STABLE CONSTANT MEAN CURVATURE HYPERSURFACES IN THE REAL PROJECTIVE SPACE

LUIS J. ALÍAS, ALDIR BRASIL JR., AND OSCAR PERDOMO

Abstract. In this paper, we prove that the only compact two-sided hypersurfaces with constant mean curvature H which are weakly stable in $\mathbb{RP}^{n+1}$ and have constant scalar curvature are (i) the twofold covering of a totally geodesic projective space; (ii) the geodesic spheres in $\mathbb{RP}^{n+1}$; and (iii) the quotient to $\mathbb{RP}^{n+1}$ of the hypersurface $\mathbb{S}^k(r)\times\mathbb{S}^{n-k}(\sqrt{1-r^2})\hookrightarrow\mathbb{S}^{n+1}$ obtained as the product of two spheres of dimensions k and $n-k$, with $k=1,\dots,n-1$, and radii r and $\sqrt{1-r^2}$, respectively, with $\sqrt{k/(n+2)}\leqslant r\leqslant\sqrt{(k+2)/(n+2)}$.

1. Introduction

Let $\psi : M^n \to \mathbb{RP}^{n+1}$ be a compact hypersurface immersed into the real proyective space $\mathbb{RP}^{n+1} = \mathbb{S}^{n+1}/\{\pm\}$. We will say that the hypersurface is two-sided if the normal bundle of M is trivial or, equivalently, if M admits a globally defined normal unit vector field. As is well-known, when $\mathbb{RP}^{n+1}$ is orientable, this property is equivalent to the orientablity of M. Assume that $\psi : M^n \to \mathbb{RP}^{n+1}$ is two-sided and denote by A its second fundamental form (with respect to a globally defined normal unit vector field N) and by H its mean curvature function, $H = (1/n)\mathrm{tr}(A)$. Then, every smooth function $u \in \mathcal{C}^\infty(M)$ induces a normal variation ψ_t of the immersion ψ, with variational normal field uN and first variation of the area functional $A(t)$ given by

$$\delta_u A = \frac{d}{dt}|_{t=0}A(t) = -n\int_M uH\mathrm{d}v.$$

As a consequence, minimal hypersurfaces ($H = 0$) are characterized as critical points of the area functional whereas constant mean curvature hypersurfaces can be viewed

Date: March 2006.

2000 *Mathematics Subject Classification.* Primary 53C42, Secondary 53A10.

Key words and phrases. constant mean curvature, $H(r)$-torus, stability operator, first eigenvalue.

L.J. Alías was partially supported by MEC/FEDER grant no. MTM2004-04934-C04-02, Spain, and by grant no. 00625/PI/04, Fundación Séneca, Spain.

A. Brasil Jr. was partially supported by CNPq.

O. Perdomo was partially supported by Colciencias.

as critical points of the area functional restricted to variations that preserve a certain volume function, that is, to smooth functions u with mean value zero, $\int_M u\mathrm{d}v = 0$.

For such critical points, the stability equation of the corresponding variational problem is given by the second variation of the area functional,

$$\delta_u^2 A = \frac{d^2}{dt^2}|_{t=0} A(t) = -\int_M uJ(u)\mathrm{d}v,$$

with $J(u) = \Delta u + (|A|^2 + n))u$, where Δ stands for the Laplacian operator on M. The operator

$$J = \Delta + |A|^2 + n$$

is called the Jacobi or stability operator of the hypersurface (for the details see [2]).

The Jacobi operator induces the quadratic form $Q(u) = -\int_M uJ(u)\mathrm{d}v$ acting on the space $\mathcal{C}^\infty(M)$ of smooth functions on M. In the case of minimal hypersurfaces, the index of a minimal hypersurface M, $\mathrm{Ind}(M)$, is defined as the maximum dimension of any subspace V of $\mathcal{C}^\infty(M)$ on which Q is negative definite. Equivalently, $\mathrm{Ind}(M)$ is the number of negative eigenvalues of J (counted with multiplicity), which is necessarily finite and, intuitively, it measures the number of independent directions in which the hypersurface fails to minimize area[1]. If $\mathrm{Ind}(M) = 0$, then M is said to be stable.

