CUBIC DYNAMICS ON THE HÉNON FAMILY

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Abstract. In this paper, we study a two-parameter family $\{\psi_{\mu,\nu}\}$ of two-dimensional diffeomorphisms such that $\psi_{0,0} = \psi$ has two generic unfolding quadratic heteroclinic tangencies which are cyclically associated with dissipative saddle points p^+, p^-. With moderate extra conditions, it is proved that there exists a parameter value (μ_0, ν_0) arbitrarily close to $(0,0)$ such that ψ_{μ_0,ν_0} has a generic unfolding cubic homoclinic tangency associated with p^+. Applying this result to the (original) Hénon family $\{f_{a,b}\}$, we show that f_{a_1,b_1} has such a cubic tangency for some (a_1, b_1) arbitrarily close to $(-2, 0)$. Combining this fact with theorems in Kiriki-Soma [12] based on results in Palis-Takens [15] and Wang-Young [20], one can observe the new phenomena in the Hénon family, appearance of persistent antimonotonic tangencies and cubic polynomial-like strange attractors.

0. Introduction

As was seen in [12], a generic unfolding cubic tangency with respect to a two-parameter family of 2-dimensional diffeomorphisms exhibits various phenomena on chaotic dynamical systems. In fact, there are some examples of 2-dimensional diffeomorphisms which admit cubic homoclinic tangencies, see Gonchenko-Shil'nikov-Turaev [7], Kaloshin [9] and so on. However, for our purpose, we need to detect such a diffeomorphism ψ in a given two-parameter family without perturbing it in the infinite dimensional space $\mathrm{Diff}^\infty(\mathbb{R}^2)$, and moreover to show that the cubic tangency of ψ unfolds generically with respect to the two-parameter family.

The following theorem presents sufficient conditions for guaranteeing that the *original* two-parameter family has infinitely many diffeomorphisms admitting such cubic tangencies.

Theorem A. *Suppose that ψ is any C^∞-diffeomorphism on the plane $\mathbb{R}^2$ with two dissipative saddle fixed points p^+, p^- such that p^- satisfies the Sternberg-Takens C^8 condition. Let $\{\psi_{\mu,\nu}\}$ be a two-parameter family in $\mathrm{Diff}^\infty(\mathbb{R}^2)$ with $\psi_{0,0} = \psi$ and let $\{p^\pm_{\mu,\nu}\}$ be continuations of dissipative saddle fixed points of $\psi_{\mu,\nu}$ with $p^+_{0,0} = p^+$, $p^-_{0,0} = p^-$. Suppose the following conditions.*

(i) *$W^s(p^+)$ and $W^u(p^-)$ have a heteroclinic quadratic tangency q^+ unfolding generically with respect to $\{\psi_{\mu,0}\}$.*
(ii) *$W^s(p^-)$ and $W^u(p^+)$ have a heteroclinic quadratic tangency q^- unfolding generically with respect to $\{\psi_{0,\nu}\}$.*
(iii) *There is a ν-continuation q^+_ν of heteroclinic tangencies of $W^s(p^+_{0,\nu})$ and $W^u(p^-_{0,\nu})$ with $q^+_0 = q^+$.*

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Then, there exists an element $(\mu_0, \nu_0) \neq (0,0)$ *in the* $\mu\nu$*-space arbitrarily close to* $(0,0)$ *such that* ψ_{μ_0,ν_0} *has a cubic homoclinic tangency associated with* $p^+_{\mu_0,\nu_0}$ *which unfolds generically with respect to* $\{\psi_{\mu,\nu}\}$.

Here, we say that a saddle fixed point p of a 2-dimensional diffeomorphism f is *dissipative* if the differential Df_p has the eigenvalues λ, σ satisfying

$$0 < |\lambda| < 1 < |\sigma| \quad \text{and} \quad |\lambda\sigma| < 1. \tag{0.1}$$

The Sternberg-Takens C^k condition given in [18] is a sufficient condition for C^k linearizing a C^∞ diffeomorphism of $\mathbb{R}^2$ in a neighborhood of a saddle point. This is a refinement of the generic condition given by Sternberg [17]. Though the Sternberg-Takens condition is rather technical, it is very useful to find locally linearizable diffeomorphisms in *fixed* two-parameter families, see Kiriki-Li-Soma [11] and Theorem B below. Refer to Subsections 1.1, 1.2 for the definitions of quadratic and cubic tangencies unfolding generically with respect to given families. In particular, the right hand side of Figure 1.1 in Subsection 1.1 (resp. Figure 1.2 in Subsection 1.2) illustrates a typical behaviour of unstable manifolds near a generic unfolding quadratic (resp. cubic) tangency with respect to coordinates fixing stable manifolds on the x-axis. When $q^\pm$ are related to $p^\pm$ as in (i) and (ii), q^+, q^- are said to be *heteroclinic tangencies cyclically associated with* p^+, p^-, see Fig. 2.1 in Subsection 2.1.

Now, we apply Theorem A to the existence of generic unfolding cubic tangencies in the *Hénon family*, which is the two-parameter family of diffeomorphisms $f_{a,b}$ of $\mathbb{R}^2$, called *Hénon maps*, defined as

$$f_{a,b}(x,y) = (1 + y - ax^2, bx), \quad b \neq 0. \tag{0.2}$$

We sometimes call these maps *original* Hénon maps consciously to distinguish them from Hénon-like maps. The Hénon family is one of the most important research subjects in the modern chaotic dynamical systems. Benedicks and Carleson [1] found a positive Lebesgue measure subset J_b of a-values near 2 for any sufficiently small $b > 0$ such that $f_{a,b}$ has a strange attractor if $a \in J_b$. Afterward, Luzzatto and Viana [13] filled gaps of some arguments in [1] in much more general contexts. Their result was generalized by Mora-Viana [14] and Viana [19] to Hénon-like families which admit renormalizations near quadratic homoclinic tangencies associated with dissipative saddle periodic points. Despite these facts, we have not yet had any mathematical proof of the existence of a strange attractor for the original Hénon map $f_{a,b}$ with $(a,b) = (1.4, 0.3)$ observed by Hénon [8]. We refer to [2, 3, 20, 21] for ergodic results concerning such strange attractors, see also [4] for comprehensive references related to these topics.

Carvalho [5, p. 769] presents a supporting evidence as numerical results for the existence of generic unfolding cubic homoclinic tangencies in the Hénon family at parameters (a,b) near $(1.203, 0.417)$ and $(1.095, 0.388)$. Figure 0.1 illustrates the stable and unstable manifolds of Hénon maps $f_{a,b}$ with (a,b) around $(1.2027, 0.41722)$, which are depicted by using the software *Janet*[1] produced by Knudsen et al. However, as far as the authors know, any strict proof of the existence of such tangencies have not been obtained.

[1]Available from `http://dcwww.fys.dtu.dk/~janet/`

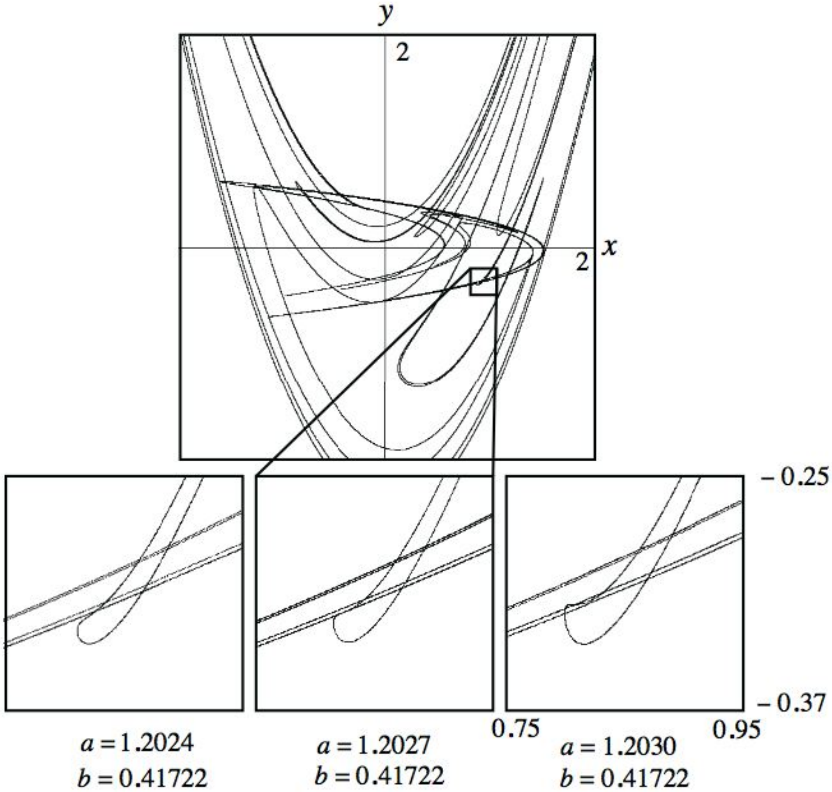

FIGURE 0.1

For convenience in our arguments, we adopt the following topologically conjugated formula of the Hénon map $f_{a,b}$:

$$\varphi_{a,b}(x, y) = (y, a - bx + y^2)$$

which is obtained from the classical formula (0.2) by the reparametrization $(a, b) \mapsto (-a, -b)$ and the coordinate change $(x, y) \mapsto (-ab^{-1}y, -ax)$. The fixed points of $\varphi_{a,b}$ are $p^{\pm}_{a,b} = (y^{\pm}_{a,b}, y^{\pm}_{a,b}) \in \mathbb{R}^2$ with $y^{\pm}_{a,b} = \big(1 + b \pm \sqrt{(1 + b)^2 - 4a}\big)/2$. Note that $p^{+}_{a,b}$ and $p^{-}_{a,b}$ converge respectively to $(2, 2)$ and $(-1, -1)$ as $(a, b) \to (-2, 0)$.

