Generalization of the incenter subdivision scheme

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Abstract

In this paper we present an interpolatory Hermite subdivision scheme depending on a free parameter, which generalizes in certain way the incenter subdivision scheme [DW10]. We prove that for any value of the free parameter the limit curve is G^1 continuous. Moreover, if vertices of the initial polygon and the tangent vectors are sampled from a circle with any arbitrary spacing, then the subdivision curve is the circle. The proposed scheme is shape preserving limiting the oscillations of the subdivision curve and introducing inflection points only in those regions of the curve where the control polygon suggests a change of convexity. Several examples are presented to demonstrate the performance of the scheme and we formulate some conjectures supported by numerical experiments.

1 Introduction

Subdivision methods are very popular currently in CAGD and related areas due to their simplicity, and flexibility. Starting with an initial control polygon, curve subdivision algorithms are iterative methods for refining a polygon according to prescribed rules. The main problem concerning these algorithms is to propose rules for subdivision such that the iterative process converges, the limit curve is smooth and the scheme has attractive properties such as preserving the convexity of the initial polygon. Interpolatory subdivision schemes have got a lot of attention since they produce curves passing through the vertices of the control polygon. This is possible defining subdivision rules that at any step keep all the vertices of the previous step polygon.

Hermite subdivision schemes are a useful tool for curve design since they take into account not only the control polygon, but also the prescribed tangent vectors at its vertices. They were introduced in [Mer92], where it is proposed a family of interpolatory schemes that depends on two parameters and generates a C^1 limit function. Merrien's family includes as a particular case the C^1 cubic spline interpolating the prescribed values and first derivatives. To study interpolatory Hermite subdivision schemes, [DL95] and [DL98]

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consider an associated point subdivision scheme generating the divided differences of the vectors containing function values and the corresponding derivatives. In [JS02] a different approach is proposed using the splines interpolating function values and derivatives.
Inspired in the method introduced recently in [DW10], in this paper we propose a family of interpolatory Hermite subdivision schemes depending on a free parameter. As in [DW10], the new point inserted from a given edge is on the triangle defined by the edge and the tangent lines at its extreme points. The free parameter $0 < \rho < 1$ controls the position of the new point, which is the incenter of the triangle for $\rho = 0.5$. Nevertheless, the subdivision curve of our scheme is different from the one obtained in [DW10], even if we take $\rho = 0.5$, since our scheme preserve prescribed tangent vectors, while the scheme proposed [DW10] updates the tangent vectors at each step of the subdivision.
The subdivision curve of our method is G^1 continuous for any value of the free parameter ρ and we conjecture that it is also G^2 continuous for $\rho = 0.5$. Moreover, the proposed scheme preserves the shape of the initial polygon, limiting the oscillations of the subdivision curve and introducing inflection points only in those regions of the curve, where the control polygon suggests a change of convexity. Furthermore, if vertices of the initial polygon and the tangent vectors are sampled from a circle with any arbitrary spacing, then the subdivision curve is the circle. The algorithms proposed by [BCR07], [CJ07], [SD05] also reproduce circles and generate a G^1 continuous subdivision curve. In [Rom09] the limit curve is C^2 continuous, but circles are reproduced only if the vertices of the initial polygon are uniformly distributed. Finally, in a recently published paper [Rom10], a new Hermite interpolatory subdivision scheme is presented. This method requires that first and second derivatives vectors are defined in the vertices of the initial polygon. The subdivision scheme reproduces circles starting from a sequence of sample points with any distribution. Moreover, it depends on a tension parameter and the limit curve is C^2 continuous if the tension parameter is set equal to 1.
The rest of the paper is organized as follows. In section 2 we introduce the notation, define the subdivision scheme rules and show some properties of the scheme. In section 3 we prove that the subdivision scheme converges and the limit curve is G^1 continuous. Section 4 deals with examples and we conclude in section 5 considering some conjectures supported by numerical experiments.

2 The subdivision scheme

Let $P_i^0 \in \mathbb{R}^2,\ i = 0, ..., n$ be the vertices of the initial polygon enumerated in clockwise order and denote by t_i^0 the normalized tangent vector associated to the vertex P_i^0. We assume that the initial polygon doesn't contain 3 consecutive collinear points and also that tangent vector t_i^0 is contained in the angular sector defined by $P_{i+1}^0 - P_i^0$ and $P_i^0 + (P_i^0 - P_{i-1}^0)$. The last hypothesis will be used to obtain a shape preserving subdivision curve. Tangent vectors may be defined by the user or computed automatically, see [ABFH08] and also section 4.

2.1 Preprocessing

With slight differences, our notation is similar to [DW10] to make the comprehension easier for the reader. We denote by α_i^k the angle from $P_{i+1}^k - P_i^k$ to t_i^k and denote by β_{i+1}^k the angle from t_{i+1}^k to $P_{i+1}^k - P_i^k$, see Figure 2. Angles are considered positive if they are counterclockwise and negative otherwise. According with the signs of angles $\alpha_i^k, \beta_{i+1}^k$ the edge $P_i^k P_{i+1}^k$ is classified into three types.

Definition 1 *[DW10], see Figure 1.*
The edge $P_i^k P_{i+1}^k$ is a convex edge if $\alpha_i^k \beta_{i+1}^k > 0$. It is a straight edge if $\alpha_i^k \beta_{i+1}^k = 0$ and it is an inflection edge if $\alpha_i^k \beta_{i+1}^k < 0$.