Using $u \equiv 1$ as a test function, one observes that

$$Q(1) = -\int_M (|A|^2 + n)\mathrm{d}v \leqslant -n\,\mathrm{area}(M) < 0.$$

In particular, $\mathrm{Ind}(M) \geqslant 1$ for every compact minimal hypersurface in $\mathbb{RP}^{n+1}$, which means that there is no stable one. In [4], do Carmo, Ritoré and Ros proved that $\mathrm{Ind}(M) = 1$ if and only if M is either a totally geodesic sphere (that is, the twofold covering of a totally geodesic projective space in $\mathbb{RP}^{n+1}$) or the quotient to $\mathbb{RP}^{n+1}$ of any of the minimal Clifford tori, $\mathbb{S}^k(\sqrt{k/n}) \times \mathbb{S}^{n-k}(\sqrt{n-k/n}) \subset \mathbb{S}^{n+1}$, $k = 1, \ldots, n-1$.

Our objective in this paper is to consider the same kind of problems for the case of constant mean curvature hypersurfaces. In contrast to the case of minimal hypersurfaces, in the case of hypersurfaces with constant mean curvature one can consider two different eigenvalue problems: the usual Dirichlet problem, associated with the quadratic form Q acting on the whole space of smooth functions on M, and the so called *twisted* Dirichlet problem, associated with the same quadratic form Q, but restricted to the subspace of smooth functions $u \in \mathcal{C}^\infty(M)$ satisfying the additional condition $\int_M u\mathrm{d}v = 0$.

[1]Observe that with our criterion, a real number λ is an eigenvalue of J if and only if $Ju + \lambda u = 0$ for some smooth function $u \in \mathcal{C}^\infty(M)$, $u \not\equiv 0$.

Similarly, there are two different notions of stability and index, the *strong stability* and *strong index*, denoted by $\mathrm{Ind}(M)$ and associated to the usual Dirichlet problem, and the *weak stability* and *weak index*, denoted by $\mathrm{Ind}_T(M)$ and associated to the twisted Dirichlet problem. Specifically, the strong index is simply the maximum dimension of any subspace V of $\mathcal{C}^\infty(M)$ on which Q is negative definite, and M is called strongly stable if and only if $\mathrm{Ind}(M) = 0$. On the other hand, the weak index is the maximum dimension of any subspace V of $\mathcal{C}_T^\infty(M)$ on which Q is negative definite, where $\mathcal{C}_T^\infty(M) = \{u \in \mathcal{C}^\infty(M) : \int_M u\mathrm{d}v = 0\}$, and M is called weakly stable if and only if $\mathrm{Ind}_T(M) = 0$ (see [1] for a detailed study of the relationship between these two eigenvalue problems and their corresponding stability and index notions).

From a geometrical point of view, the weak index is more natural than the strong index. However, from an analytical point of view, the strong index is more natural and easier to use. For instance, observe that for a hypersurface with constant mean curvature H in $\mathbb{RP}^{n+1}$, the Jacobi operator can be written as

$$J = \Delta + |A|^2 + n = \Delta + |\phi|^2 + n(1 + H^2),$$

where $|\phi|^2 = |A|^2 - nH^2 \geqslant 0$, and $|\phi|^2 \equiv 0$ if and only of M is totally umbilical. Therefore, using again $u \equiv 1$ as a test function for estimating $\mathrm{Ind}(M)$, one observes that

$$\begin{aligned} Q(1) &= -\int_M (|\phi|^2 + n(1 + H^2))\mathrm{d}v = -n(1 + H^2)\mathrm{area}(M) - \int_M |\phi|^2\mathrm{d}v \\ &\leqslant -n(1 + H^2)\mathrm{area}(M) < 0. \end{aligned}$$

In particular, $\mathrm{Ind}(M) \geqslant 1$ for every constant mean curvature hypersurface in $\mathbb{RP}^{n+1}$, which means that no constant mean curvature hypersurface is strongly stable.

