(\Omega)} \\ &= \big\| c^1_k(z_k - z) + (c^1_k - c^1) z + c^2_k \big[(1 - z_k) - (1 - z) \big] + (c^2_k - c^2)(1 - z) \big\|_{L^p(\Omega)} \\ &\le |c^1_k| \|z_k - z\|_{L^p(\Omega)} + |c^1_k - c^1| \|z\|_{L^p(\Omega)} + |c^2_k| \|z_k - z\|_{L^p(\Omega)} + |c^2_k - c^2| \|1 - z\|_{L^p(\Omega)} \end{aligned}$$

and the assertion follows. □

Lemma 2. *Let (z_k, ϕ_k, c_k^j) be a sequence of admissible vectors converging in $L^p(\Omega)\times H^{-1}(\Omega)\times \mathbb{R}^2$ to some (z, ϕ, c^j) in $L^\infty(\Omega)\times L^2(\Omega)\times\mathbb{R}^2$. Then (z,ϕ,c^j) is also an admissible vector.*

Sketch of the proof. For each $k \in \mathbb{N}$, it follows from Definition 1 that there exists a sequence $\{\phi_k^l\}$ in $L^2(\Omega)$ and a sequence $\{\varepsilon_k^l\}$ in $\mathbb{R}^+$ such that as $l \to \infty$ we have $\phi_k^l \to \phi_k$ in $H^{-1}(\Omega)$ and $H_{\varepsilon_k^l}(\phi_k^l) \to z_k$ in $L^p(\Omega)$. Thus, we can select a monotone increasing index function $\gamma : \mathbb{N} \to \mathbb{N}$ such that

$$\varepsilon_k^{\gamma(k)} \le \frac{1}{2}\varepsilon_{k-1}^{\gamma(k-1)}\,, \qquad \|\phi_k^{\gamma(k)} - \phi_k\|_{H^{-1}(\Omega)} \le k^{-1}\,, \qquad \|H_{\varepsilon_k^{\gamma(k)}}(\phi_k^{\gamma(k)}) - z_k\|_{L^p(\Omega)} \le k^{-1}\,,$$

for every $k\in\mathbb{N}$. Now, the lemma follows arguing with the triangular inequality. □

In the sequel we prove coercivity and weak lower semi-continuity of the penalization functional R. There properties are fundamental for the convergence analysis in Section 2.3. First however, we briefly recall some facts about the space $\mathtt{BV}(\Omega)$. For a proof we refer the reader to [12, Chapter 5].

Lemma 3. *The following assertions hold true:*

i) The semi-norm $|\cdot|_{\mathtt{BV}}$ is weakly lower semi-continuous with respect to L^p-convergence, i.e., if $\{x_k\}\in \mathtt{BV}(\Omega)$ converges to x in the L^p-norm, then $x\in\mathtt{BV}(\Omega)$ and $|x|_{\mathtt{BV}} \le \liminf_{k\to\infty}|x_k|_{\mathtt{BV}}$.

ii) $\mathtt{BV}(\Omega)$ is compactly embedded in $L^p(\Omega)$ for $1 \le p < d/(d-1)$. Consequently, any bounded sequence $\{x_k\}\in\mathtt{BV}(\Omega)$ has a subsequence converging in $L^p(\Omega)$ to some $x\in\mathtt{BV}(\Omega)$.

Lemma 4. *The functional R in (12) is coercive on the set of admissible vectors.*

Sketch of the proof. Let (z,ϕ,c^j) be an admissible vector. From definition of $\rho(z,\phi)$ and the definition of admissible vectors, we can guarantee the existence of sequences $\{\phi_k\}\in L^2(\Omega)$ and $\{\varepsilon_k\}\in\mathbb{R}^+$ such that $\varepsilon_k\to 0$, $\phi_k\to\phi$ in $H^{-1}(\Omega)$, $H_{\varepsilon_k}(\phi_k)\to z$ in $L^p(\Omega)$, and

$$\rho(z,\phi) \;=\; \liminf_{k\to\infty}\left\{\beta_1|H_{\varepsilon_k}(\phi_k)|_{\mathtt{BV}} + \beta_2\|\phi_k\|^2_{L^2(\Omega)}\right\} \tag{13}$$

(see [8, Lemma 3]). From (13), the weak lower semi-continuity of the L^2-norm, and Lemma 3 i), it follows that

$$\rho(z,\phi) \;\ge\; \beta_1\liminf_{k\to\infty}|H_{\varepsilon_k}(\phi_k)|_{\mathtt{BV}} + \beta_2\liminf_{k\to\infty}\|\phi_k\|^2_{L^2(\Omega)} \;\ge\; \beta_1|z|_{\mathtt{BV}} + \beta_2\|\phi\|^2_{L^2(\Omega)}\,. \tag{14}$$

Thus, it follows from (12), (14) that $\beta_1|z|_{\mathtt{BV}} + \beta_2\|\phi\|^2_{L^2(\Omega)} + \beta_3\|c^j\|^2_{\mathbb{R}^2} \le R(z,\phi,c^j)$, concluding the proof. □

Lemma 5. *The functional R in (12) is weak lower semi-continuous on the set of admissible vectors, i.e. given a sequence $\{(z_k,\phi_k,c_k^j)\}$ of admissible vectors such that $z_k\to z$ in $L^p(\Omega)$, $\phi_k\rightharpoonup\phi$ in $L^2(\Omega)$, $c_k^j\to c^j$ in $\mathbb{R}$, for some admissible vector (z,ϕ,c^j), then*

$$R(z,\phi,c^j) \;\le\; \liminf_{k\to\infty} R(z_k,\phi_k,c_k^j)\,.$$

Sketch of the proof. Since the norm in $\mathbb{R}^2$ is lower semi-continuous, it is enough to prove the weak lower semi-continuity of ρ. We argue by contradiction. Let $\{(z_k,\phi_k,c_k^j)\}$ and (z,ϕ,c^j) be given as above and assume that $\rho(z,\phi) > \liminf_{k\to\infty}\rho(z_k,\phi_k)$. Consequently, there exists a constant $\underline{c}>0$ such that $\rho(z,\phi)\ge\underline{c}>\liminf_{k\in\mathbb{N}}\rho(z_k,\phi_k)$. Arguing as in [8, Lemma 5] we prove the following
Claim: For every sequence $\{(z_l,\phi_l,c_l^j)\}$ of admissible vectors satisfying $z_l\to z$ in $L^p(\Omega)$ and $\phi_l\to\phi$ in $H^{-1}(\Omega)$ such that $\rho(z_l,\phi_l)\le\bar{c}$ we have $\rho(z,\phi)\le\bar{c}$.
Notice that this claim is a sufficient condition for the weak lower semi-continuity of ρ. Indeed, if the claim holds true, the constant $\underline{c}$ above can not exist. □

2.3 Convergence Analysis

Our first goal is to prove that for any positive parameters α, β_1, β_2, β_3, the functional $\mathcal{G}_{\alpha,s}$ in (5) is well posed.