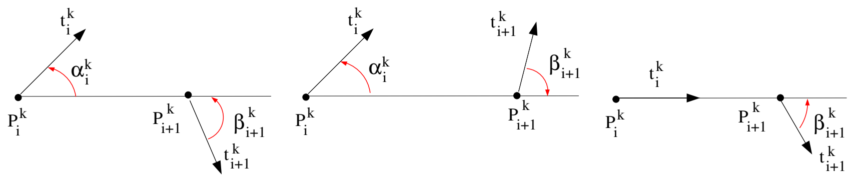

Figure 1: Left: Convex edge, Middle: inflexion edge, Right: straight edge

In this paper we assume that there are not straight edges in the control polygon. In a future version we will eliminate this restriction in order to include straight lines in the subdivision curve. Inflection edges of the initial polygon are split into two convex edges in a preprocessing step. More precisely, if $P_i^0 P_{i+1}^0$ is an inflection edge, then we introduce a new point in the initial polygon

$$P_{i+1/2}^0 = \frac{P_i^0 + P_{i+1}^0}{2}$$

The corresponding tangent vector $t_{i+1/2}^0$ is defined as the vector perpendicular to $\frac{t_i^0 + t_{i+1}^0}{2}$ such that $P_i^0 P_{i+1/2}^0$ and $P_{i+1/2}^0 P_{i+1}^0$ are convex edges. After the preprocessing step we have not inflection edges in the initial polygon. In the next section we show that our scheme preserves the convexity of the edges. Therefore, we define the subdivision rules assuming that all edges are convex.

2.2 Insertion rules

In the step $k+1$ of the subdivision we refine the control polygon of the step k, preserving the points P_i^k already generated and inserting a new point P_{2i+1}^k inside the triangle with vertices P_i^k, P_{i+1}^k and Q_i^k, where Q_i^k is the intersection point between the line passing through P_i^k in the direction of t_i^k and the line passing through P_{i+1}^k in the direction of t_{i+1}^k, see Figure 2. The position of P_{2i+1}^{k+1} depends on a free parameter $0 < \rho < 1$. More precisely,

$$
\begin{aligned}
P_{2i}^{k+1} &= P_i^k && (1)\\
P_{2i+1}^{k+1} &= P_i^k + \frac{\sin(\rho\beta_{i+1}^k)}{\sin(\rho(\alpha_i^k + \beta_{i+1}^k))} R(\rho\alpha_i^k)(P_{i+1}^k - P_i^k) && (2)
\end{aligned}
$$

where R is the rotation matrix

$$
R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}
$$

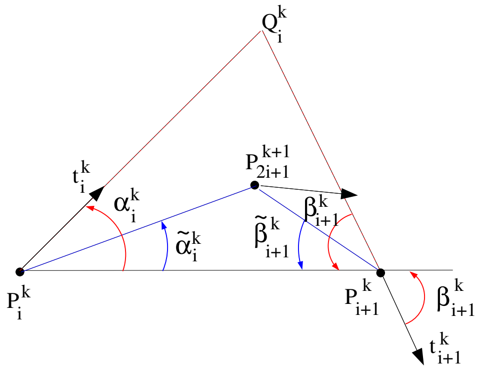

Figure 2: Notation for the $k-th$ step

The tangent vector t_{2i+1}^{k+1} at the new point P_{2i+1}^{k+1} is defined as the perpendicular to the bisector of the angle $\angle P_i^k P_{2i+1}^{k+1} P_{i+1}^k$, while we keep the old tangent vectors, i.e.

$$
\begin{aligned}
t_{2i}^{k+1} &= t_i^k && (3)\\
t_{2i+1}^{k+1} &= \frac{t_l^k + t_r^k}{\|t_l^k + t_r^k\|} && (4)
\end{aligned}
$$

with

$$
\begin{aligned}
t_l^k &= \frac{P_i^k - P_{i-1}^k}{\|P_i^k - P_{i-1}^k\|} && (5)\\
t_r^k &= \frac{P_{i+1}^k - P_i^k}{\|P_{i+1}^k - P_i^k\|} && (6)
\end{aligned}
$$

2.3 Properties

Proposition 1 *If the vertices P_i^0 of the initial polygon and the corresponding tangent vectors t_i^0 are sampled from a circular arc, then the subdivision curve generated by the scheme (1)-(4) with $\rho = 0.5$ is the circular arc.*

Proof
See the Appendix

Proposition 2 *see Figure 3*
If $P_i^k P_{i+1}^k$ is a convex edge then the edges $P_{2i}^{k+1} P_{2i+1}^{k+1}$ and $P_{2i+1}^{k+1} P_{2i+2}^{k+1}$ generated by the scheme (1)-(4) are also convex.

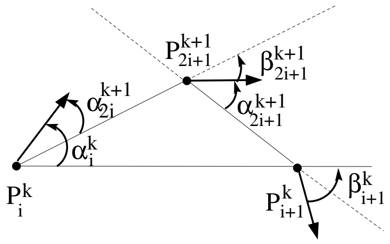

Figure 3: Convexity preserving property

Proof
By construction $\alpha_{2i}^{k+1} = (1-\rho)\alpha_i^k$. Therefore α_{2i}^{k+1} has the same sign that α_i^k. Moreover, from (4) the tangent vector t_{2i+1}^{k+1} at the new point P_{2i+1}^{k+1} is contained in the angular sector defined by $P_{2i+2}^{k+1} - P_{2i+1}^{k+1}$ and $P_{2i+1}^{k+1} + (P_{2i+1}^{k+1} - P_{2i}^{k+1})$. Therefore, the points $P_{2i}^{k+1} + t_{2i}^{k+1}$ and $P_{2i+1}^{k+1} + t_{2i+1}^{k+1}$ are in different semiplanes with respect to the line passing through $P_{2i}^{k+1} = P_i^k$ and P_{2i+1}^{k+1}, see Figure 3. Hence, $\alpha_{2i}^{k+1}\beta_{2i+1}^{k+1} > 0$, i.e. the edge $P_{2i}^{k+1}P_{2i+1}^{k+1}$ is convex. The proof for the edge $P_{2i+1}^{k+1}P_{2i+2}^{k+1}$ is similar and we omit it.