The following is our second main theorem which is proved by invoking Theorem A. In fact, we will find a parameter value (a_0, b_0) with $b_0 > 0$, arbitrarily close to $(-2, 0)$ and such that φ_{a_0,b_0} has two quadratic tangencies cyclically associated with $p^{\pm}_{a_0,b_0}$ one of which unfolds generically with respect to a and the other with respect to a certain b-parameter subfamily. Then, one can detect our desired parameter value (a_1, b_1) in any neighborhood of (a_0, b_0).

Theorem B. *There exists (a_1, b_1) with $b_1 > 0$, arbitrarily close to $(-2, 0)$ and such that the Hénon map φ_{a_1,b_1} has a cubic homoclinic tangency associated with*

$p^+_{a_1,b_1}$ which unfolds generically with respect to $\{\varphi_{a,b}\}$. Moreover, $p^+_{a_1,b_1}$ satisfies the Sternberg-Takens C^4 condition.

In the proof of Theorem B, we will show that $p^-_{a_0,b_0}$ and $p^+_{a_0,b_0}$ satisfy the Sternberg-Takens C^8 and C^4 conditions respectively. The condition for $p^-_{a_0,b_0}$ is necessary for applying Theorem A to the proof of Theorem B. The condition for $p^+_{a_0,b_0}$ implies the same condition for $p^+_{a_1,b_1}$ if (a_1,b_1) is sufficiently near (a_0,b_0), which is in turn used to apply Theorems in [12]. In fact, by our Theorem B together with [12, Theorems A and B], we have the following corollary which presents the two new phenomena for certain Hénon subfamilies, so called persistent antimonotonic tangencies and cubic polynomial-like strange attractors.

Corollary C. *There exist subsets $\mathcal{O}$ and $\mathcal{Z}$ in the ab-space such that the Hénon subfamilies $\{\varphi_{a,b}\}_{(a,b)\in\mathcal{O}}$ and $\{\varphi_{a,b}\}_{(a,b)\in\mathcal{Z}}$ satisfy the following conditions.*

(i) *$\mathcal{O}$ is an open set with $\mathrm{Cl}(\mathcal{O}) \ni (-2,0)$. For any $(a,b)\in\mathcal{O}$ and a sufficiently small $\varepsilon>0$, there exists a regular curve $c:(-\varepsilon,\varepsilon)\to\mathcal{O}$ with $c(0)=(a,b)$ such that the one-parameter family $\{\varphi_{c(t)}\}$ exhibits persistent antimonotonic tangencies.*

(ii) *For any open neighborhood U of $(-2,0)$ in the ab-space, $\mathcal{Z}\cap U$ has positive 2-dimensional Lebesgue measure. For any $(a,b)\in\mathcal{Z}$, there exists an integer $n+N>0$ so that $\varphi^{n+N}_{a,b}$ exhibits a cubic polynomial-like strange attractor supported by an SRB measure.*

A parametrized curve $c(t)$ is *regular* if $dc/dt(t)\neq(0,0)$ for any $t\in(-\varepsilon,\varepsilon)$. A one-parameter family $\{\psi_t\}$ of 2-dimensional diffeomorphisms is said to exhibit *contact-making tangencies* (resp. *contact-breaking tangencies*) at $t=t_0$ if there exist continuations of basic sets $\Lambda_{1,t},\Lambda_{2,t}$ of ψ_t and a quadratic tangency r_{t_0} associated with Λ_{1,t_0} and Λ_{2,t_0} such that, for a small neighborhood $\mathcal{N}(r_{t_0})$ of r_{t_0} in $\mathbb{R}^2$, there are continuations of curves $l^u_t\subset W^u(\Lambda_{1,t})\cap\mathcal{N}(r_{t_0})$, $l^s_t\subset W^s(\Lambda_{2,t})\cap\mathcal{N}(r_{t_0})$ and a sufficiently small $\delta>0$ such that (i) $l^u_t\cap l^s_t=\varnothing$ for $t<t_0$ (resp. $t>t_0$) with $|t-t_0|<\delta$, (ii) $l^u_{t_0}\cap l^s_{t_0}=\{r_{t_0}\}$ and (iii) l^u_t meets l^s_t non-trivially and transversely for $t>t_0$ (resp. $t<t_0$) with $|t-t_0|<\delta$, see Fig. 0.2. Contact-making and breaking tangencies associated with the same pair of basic sets and occurring simultaneously are called *antimonotonic tangencies*. The family $\{\psi_t\}$ is said to exhibit *persistent antimonotonic tangencies* if each ψ_t has antimonotonic tangencies.

An invariant set Ω of a 2-dimensional diffeomorphism ψ is called a *strange attractor* if (a) there exists a saddle point $p\in\Omega$ such that the unstable manifold $W^u(p)$ has dimension 1 and $\mathrm{Cl}(W^u(p))=\Omega$, (b) there exists an open neighborhood U of Ω such that $\{\psi^n(U)\}_{n=1}^\infty$ is a decreasing sequence with $\Omega=\bigcap_{n=1}^\infty\psi^n(U)$, and (c) there exists a point $z_0\in\Omega$ whose positive orbit is dense in Ω and a non-zero vector $v_0\in T_{z_0}(\mathbb{R}^2)$ with $\|D\psi^n_{z_0}(v_0)\|\geq e^{cn}\|v_0\|$ for any integer $n\geq 0$ and some constant $c>0$. The strange attractor is *cubic polynomial-like* if there exists an integer $m>0$ such that $\psi^m|\Omega$ is close (up to scale) to the one-dimensional map $x\mapsto -x^3+ax$ with $a\in(3\sqrt{3}/2,3)$ and has three saddle fixed points, see Fig. 0.3.

Here, an *SRB measure* means a ψ-invariant Borel probability measure which is ergodic, has a compact support and has absolutely conditional measures on unstable manifolds.

We finish this section by proposing the following problem asking if one can generalize Theorem B.

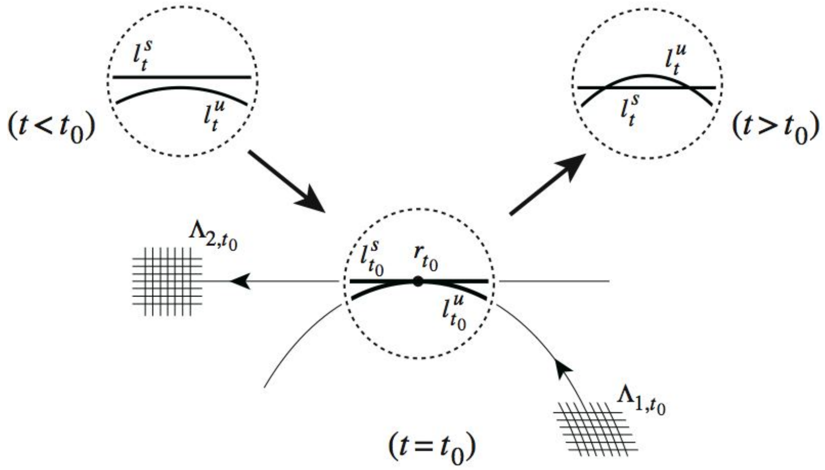

FIGURE 0.2. Each region encircled by one of the dotted loops represents the same neighborhood $\mathcal{N}(r_{t_0})$. The point r_{t_0} is a contact-making tangency.