Theorem 6. *The functional $\mathcal{G}_{\alpha,s}$ in (5) attains minimizers on the set of admissible vectors.*

Proof. Notice that the set of admissible vectors is not empty, since $(0, 0, 0, 0)$ is admissible. Let $\{(z_k, \phi_k, c^j_k)\}$ be a minimizing sequence for $\mathcal{G}_\alpha$, i.e. a sequence of admissible vectors satisfying $\mathcal{G}_\alpha(z_k, \phi_k, c^j_k) \to \inf \mathcal{G}_\alpha \le \mathcal{G}_\alpha(0, 0, 0, 0) < \infty$. Then, $\{\mathcal{G}_\alpha(z_k, \phi_k, c^j_k)\}$ is a bounded sequence of real numbers. Therefore, $\{(z_k, \phi_k, c^j_k)\}$ is uniformly bounded in $\mathtt{BV} \times L^2 \times \mathbb{R}^2$. Thus, Lemma 3, the Sobolev compact embedding theorem [1] and the Bolzano-Weierstraß theorem guarantee the existence of a subsequence (denoted again by $\{(z_k, \phi_k, c^j_k)\}$) and the existence of $(z, \phi, c^j) \in L^p(\Omega) \times L^2(\Omega) \times \mathbb{R}^2$ such that $\phi_k \rightharpoonup \phi$ in $L^2(\Omega)$, $\phi_k \to \phi$ in $H^{-1}(\Omega)$, $z_k \to z$ in $L^p(\Omega)$ and $c^j_k \to c^j$ in $\mathbb{R}$.

From Lemma 2 we conclude that (z, ϕ, c^j) is an admissible vector. Moreover, from Lemma 5 together with the continuity of F and q we obtain

$$\begin{aligned}\inf\, \mathcal{G}_\alpha &= \lim_{k\to\infty} \mathcal{G}_\alpha(z_k, \phi_k, c^j_k) = \liminf_{k\to\infty} \left\{\|F(q(z_k, c^j_k)) - y^\delta\|^2_Y + \alpha R(z_k, \phi_k, c^j_k)\right\} \\ &\ge \|F(q(z, c^j)) - y^\delta\|^2_Y + \alpha R(z, \phi, c^j) = \mathcal{G}_\alpha(z, \phi, c^j)\,,\end{aligned}$$

proving that (z, ϕ, c^j) minimizes $\mathcal{G}_\alpha$. □

In the next theorems we present the main convergence and stability results. The proofs use classical techniques from the analysis of Tikhonov type regularization methods (see, e.g., [11, 10]) and will be omitted.

Theorem 7 (**Convergence for exact data**)**.** *Assume that we have exact data, i.e. $y^\delta = y$ and $\beta_j > 0$, $j = 1, 2, 3$. For every $\alpha > 0$ denote by $(z_\alpha, \phi_\alpha, c^j_\alpha)$ a minimizer of $\mathcal{G}_\alpha$ on the set of admissible vectors. Then, for every sequence of positive numbers $\{\alpha_k\}$ converging to zero there exists a subsequence, denoted again by $\{\alpha_k\}$, such that $(z_{\alpha_k}, \phi_{\alpha_k}, c^j_{\alpha_k})$ is strongly convergent in $L^p(\Omega) \times H^{-1}(\Omega) \times \mathbb{R}^2$. Moreover, the limit is a solution of (4).*

Theorem 8 (**Convergence for noisy data**)**.** *Let $\alpha = \alpha(\delta)$ be a function satisfying* $\lim_{\delta\to 0}$ $\alpha(\delta) = 0$ *and* $\lim_{\delta\to 0} \delta^2 \alpha(\delta)^{-1} = 0$*. Moreover, let $\{\delta_k\}$ be a sequence of positive numbers converging to zero and $\{y^{\delta_k}\} \in Y$ be corresponding noisy data satisfying (2). Then, there exist a subsequence, denoted again by $\{\delta_k\}$, and a sequence $\{\alpha_k := \alpha(\delta_k)\}$ such that $(z_{\alpha_k}, \phi_{\alpha_k}, c^j_{\alpha_k})$ converges in $L^p(\Omega) \times H^{-1}(\Omega) \times \mathbb{R}^2$ to solution of (4).*

3 The pcLS approaches

In these level-set approaches we consider the model problem described in the introduction under assumptions (A1) – (A3). In the case of the penalized-pcLS approach, we also require that

(A4') Equation (1) has a solution, i.e. there exists $u \in L^\infty(\Omega)$ satisfying $F(u) = y$; there exists a function $\phi \in \mathtt{BV}(\Omega) \subset L^2(\Omega)$ and constants $c^1 \ne c^2 \in \mathbb{R}$ such that $P_{pc}(\phi, c^j) = u$.

In the case of the strict-pcLS approach however, we require that

(A4") There exists $u \in L^\infty(\Omega)$ satisfying $F(u) = y$. Moreover, there exists a function $\phi \in \mathtt{BV}(\Omega) \subset L^2(\Omega)$ and constants $c^1 \ne c^2 \in \mathbb{R}$ such that $P_{pc}(\phi, c^j) = u$ and $\mathcal{K}(\phi) = 0$.

Differently from the P_s, for fixed constants c^j the operator $P_{pc}(\cdot, c^j)$ is 1-1, continuous and continuously differentiable from $L^2(\Omega)$ onto $L^2(\Omega)$. Consequently, the set of admissible vectors for the Tikhonov functionals in (8) and (9) is defined in a different manner.

Definition 2. *Let the operator P_{pc} be defined as in (6) and $\tau > 0$. A* **vector** $(\phi, c^j) \in L^2(\Omega) \times \mathbb{R}^2$ *is called* **admissible** *when $\phi \in \mathtt{BV}(\Omega)$ and $|c^2 - c^1| \geq \tau$.*

From (6), it follows that P_{pc} maps admissible vectors to $\mathtt{BV}(\Omega)$. The next lemmas are devoted to the investigation of relevant properties of P_{pc}, K and $\mathcal{K}$.