■

3 Smoothness analysis

In this section we prove that for any fixed $\rho \in (0,1)$ the subdivision scheme (1)-(4) *converges* and the limit curve is G^1*-continuous.*
Let $\widetilde{\alpha}_i^k$ be the angle $\angle P_{i+1}^k P_i^k P_{2i+1}^{k+1}$ and denote by $\widetilde{\beta}_{i+1}^k$ the angle $\angle P_i^k P_{i+1}^k P_{2i+1}^{k+1}$, see Figure 2.

Lemma 1 *Consider an interpolatory subdivision scheme with tangent vectors defined by (3)-(4). If the new points P_{2i+1}^{k+1} are selected in such a way that following inequalities hold,*

$$r^{'}\alpha_i^k \leq \widetilde{\alpha}_i^k \leq r\alpha_i^k \quad (7)$$

$$r^{'}\beta_{i+1}^k \leq \widetilde{\beta}_{i+1}^k \leq r\beta_{i+1}^k \quad (8)$$

for $0 < r^{'} \leq r < 1$, then, there is $\widetilde{r} \in (0,1)$ such that

$$\theta^{k+1} < \widetilde{r}\theta^k \quad (9)$$

where

$$\theta^k := \max_i\{\alpha_i^k, \beta_i^k\} \tag{10}$$

Proof

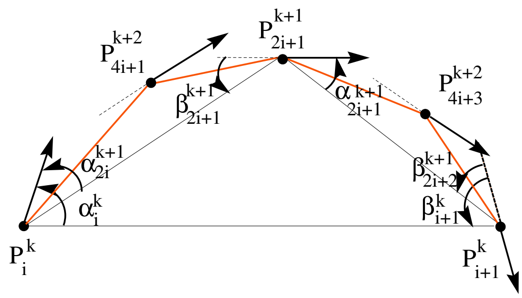

Figure 4: Two steps of the subdivision

We divide the proof in two cases, see Figure 4.
a) *Case: New vertex* P_{2i+1}^{k+1}
From (4)

$$\alpha_{2i+1}^{k+1} = \beta_{2i+1}^{k+1} \tag{11}$$

Moreover,

$$\alpha_{2i+1}^{k+1} + \angle P_i^k P_{2i+1}^{k+1} P_{i+1}^k + \beta_{2i+1}^{k+1} = \pi \tag{12}$$

$$\widetilde{\alpha}_i^k + \angle P_i^k P_{2i+1}^{k+1} P_{i+1}^k + \widetilde{\beta}_{i+1}^k = \pi \tag{13}$$

Hence, from (11),(12) and (13) we have,

$$\alpha_{2i+1}^{k+1} = \beta_{2i+1}^{k+1} = \frac{\widetilde{\alpha}_i^k + \widetilde{\beta}_{i+1}^k}{2}$$

From the right hand side of the hypothesis (7)-(8) we obtain,

$$\alpha_{2i+1}^{k+1} = \beta_{2i+1}^{k+1} \leq r\frac{\alpha_i^k + \beta_{i+1}^k}{2} \leq r\max_i\{\alpha_i^k, \beta_{i+1}^k\} = r\theta^k \tag{14}$$

b) *Case: Old vertex* P_{2i}^{k+1}
From (3) it is clear that

$$\alpha_{2i}^{k+1} = \alpha_i^k - \widetilde{\alpha}_i^k$$

$$\beta_{2i}^{k+1} = \beta_i^k - \widetilde{\beta}_i^k$$

Using the left hand side of the hypothesis (7)-(8) we obtain,

$$\begin{aligned} \alpha_{2i}^{k+1} &\leq (1-r^{'})\alpha_i^k \\ \beta_{2i}^{k+1} &\leq (1-r^{'})\beta_i^k \end{aligned}$$

Hence,

$$\max_i\{\alpha_{2i}^{k+1}, \beta_{2i}^{k+1}\} \leq (1-r^{'}) \max_i\{\alpha_i^k, \beta_i^k\} = (1-r^{'})\theta^k \tag{15}$$

Summarizing Case a) and b) we obtain from (14) and (15)

$$\begin{aligned} \alpha_i^{k+1} \leq \widetilde{r}\theta^k \\ \beta_i^{k+1} \leq \widetilde{r}\theta^k \end{aligned}$$

for all i, with $\widetilde{r} = \max\{r, 1-r^{'}\}$. Thus,

$$\max_i\{\alpha_i^{k+1}, \beta_i^{k+1}\} = \theta^{k+1} < \widetilde{r}\theta^k$$

■

By construction, the following equalities hold for the subdivision scheme (1)-(2)

$$\widetilde{\alpha}_i^k = \rho\alpha_i^k \tag{16}$$

$$\widetilde{\beta}_{i+1}^k = \rho\beta_{i+1}^k \tag{17}$$

Thus, the proposed subdivision scheme satisfies the hypothesis of the previous Lemma and we have the following theorem.

Theorem 1 *The polygon sequence of the subdivision scheme (1)-(4), converges to a continuous curve c.*

Proof
Let Γ^k be the k-th subdivision polygon. We compute first the distance d^k between Γ^{k+1} and Γ^k. From Lemma 1, we know that exists an integer k_0 such that $2\rho\theta^k < \frac{\pi}{2}$, for $0 < \rho < 1$. Then, according to (2) it holds,

$$\|P_{2i+1}^{k+1} - P_i^k\| = \frac{\sin(\rho\beta_{i+1}^k)}{\sin(\rho(\alpha_i^k + \beta_{i+1}^k))}\|P_{i+1}^k - P_i^k\| < \|P_{i+1}^k - P_i^k\| \tag{18}$$

since $\sin(\vartheta)$ is a monotonic increasing function for $\vartheta \in (0, \frac{\pi}{2})$. In a similar way we can prove that

$$\|P_{2i+1}^{k+1} - P_{i+1}^k\| < \|P_{i+1}^k - P_i^k\| \tag{19}$$

The distance d_i^k from P_{2i+1}^{k+1} to the edge $P_{i+1}^k P_i^k$ is equal to

$$d_i^k = \|P_{2i+1}^{k+1} - P_i^k\| \sin(\rho\alpha_i^k)$$

Therefore, it is bounded by

$$d_i^k < \rho \alpha_i^k \|P_{i+1}^k - P_i^k\| \leq \rho \theta^k \|P_{i+1}^k - P_i^k\| \tag{20}$$