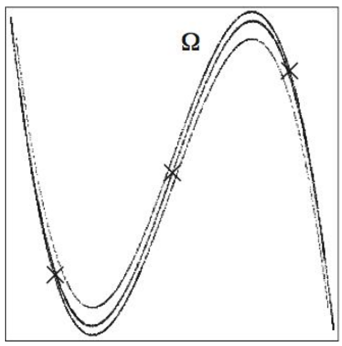

FIGURE 0.3

Problem 0.1. Does the original Hénon family unfold generically in a reasonable sense an arbitrarily higher order homoclinic tangency associated with $p_{a,b}^+$ for some (a,b)? Moreover, with respect to the parameter value (a,b), does $p_{a,b}^+$ satisfy the Sternberg-Takens smooth linearizing condition?

1. GENERIC UNFOLDING TANGENCIES

In this section, we will review some properties of quadratic and cubic tangencies which are associated with dissipative saddle points and unfold generically with respect to two-parameter families of 2-dimensional diffeomorphisms.

1.1. **Generic unfolding quadratic tangencies.** A diffeomorphism ψ on $\mathbb{R}^2$ has a *transverse point* r associated with saddle fixed points p_1, p_2 if

- $r \in W^u(p_1) \cap W^s(p_2) \setminus \{p_1, p_2\}$,
- $\dim(T_r W^u(p_1) + T_r W^s(p_2)) = 2$.

We also say that ψ has a *tangency* q of order n associated with saddle fixed points p_1, p_2 if it satisfies the following conditions.

- $q \in W^u(p_1) \cap W^s(p_2) \setminus \{p_1, p_2\}$.
- $\dim(T_q W^u(p_1) + T_q W^s(p_2)) = 1$.
- There exists a local C^{n+1} coordinate (x, y) in a neighborhood of q such that $q = (0, 0)$, $\{(x, y); y = 0\} \subset W^s(p_1)$ and $\{(x, y); y = \alpha(x)\} \subset W^u(p_2)$, where α is a C^{n+1}-function satisfying

$$\alpha(0) = \alpha'(0) = \cdots = \alpha^{(n)}(0) = 0 \quad \text{and} \quad \alpha^{(n+1)}(0) \neq 0. \tag{1.1}$$

In the case when $p_1 = p_2$, the transverse point or the tangency is called to be *homoclinic*, and otherwise *heteroclinic*. The definition of a tangency of order n is independent of the choice of a local C^{n+1} coordinate satisfying the condition as above. Usually, the first order tangency is called to *quadratic*, and the second order is *cubic*. In particular, the tangency q is quadratic if and only if $W^u(p_1)$ and $W^s(p_2)$ have distinct curvatures at q.

Let $\{\psi_\mu\}_{\mu \in J}$ be a one-parameter family in $\mathrm{Diff}^\infty(\mathbb{R}^2)$ such that the parameter space J is an interval, and $p_{1,\mu}, p_{2,\mu}$ (possibly $p_{1,\mu} = p_{2,\mu}$) continuations of saddle fixed point of ψ_μ such that $W^s(p_{1,\mu_0})$ and $W^u(p_{2,\mu_0})$ have a quadratic tangency q_{μ_0} at $\mu_0 \in J$. We say that the tangency q_{μ_0} *unfolds generically* with respect to $\{\psi_\mu\}_{\mu \in J}$ if there exist local coordinates (x, y) on $\mathcal{N}_\mu$ and C^2 functions $\alpha_\mu(x)$ which C^2 depend on μ and satisfy the following conditions, where $\{\mathcal{N}_\mu\}$ is a continuation of small open neighborhoods of q_μ in $\mathbb{R}^2$.

- $\alpha_{\mu_0}(x)$ satisfies $(1.1)_{n=1}$ and $\alpha_{\mu_0}(0) = q_{\mu_0}$.
- $\{(x, y); y = 0\} \subset W^s(p_{1,\mu})$ and $\{(x, y); y = \alpha_\mu(x)\} \subset W^u(p_{2,\mu})$ for any $\mu \in J$ near μ_0.
- For the two variable function $\alpha(\mu, x) := \alpha_\mu(x)$,

$$\frac{\partial \alpha}{\partial \mu}(\mu_0, 0) \neq 0. \tag{1.2}$$

It is not hard to see that the definition of this generic condition is independent of the choice of the coordinate neighbhorhoods $\mathcal{N}_\mu$ as above.

Now, we study the generic condition under more general coordinates $\tilde{\mathcal{N}}_\mu$ of a neighborhood of q_{μ_0}. With respect to $\tilde{\mathcal{N}}_\mu$, suppose that there exists a continuation S_μ of curves in $W^s(p_{1,\mu})$ with $q_{\mu_0} \in \mathrm{Int} S_{\mu_0}$ and which are represented as graphs of C^2 functions $\eta_\mu(x)$ of x with $|x| \le \delta$ for some $\delta > 0$, that is,

$$S_\mu = \{(x, \eta_\mu(x));\ |x| \le \delta\}.$$

The η_μ is called the *holding function* of S_μ. Let U_μ be a continuation of curves in $W^u(p_{2,\mu})$ with $q_{\mu_0} \in \mathrm{Int} U_{\mu_0}$, and σ a vertical segment passing through S_{μ_0} at q_{μ_0}. The intersection $S_\mu \cap \sigma$ (resp. $U_\mu \cap \sigma$) defines a continuation of points $r^s(\mu)$ (resp. $r^u(\mu)$). We denote the velocity vectors of $r^s(\mu)$ and $r^u(\mu)$ at $\mu = \mu_0$ by $\boldsymbol{v}_\mu^{s,\perp}(q_{\mu_0})$, $\boldsymbol{v}_\mu^{u,\perp}(q_{\mu_0})$ respectively. That is,

$$\boldsymbol{v}_\mu^{s,\perp}(q_{\mu_0}) = \frac{d}{d\mu} r^s(\mu_0), \quad \boldsymbol{v}_\mu^{u,\perp}(q_{\mu_0}) = \frac{d}{d\mu} r^u(\mu_0).$$

Let Φ_μ be the coordinate change of $\tilde{\mathcal{N}}_\mu$ defined by $\Phi_\mu(x, y) = (x, y - \eta_\mu(x))$. Then, $\tilde{S}_\mu = \Phi_\mu(S_\mu)$ is contained in the x-axis. Let $(x_\mu(t), y_\mu(t))$ be a C^2 regular curve parametrization of U_μ which C^2 depends on μ and such that the curve passes through q_{μ_0} at $t=0$. Then, $y_\mu(t)=\alpha_\mu(x_\mu(t))$. Set

$$\theta_\mu(t)=y_\mu(t)-\eta_\mu(x_\mu(t)). \tag{1.3}$$

Then, $(x_\mu(t),\theta_\mu(t))$ is a parametrization of $\tilde{U}_\mu=\Phi_\mu(U_\mu)$. Since $\tilde{U}_\mu$ is a quadratic curve for any μ close to μ_0, there exists a unique t_μ near 0 such that $\theta_\mu(t)$ has an extremal point at $t=t_\mu$ which C^1 depends on μ. Similarly, since $\tilde{U}_\mu$ meets the y-axis transversely in a single point $\tilde{r}^u(\mu)$, there exists a unique $\hat{t}_\mu$ near 0 with $(0,\theta_\mu(\hat{t}_\mu))=\tilde{r}^u(\mu)$ which C^2 depends on μ, see Fig. 1.1. The generic condition

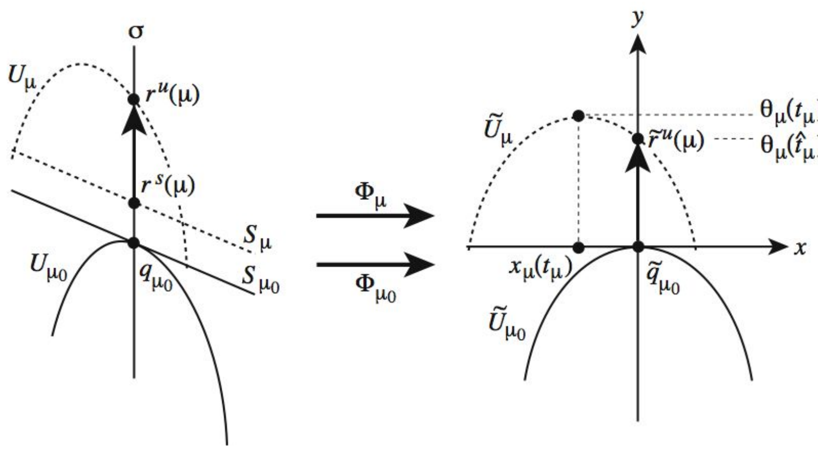

FIGURE 1.1

(1.2) for $\tilde{U}_{\mu_0}$ with respect to the new coordinate is $d\theta_\mu(\hat{t}_\mu)/d\mu(\mu_0)\neq 0$. Since $\tilde{r}^u(\mu)=r^u(\mu)-r^s(\mu)$ as a vector,

$$\Big(0,\frac{d\theta_\mu(\hat{t}_\mu)}{d\mu}(\mu_0)\Big)=\frac{d\tilde{r}^u}{d\mu}(\mu_0)=\boldsymbol{v}_\mu^{u,\perp}(q_{\mu_0})-\boldsymbol{v}_\mu^{s,\perp}(q_{\mu_0}).$$