Lemma 9. *Let K be the operator defined in Section 1. The following assertions hold true:*
i) K maps $L^2(\Omega)$ to $L^2(\Omega)$.
ii) $\|K(\cdot)\|^2_{L^2(\Omega)}$ is coercive.
iii) $\|K(\cdot)\|^2_{L^2(\Omega)}$ is weak lower semi-continuous w.r.t the weak convergence in $L^2(\Omega)$.

Proof. Let $\phi \in L^2(\Omega)$ be given. Assertion (i) is a consequence of

$$\|K(\phi)\|^2_{L^2} \;=\; \|(\prod_{j=1}^{2}(\phi - j) + 2\phi^2)^{1/2}\|^2_{L^2} \;=\; \int_\Omega (3\phi^2 - 3\phi + 2) \;\leq\; 3\|\phi\|^2_{L^2} + \|3\phi - 2\|^2_{L^2}\,.$$

Ad ii): Notice that $(3t^2 - 3t + 2) - t^2 > 0$, $t \in \mathbb{R}$. Therefore, the desired coercivity follows from

$$\|K(\phi)\|^2_{L^2} \;=\; \int_\Omega (3\phi^2 - 3\phi + 2) \;\geq\; \|\phi\|^2_{L^2}\,.$$

Ad iii): One can easily check that $F(t) = 3t^2 - 3t + 2$ is a convex real function. Therefore, the functional $\phi \mapsto \int_\Omega F(\phi(x))dx$ is weak l.s.c. with respect to the weak convergence in L^1 [6]. The assertion follows now from the fact that weak convergence in L^1 is implied by weak convergence in L^2. □

Lemma 10. *Let $\mathcal{K}$ be the operator defined in Section 1. The following assertions hold true:*
i) $\mathcal{K}$ is a continuous map from $L^2(\Omega)$ to $L^1(\Omega)$.
ii) If $\|K(\phi)\|_{L^1(\Omega)} = 0$ for some $\phi \in L^2(\Omega)$, then $\phi(x) \in \{1, 2\}$ a.e. in Ω.

Proof. Assertion i) follows from

$$\int_\Omega |\mathcal{K}(\phi) - \mathcal{K}(\psi)| \;\leq\; \int_\Omega |(\phi - 1)(\phi - \psi)| + \int_\Omega |(\psi - 2)(\psi - \phi)|\,,$$

together with the Cauchy-Schwarz inequality. Assertion ii) follows directly from the definitions of $\mathcal{K}$ and the L^1-norm. □

Lemma 11. *Let P_{pc} be the operator defined in (6). The following assertions hold true:*
i) For every admissible vector (ϕ, c^j) it holds $|P_{pc}(\phi, c^j)|_{\mathtt{BV}} \geq \tau|\phi|_{\mathtt{BV}}$.
Moreover, if (ϕ_k, c^j_k) is a sequence of admissible vectors converging in $L^p(\Omega) \times \mathbb{R}^2$ to some admissible vector (ϕ, c^j), then
ii) $P_{pc}(\phi_k, c^j_k)$ converges to $P_{pc}(\phi, c^j)$ in $L^p(\Omega)$.
iii) $|P_{pc}(\phi, c^j)|_{\mathtt{BV}} \leq \liminf_{k\to\infty} |P_{pc}(\phi_k, c^j_k)|_{\mathtt{BV}}$.
iv) $|P_{pc}(\phi, c^j)|_{\mathtt{BV}} \geq \tau\|\phi\|_{L^2}$.

Proof. Assertion i) follows from the identity $|P_{pc}(\phi, c^j)|_{\mathtt{BV}} \;=\; |c^2 - c^1|\, |\phi|_{\mathtt{BV}}$.
Ad ii): Since Ω is bounded, we have $c^j_k \to c^j$ in $L^p(\Omega)$. Therefore, $c^j_k\phi_k \to c^j\phi$ in $L^p(\Omega)$ and we conclude that $P_{pc}(\phi_k, c^j_k) = c^1_k(2 - \phi_k) + c^2_k(\phi_k - 1) \to P_{pc}(\phi, c^j)$ in $L^p(\Omega)$.
Assertion iii) follows from part ii) together with Lemma 3 (i), while assertion iv) is a corollary of part i). □

3.1 Convergence Analysis: penalized-pcLS

Let $R_{ppc}(\phi, c^j) := \beta_1 |P_{pc}(\phi, c^j)|_{\mathtt{BV}} + \beta_2 \|K(\phi)\|^2_{L^2} + \beta_3 \|c^j\|^2_{\mathbb{R}^2}$ be the penalization term of $\mathcal{G}_{\alpha,ppc}$. In the sequel we prove that for any positive parameters α, β_1, β_2, β_3, the functional $\mathcal{G}_{\alpha,ppc}$ in (8) is well posed.

Theorem 12. *The functional $\mathcal{G}_{\alpha,ppc}$ in (8) attains minimizers on the set of admissible vectors.*

Proof. Let $\{(\phi_k, c^j_k)\}$ be a minimizing sequence for $\mathcal{G}_{\alpha,ppc}$, i.e. a sequence of admissible vectors satisfying $\mathcal{G}_{\alpha,ppc}(\phi_k, c^j_k) \to \inf \mathcal{G}_{\alpha,ppc}$, $k \to \infty$. Then, $\{R_{ppc}(\phi_k, c^j_k)\}$ is a bounded sequence of real numbers. Therefore, it follows from Lemma 9 ii) the existence of a subsequence $\{\phi_k\}$ and $\overline{\phi} \in L^2(\Omega)$ such that $\phi_k \rightharpoonup \overline{\phi}$ in $L^2(\Omega)$. Moreover, from Lemma 11 i) and Lemma 3 ii) we conclude that $\overline{\phi} \in \mathtt{BV}(\Omega)$ and that this subsequence also satisfies $\phi_k \to \overline{\phi}$ in $L^p(\Omega)$.
On the other hand, the boundedness of $\{R_{ppc}(\phi_k, c^j_k)\}$ also guarantees the existence of subsequences $\{c^j_k\}$ converging to $\overline{c}^j$ in $\mathbb{R}^2$.
Clearly $(\overline{\phi}, \overline{c}^j)$ is an admissible vector. Moreover, from (A2), Lemma 11 iii), Lemma 9 iii) it follows that

$$\begin{aligned} \inf\ \mathcal{G}_{\alpha,ppc} &= \lim_{k\to\infty} \mathcal{G}_{\alpha,ppc}(\phi_k, c^j_k) \;=\; \liminf_{k\to\infty} \left\{ \|F(P_{pc}(\phi_k, c^j_k)) - y^\delta\|^2_Y + \alpha R(\phi_k, c^j_k) \right\} \\ &\geq \|F(P_{pc}(\overline{\phi}, \overline{c}^j)) - y^\delta\|^2_Y + \alpha R(\phi, \overline{c}^j) \;=\; \mathcal{G}_{\alpha,ppc}(\overline{\phi}, \overline{c}^j)\,, \end{aligned}$$

proving that $(\overline{\phi}, \overline{c}^j)$ minimizes $\mathcal{G}_{\alpha,ppc}$. □

The convergence and stability results in Theorems 7 and 8 hold true for the pcLS approach. As before, the proofs are based on classical techniques from the analysis of Tikhonov regularization.