When $n = 2$, Ritoré and Ros [6] proved that the only compact (indeed, complete) two-sided surfaces with constant mean curvature H which are weakly stable in $\mathbb{RP}^3$ are the twofold covering of a totally geodesic projective plane, the geodesic spheres, and the flat tubes of radius ϱ about a geodesic in $\mathbb{RP}^3$, with $\pi/6 \leq \varrho \leq \pi/3$. In this paper we consider the case of higher dimension $n \geq 3$ and, under the additional hypothesis of constant scalar curvature, we obtain the following classification.

Theorem 1. *The only compact two-sided hypersurfaces with constant mean curvature H which are weakly stable in $\mathbb{RP}^{n+1}$ and have constant scalar curvature are:*

(i) *the twofold covering of a totally geodesic projective space;*
(ii) *the geodesic spheres in $\mathbb{RP}^{n+1}$; and*
(iii) *the Clifford hypersurfaces $\mathbb{M}^n(k, r)$ with $k = 1, \ldots, n-1$ and $\sqrt{k/(n+2)} \leqslant r \leqslant \sqrt{(k+2)/(n+2)}$.*

The first ones, (i), are immersed while the two other, (ii) and (iii), are embedded. The first ones are minimal and they are the twofold covering of the totally geodesic projective spaces $\mathbb{RP}^n \subset \mathbb{RP}^{n+1}$ obtained as the quotient to $\mathbb{RP}^{n+1}$ of the totally geodesic equators of the sphere $\mathbb{S}^{n+1}$. The second ones are totally umbilical spheres in $\mathbb{RP}^{n+1}$ with non-zero constant mean curvature. The third ones are the quotient to $\mathbb{RP}^{n+1}$ of the product of two spheres of dimensions k and $n-k$, with $k=1,\dots,n-1$, and radii r and $\sqrt{1-r^2}$, respectively, with $\sqrt{k/(n+2)} \leqslant r \leqslant \sqrt{(k+2)/(n+2)}$ (see Section 2 for further details). In particular, when $n=2$ they can be seen also as flat tubes of radius ϱ about a geodesic in $\mathbb{RP}^3$, with $\pi/6 \leq \varrho \leq \pi/3$ (case (ii) in [6, Corollary 7]).

2. Stability index of CMC Clifford hypersurfaces in $\mathbb{RP}^{n+1}$

Apart from the totally umbilical spheres, the easiest CMC hypersurfaces in the unit sphere $\mathbb{S}^{n+1} \subset \mathbb{R}^{n+2}$ are the CMC Clifford hypersurfaces. A CMC Clifford hypersurface in $\mathbb{S}^{n+1}$ is obtained by considering the standard immersions $\mathbb{S}^k(r) \hookrightarrow \mathbb{R}^{k+1}$ and $\mathbb{S}^{n-k}(\sqrt{1-r^2}) \hookrightarrow \mathbb{R}^{n-k+1}$, for a given radius $0<r<1$ and integer $k \in \{1,\dots,n-1\}$, and taking the product immersion $\mathbb{T}^n(k,r) = \mathbb{S}^k(r) \times \mathbb{S}^{n-k}(\sqrt{1-r^2}) \hookrightarrow \mathbb{S}^{n+1} \subset \mathbb{R}^{n+2}$.

At a point $(x,y) \in \mathbb{T}^n(k,r)$, the vector field

$$N(x,y) = \left(\frac{\sqrt{1-r^2}}{r} x, -\frac{r}{\sqrt{1-r^2}} y \right)$$

defines a unit vector normal to $\mathbb{T}^n(k,r)$ at the point (x,y). With respect to this orientation, the principal curvatures of $\mathbb{T}^n(k,r)$ are given by

$$\kappa_1 = \cdots = \kappa_k = -\frac{\sqrt{1-r^2}}{r}, \quad \kappa_{k+1} = \cdots = \kappa_n = \frac{r}{\sqrt{1-r^2}},$$

and its constant mean curvature $H = H(r)$ is given by

$$nH(r) = \frac{nr^2-k}{r\sqrt{1-r^2}}.$$

In particular, $H(r)=0$ precisely when $r=\sqrt{k/n}$, which corresponds to the minimal Clifford hypersurface.