Moreover, from (18) and (19) there is $L > 0$ independent of k such that for $k > k_0$,

$$\max_i \{\|P_{i+1}^k - P_i^k\|\} < \max_i \{\|P_{i+1}^{k_0} - P_i^{k_0}\|\} < L \tag{21}$$

On the other hand, from Lemma 1 we get,

$$\theta^k < \widetilde{r}^{(k-k_0)} \theta^{k_0} \tag{22}$$

with $\widetilde{r} < 1$. Thus, substituting (21) and (22) in (20) we obtain,

$$d_i^k < \rho L \widetilde{r}^{(k-k_0)} \theta^{k_0}$$

Since $d^k = \max_i \{d_i^k\}$, it holds

$$\lim_{k \to \infty} d^k \leq \rho L \theta^{k_0} \lim_{k \to \infty} \widetilde{r}^{(k-k_0)} = 0$$

This means that the polygons $\{\Gamma^k\}$ form a Cauchy sequence and this sequence of polygons *converges uniformly.* Since each polygon is a piecewise linear curve, the limit curve c is continuous.

■

The Lemma 1 and Theorem 1 are useful to prove that the subdivision scheme converges to a G^1 continuous limit curve.
Without loss of generality, we restrict our analysis to the section of the limit curve c corresponding to a convex edge with endpoints P_i^k and P_{i+1}^k. The points generated applying the subdivision scheme to the edge $P_i^k P_{i+1}^k$, a finite number of times, may be written as $P_{2^r i+s}^{k+r}$, $0 \leq s \leq 2^r$, for some integer $r \geq 0$.

Clearly the proposed subdivision scheme satisfies the following

Convex hull property:
The segment of limit curve c corresponding to a convex edge with endpoints $P_{2^r i+s}^{k+r}$ and $P_{2^r i+s+1}^{k+r}$, with $0 \leq s \leq 2^r - 1$ is contained in the triangle with vertices $P_{2^r i+s}^{k+r}, Q_{2^r i+s}^{k+r}$ and $P_{2^r i+s+1}^{k+r}$.

Theorem 2 *The polygon sequence of the subdivision scheme (1)-(4) converges to a G^1 continuous curve c.*

Proof
The proof of the convergence is divided in three parts. In the first part, for a given sequence of points on c generated by the subdivision scheme in the iterations posterior to k

that converges to the point P_i^k, we show that the sequence of tangent vectors associated by our scheme converges to the tangent vector associated by our scheme to the point P_i^k. In the second part, we prove that the same result holds true for any other sequence of points on c converging to P_i^k. In the third part, we sketch the proof for any sequence of points on c converging to a given point R on c.

a) First part:
For any strictly monotonic increasing sequence $\{l_j \geq 0\}$, consider the sequence of lines $P_i^k P_{2^{l_j}i+1}^{k+l_j}$. From (3) and (16) it is clear that the angle $\alpha_{2^{l_j}i}^{k+l_j}$ between the line $P_i^k P_{2^{l_j}i+1}^{k+l_j}$ and the tangent vector t_i^k at P_i^k can be written as

$$\begin{aligned} \alpha_{2^{l_j}i}^{k+l_j} &= \alpha_{2^{l_j-1}i}^{k+l_j-1} - \tilde{\alpha}_{2^{l_j-1}i}^{k+l_j-1} \leq (1-\rho)\alpha_{2^{l_j-1}i}^{k+l_j-1} \\ &\leq \ldots \leq (1-\rho)^{l_j}\alpha_i^k \end{aligned}$$

Since $0 < \rho < 1$, it holds

$$\lim_{j\to\infty} \alpha_{2^{l_j}i}^{k+l_j} = 0$$

Therefore, we have shown that

$$\lim_{j\to\infty} \frac{P_i^k - P_{2^{l_j}i+1}^{k+l_j}}{\|P_i^k - P_{2^{l_j}i+1}^{k+l_j}\|} = t_i^k$$

On the other hand, considering the sequence of lines $P_{2^{l_j}i-1}^{k+l_j} P_i^k$, and using similar arguments as above, it can be shown that for $i > 0$, and $0 < \rho < 1$ the angle $\beta_{2^{l_j}i}^{k+l_j}$ between the line $P_{2^{l_j}i-1}^{k+l_j} P_i^k$ and the tangent vector t_i^k at P_i^k tends to 0. Consequently it holds

$$\lim_{j\to\infty} \frac{P_{2^{l_j}i-1}^{k+l_j} - P_i^k}{\|P_{2^{l_j}i-1}^{k+l_j} - P_i^k\|} = t_i^k$$

b) Second part:
Let be $\{R^n, n \geq 0\}$ a sequence of points on the limit curve c converging to P_i^k. We may assume that for $n > n_0$ the following conditions hold:

i) $R^n \neq P_i^k$, otherwise the secant lines $P_i^k R^n$ collapse to a single point.

ii) $R^n \neq P_{2^{l_n}i\pm1}^{k+l_n}$ for any strictly monotonic increasing sequence $\{l_n, n \geq 0\}$, otherwise this coincides with the first part of the proof treated above.