Theorem 13 (Convergence analysis: pcLS). *Assume that we have exact data and $\beta_j > 0$, $j = 1, 2, 3$. For every $\alpha > 0$ denote by $(\phi_\alpha, c^j_\alpha)$ a minimizer of $\mathcal{G}_{\alpha,ppc}$ on the set of admissible vectors. Then, for every sequence of positive numbers $\{\alpha_k\}$ converging to zero there exists a subsequence such that $(\phi_{\alpha_k}, c^j_{\alpha_k})$ is strongly convergent in $L^p(\Omega) \times H^{-1}(\Omega) \times \mathbb{R}^2$. Moreover, the limit is a solution of (7).*
In the case of noisy data, let $\alpha = \alpha(\delta)$ be a function chosen as in Theorem 8. Given a sequence $\{\delta_k\}$ of positive numbers converging to zero and $\{y^{\delta_k}\} \in Y$ be corresponding noisy data satisfying (2), there exist a subsequence, denoted again by $\{\delta_k\}$, and a sequence $\{\alpha_k := \alpha(\delta_k)\}$ such that $(\phi_{\alpha_k}, c^j_{\alpha_k})$ converges in $L^p(\Omega) \times H^{-1}(\Omega) \times \mathbb{R}^2$ to solution of (7).

Notice that the limit elements (ϕ, c^j) obtained in Theorem 13 (as limit of the sequence $\{(\phi_{\alpha_k}, c^j_{\alpha_k})\}_k$) satisfy $F(P_{pc}(\phi, c^j)) = y$. However, we cannot guarantee that $\mathcal{K}(\phi) = 0$.

3.2 Convergence Analysis: strict-pcLS

Let $R_{spc}(\phi, c^j) := \beta_1 |P_{pc}(\phi, c^j)|_{\mathtt{BV}} + \beta_2 \|c^j\|^2_{\mathbb{R}^2}$ be the penalization term of $\mathcal{G}_{\alpha,spc}$. Given α, β_1, $\beta_2 > 0$ it is possible to prove that the functional $\mathcal{G}_{\alpha,spc}$ in (9) attains minimizers on the set of admissible vectors. The proof follows the lines of the proof of Theorem 12 and only a few changes are needed, namely:
— In order to guarantee the existence of a weak convergent subsequence, Lemma 9 ii) has to be substituted by Lemma 11 iv), which guarantees the coercivity of the functional $|P_{pc}(\cdot, c^j)|_{\mathtt{BV}}$ (w.r.t. the L^2-norm) on the set of admissible parameters.

— Moreover, in order to guarantee the inf $\mathcal{G}_{\alpha,spc} = \mathcal{G}_{\alpha,spc}(\overline{\phi}, \overline{c}^j)$, one has to argue with Lemma 11 iii) and Lemma 10 1).

It is worth noticing that the convergence and stability results in Theorem 13 hold also for the strict-pcLS approach, after obvious changes.

Differently from Theorem 13, the limit elements (ϕ, c^j) obtained from the convergence-stability theorem for the strict-pcLS approach satisfy not only $F(P_{pc}(\phi, c^j)) = y$, but also $\|\mathcal{K}(\phi)\|_{L^1} = 0$. Therefore, due to Lemma 10 ii), we conclude that the limit level-set function ϕ is piecewise constant.

4 Numerical results

In this section we discuss the numerical implementations of iterative methods based on the sLS and pcLS approaches. As test problem we use an inverse potential problem, similar to the one considered in [13, 23, 8, 14, 24].

The forward problem consists of solving on a given Lipschitz domain $\Omega \subset \mathbb{R}^n$, for a given source function $u \in L^2(\Omega)$, the Poisson boundary value problem

$$-\Delta w \;=\; u\,,\ \text{in}\,\Omega\,, \quad w \;=\; 0\ \text{on}\ \partial\Omega\,. \tag{15}$$

This problem can be modeled by the operator $F : L^2(\Omega) \to L^2(\partial\Omega)$, $F(u) := w_\nu|_{\partial\Omega}$ [16].

The corresponding inverse problem is the so called em inverse potential problem, which consists of recovering an L^2–function u, from measurements of the Cauchy data of its corresponding potential on the boundary of Ω. Using the notation introduced above, the inverse potential problem can be written in the abbreviated form $F(u) = y^\delta$, where the available noisy data $y^\delta \in L^2(\partial\Omega)$ have the same meaning as in (2).

It is worth noticing that this inverse problem has, in general, non unique solution [14]. Sufficient conditions for identifiability are given in [15]. For issues related to redundancy of data as well as for an example of non identifiability we refer the reader to [14]. A generalization of this inverse problem, with the Laplacian replaced by a general elliptic operator, appears in many relevant applications including: inverse gravimetry [19, 16], EEG [9], and EMG [25].

In our experiments we follow [8] in the experimental setup, selecting $\Omega = (0,1) \times (0,1)$ and assuming that the unknown parameter is a piecewise constant function of the form $u = 1 + \chi_D$, where $D \subset\subset \Omega$. In particular, we allow piecewise constant functions u supported at domains, which consist of a number of connected inclusions. For this class of parameters no unique identifiability result is known and we restrict our attention to minimum-norm solutions [10].