Observe that a CMC Clifford hypersurface $\mathbb{T}^n(k,r) \subset \mathbb{S}^{n+1}$ is invariant under the antipodal map and its normal vector field N is odd with respect to the antipodal map, that is, it satisfies $N(-x,-y) = -N(x,y)$. Therefore, it induces an embedded two-side CMC hypersurface $\mathbb{M}^n(k,r) = \mathbb{T}^n(k,r)/\{\pm\}$ in the real projective $\mathbb{RP}^{n+1} = \mathbb{S}^{n+1}/\{\pm\}$ with the same mean curvature $H=H(r)$, which we will also call a CMC Clifford hypersurface.

For a CMC Clifford hypersurface, one gets

$$|A|^2 + n = \frac{k}{r^2} + \frac{n-k}{1-r^2}$$

and its Jacobi operator, as a CMC hypersurface in $\mathbb{RP}^{n+1}$, reduces to

$$J = \Delta + \left(\frac{k}{r^2} + \frac{n-k}{1-r^2} \right),$$

where Δ denotes here the Laplacian on the quotient manifold $\mathbb{T}^n(k,r)/\{\pm\}$. In particular, the spectrum of J is directly related to the spectrum of Δ; specifically, they have the same eigenfunctions, and their eigenvalues are related by

$$\lambda_i^J = \lambda_i^\Delta - \left(\frac{k}{r^2} + \frac{n-k}{1-r^2} \right), \qquad i = 1, 2, \ldots.$$

Therefore, taking into account that $\lambda_1^\Delta = 0$ with constant eigenfunctions, we know that $\lambda_1^J = -\left(\frac{k}{r^2} + \frac{n-k}{1-r^2}\right)$ with multiplicity 1 and its corresponding eigenfunctions are the constant functions. Observe that $\lambda_1^J = -\left(\frac{k}{r^2} + \frac{n-k}{1-r^2}\right) < 0$ contributes to $\mathrm{Ind}(M)$ but not to $\mathrm{Ind}_T(M)$, because its eigenfunctions do not satisfy the restriction $\int_M u \mathrm{d}v = 0$. Even more, since all the rest of eigenfunctions of J are orthogonal to the constant functions, they do satisfy the restriction $\int_M u \mathrm{d}v = 0$ and do contribute to $\mathrm{Ind}_T(M)$. Therefore, in this case we have $\mathrm{Ind}(M) = \mathrm{Ind}_T(M) + 1$, and $\mathrm{Ind}_T(M)$ is given by the number of positive eigenvalues of the Laplacian operator on the quotient manifold $\mathbb{M}^n(k,r)$ (counted with multiplicity) which are strictly less than $\frac{k}{r^2} + \frac{n-k}{1-r^2}$.

To compute it, first recall that the eigenvalues of the Laplacian on the product manifold $\mathbb{T}^n(k,r)$ are given by

$$\frac{\ell(k+\ell-1)}{r^2} + \frac{m(n-k+m-1)}{1-r^2}, \tag{1}$$

where $\ell \geq 0$ and $m \geq 0$ are nonnegative integers, with multiplicity

$$\sum_{i,j} \mu_i \nu_j, \tag{2}$$

where

$$\mu_i = \binom{k+i}{i} - \binom{k+i-2}{i-2}, \quad i \geq 0,$$

$$\nu_j = \binom{n-k+j}{j} - \binom{n-k+j-2}{j-2}, \quad j \geq 0,$$

with the convention that $\binom{p}{q} = 0$ when $q < 0$, and the sum in (2) extends to all possible values of $i, j \geq 0$ for which

$$\frac{i(k+i-1)}{r^2} + \frac{j(n-k+j-1)}{1-r^2} = \frac{\ell(k+\ell-1)}{r^2} + \frac{m(n-k+m-1)}{1-r^2}.$$

Since $\mathbb{M}^n(k,r) = \mathbb{T}^n(k,r)/\{\pm\}$, then the eigenvalues of the Laplacian on $\mathbb{M}^n(k,r)$ are only those in (1) whose corresponding eigenfunctions on $\mathbb{T}^n(k,r)$ are invariant under the antipodal map. Recall now that the eigenfunctions on $\mathbb{T}^n(k,r)$ associated to an eigenvalue given by (1) are given by

$$P_\ell(x)Q_m(y)$$

where $P_\ell(x)$ (resp. $Q_m(y)$) denotes a homogeneous harmonic polynomial on $\mathbb{R}^{k+1}$ (resp. $\mathbb{R}^{n-k+1}$) of degree ℓ (resp. m). Therefore, the eigenvalues of the Laplacian on the quotient $\mathbb{M}^n(k,r) = \mathbb{T}^n(k,r)/\{\pm\}$ are only those in (1) for which $\ell + m$ is even (for the details, see [3]).