Since $\lim_{n\to\infty} R^n = P_i^k = P_{2^j i}^{k+j}$, for each $j \geq 0$, exists an index $0 \leq n_j \leq 2^j$, such that R^{n_j} is in the interior of the section of c with endpoints $P_{2^j i-1}^{k+j}$ and $P_{2^j i+1}^{k+j}$. Then, we get

$$R^{n_j} \in \triangle_- \bigcup \triangle_+$$

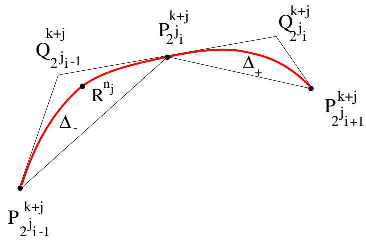

Figure 5: Proof of Theorem 2, second part.

where $\triangle_-$ is the triangle with vertices $P^{k+j}_{2^j i-1}, Q^{k+j}_{2^j i-1}, P^{k+j}_{2^j i}$ and $\triangle_+$ is the triangle with vertices $P^{k+j}_{2^j i}, Q^{k+j}_{2^j i}, P^{k+j}_{2^j i+1}$. According to the convex hull property of our subdivision scheme,

- if $R^{n_j} \in \triangle_-$, then the angle between the line $R^{n_j} P^k_i$ and the tangent vector t^k_i is smaller than $\beta^{k+j}_{2^j i}$
- if $R^{n_j} \in \triangle_+$, then the angle between the line $P^k_i R^{n_j}$ and the tangent vector t^k_i is smaller than $\alpha^{k+j}_{2^j i}$

In the previous Lemma, we have shown that $\theta^k = \max_i\{\alpha^k_i, \beta^k_i\}$ tends to 0 for $k \to \infty$, thus we get that for $j \to \infty$, the secant lines $P^k_i - R^{n_j}$ tend to have the direction t^k_i.

c) Third part:
Let be R an arbitrary point on c. Without loss of generality, we may assume that R belongs to the section of c with endpoints P^k_i and P^k_{i+1}, for some indexes i, k.
For each $j \geq 0$, exists an index $0 \leq s \leq 2^j$, such that R is in the interior of the section of c with endpoints $P^{k+j}_{2^j i+s}$ and $P^{k+j}_{2^j i+s+1}$. Then, according to the convex hull property of our subdivision scheme, R is in the triangle with vertices $P^{k+j}_{2^j i+s} Q^{k+j}_{2^j i+s} P^{k+j}_{2^j i+s+1}$. Hence the angle δ^{k+j}_R from vector $t_{R_l} = R - P^{k+j}_{2^j i+s}$ to vector $t_{R_r} = P^{k+j}_{2^j i+s+1} - R$ holds

$$0 < \delta^{k+j}_R < \alpha^{k+j}_{2^j i+s} + \beta^{k+j}_{2^j i+s+1} \leq 2\theta^{k+j}$$

Since $\lim_{j\to\infty} \theta^{k+j} = 0$, then we get that the two sequences of unitary vectors

$$\frac{t_{R_l}}{\|t_{R_l}\|}, \quad \frac{t_{R_r}}{\|t_{R_r}\|}$$

converge to the same limit, for $j \to \infty$. This permits us to associate to the point R the unitary vector t_R defined as the common limit of the two sequences above.
In the previous parts of the proof, we have shown that for the points $P^k_{2^k i+s}$ computed in the k-th iteration step, the vector $t^k_{2^k i+s}$ associated to $P^k_{2^k i+s}$ by our subdivision algorithm coincides with the tangent vector at $P^k_{2^k i+s}$ that we denote by $t_{P^k_{2^k i+s}}$.

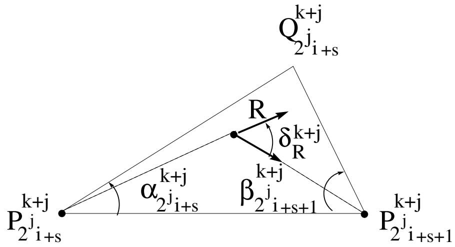

Figure 6: Proof of Theorem 2, third part.

Therefore, it remains to show that for a given point R on c and for any sequence of points $\{R^n = P^n_{2^n i+s_n}, 0 \le s_n \le 2^n\}$ on c, such that $\lim_{n\to\infty} R^n = R$, the sequence of unitary vectors $t_{R^n} = t^n_{2^n i+s_n}$ converges to the vector t_R. Since $\lim_{n\to\infty} R^n = R$, for any $j \ge 0$ we may assume that for $n \ge n_j$ and some index $0 \le s \le 2^j$, the points R and R^{n_j} are in the interior of the section of c with endpoints $P^{k+j}_{2^j i+s}$ and $P^{k+j}_{2^j i+s+1}$. We may apply now the argument of the convex hull property to the points R^{n_j} and we get as above that the two sequences of unitary vectors

$$\frac{R^{n_j} - P^{k+j}_{2^j i+s}}{\|R^{n_j} - P^{k+j}_{2^j i+s}\|}, \quad \frac{P^{k+j}_{2^j i+s+1} - R^{n_j}}{\|P^{k+j}_{2^j i+s+1} - R^{n_j}\|}$$

converge to the same limit vector, for $j \to \infty$. Since $\lim_{j\to\infty} R^{n_j} = R$, this common limit vector is equal to t_R.

■

4 Numerical examples

In this section we show the performance of the proposed subdivision scheme for several values of the parameter $\rho \in (0, 1)$ and also for different methods of assigning tangent vectors t^0_i to the vertices of the initial polygon. Among others, we consider the the following methods

1. Catmull:
$$t^0_i = P^0_{i+1} - P^0_{i-1} \tag{23}$$

2. Bessel:
$$t^0_i = \|t^0_r\| \frac{t^0_l}{\|t^0_l\|} + \|t^0_l\| \frac{t^0_r}{\|t^0_r\|} \tag{24}$$

   where t^0_l and t^0_r are given by (5) and (6) respectively with $k = 0$.