4.1 A level set algorithm based on the sLS approach

The iterative algorithm based on the sLS approach proposed in this article is an explicit iterative method derived from the conditions of optimality for the Tikhonov functional $\mathcal{G}_{\alpha,s}$ in (5). These optimality conditions can be written in in the form of the system

$$\alpha\phi \;=\; L_{\varepsilon,\alpha,\beta}(\phi, c^1, c^2)\,, \qquad \alpha\, c^j = L^j_{\varepsilon,\alpha,\beta}(\phi, c^1, c^2)\,,\ j = 1, 2\,, \tag{16}$$

where

$$\begin{aligned} L_{\varepsilon,\alpha,\beta}(\phi, c^1, c^2) \;&=\; (c^1 - c^2)\beta_2^{-1} H'_\varepsilon(\phi)^* F'(P_{s,\varepsilon}(\phi, c^1, c^2))^*(F(P_{s,\varepsilon}(\phi, c^1, c^2)) - y^\delta) \\ &\qquad - \beta_1(2\beta_2)^{-1} H'_\varepsilon(\phi)\, \nabla\cdot\big[\nabla H_\varepsilon(\phi)/|\nabla H_\varepsilon(\phi)|\big], && \text{(17a)} \\ L^1_{\varepsilon,\alpha,\beta}(\phi, c^1, c^2) \;&=\; (2\beta_3)^{-1}\big(F'(P_{s,\varepsilon}(\phi, c^1, c^2))H_\varepsilon(\phi)\big)^*(F(P_{s,\varepsilon}(\phi, c^1, c^2)) - y^\delta), && \text{(17b)} \\ L^2_{\varepsilon,\alpha,\beta}(\phi, c^1, c^2) \;&=\; (2\beta_3)^{-1}\big(F'(P_{s,\varepsilon}(\phi, c^1, c^2))(1 - H_\varepsilon(\phi))\big)^*(F(P_\varepsilon(\phi, c^1, c^2)) - y^\delta). && \text{(17c)} \end{aligned}$$

Each iteration of this method consists of three steps: i) The residual $F(\phi_k, c^j) - y^\delta \in L^2(\partial\Omega)$ of the iterate (ϕ_k, c_k^j) is evaluated (this requires solving one elliptic BVP of Dirichlet type); ii) The H^1–solution of the adjoint problem for the residual is evaluated (this corresponds to solving one elliptic BVP of Dirichlet type); iii) The updates for the level-set function ϕ_k and for the levels c_k^j are evaluated (this corresponds to multiplying two functions).

A detailed description of the explicit iterative step above is given in Table 1. In order to improve the regularity of the update $\delta\phi_k$ we suggest substituting step 3. by

3'. `Evaluate the update` $\delta\phi_k \in H^1(\Omega)$, `solving`

$$(I - \mu\Delta)\delta\phi_k = L_{\varepsilon,\alpha,\beta}(\phi_k, c_k^1, c_k^2)\,, \ \text{in } \Omega\,; \qquad (\delta\phi_k)_\nu = 0\,, \ \text{at } \partial\Omega\,.$$

where the positive constant μ satisfies $\mu << 1$. Notice that this corresponds to the optimality condition for the functional $\mathcal{G}_{\alpha,s}$ if we add $\beta_2\mu\|\nabla\phi\|_{L^2(\Omega)}$ to the the penalization term in (5). In [7] a similar Tikhonov functional (with $\mu = 1$) based on BV–H^1 regularization was proposed. The corresponding update $\delta\phi_k$ was very smooth and lead to a slow convergence of the iteration.

4.2 A level set algorithm based on the pcLS approach

The iterative algorithm based on the pcLS approach proposed in this article is an explicit iterative method based on the operator splitting technique [18] and derived from the optimality conditions for the Tikhonov functional $\mathcal{G}_{\alpha,ppc}$ in (8). First the operator $\mathcal{G}_{\alpha,ppc}$ is splitted in the sum $\mathcal{G}_{\alpha,ppc}(\phi, c^j) = \mathcal{G}^1_{\alpha,ppc}(\phi, c^j) + \mathcal{G}^2_{\alpha,ppc}(\phi)$, where

$$\begin{aligned} \mathcal{G}^1_{\alpha,ppc}(\phi, c^j) &:= \|F(P_{pc}(\phi, c^j)) - y^\delta\|_Y^2 + \alpha\beta_1 |P_{pc}(\phi, c^j)|_{\mathtt{BV}} + 2\alpha\beta_2\|\phi\|^2_{L^2(\Omega)} + \alpha\beta_3\|c^j\|^2_{\mathbb{R}^2} \\ \mathcal{G}^2_{\alpha,ppc}(\phi) &:= \alpha\beta_2\|\mathcal{K}(t)\|^2_{L^2(\Omega)}\,. \end{aligned}$$

Each step of the iterative method consists of two parts: i) The iterate (ϕ_k, c_k^j) is updated using an explicit gradient step w.r.t. the operator $\mathcal{G}^1_{\alpha,ppc}$, i.e.

$$\phi_{k+1/2} := \phi_k - \frac{\partial}{\partial\phi}\mathcal{G}^1_{\alpha,ppc}(\phi_k, c_k^j)\,, \quad c^j_{k+1/2} := c_k^j - \frac{\partial}{\partial c^j}\mathcal{G}^1_{\alpha,ppc}(\phi_k, c_k^j)\,.$$

It is worth noticing that this first part is analog to steps $1-4$ in Table 1.
ii) The obtained approximation $(\phi_{k+1/2}, c^j_{k+1/2})$ is improved by giving a gradient step w.r.t.

1. `Evaluate the residual` $r_k := F(P_{s,\varepsilon}(\phi_k, c_k^1, c_k^2)) - y^\delta = (w_k)_\nu|_{\partial\Omega} - y^\delta$, `where` w_k `solves`

$$\Delta w_k = P_{s,\varepsilon}(\phi_k, c_k^1, c_k^2)\,, \ \text{in } \Omega\,; \qquad w_k = 0\,, \ \text{at } \partial\Omega\,.$$

2. `Evaluate` $h_k := F'(P_{s,\varepsilon}(\phi_k, c_k^1, c_k^2))^*(r_k) \in L^2(\Omega)$, `solving`

$$\Delta h_k = 0\,, \ \text{in } \Omega\,; \qquad h_k = r_k\,, \ \text{at } \partial\Omega\,.$$

3. `Calculate` $\delta\phi_k := L_{\varepsilon,\alpha,\beta}(\phi_k, c_k^1, c_k^2)$ `and` $\delta c_k^j := L^j_{\varepsilon,\alpha,\beta}(\phi_k, c_k^1, c_k^2)$, `as in` (17).

4. `Update the level set function` ϕ_k `and the level values` c_k^j, $j = 1, 2$:

$$\phi_{k+1} = \phi_k + \tfrac{1}{\alpha}\,\delta\phi_k\,, \qquad c^j_{k+1} = c_k^j + \tfrac{1}{\alpha}\,\delta c_k^j\,.$$

Table 1: Iterative algorithm based on the sLS approach for the inverse potential problem.