It follows from here that the computation of the index of a CMC Clifford hypersurface in $\mathbb{RP}^{n+1}$ reduces to count when

$$0 < \frac{\ell(k+\ell-1)}{r^2} + \frac{m(n-k+m-1)}{1-r^2} < \frac{k}{r^2} + \frac{n-k}{1-r^2}$$

with $\ell + m$ even. Observe that

$$\frac{\ell(k+\ell-1)}{r^2} + \frac{m(n-k+m-1)}{1-r^2} \geq \frac{k}{r^2} + \frac{n-k}{1-r^2}$$

when $\ell \geq 1$ and $m \geq 1$. In particular, a CMC Clifford hypersurface in $\mathbb{RP}^{n+1}$ is stable if and only if

$$\frac{2(n-k+1)}{1-r^2} \geq \frac{k}{r^2} + \frac{n-k}{1-r^2} \quad \text{and} \quad \frac{2(k+1)}{r^2} \geq \frac{k}{r^2} + \frac{n-k}{1-r^2},$$

that is, if and only if

$$\sqrt{\frac{k}{n+2}} \leqslant r \leqslant \sqrt{\frac{k+2}{n+2}}.$$

Observe that, in particular, this happens when $r = \sqrt{k/n}$, so that the minimal Clifford hypersurfaces in $\mathbb{RP}^{n+1}$ are stable when regarded as CMC hypersurfaces.

3. Proof of Theorem 1

We will follow closely the ideas of do Carmo, Ritoré and Ros in their proof of [4, Theorem 3] for the case of minimal hypersurfaces with index one. Actually, when $H = 0$ our result follows directly from their Theorem 3 because, under our assumptions, we get that the hypersurfaces in question are minimal hypersurfaces with index one, that is $\mathrm{Ind}(M) = 1$. In fact, since the scalar curvature is constant, the

Jacobi operator reduces to $J = \Delta + q$, where $q = |A|^2 + n > 0$ is a positive constant. In particular $\lambda_1^J = -q < 0$ with multiplicity 1 and constant eigenfunctions. Observe that λ_1^J contributes to $\mathrm{Ind}(M)$ but not to $\mathrm{Ind}_T(M)$, because its eigenfunctions do not satisfy the restriction $\int_M u\mathrm{d}v = 0$. Even more, since all the rest of eigenfunctions of J are orthogonal to the constant functions, they do satisfy the restriction $\int_M u\mathrm{d}v = 0$ and do contribute to $\mathrm{Ind}_T(M)$. Therefore, in the case where the scalar curvature is constant we obtain that $\mathrm{Ind}(M) = \mathrm{Ind}_T(M) + 1$. In particular, a minimal M is weakly stable (as a constant mean curvature hypersurface) if and only if $\mathrm{Ind}(M) = 1$, that is, M has index one as a minimal hypersurface.

Therefore, we will assume in what follows that $H \neq 0$. Consider $\psi : M^n \to \mathbb{RP}^{n+1}$ a compact two-sided hypersurface immersed with constant mean curvature in $\mathbb{RP}^{n+1}$, and assume that it is weakly stable and has constant scalar curvature. From the stability assumption, we conclude that M must be connected by the following standard argument. Suppose that M is not connected and take a locally constant test function u given by $u \equiv 1$ on a connected component Ω and $u \equiv c$ on $M - \Omega$, where the constant c is chosen such that $\int_M u\mathrm{d}v = 0$. But then, we would get $J(u) = (|A|^2 + n)u$ and $Q(u) = -\int_M (|A|^2 + n)u^2\mathrm{d}v < 0$, in contradiction to the fact that M is weakly stable.