From the definitions as above, $t_{\mu_0}=\hat{t}_{\mu_0}$. Thus, $t_\mu-\hat{t}_\mu=O(\Delta\mu)$ for $\mu=\mu_0+\Delta\mu$. Since $\theta_\mu(t)$ has an extremal value at $t=t_\mu$ and $x_\mu(\hat{t}_\mu)=0$,

$$\theta_\mu(\hat{t}_\mu)-\theta_\mu(t_\mu)=O(x_\mu(t_\mu)^2)=O((\hat{t}_\mu-t_\mu)^2)=O(\Delta\mu^2).$$

This shows that $d\theta_\mu(\hat{t}_\mu)/d\mu|_{\mu=\mu_0}=d\theta_\mu(t_\mu)/d\mu|_{\mu=\mu_0}$. Hence, the generic condition (1.2) is equivalent to

$$\boldsymbol{v}_\mu^{u,\perp}(q_{\mu_0})-\boldsymbol{v}_\mu^{s,\perp}(q_{\mu_0})=\Big(0,\frac{d\theta_\mu(t_\mu)}{d\mu}(\mu_0)\Big)\neq(0,0). \tag{1.4}$$

1.2. **Generic unfolding cubic tangencies.** Suppose that ψ is a C^∞ diffeomorphism of $\mathbb{R}^2$ with a dissipative saddle fixed point p. A cubic homoclinic tangency q of ψ associated with p is said to *unfold generically* with respect to a two-parameter family $\{\psi_{u,v}\}$ in $\mathrm{Diff}^\infty(\mathbb{R}^2)$ with $\psi_{0,0}=\psi$ if there exist (u,v)-dependent local coordinates (x,y) on a neighborhood of q with $q=(0,0)$ such that $W^s(p_{u,v})=\{(x,y);y=0\}$ and $W^u(p_{u,v})=\{(x,y);y=y_{u,v}(x)\}$. Here, $\{p_{u,v}\}$

is a continuation of saddle fixed points of $\psi_{u,v}$ with $p_{0,0} = p$, and $y_{u,v}(x) = y(u, v, x)$ is a C^4 function satisfying

$$(\partial_u y \cdot \partial_{vx} y - \partial_v y \cdot \partial_{ux} y)(0, 0, 0) \neq 0. \tag{1.5}$$

Since $y_{0,0}(0) = y'_{0,0}(0) = y''_{0,0}(0) = 0$, $y_{\mu,\nu}$ has the Taylor expansion

$$y_{u,v}(x) = a_1 u + a_2 v + a_3 ux + a_4 vx + a_5 uv + h(x, u, v), \tag{1.6}$$

where $a_1, \dots, a_5$ are constants and $h(t, u, v)$ is a C^4 function with

$$h = \partial_u h = \partial_v h = \partial_x h = \partial_{ux} h = \partial_{vx} h = \partial_{xx} h = 0$$

at $(x, u, v) = (0, 0, 0)$. Then, the generic condition (1.5) is rewritten as follows.

$$a_1 a_4 - a_2 a_3 \neq 0. \tag{1.7}$$

Let $F : (u, v) \mapsto (\hat{u}, \hat{v})$ is a C^4-diffeomorphism with $F(0, 0) = (0, 0)$, and let

$$y_{F^{-1}(\hat{u},\hat{v})}(x) = b_1 \hat{u} + b_2 \hat{v} + b_3 \hat{u} x + b_4 \hat{v} x + b_5 \hat{u}\hat{v} + \hat{h}(x, F^{-1}(\hat{u}, \hat{v}))$$

be the expansion of $y_{F^{-1}(\hat{u},\hat{v})}$, where $\hat{h}$ is a C^4 function as h. Then, we have

$$\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} DF_{(0,0)}.$$

This equation implies the following.

Lemma 1.1. *With the notation as above, $q = (0, 0)$ is a cubic tangency unfolding generically with respect to $\{\psi_{u,v}\}$ if and only if unfolding generically with respect to $\{\psi_{F^{-1}(\hat{u},\hat{v})}\}$.*

Now, we show that this generic condition is preserved under coordinate changes of the xy-plane fixing the x-axis as a set. Let U be a small neighborhood of $(0, 0)$ in the uv-space. Suppose that $\{\Phi_{u,v}\}_{(u,v)\in U}$ is a two-parameter family of C^4 diffeomorphisms of the xy-plane which C^4 depends on (u, v) and such that each $\Phi_{u,v}$ fixes the x-axis as a set and $\Phi_{0,0}(0, 0) = (0, 0)$. Let $\rho_{u,v}$ is a continuation of curves in $W^u(p_{u,v})$ with $\mathrm{Int}\rho_{u,v} \ni q$. We set

$$\tilde{\rho}_{u,v} = \Phi_{u,v}(\rho_{u,v}), \quad \tilde{\psi}_{u,v} = \Phi_{u,v} \circ \psi_{u,v} \circ \Phi^{-1}_{u,v}.$$

Lemma 1.2. *With the notation as above, if $q = (0, 0)$ is a cubic tangency of $\rho_{0,0}$ and the x-axis which unfolds generically with respect to $\{\psi_{u,v}\}$, then $\tilde{q} = \Phi_{0,0}(q)$ is also a cubic tangency of $\tilde{\rho}_{0,0}$ and the x-axis which unfolds generically with respect to $\{\tilde{\psi}_{u,v}\}$.*

Proof. Since $\Phi_{u,v}$ preserves the x-axis, $\Phi_{u,v}$ is represented as

$$\Phi_{u,v}(x, y) = (\beta_{u,v}(x, y), y\gamma_{u,v}(x, y)),$$

where $\beta_{u,v}$ (resp. $\gamma_{u,v}$) is a C^4 (resp. C^3) function. Moreover, the condition $\Phi_{0,}(0, 0) = (0, 0)$ implies

$$\beta_{0,0}(0, 0) = 0.$$

Since the differential of $\Phi_{u,v}$ is

$$D\Phi_{u,v} = \begin{pmatrix} \partial_x \beta_{u,v} & \partial_y \beta_{u,v} \\ y\partial_x \gamma_{u,v} & \gamma_{u,v} + y\partial_y \gamma_{u,v} \end{pmatrix},$$

$\det(D\Phi_{0,0})_{(x,y)=(0,0)} = \partial_x \beta_{0,0}(0, 0) \cdot \gamma_{0,0}(0, 0)$. Since $\Phi_{0,0}$ is a diffeomorphism,

$$b = \partial_x \beta_{0,0}(0, 0) \neq 0, \quad c = \gamma_{0,0}(0, 0) \neq 0.$$

The curve $\tilde{\rho}_{u,v}$ is parametrized as

$$\Phi_{u,v}(x, y_{u,v}(x)) = (\beta_{u,v}(x, y_{u,v}(x)), y_{u,v}(x) \cdot \gamma_{u,v}(x, y_{u,v}(x))).$$

Set $\tilde{x} = \tilde{x}(x, u, v) = \beta_{u,v}(x, y_{u,v}(x))$. Then, $\tilde{x}(0, 0, 0) = \beta_{0,0}(0, 0) = 0$. Differentiating $\tilde{x}$ by x,

$$\partial_x \tilde{x} = \partial_x \beta_{u,v}(x, y_{u,v}(x)) + \partial_y \beta_{u,v}(x, y_{u,v}(x)) \partial_x y_{u,v}(x).$$