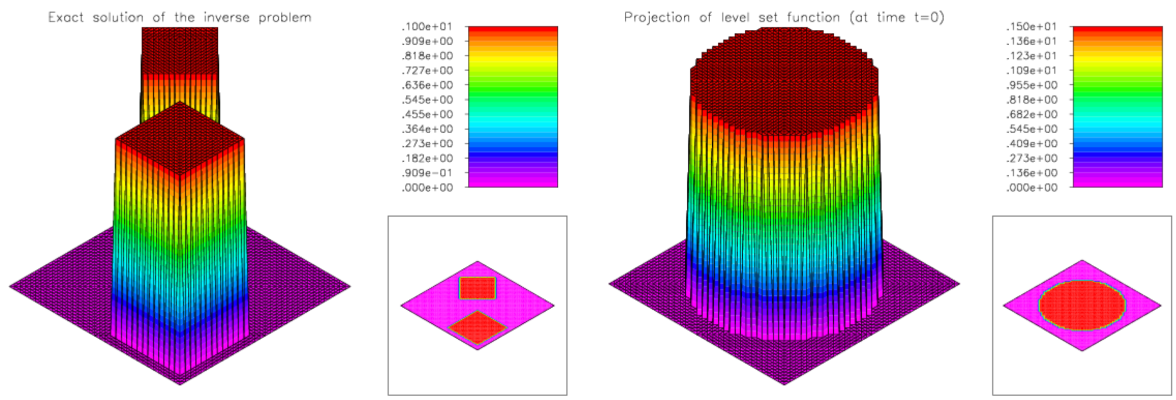

Figure 1: First experiment: The picture on the left hand side shows the coefficient u_{exact} to be reconstructed. On the right hand side, the initial condition for the sLS level-set method.

the operator $\mathcal{G}^2_{\alpha,ppc}$, i.e.

$$\phi_{k+1} := \phi_{k+1/2} - \frac{d}{d\phi}\mathcal{G}^2_{\alpha,ppc}(\phi_{k+1/2}) \,, \quad c^j_{k+1} := c^j_{k+1/2} \,.$$

In [22] a similar operator splitting strategy was use to minimize a Tikhonov functional related to an elliptic inverse problem in EIT.

4.3 First numerical example: exact data

In this first numerical experiment we aim to identify the right hand side u of (15) from the knowledge of the exact data $y = w_\nu|_{\partial\Omega}$. We assume that the level value $c^2 = 0$ is given, and that we have to identify only the support of u and the level value $c^1 > 0$.

The exact data $y = F(u)$ is obtained by solving numerically the elliptic boundary value problem in (15) at a very fine grid (the word 'exact' here means: up to the precision of the numerical method used for solving the direct problem). In order to avoid inverse crimes, the direct problem (15) is solved on an adaptively refined finite element grid with 8.804 nodes. However, in the numerical implementation of the level-set method, all boundary value problems are solved at an uniform grid with 545 nodes (33 nodes at each boundary side).

For this experiment with exact data, the level-set method was tested without the BV regularization term, i.e. we set $\beta_1 = 0$. Moreover, we chose $\varepsilon = 2^{-4}$ in (10).

In Figure 1 the solution u_{exact} of the inverse problem and the initial guess for the iterative method based on the sLS approach are presented (the initial guess $c^1_0 = 1.5$ is used for the unknown level value). Notice that the support of u is a non-connected proper subset of Ω. In Figure 2 the evolution of the sLS level-set method for the first 1500 iterative steps is presented. Notice, the shapes of both inclusions are reasonably reconstructed, and the level value c^1 is accurately reconstructed as well. The iteration is stopped when the residual drops below the predefined precision $\|F(P_{s,\varepsilon}(\phi_k, c^1_k)) - y\|_{L^2} < 10^{-2}$. For comparison purposes we present in the second line of this figure the evolution of the BV–H^1 level-set method [7] for the same initial guess.

The same stop criteria is used. Both methods deliver good approximations for the support of u as well as for the unknown level c^1. However, the sLS level-set method uses a less regular update and converges much faster. In Figure 2 (last line) we present the iteration error after

$k = 200$ steps for sLS level-set method, and after $k = 1900$ steps for the BV–H^1 level-set method.

We performed other numerical simulations with different choice of initial guess (ϕ_0, c_0^1), and observed that the number of iterative steps required in order to obtain a reasonable approximation (up to the predefined precision of 10^{-2} in the L^2-norm) strongly depends on the choice of the initial guess c_0^1. On the other hand, the final result is not sensitive with respect to the choice of the initial guess ϕ_0.

What concerns the level-set method based on the pcLS approach, in Figure 3 we present the results obtained for the exact data case. The initial guess is a smooth (polynomial) function attaining values in the interval $(1, 2)$. The initial guess for c_0^1 is the same as before. The evolution of the pcLS level-set method is shown for the first 1000 iterative steps of the algorithm presented in Subsection 4.2. As in the previous methods the shape of the inclusions could be well reconstructed. The level value c^1 could be accurately reconstructed as well. For