Similarly to the minimal case [4, Theorem 3], when ψ lifts to an immersion of M into the sphere $\mathbb{S}^{n+1}$, then the lifting $\psi : M^n \to \mathbb{S}^{n+1}$ defines an orientable weakly stable compact hypersurface with the same constant mean curvature in $\mathbb{S}^{n+1}$, and by [2] it is a totally umbilical sphere. Therefore, from now on we assume that ψ does not lift to an immersion into the sphere. In that case, following the ideas in the proof of [4, Theorem 3], there is a connected twofold covering $\pi : \widetilde{M} \to M$ and an isometric immersion $\widetilde{\psi} : \widetilde{M}^n \to \mathbb{S}^{n+1}$ with the same constant mean curvature H which is locally congruent to ψ and such that

$$\widetilde{\psi} \circ \tau = -\widetilde{\psi}, \tag{3}$$

where $\tau : \widetilde{M} \to \widetilde{M}$ is the isometric involution induced by the covering map. Moreover, the two-sidedness of M implies that $\widetilde{M}$ is orientable and that its Gauss map $\widetilde{N} : \widetilde{M} \to \mathbb{S}^{n+1}$ also satisfies

$$\widetilde{N} \circ \tau = -\widetilde{N}. \tag{4}$$

In particular, $\widetilde{M}$ is not totally umbilical in $\mathbb{S}^{n+1}$.

Our stability hypothesis on M, when translated to $\widetilde{M}$, says that

$$Q(u) = -\int_{\widetilde{M}} uJ(u)\mathrm{d}v \geq 0$$

for every smooth function u on $\widetilde{M}$ such that $u \circ \tau = u$ and $\int_{\widetilde{M}} u dv = 0$. Moreover, if $Q(u) = 0$ for such a function u, then u is a Jacobi function, that is, $J(u) = 0$.

For a fixed arbitrary vector $\mathbf{v} \in \mathbb{R}^{n+2}$, we will consider with the functions $\ell_{\mathbf{v}} = \langle \widetilde{\psi}, \mathbf{v} \rangle$ and $f_{\mathbf{v}} = \langle \widetilde{N}, \mathbf{v} \rangle$ defined on $\widetilde{M}$. Observe that their gradients are given by

$$\nabla \ell_{\mathbf{v}} = \mathbf{v}^{\top} \quad \text{and} \quad \nabla f_{\mathbf{v}} = -A(\mathbf{v}^{\top}), \tag{5}$$

where $\mathbf{v}^{\top} \in \mathcal{X}(\widetilde{M})$ denotes the tangent component of $\mathbf{v}$ along the immersion $\widetilde{\psi}$,

$$\mathbf{v} = \mathbf{v}^{\top} + f_{\mathbf{v}} \widetilde{N} + \ell_{\mathbf{v}} \widetilde{\psi}.$$

Therefore, for every $X \in \mathcal{X}(\widetilde{M})$ we have

$$\nabla_X \nabla \ell_{\mathbf{v}} = \nabla_X \mathbf{v}^{\top} = -\ell_{\mathbf{v}} X + f_{\mathbf{v}} A(X),$$

so that

$$\Delta \ell_{\mathbf{v}} = \mathrm{tr}(\nabla^2 \ell_{\mathbf{v}}) = -n\ell_{\mathbf{v}} + nHf_{\mathbf{v}}, \tag{6}$$

and

$$J\ell_{\mathbf{v}} = |A|^2 \ell_{\mathbf{v}} + nHf_{\mathbf{v}}, \tag{7}$$

On the other hand, using Codazzi equation,

$$\begin{aligned} \nabla_X \nabla f_{\mathbf{v}} &= -\nabla_X (A(\mathbf{v}^{\top})) = -(\nabla_X A)(\mathbf{v}^{\top}) - A(\nabla_X \mathbf{v}^{\top}) \\ &= -(\nabla_{\mathbf{v}^{\top}} A)(X) + \ell_{\mathbf{v}} A(X) - f_{\mathbf{v}} A^2(X). \end{aligned}$$