Since $\partial_x y_{0,0}(0) = 0$, $\partial_x \tilde{x}(0, 0, 0) = b \neq 0$. Thus, $\tilde{x}(x, u, v)$ has a local inverse function $x = \delta_{u,v}(\tilde{x})$ with $\delta_{0,0}(0) = 0$ defined for any (u, v) near $(0, 0)$ and any $\tilde{x}$ near 0. Then,

$$\Phi_{u,v}(x, y_{u,v}(x)) = (\tilde{x}, \tilde{y}_{u,v}(\tilde{x})) = (\tilde{x}, \tilde{w}_{u,v}(\tilde{x}) \cdot \tilde{\gamma}_{u,v}(\tilde{x})),$$

where $\tilde{w}_{u,v}(\tilde{x}) = y_{u,v} \circ \delta_{u,v}(\tilde{x})$ and $\tilde{\gamma}_{u,v}(\tilde{x}) = \gamma_{u,v}(\delta_{u,v}(\tilde{x}), \tilde{w}_{u,v}(\tilde{x}))$. Then,

$$\partial_u \tilde{w}_{u,v}(\tilde{x}) = (\partial_u y_{u,v})(\delta_{u,v}(\tilde{x})) + (\partial_x y_{u,v})(\delta_{u,v}(\tilde{x})) \partial_u \delta_{u,v}(\tilde{x}). \tag{1.8}$$

Since $\partial_x y_{0,0}(0) = 0$, $\partial_u \tilde{w}_{u,0}(0)|_{u=0} = a_1$. We have as well $\partial_v \tilde{w}_{0,v}(0)|_{v=0} = a_2$. From $\tilde{y}_{u,v}(\tilde{x}) = \tilde{w}_{u,v}(\tilde{x}) \cdot \tilde{\gamma}_{u,v}(\tilde{x})$,

$$\partial_u(\tilde{y}_{u,v}(\tilde{x})) = \partial_u \tilde{w}_{u,v}(\tilde{x}) \cdot \tilde{\gamma}_{u,v}(\tilde{x}) + \tilde{w}_{u,v}(\tilde{x}) \cdot \partial_u \tilde{\gamma}_{u,v}(\tilde{x}). \tag{1.9}$$

Since $\tilde{\gamma}_{0,0}(0) = \gamma_{0,0}(0) = c$ and $\tilde{w}_{0,0}(0) = 0$, $\partial_u(\tilde{y}_{u,0}(0))|_{u=0} = a_1 c$. A similar argument shows $\partial_v(\tilde{y}_{0,v}(0))|_{v=0} = a_2 c$.

Differentiating the both sides of (1.8) by $\tilde{x}$ and putting $(u, \tilde{x}) = (0, 0)$, we have

$$\partial_{u\tilde{x}}(\tilde{w}_{u,0}(\tilde{x}))|_{(u,\tilde{x})=(0,0)} = \frac{a_3}{b}.$$

Then, from this equation together with the differentiation of (1.9) by $\tilde{x}$, we have

$$\partial_{u\tilde{x}}(\tilde{y}_{u,0}(\tilde{x}))|_{(u,\tilde{x})=(0,0)} = \frac{a_3 c}{b} + a_1 d,$$

where $d = \partial_{\tilde{x}} \tilde{\gamma}_{0,0}(\tilde{x})|_{\tilde{x}=0}$. Similarly,

$$\partial_{v\tilde{x}}(\tilde{y}_{0,v}(\tilde{x}))|_{(v,\tilde{x})=(0,0)} = \frac{a_4 c}{b} + a_2 d.$$

By using the equalities as above,

$$\begin{aligned}
\partial_u(\tilde{y}_{u,0}(0)) \cdot \partial_{v\tilde{x}}(\tilde{y}_{0,v}(\tilde{x}))|_{(u,v,\tilde{x})=(0,0,0)} &- \partial_v(\tilde{y}_{0,v}(0)) \cdot \partial_{u\tilde{x}}(\tilde{y}_{u,0}(\tilde{x}))|_{(u,v,\tilde{x})=(0,0,0)} \\
&= a_1 c\Big(\frac{c}{b} a_4 + a_2 d\Big) - a_2 c\Big(\frac{c}{b} a_3 + a_1 d\Big) \\
&= \frac{c^2}{b}(a_1 a_4 - a_2 a_3) \neq 0.
\end{aligned}$$

It follows that $(0, 0)$ is a cubic tangency of $\tilde{\rho}_{0,0}$ and the $\tilde{x}$-axis which unfolds generically with respect to $\{\tilde{\psi}_{u,v}\}$. □

The following lemma presents sufficient conditions for the generic unfolding of a cubic tangency of a two-parameter family $\{\psi_{u,v}\}$ in $\mathrm{Diff}^4(\mathbb{R}^2)$.

Lemma 1.3. *Let U be an open neighborhood of $(0, 0)$ in the uv-space, and $\rho_{u,v}$ a continuation of curves in $W^u(p_{u,v})$. Suppose that these $\rho_{u,v}$ have regular curve parametrizations $\rho_{u,v}(t) = (x_{u,v}(t), y_{u,v}(t))$ for any t near 0 which C^4 vary with respect to (u, v) and satisfy the following conditions.*

(i) $x_{0,0}(0) = 0$ *and* $y_{0,0}(0) = \dot{y}_{0,0}(0) = \ddot{y}_{0,0}(0) = 0$, $\dddot{y}_{0,0}(0) \neq 0$.

(ii) *There exists a C^2 function $t_{u,v}$ on U with $t_{0,0} = 0$ and $\ddot{y}_{u,v}(t_{u,v}) = 0$ for any $(u, v) \in U$.*

(iii) *There exists a C^2 function $v = v(u)$ of u with $v(0) = 0$ and $\dot{y}_{u,v(u)}(t_{u,v(u)}) = 0$ for any u near 0.*
(iv) $(\partial_u y_{u,v(u)})(0)|_{u=0} \neq 0$ *and* $(\partial_v \dot{y}_{0,v})(0)|_{v=0} \neq 0$.

Then, the origin $(0,0)$ in the xy-plane is a cubic tangency of $\rho_{0,0}$ and the x-axis which unfolds generically with respect to $\{\psi_{u,v}\}$.

The condition (ii) means that $r_{u,v} = \rho_{u,v}(t_{u,v})$ is a reflection point of $\rho_{u,v}$. The condition (iv) implies that the height of $r_{u,v(u)}$ varies with non-zero speed at $u = 0$ and the slope of the line tangent to $\rho_{0,v}$ at $r_{0,v}$ also varies with non-zero speed at $v = 0$. Figure 1.2 illustrates a typical movement of $\rho_{u,v}$ obtained by the combination of these two variations.

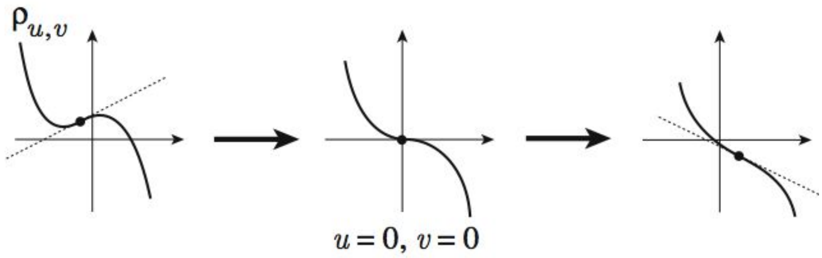

FIGURE 1.2

Proof. Since $\rho_{0,0}(t)$ is a regular curve, $\dot{\rho}_{0,0}(0) = (\dot{x}_{0,0}(0), 0) \neq (0,0)$. Thus, for any (u, v) near $(0,0)$, there exists a C^4 inverse function $t = \eta_{u,v}(x)$ of $x_{u,v}(t)$ defined on any t close to 0. Define the coordinate change $\Psi_{u,v}$ from a neighborhood of $(0,0)$ in the xy-plane to that in the ty-plane by $\Psi_{u,v}(x, y) = (\eta_{u,v}(x), y)$. Set $\hat{\rho}_{u,v} = \Psi_{u,v}(\rho_{u,v})$. Then, $\hat{\rho}_{u,v}$ is the regular curve parametrized by $(t, y_{u,v}(t))$. By Lemmas 1.1 and 1.2, it suffices to show that $(0,0)$ is a cubic tangency of $\hat{\rho}_{0,0}$ and the t-axis which unfolds generically with respect to the uw-parameter family $\{\hat{\rho}_{u,w+v(u)}\}$.