3. Weighted, $m > 0$:

$$t_i^0 = \|t_l^0\|^{(\frac{m-1}{m})} \frac{t_l^0}{\|t_l^0\|} + \|t_r^0\|^{(\frac{m-1}{m})} \frac{t_r^0}{\|t_r^0\|} \tag{25}$$

Example 1

In this example we show the influence of the tangent vector at the vertices of the initial polygon in the shape of the subdivision curve for $\rho = 0.5$. The plots in Figure 7 show that the distance between the subdivision curve and the initial polygon depends on the initial tangent vectors. In particular we observe that using Catmull's method (23) the limit curve tends to be close to the long edges and far from the short edges. On the other hand, Bessel method (24) produces curves that are very roundish and close to the short edges while far a way from the long ones. Finally, the curves obtained with the weighted method (25) lie between these extremes and have a nice shape.

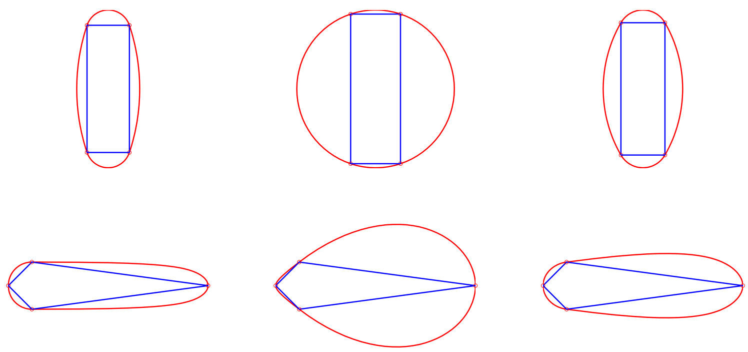

Figure 7: Initial polygon and the subdivision curve ($\rho = 0.5$) after 5 steps. Tangent vectors at the initial polygon are computed using left:(23),middle: (24)and right: (25) with $m = 2$ (Right) respectively.

Example 2

The influence of the free parameter ρ in the subdivision curve is shown is this example. The results were obtained using $\rho = 0.35, 0.5, 0.65$. The tangent vector t_i^0 at the vertices of the initial polygon were computed as the perpendicular to the bisector of the angle $\angle P_{i-1}^0 P_i^0 P_{i+1}^0$ using (4) for $k = 0$.
Figure 8 shows that the distance between the limit curve and the initial polygon is proportional to ρ. Moreover, the value of ρ also has an influence in the shape of the subdivision curve. For $\rho = 0.5$ the limit curve is rounded. On the other hand, for $\rho < 0.5$ and $\rho > 0.5$ the subdivision curve tends to include rounded corners and flat areas respectively.

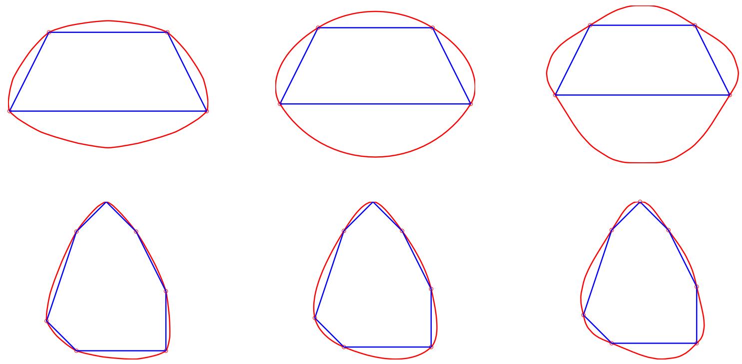

Figure 8: From left to right: subdivision curves after 5 steps computed using $\rho = 0.35, 0.5, 0.65$ respectively.

Example 3

The main purpose of this example is to show the performance of the proposed scheme when the initial polygon is nonconvex. We compute the tangent vectors at t_i^0 using (4) for $k = 0$. Since the initial polygons of this example are nonconvex, we introduce new vertices which will be inflexion points of the subdivision curves. More precisely, if $P_i^0 P_{i+1}^0$ is an inflexion edge, then the tangent vector at the inflexion point $\frac{P_i^0 + P_{i+1}^0}{2}$ is defined as the perpendicular to the vector $\frac{t_i^0 + t_{i+1}^0}{2}$.

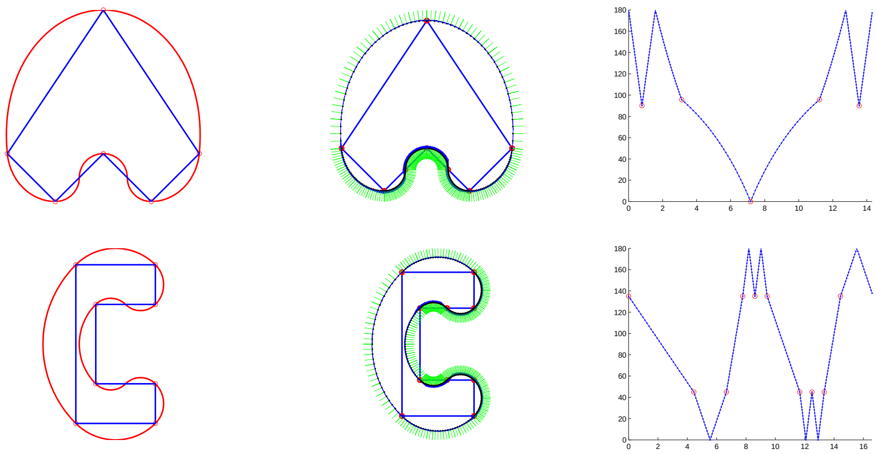

Figure 9: Left: Initial polygon and the subdivision curve ($\rho = 0.5$) after 5 steps, Center: Tangent and normal vectors at the vertices of the subdivision curve, Right: Absolute value of the tangent vector angle vs. arc-length of the subdivision curve.