Therefore

$$\begin{aligned} \Delta f_{\mathbf{v}} &= -\mathrm{tr}(\nabla_{\mathbf{v}^{\top}} A) + nH\ell_{\mathbf{v}} - |A|^2 f_{\mathbf{v}} = -n\langle \mathbf{v}^{\top}, \nabla H \rangle + nH\ell_{\mathbf{v}} - |A|^2 f_{\mathbf{v}} \\ &= nH\ell_{\mathbf{v}} - |A|^2 f_{\mathbf{v}}, \end{aligned} \tag{8}$$

since the mean curvature H is constant, and

$$Jf_{\mathbf{v}} = nH\ell_{\mathbf{v}} + nf_{\mathbf{v}}, \tag{9}$$

We will consider test functions of the form

$$g_{\mathbf{v},\mathbf{w}} = f_{\mathbf{v}} \Delta \ell_{\mathbf{w}} - \ell_{\mathbf{w}} \Delta f_{\mathbf{v}}.$$

Observe that the functions $g_{\mathbf{v},\mathbf{w}}$ clearly satisfy the condition $\int_{\widetilde{M}} g_{\mathbf{v},\mathbf{w}} dv = 0$ because of the self-adjointness of Δ. Moreover, a straightforward computation using (6) and (8) gives

$$g_{\mathbf{v},\mathbf{w}} = (|A|^2 - n)\ell_{\mathbf{w}} f_{\mathbf{v}} + nH(f_{\mathbf{v}} f_{\mathbf{w}} - \ell_{\mathbf{v}} \ell_{\mathbf{w}}). \tag{10}$$

In particular, by (3) and (4), the functions $g_{\mathbf{v},\mathbf{w}}$ also satisfy the condition $g_{\mathbf{v},\mathbf{w}} \circ \tau = g_{\mathbf{v},\mathbf{w}}$. Therefore

$$Q(g_{\mathbf{v},\mathbf{w}}) \geq 0$$

for every fixed arbitrary directions $\mathbf{v}, \mathbf{w} \in \mathbb{R}^{n+2}$, with equality if and only if $J(g_{\mathbf{v},\mathbf{w}}) = 0$.

On the other hand, a tedious but straightforward computation from (10), using (5), (6) and (8), gives

$$\frac{1}{2}J(g_{\mathbf{v},\mathbf{w}}) = \langle L\mathbf{v}^\top, \mathbf{w}^\top\rangle = \langle L\mathbf{v}^\top, \mathbf{w}\rangle, \tag{11}$$

where $L : \mathcal{X}(\widetilde{M}) \to \mathcal{X}(\widetilde{M})$ is given by

$$L = nHA^2 + (n - |A|^2)A - nHI.$$

Take $\mathbf{e}_1, \ldots, \mathbf{e}_{n+2}$ an orthonormal basis in $\mathbb{R}^{n+2}$. Then for every $\mathbf{v} \in \mathbb{R}^{n+2}$ we obtain by (10) and (11) that

$$\begin{aligned} \frac{1}{2}g_{\mathbf{v},\mathbf{e}_i}J(g_{\mathbf{v},\mathbf{e}_i}) &= (|A|^2 - n)f_{\mathbf{v}}\langle L\mathbf{v}^\top, \mathbf{e}_i\rangle\langle\widetilde{\psi}, \mathbf{e}_i\rangle \\ &\quad + nHf_{\mathbf{v}}\langle L\mathbf{v}^\top, \mathbf{e}_i\rangle\langle\widetilde{N}, \mathbf{e}_i\rangle - nH\ell_{\mathbf{v}}\langle L\mathbf{v}^\top, \mathbf{e}_i\rangle\langle\widetilde{\psi}, \mathbf{e}_i\rangle, \end{aligned}$$

and

$$\begin{aligned} \frac{1}{2}\sum_{i=1}^{n+2} g_{\mathbf{v},\mathbf{e}_i}J(g_{\mathbf{v},\mathbf{e}_i}) &= (|A|^2 - n)f_{\mathbf{v}}\langle L\mathbf{v}^\top, \widetilde{\psi}\rangle \\ &\quad + nHf_{\mathbf{v}}\langle L\mathbf{v}^\top, \widetilde{N}\rangle - nH\ell_{\mathbf{v}}\langle L\mathbf{v}^\top, \widetilde{\psi}\rangle = 0. \end{aligned}$$