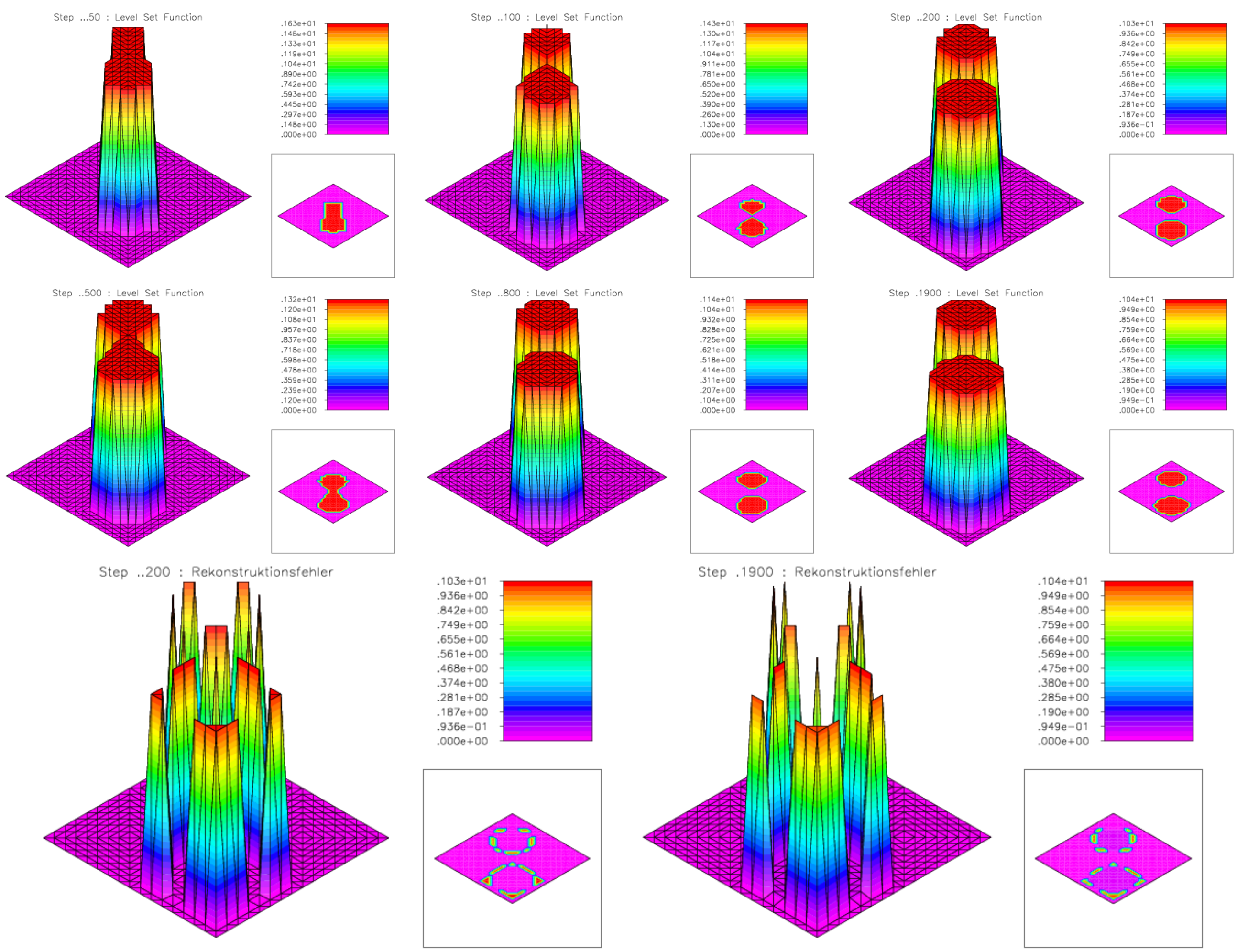

Figure 2: First experiment sLS: On the first line, plots of $P_{s,\varepsilon}(\phi_k, c_k^1)$, $k = 50, 100, 200$, for the sLS level-set method. The pictures on the second line show $P_{s,\varepsilon}(\phi_k, c_k^1)$, $k = 500, 800, 1900$, for the BV–H^1 level-set method in [7]. On the third line, the picture on the left hand side shows the iteration error for the sLS level-set method after $k = 200$ iterations, while the other picture shows the iteration error for the BV–H^1 level-set method after $k = 1900$ iterations.

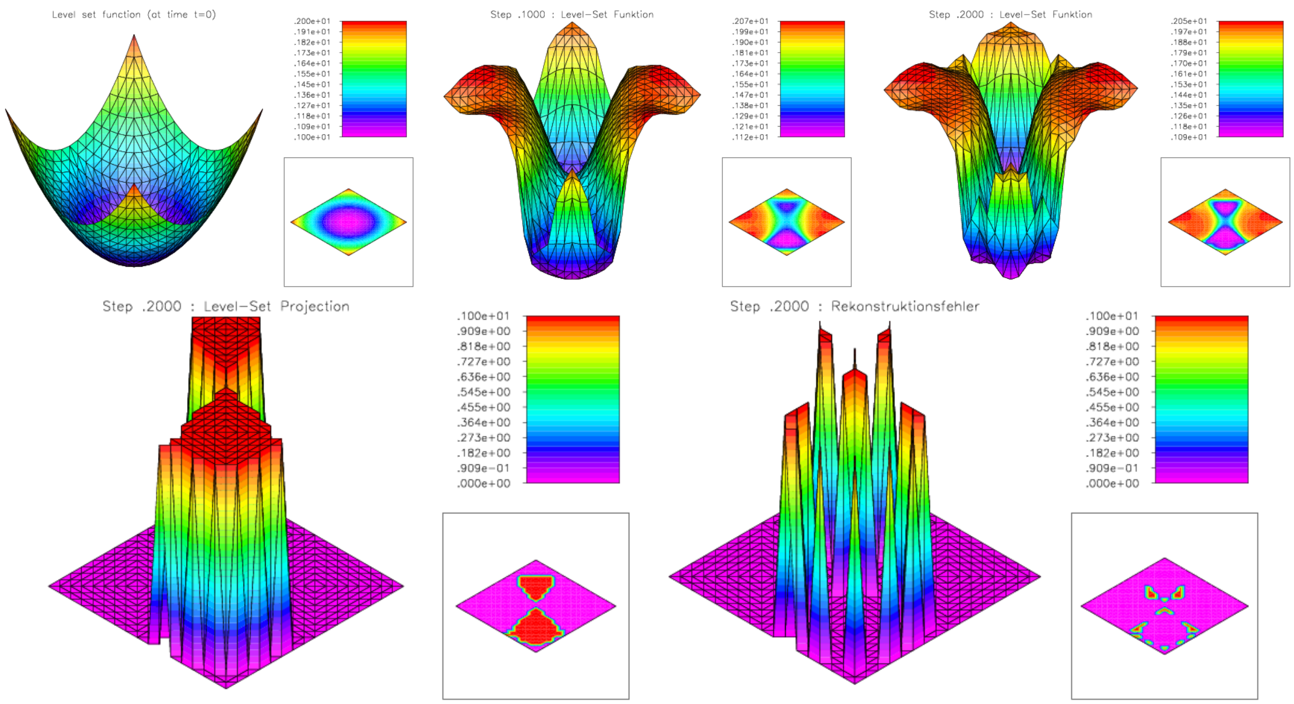

Figure 3: First experiment pcLS: The picture on the top left shows the initial condition for the pcLS level-set method. On the 2 subsequent pictures of the first line, plots of ϕ_k, for $k = 1000, 2000$. The bottom left picture shows $P_{pc}(\phi_k, c_k^1)$ for $k = 2000$. The bottom right picture shows the iteration error after $k = 2000$ iterations.

comparison purposes, we used the same stop criterion as before, i.e. $\|F(P_{pc}(\phi_k, c_k^1)) - y\|_{L^2} < 10^{-2}$.

In the numerical implementation of the level-set method based on the pcLS approach, some facts have to be observed:
— Due to the operator splitting technique, we compute several times the step-part (i) before a single calculation of step-part (ii) is performed.
— Step-part (i) aims to minimize the misfit in the iteration and is the most relevant component of the iteration step described in Subsection 4.2.
— Step-part (ii) aims to drag the iterate ϕ_k to a piecewise constant (integer valued) function. If step-part (ii) is implemented too often, all the iterates ϕ_k become piecewise constant functions and the misfit never decreases. If step-part (ii) is implemented only seldom, the iterates ϕ_k become too smooth and may be trapped in some local minimizer (due to the non-uniqueness of the inverse potential problem). — The constants α and β_2 should be chosen in such a way that the factor $\alpha\beta_2 << 1$ in step-part (ii). This choice guarantees that the dragging effect resulting from step-part (ii) is not enforced too strongly. If $\alpha\beta_2 \approx 2$ the iterates once again become piecewise constant functions and the misfit never decreases.