Figure 9 shows the subdivision curve after 5 steps and tangent and normal vectors at the vertices. We also plot the absolute value of the tangent vector angle as function of the arclength (Figure 9 right) as a numerical evidence of the G^1 continuity of the subdivision curve.

5 Conjectures

In [DW10] a conjecture about the G^2 continuity of the incenter subdivision curve is presented. As we already have mentioned in the introductory section, our scheme for $\rho = 0.5$ is not the same as the proposed in [DW10], since we do not correct the position of the tangent vectors. In consequence, the subdivision curve generated by our scheme interpolates the prescribed tangent vectors. Nevertheless, our numerical experiments also lead us to several conjectures concerning the subdivision curve generated by (1)-(4)with $\rho = 0.5$.

Conjectures

1. The subdivision curve is C^1 continuous with respect to the arc-length parametrization.

2. The subdivision curve is G^2 continuous in all the points, except the vertices P_i^0 of the initial polygon such that the lengths of the edges $P_{i-1}^0 P_i^0$ and $P_i^0 P_{i+1}^0$ are different.

3. It is possible to define tangent vectors at the vertices P_i^0 of the initial polygon in such a way that the subdivision curve is G^2 everywhere.

4. The points $P_{2^k i}^k, ..., P_{2^k(i+1)}^k$ on the subdivision curve obtained from the edge $P_i^0 P_{i+1}^0$ of the initial polygon tends to be uniformly distributed with respect to the arc-length when $k \to \infty$, if the angles between the edge and tangent vectors at its extremes is the same.

In the following experiments we always take $\rho = 0.5$ to support the previous conjectures.

Experiment 1

We compute the tangent vector at each vertex P_i^0 of initial polygon as the tangent to the circle interpolating $P_{i-1}^0, P_i^0, P_{i+1}^0$. The curvature at each vertex of the subdivision curve after 5 steps is approximated by the curvature of the circle interpolating the vertex and its left and right neighbors. The results are shown in Figure 10. The curvature plots confirm our conjecture about the G^2 continuity of the subdivision curve at all points but the vertices of the initial polygon.

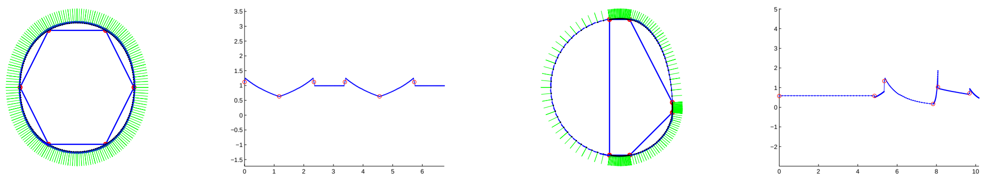

Figure 10: Left: Initial polygon, the subdivision curve ($\rho = 0.5$) and the normal vectors after 5 step, Right: Plot of curvature vs. arc-length of the subdivision curve.

Experiment 2

In this experiment we compute the tangent vector at each vertex P_i^0 of initial polygon using method (25) with $m = 3$. The results are shown in Figure 11. Since tangent vectors t_i^0 are defined in such a way that $\alpha_i^0 = \beta_{i+1}^0$, the segment of the subdivision curve between P_i^0 and P_{i+1}^0 is an arc of the circle interpolating P_i^0, P_{i+1}^0 and the corresponding tangent vectors. The limit is a G^1 piecewise circle curve, as confirms the curvature plot (curvature is constant for each section of the subdivision curve corresponding to an edge of the initial polygon). Therefore, it will be G^2 continuous only if it is a circle interpolating all points. This is an undesirable property when the edges of the initial polygon have different sizes, as the rectangle of this example.

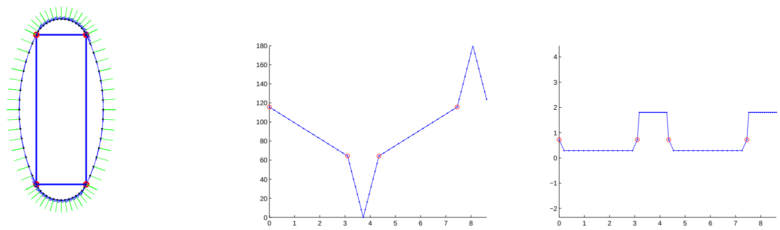

Figure 11: Left: Initial polygon, the subdivision curve ($\rho = 0.5$) and the tangent and normal vectors after 4 step, Middle: Absolute value of the tangent vector angle vs. arc-length of the subdivision curve, Right: Plot of curvature vs. arc-length of the subdivision curve. Tangent vectors at initial polygon vertices are preserved.

To obtain a G^2 continuous curve, which is not a circle, we *update* the tangent vectors at the vertices P_i^0 of the initial polygon after the first step, as the perpendicular to the bisector of the angle $P_{2i-1}^1 P_i^0 P_{2i+1}^1$, i.e

$$t_{2i}^1 = \frac{P_i^0 - P_{2i-1}^1}{\|P_i^0 - P_{2i-1}^k\|} + \frac{P_{2i+1}^1 - P_i^0}{\|P_{2i+1}^1 - P_i^0\|}$$

Figure 12 shows that this modification produces a nice subdivision curve which is G^2 continuous, as we observe in the curvature plot.

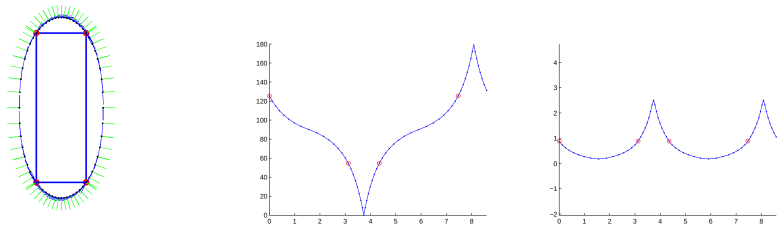

Figure 12: Left: Initial polygon, the subdivision curve ($\rho = 0.5$) and normal vectors after 4 step, Middle: Absolute value of the tangent vector angle vs. arc-length of the subdivision curve, Right: Plot of curvature vs. arc-length of the subdivision curve. Tangent vectors at initial polygon vertices are updated.