Therefore,

$$\sum_{i=1}^{n+2} Q(g_{\mathbf{v},\mathbf{e}_i}) = -\int_M \sum_{i=1}^{n+2} g_{\mathbf{v},\mathbf{e}_i}J(g_{\mathbf{v},\mathbf{e}_i})\mathrm{d}v = 0,$$

but being $Q(g_{\mathbf{v},\mathbf{e}_i}) \geq 0$ for every i, this implies that $J(g_{\mathbf{v},\mathbf{e}_i}) = 0$ for every $i = 1, \ldots, n+2$. In other words, $L\mathbf{v}^\top = 0$ on $\widetilde{M}$ for every arbitrary fixed vector $\mathbf{v} \in \mathbb{R}^{n+2}$. It follows from here that $L = 0$; that is, the second fundamental form of $\widetilde{\psi} : \widetilde{M}^n \to \mathbb{S}^{n+1}$ satisfies the quadratic equation

$$A^2 + \frac{n - |A|^2}{nH}A - I = 0.$$

As a consequence, since $\widetilde{M}$ is not totally umbilical, we obtain that $\widetilde{M}$ is a compact hypersurface of $\mathbb{S}^{n+1}$ with two different principal curvatures, and by a well known result by Cartan [5] we deduce that $\widetilde{M}$ is a CMC Clifford torus in $\mathbb{S}^{n+1}$ of the form $\mathbb{S}^k(r) \times \mathbb{S}^{n-k}(\sqrt{1-r^2})$ with radius $0 < r < 1$. Finally, from our previous discussion about the values of the weak index for the quotients to $\mathbb{RP}^{n+1}$ of the products $\mathbb{S}^k(r) \times \mathbb{S}^{n-k}(\sqrt{1-r^2})$, we conclude that it must be $\sqrt{k/(n+2)} \leqslant r \leqslant \sqrt{(k+2)/(n+2)}$. This finishes the proof of Theorem 1.

Acknowledgements

This work was started while the first and third authors were visiting the Departamento de Matemtica of the Universidade Federal do Cear in Fortaleza, Brasil. They would like to thank that institution and the members of the department for their hospitality. This paper was written while the first author was visiting the IHÉS. He thanks IHÉS for its hospitality and support.

The first author would like to thank M. Ritoré for his useful comments and explanations during the preparation of this paper.

References

[1] J.L. Barbosa and P. Bérard, Eigenvalue and "twisted" eigenvalue problems. Applications to CMC surfaces, *J. Math. Pures Appl. (9)* **79** (2000), 427–450.

[2] J.L. Barbosa, M. do Carmo and J. Eschenburg, Stability of hypersurfaces with constant mean curvature in Riemannian manifolds, *Math. Z.* **197** (1988), 123–138.

[3] M. Berger, P. Gauduchon, and E. Mazet, Le spectre d'une varit riemannienne. Lecture Notes in Mathematics, Vol. 194 Springer-Verlag, Berlin-New York 1971.

[4] M. do Carmo, M. Ritoré and A. Ros, Compact minimal hypersurfaces with index one in the real projective space, *Comment. Math. Helv.* **75** (2000), 247–254.

[5] E. Cartan, Familles de surfaces isoparamtriques dans les espaces courbure constante, *Annali di Mat.* **17** (1938), 177–191.

[6] M. Ritoré and A. Ros, Stable constant mean curvature tori and the isoperimetric problem in three space forms, *Comment. Math. Helv.* **67** (1992), 293–305. 247–254.

Departamento de Matemáticas, Universidad de Murcia, Campus de Espinardo, E-30100 Espinardo, Murcia, Spain

E-mail address: ljalias@um.es

Departamento de Matemática, Universidade Federal do Ceará, Campus do Pici, 60455-760 Fortaleza-Ce, Brazil

E-mail address: aldir@mat.ufc.br

Departamento de Matemáticas, Universidad del Valle, Cali, Colombia

E-mail address: osperdom@mafalda.univalle.edu.co