By the conditions of (i), the origin $(0,0)$ is a cubic tangency of $\hat{\rho}_{0,0}$ and the t-axis. Set $y_{u,w+v(u)} = \hat{y}_{u,w}$ and consider the Taylor expansion

$$\hat{y}_{u,w}(t) = a_1 u + a_2 w + a_3 ut + a_4 wt + a_5 uw + \hat{h}(t, u, w)$$

of $\hat{y}_{u,w}$ as (1.6). By the former condition of (iv),

$$a_1 = (\partial_u \hat{y}_{u,0})(0)|_{u=0} = (\partial_u y_{u,v(u)})(0)|_{u=0} \neq 0.$$

Since $\dot{y}_{u,v(u)}(t_{u,v(u)}) = 0$ for any u near 0 by the condition (iii),

$$\partial_u(\dot{y}_{u,v(u)}(t_{u,v(u)})) = (\partial_u \dot{y}_{u,v(u)})(t_{u,v(u)}) + \ddot{y}_{u,v(u)}(t_{u,v(u)})\partial_u(t_{u,v(u)}) = 0.$$

Since $\ddot{y}_{u,v(u)}(t_{u,v(u)}) = 0$ by the condition (ii), $(\partial_u \dot{y}_{u,v(u)})(t_{u,v(u)}) = 0$. Thus,

$$a_3 = (\partial_u \dot{\hat{y}}_{u,0})(0)|_{u=0} = (\partial_u \dot{y}_{u,v(u)})(0)|_{u=0} = 0.$$

Suppose that $u = 0$. Then, $v = w$. Hence, the latter condition of (iv) implies

$$a_4 = (\partial_w \dot{\hat{y}}_{0,w})(0)|_{w=0} = (\partial_v \dot{y}_{0,v})(0)|_{v=0} \neq 0.$$

It follows that $\hat{y}_{u,v}$ satisfies the generic condition (1.7), and hence the point $(0,0)$ is a cubic tangency unfolding generically with respect to $\{\psi_{u,w+v(u)}\}$. □

2. Existence of generic unfolding cubic tangencies

In this section, we give the proof of Theorem A.

2.1. **Outline of proof of Theorem A.** Let $\{\psi_{\mu,\nu}\}$ be a two-parameter family in $\mathrm{Diff}^\infty(\mathbb{R}^2)$ with $\psi_{0,0} = \psi$, and $\{p^{\pm}_{\mu,\nu}\}$ continuations of dissipative saddle fixed points of $\psi_{\mu,\nu}$ with $p^+_{0,0} = p^+$, $p^-_{0,0} = p^-$ satisfying the conditions of Theorem A. In particular, there exist heteroclinic quadratic tangencies q^+, q^- cyclically associated with p^+, p^- such that q^+ (resp. q^-) unfolds generically with respect to $\{\psi_{\mu,0}\}$ (resp. $\{\psi_{0,\nu}\}$). Moreover, there is a ν-continuation q^+_ν of heteroclinic tangencies of $W^s(p^+_{0,\nu})$ and $W^u(p^-_{0,\nu})$ with $q^+_0 = q^+$. Thus, when $(\mu,\nu) = (0,0)$, we have the situation as illustrated in Fig. 2.1, where r is a point in $W^u(p^-_{0,0})$ with $\psi^N_{0,0}(r) = q^+$ for some integer $N > 0$.

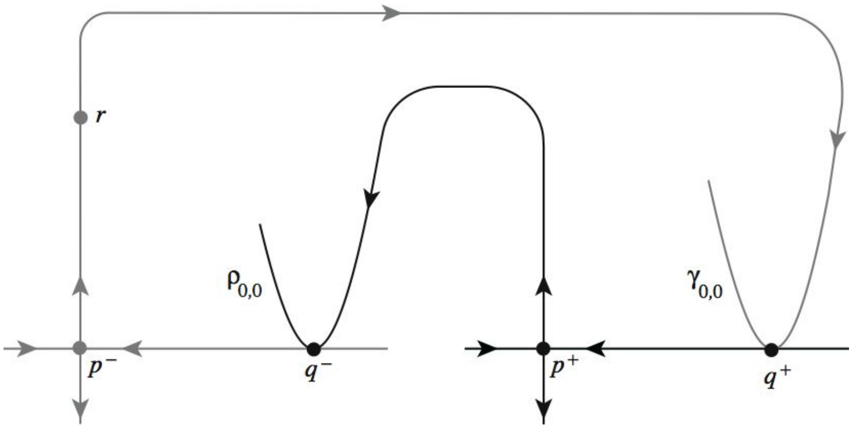

Figure 2.1

Let $\rho_{\mu,\nu}, \gamma_{\mu,\nu}$ be continuations of quadratic curves in $W^u(p^+_{\mu,\nu})$, $W^u(p^-_{\mu,\nu})$ respectively with $\mathrm{Int}\rho_{0,0} \ni q^-$, $\mathrm{Int}\gamma_{0,0} \ni q^+$, and let $q^-_{\mu,\nu}$, $q^+_{\mu,\nu}$ be continuations of minimal points of $\rho_{\mu,\nu}$, $\gamma_{\mu,\nu}$ based at q^- and q^+ respectively. From the generic conditions for the cubic tangencies $q^{\pm}$, we may suppose that, the level of $q^+_{\mu,\nu}$ (resp. $q^-_{\mu,\nu}$) rises as μ (resp. ν) increases. Now, we consider the situation where μ decreases and ν increases slightly from 0, see Fig. 2.2. Then, $\rho^{(n)}_{\mu,\nu} = \psi^n_{\mu,\nu}(\rho_{\mu,\nu})$ is a pinched quadratic curve with a unique minimal point $s_{\mu,\nu}$ such that $\psi^N_{\mu,\nu}(s_{\mu,\nu})$ is sufficiently close to $\mathrm{F}q^+_{\mu,\nu}$. Figure 2.2 suggests that, if we choose $\mu_0 < 0$ and $\nu_0 > 0$ suitably, then $\rho^{(n+N)}_{\mu_0,\nu_0}$ has a cubic tangency. From the fact that q^+ unfolds generically with respect to the μ-parameter $\{\psi_{\mu,\nu(\text{fixed})}\}$, the reflection point of $\rho^{(n+N)}_{\mu,\nu}$ moves upward or downward together with $q^+_{\mu,\nu}$ when μ varies. Moreover, as is suggested in Fig. 2.3, the slope of the line tangent to $\rho^{(n+N)}_{\mu,\nu}$ at its reflection point decreases as ν increases. Then, by using Lemma 1.3 (see also Fig. 1.2), one can prove that the reflection point of $\rho^{(n+N)}_{\mu_0,\nu_0}$ is a cubic tangency unfolding generically with respect to $\{\psi_{\mu,\nu}\}$.

Now, we have known that the idea of our proof is simple and elementary. However, in the actual argument below, we need to deal higher order terms appeared

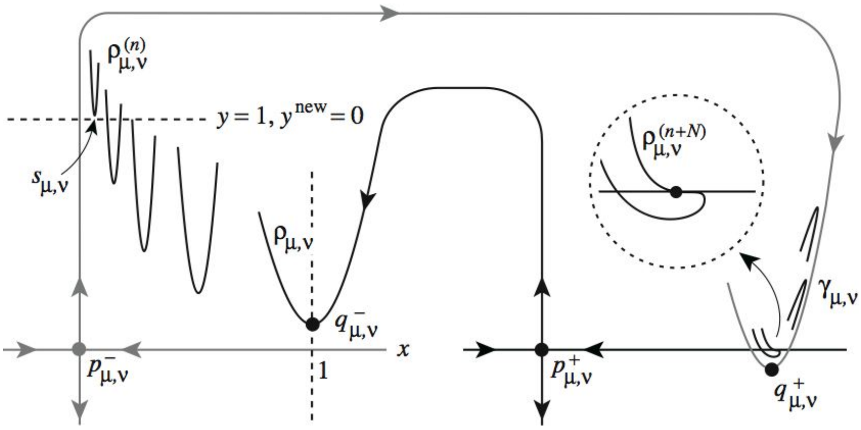

FIGURE 2.2

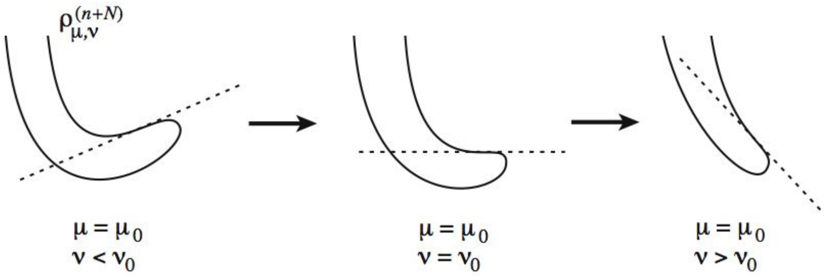

FIGURE 2.3

in the Taylor expansions of the holding functions of $\rho_{\mu,\nu}$ and $\gamma_{\mu,\nu}$ much precisely and carefully.