4.4 Second numerical example: noisy data

In the sequel we consider once again the inverse potential problem in (15) with the solution shown in Figure 1. This time, the data y^δ for the inverse problem is obtained by adding to the exact data $y = F(u)$ random generated noise of 25%.

As in the previous experiment, the direct problem is solved at a grid that is finer than the one used in the numerical implementation of the level-set method. The initial guess (ϕ_0, c_0^1) is the same as in the experiment with exact data (see Figure 1), as well as the value used for ε. For this experiment with noisy data, the level-set method was tested with the BV regularization term and $\beta_1 = 10^{-3}$. As stop criteria, we used the generalized discrepancy with $\tau = 2$, i.e. the iteration was stopped when for the first time $\|F(P_{s,\varepsilon}(\phi_k, c_k^1)) - y^\delta\|_{L^2} < \tau\delta$.

In Figure 4 we show the evolution of the level-set method based on the sLS approach, white in Figure 5 the evolution of the level-set method based on the pcLS approach is shown.

5 Conclusions

Two distinct level-set type approaches for solving ill-posed problems are considered, where the level-set functions are chosen in L^2-spaces. The first approach (sLS) corresponds to an extension of the results obtained in [8] for H^1 level-set functions. In the second approach (pcLS) the parameter space consists of piecewise constant level-set functions.

Based on each one of the level-set approaches above, Tikhonov functionals are proposed and we provide convergence analysis for the Tikhonov reconstruction methods.

In the pcLS approach, the piecewise constant constraint in 7 is enforced in two different manners, which are called penalized-pcLS and strict-pcLS respectively. We observe that the properties of the minimizers of the Tikhonov functionals corresponding to the penalized-pcLS and strict-pcLS methods are slightly different. However, for a fixed noise level $\delta > 0$, the approximate solutions obtained by each of the pcLS methods are comparable.

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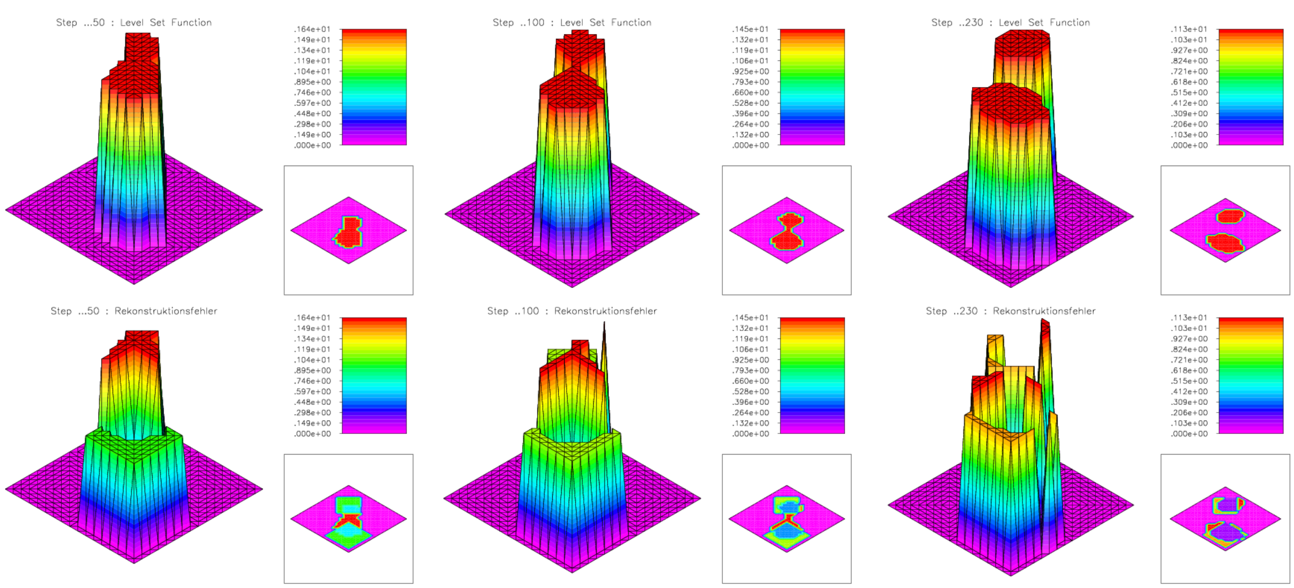

Figure 4: Second experiment sLS: On the first line, plots of $P_{s,\varepsilon}(\phi_k, c_k^1)$, $k = 50, 100, 230$, for the sLS level-set method. On the second line the corresponding iterative error $e_k := |P_{s,\varepsilon}(\phi_k, c_k^1) - u_{exact}|$.

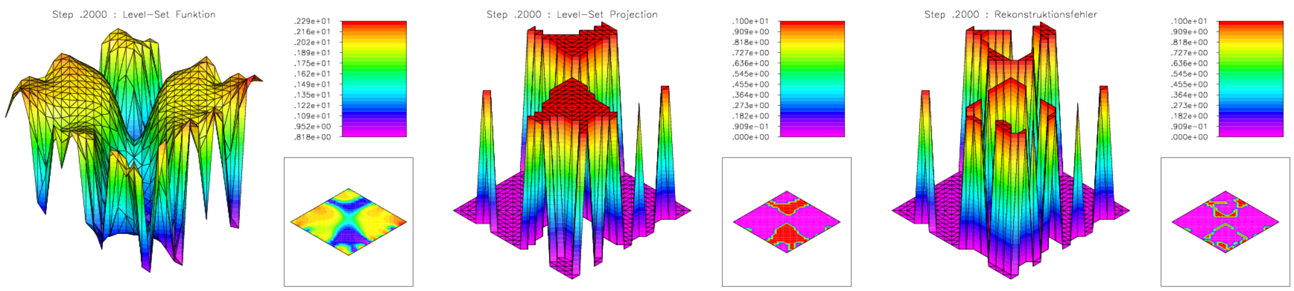

Figure 5: Second experiment pcLS: On the left, plots of ϕ_k for $k = 2000$, for the pcLS level-set method. On the center the corresponding projection $P_{pc}(\phi_k)$. On the right hand side, the iterative error $e_k := |P_{s,\varepsilon}(\phi_k, c_k^1) - u_{exact}|$.

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