Experiment 3

Given two consecutive points P_i^0, P_{i+1}^0 of the starting polygon, in this experiment we compute the sequence of radii r_i^k of the circle interpolating the point $P_{2i+1}^1 = P_{4i+2}^2 = \ldots = P_{2^{k-1}(2i+1)}^k$ and its 2 neighbors $P_{2^{k-1}(2i+1)-1}^k, P_{2^{k-1}(2i+1)+1}^k$ in the step $k \geq 1$, see Figure 13. Recall that except the vertices of the initial polygon, all the points generated in any step of subdivision are included in the sequence $P_{2^{k-1}(2i+1)+1}^k$ for some i and k. The purpose is to show that the sequence $r_i^1, r_i^2, \ldots.$ converges for $\rho = 0.5$, see the first and second row of Tables 1 and 2. This is a numerical evidence of the G^2 continuity of the subdivision curve at all but the vertices of the initial polygon.

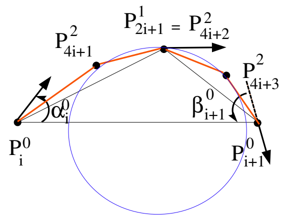

Figure 13: Circle interpolating P_{2i+1}^1 and its two neighbors in the step $k = 2$ for $\rho = 0.5$.

We also compute the ratio

$$l_i^k = \frac{\|P_{2^{k-1}(2i+1)-1}^k - P_{2^{k-1}(2i+1)}^k\|}{\|P_{2^{k-1}(2i+1)}^k - P_{2^{k-1}(2i+1)+1}^k\|}$$

and observe experimentally that the sequence $l_i^1, l_i^2, \ldots.$ also converges, see rows 3 and 4 of Tables 1 and 2. In particular, we observe in Table 2 that $\lim_{k\to\infty} l_i^k = 1$ when $\alpha_i^0 = \beta_{i+1}^0$, i.e. the points on the subdivision curve corresponding to an edge of the initial polygon tend to be uniformly distributed with respect to the arclength if the angles between the edge and tangent vectors at its extremes are the same.

Parameters $\rho = 0.5$, $\|P_i^0 - P_{i+1}^0\| = 1$, $\alpha_i^0 = \frac{\pi}{10}$, $\beta_{i+1}^0 = \frac{3\pi}{10}$							
r_i^k	0.6748	0.6439	0.6371	0.6357	0.6354	0.6354	0.6354
	0.6354	0.6355	0.6355	0.6355	0.6355	0.6355	0.6355
l_i^k	1.5989	1.2322	1.0860	1.0200	0.9886	0.9733	0.9657
	0.9620	0.9601	0.9592	0.9587	0.9585	0.9583	0.9583

Table 1: Sequence of curvature radii r_i^k and ratio of distances l_i^k, $k = 1, ..., 14$.

Parameters $\rho = 0.5$, $\|P_i^0 - P_{i+1}^0\| = 1$, $\alpha_i^0 = \beta_{i+1}^0 = \frac{\pi}{4}$							
r_i^k	0.7071	0.7071	0.7071	0.7071	0.7071	0.7071	0.7071
	0.7071	0.7071	0.7071	0.7071	0.7071	0.7071	0.7071
l_i^k	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.000
	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.000

Table 2: Sequence of curvature radii r_i^k and ratio of distances l_i^k,$k = 1, ..., 14$.

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6 Appendix

Proposition 3 *see Figure 14*
Let P_0, P_1 be two points on the circle with center C and radius r. Denote by Q be the intersection point of the line passing through P_0 in the direction of the tangent to the circle at P_0 and line passing through P_1 in the direction of the tangent to the circle at P_1. Then the intersection point R of the bisectors of the angles $\angle P_1P_0Q$ and $\angle P_0P_1Q$ is on the circle with center C and radius r.

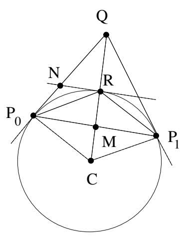

Figure 14: Notation for Proposition 3

Proof
Since P_0, P_1 are on the circle, $\|P_0 - C\| = \|P_1 - C\| = r$. Therefore,

$$\angle CP_0P_1 = \angle CP_1P_0$$

Moreover, $CP_0 \perp P_0Q$ and $CP_1 \perp P_1Q$. Hence,

$$\angle P_1P_0Q = 90^\circ - \angle CP_0P_1 = 90^\circ - \angle CP_1P_0 = \angle P_0P_1Q$$

i.e. the triangle with vertices P_0, P_1, Q is isosceles. In consequence Q and C are on the line passing through R and the midpoint M of the segment P_0P_1. Moreover, $P_0P_1 \perp QM$, i.e $\angle QMP_0 = 90^\circ$.
Denote by N the intersection point of the the line passing through P_0 and Q and the line passing through R perpendicular to QM (and in consequence parallel to P_0P1). Then.

$$\angle P_0RN = \angle RP_0M = \angle RP_0N = (\angle QP_0P_1)/2 \tag{26}$$

Furthermore,

$$\begin{aligned} \angle CP_0R &= 90^\circ - \angle RP_0N && (27) \\ \angle CRP_0 &= 90^\circ - \angle P_0RN && (28) \end{aligned}$$

Substituting (26) in (27) and (28) we obtain $\angle CP_0R = \angle CRP_0$. Thus,

$$r = \|P_0 - C\| = \|R - C\|$$

i.e, the point R is on the circle with center C and radius r.

■

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