2.2. **Rearrangements.** Take any neighborhood U of $(0,0)$ in the $\mu\nu$-space such that, for any $(\mu,\nu) \in U$, $p^-_{\mu,\nu}$ is a dissipative saddle fixed point of $\psi_{\mu,\nu}$ such that $(D\psi_{\mu,\nu})_{p^-_{\mu,\nu}}$ has the eigenvalues $\lambda = \lambda_{\mu,\nu}$, $\sigma = \sigma_{\mu,\nu}$. If necessarily replacing $\psi_{\mu,\nu}$ by $\psi^2_{\mu,\nu}$, we may assume that

$$0 < \lambda < 1 < \sigma, \quad \lambda\sigma < 1. \tag{2.1}$$

Since the Sternberg-Takens C^8 condition given in [18] is an open condition, one can replace U by a smaller neighborhood if necessary so that, for any $(\mu,\nu) \in U$, there exists a C^8-coordinate neighborhood $\mathcal{N}_{\mu,\nu}$ of $p^-_{\mu,\nu}$ C^8-depending on (μ,ν) with respect to which $\psi_{\mu,\nu}|\mathcal{N}_{\mu,\nu}$ is a linear, that is, $\psi_{\mu,\nu}(x,y) = (\lambda x, \sigma y)$ if both $(x,y), \psi_{\mu,\nu}(x,y)$ belong to $\mathcal{N}_{\mu,\nu}$. It follows that the x-axis is contained in $W^s(p^-_{\mu,\nu})$ and the y-axis is contained in $W^u(p^-_{\mu,\nu})$ for any $(\mu,\nu) \in U$. Moreover, one can retake that the coordinates so that the following conditions hold without violating the linearity condition for $\psi_{\mu,\nu}$.

(a.i) For any $(\mu,\nu) \in U$, $\mathcal{N}_{\mu,\nu}$ contains the square $[-2,2] \times [-2,2]$.

(a.ii) A continuation of minimal points $q^-_{\mu,\nu}$ of $W^u(p^+_{\mu,\nu}) \cap \mathcal{N}_{\mu,\nu}$ based at q^- is on the vertical line $x = 1$, see Fig. 2.2. In particular, $q^- = (1,0)$.

(a.iii) There exists an integer $N > 0$ such that $\psi_{0,0}^N(r) = q^+$, where r is the point of $\mathcal{N}_{0,0}$ with coordinate $(0,1)$.

One can suppose that a curve $\rho_{\mu,\nu}$ in $W^u(p^+_{\mu,\nu})$ passing through $q^-_{\mu,\nu}$ is parametrized as follows.

$$\rho_{\mu,\nu} : x = u + 1, \quad y = y(u,\mu,\nu) = a_1\mu + a_2\nu + bu^2 + cu^3 + h(u,\mu,\nu), \tag{2.2}$$

where the parameter u varies in a fixed open interval containing 0 for any $(\mu,\nu) \in U$. Since q^- unfolds generically with respect to $\{\psi_{0,\nu}\}$, $a_2 \neq 0$. Consider the new parametrization with $\hat{\mu} = \mu$, $\hat{\nu} = a_1\mu + a_2\nu$. It is not hard to see that the conditions (i)-(iii) of Theorem A still hold with respect to $(\hat{\mu},\hat{\nu})$. For example, the condition (iii) is derived from the fact that $(0,\hat{\nu})$ in the $\hat{\mu}\hat{\nu}$-plane corresponds to $(0,\hat{\nu}/a_2)$ in the $\mu\nu$-plane. For simplicity, we denote $(\hat{\mu},\hat{\nu})$ and $h(u,\hat{\mu},(\hat{\nu} - a_1\hat{\mu})/a_2)$ again by (μ,ν) and $h(u,\mu,\nu)$ respectively. Then, (2.2) is rewritten as follows:

$$\rho_{\mu,\nu} : x = u + 1, \quad y = y(u,\mu,\nu) = \nu + bu^2 + cu^3 + h(u,\mu,\nu). \tag{2.2$'$}$$

Since $\rho_{0,0}$ is a quadratic curve tangent to the x-axis,

$$b \neq 0. \tag{2.3}$$

Moreover, $h(u,\mu,\nu)$ is a C^8 function satisfying the following conditions.

(b.i) $q^-_{\mu,\nu} = (1, \nu + h(0,\mu,\nu))$ for any $(\mu,\nu) \in U$. In particular, $h(0,0,0) = 0$.
(b.ii) $\partial_u h(0,\mu,\nu) = 0$ for any $(\mu,\nu) \in U$.
(b.iii) $\partial_\mu h(0,0,0) = \partial_\nu h(0,0,0) = 0$.
(b.iv) $\partial_{uu} h(0,0,0) = \partial_{uuu} h(0,0,0) = 0$.

Here, the conditions (b.i) and (b.ii) are derived from (a.ii). The conditions (b.iii) and (b.iv) are derived from the form (2.2$'$) of $\rho_{\mu,\nu}$.

For any integer $n > 0$, let $\rho^{(n)}_{\mu,\nu}$ be the component of $\psi^n_{\mu,\nu}(\rho_{\mu,\nu}) \cap \mathcal{N}_{\mu,\nu}$ containing $\psi^n_{\mu,\nu}(q^-_{\mu,\nu})$. When n is very large, $\rho^{(n)}_{\mu,\nu}$ may be empty if $|\nu|$ is bounded away from zero. When $|\nu|$ is sufficiently small, $\rho^{(n)}_{\mu,\nu}$ is parametrized as follows.

$$\rho^{(n)}_{\mu,\nu} : x = \lambda^n(u+1), \quad y = \sigma^n\nu + b\sigma^n u^2 + c\sigma^n u^3 + \sigma^n h(u,\mu,\nu).$$

Consider the new parameters $\bar{u}, \bar{\mu}, \bar{\nu}$ defined as

$$\bar{u} = \sigma^{n/2}u, \quad \bar{\mu} = \sigma^n\mu, \quad \bar{\nu} = \sigma^n\nu. \tag{2.4}$$

We fix a sufficiently small constant $\varepsilon > 0$ independent of n, which will be chosen suitably later. From now on, we only consider $\bar{\mu}, \bar{\nu}$'s contained in the rectangle

$$\bar{R} = \{(\bar{\mu},\bar{\nu});\ |\bar{\mu}| \le 1, |\bar{\nu} - 1| \le \varepsilon\}.$$

In our argument, it is crucial that $\bar{R}$ does not depend on n. Since $\bar{R}$ is compact, (2.4) implies that

$$\mu = O(\sigma^{-n}), \quad \nu = O(\sigma^{-n}). \tag{2.5}$$

Here, $O(\sigma^{-n})$ and $o(\sigma^{-n})$ represent functions of $\bar{u}, \bar{\mu}, \bar{\nu}$ satisfying

$$\lim_{n\to\infty} \sup_{(\bar{\mu},\bar{\nu})\in\bar{R}, |\bar{u}|\le 2\varepsilon} \frac{|O(\sigma^{-n})|}{\sigma^{-n}} < \infty \quad \text{and} \quad \lim_{n\to\infty} \sup_{(\bar{\mu},\bar{\nu})\in\bar{R}, |\bar{u}|\le 2\varepsilon} \frac{|o(\sigma^{-n})|}{\sigma^{-n}} = 0.$$

Furthermore, $\hat{O}(\cdot)$ denotes an $O(\cdot)$-function of μ,ν which is constant on u, that is, $\partial_u \hat{O}(\cdot) = 0$ and hence $\partial_{\bar{u}} \hat{O}(\cdot) = 0$.

Now, we introduce the new coordinate $(\hat{x}, \hat{y})$ on $\mathcal{N}_{\mu,\nu}$ by

$$\hat{x} = x, \quad \hat{y} = y - 1.$$

By the condition (a.iii), we have

$$\psi_{0,0}^{N}(0,0) = q^{+} \tag{2.6}$$

with respect to the new coordinate. For saving symbols, we denote $(\hat{x}, \hat{y})$ again by (x, y). Then, $\rho_{\mu,\nu}^{(n)}$ is parametrized with respect to the new coordinate as follows.

$$\begin{aligned} x &= x_n(\bar{u}, \bar{\mu}, \bar{\nu}) = \lambda^n + \tau^n \bar{u}, \\ y &= y_n(\bar{u}, \bar{\mu}, \bar{\nu}) = -1 + \bar{\nu} + b\bar{u}^2 + c\sigma^{-n/2}\bar{u}^3 + \sigma^n h(\sigma^{-n/2}\bar{u}, \sigma^{-n}\bar{\mu}, \sigma^{-n}\bar{\nu}), \end{aligned} \tag{2.7}$$

where

$$\tau = \lambda\sigma^{-1/2}. \tag{2.8}$$

By (2.1), the τ satisfies

$$0 < \tau < \lambda < \sigma^{-1} < 1.$$

By (b.ii), $h(u, \mu, \nu)$ does not have any term the u-order of which is one. Moreover, by (b.i), (b.iii) and (b.iv), $h(u, \mu, \nu)$ is represented as

$$h(u, \mu, \nu) = \hat{O}^{(2)}(\mu, \nu) + \hat{O}^{(1)}(\mu, \nu)u^2 + \hat{O}^{(1)}(\mu, \nu)u^3 + O(u^4),$$

where $\hat{O}^{(1)}(\mu, \nu) = \hat{O}(\mu) + \hat{O}(\nu)$ and $\hat{O}^{(2)}(\mu, \nu) = \hat{O}(\mu^2) + \hat{O}(\mu\nu) + \hat{O}(\nu^2)$. It follows from this fact together with (2.7) that

$$y_n(\bar{u}, \bar{\mu}, \bar{\nu}) = K + b_1\bar{u}^2 + c_1\sigma^{-n/2}\bar{u}^3 + O(\sigma^{-n}\bar{u}^4), \tag{2.9}$$

where

$$K = -1 + \bar{\nu} + \hat{O}(\sigma^{-n}), \quad b_1 = b + \hat{O}(\sigma^{-n}), \quad c_1 = c + \hat{O}(\sigma^{-n}). \tag{2.10}$$

Since $y_n(\bar{u}, \bar{\mu}, \bar{\nu})$ is a C^8 function, K, b_1, c_1 are C^8, C^6, C^5 functions of $\bar{\mu}, \bar{\nu}$ respectively.

From the form (2.9) of y_n, we may assume that, for any sufficiently large integer $n > 0$ and any $(\bar{u}, \bar{\mu}, \bar{\nu}) \in [-2\varepsilon, 2\varepsilon] \times \bar{R}$, the point $(x_n(\bar{u}, \bar{\mu}, \bar{\nu}), y_n(\bar{u}, \bar{\mu}, \bar{\nu}))$ is contained in $\mathcal{N}_{\mu,\nu}$ if necessary replacing ε by a smaller positive number. The new parameter t of the curve $\rho_{\mu,\nu}^{(n)}$ is introduced by

$$t = \bar{u}\sqrt{1 + \sigma^{-n/2}c_2\bar{u}} \tag{2.11}$$

for $\bar{u} \in [-\varepsilon, \varepsilon]$, where $c_2 = c_1/b_1$. Then, one can suppose that $\bar{u}$ is a C^5 function of $t, \bar{\mu}, \bar{\nu}$:

$$\bar{u} = \bar{u}_n(t, \bar{\mu}, \bar{\nu}).$$

By (2.11), it is not hard to show that $\bar{u}_n$ C^5 converges to the identity of t as $n \to \infty$ uniformly. In particular, the derivative $\dot{\bar{u}}_n$ (resp. $\ddot{\bar{u}}_n$) converges uniformly to 1 (resp. 0) as $n \to \infty$, where $\dot{f}$ denotes the derivative of a function f by t, that is, $\dot{f} = df/dt$. From now on, we always assume that the domain of any function of t is $[-\varepsilon, \varepsilon]$. For short, set $x_n(\bar{u}_n(t, \bar{\mu}, \bar{\nu}), \bar{\mu}, \bar{\nu}) = x_n(t, \bar{\mu}, \bar{\nu})$ and $y_n(\bar{u}_n(t, \bar{\mu}, \bar{\nu}), \bar{\mu}, \bar{\nu}) = y_n(t, \bar{\mu}, \bar{\nu})$, which are C^5 functions of $t, \bar{\mu}, \bar{\nu}$. Then, by (2.7), (2.9) and (2.11), we have

$$\begin{aligned} x_n(t, \bar{\mu}, \bar{\nu}) &= \lambda^n + \tau^n \bar{u}_n(t, \bar{\mu}, \bar{\nu}), \\ y_n(t, \bar{\mu}, \bar{\nu}) &= K + b_1 t^2 + O(\sigma^{-n}t^4), \end{aligned} \tag{2.12}$$

where the equality $O(\bar{u}_n(t,\bar{\mu},\bar{\nu})^4) = O(t^4)$ is derived from (2.11). By the first equation of (2.10) and (2.12), for any $(\bar{\mu},\bar{\nu}) \in \bar{R}$ and for all sufficiently large $n > 0$,

$$\begin{aligned} x_n &= O(\lambda^n),\ \dot{x}_n = \tau^n \dot{\bar{u}}_n = O(\tau^n), \\ y_n &= O(\varepsilon),\ \dot{y}_n = 2b_1 t + O(\sigma^{-n}t^3), \\ y_n\dot{y}_n &= 2b_1Kt + 2b_1^2t^3 + O(\sigma^{-n}t^3). \end{aligned} \tag{2.13}$$

Since $|t| \le 2\varepsilon$, the fourth equation implies that $\dot{y}_n$ is an $O(\varepsilon)$-function, but $\ddot{y}_n = 2b_1 + O(\sigma^{-n}t^2)$ is not. This implies that the t-derivative of an $O(\varepsilon)$-function is not necessarily an $O(\varepsilon)$-function. On the other hand, since σ,λ are functions of μ,ν independent of t, we have $dO(\sigma^{-n})/dt = O(\sigma^{-n})$, $dO(\lambda^n)/dt = O(\lambda^n)$ and so on.

2.3. **Differentiations with respect to the new parameters.** In this subsection, we consider derivatives of functions of μ,ν by the variables $\bar{\mu},\bar{\nu}$ which are given by $\bar{\mu} = \sigma^n\mu, \bar{\nu} = \sigma^n\nu$ as in (2.4). Note that σ is not in general a constant but a function of μ,ν. Let $F_n : (\mu,\nu) \mapsto (\bar{\mu},\bar{\nu})$ be the function defined by (2.4). Then, the differential of F_n is

$$(DF_n)_{\mu,\nu} = \begin{pmatrix} \sigma^n + n\sigma^{n-1}\partial_\mu\sigma\cdot\mu & n\sigma^{n-1}\partial_\nu\sigma\cdot\mu \\ n\sigma^{n-1}\partial_\mu\sigma\cdot\nu & \sigma^n + n\sigma^{n-1}\partial_\nu\sigma\cdot\nu \end{pmatrix}.$$

By (2.5), $(DF_n)_{\mu,\nu}$ has the inverse matrix for all sufficiently large n such that

$$(DF_n)^{-1}_{\mu,\nu} = \begin{pmatrix} O(\sigma^{-n}) & O(n\sigma^{-2n}) \\ O(n\sigma^{-2n}) & O(\sigma^{-n}) \end{pmatrix}. \tag{2.14}$$

By the Inverses Function Theorem, F_n has a local inverse C^∞ diffeomorphism defined in a neighborhood $\bar{D}_n$ of any point of $\bar{R}$ in the $\bar{\mu}\bar{\nu}$-space. Moreover, since $\sigma^{-n}(DF_n)_{\mu,\nu}$ converges uniformly to the identity matrix as $n \to \infty$, from the Covering Estimate Theorem (see for example Robinson [16, §5.2.2, Theorem 2.4 and §5.12, Exercise 5.3]), one can take the neighborhood $\bar{D}_n$ so that $\bar{D}_n \supset \bar{R}$ for all sufficiently large n.

For $\bar{\delta} = \bar{\mu},\bar{\nu}$,

$$\partial_{\bar{\delta}}\sigma^{-n} = -n\sigma^{-n-1}\left(\partial_\mu\sigma\cdot\frac{\partial\mu}{\partial\bar{\delta}} + \partial_\nu\sigma\cdot\frac{\partial\nu}{\partial\bar{\delta}}\right).$$

Then, by (2.14), we have

$$\partial_{\bar{\delta}}\sigma^{-n} = O(n\sigma^{-n-1}\cdot\sigma^{-n}) = O(n\sigma^{-2n}) = O(\sigma^{-n}). \tag{2.15}$$

Here, the equality $O(n\sigma^{-2n}) = O(\sigma^{-n})$ means that any $O(n\sigma^{-2n})$-function is an $O(\sigma^{-n})$-function, but it does not necessarily imply that the inverse holds. Similarly,

$$\partial_{\bar{\delta}}\tau^n = O(n\tau^{n-1}\cdot\sigma^{-n}) = O(\tau^n). \tag{2.16}$$

Moreover, by using (2.11) and (2.15), one can show that both $\partial_{\bar{\delta}}\bar{u}_n$ and $\partial_{\bar{\delta}}\dot{\bar{u}}_n$ converge uniformly to the zero function as $n \to \infty$.

2.4. **The Taylor expansion of holding functions.** By the condition (i) of Theorem A, there exists a continuation $\gamma_{\mu,\nu}$ of curves in $W^u(p^-_{\mu,\nu})$ such that $\gamma = \gamma_{0,0}$ and $W^s(p^+)$ have a quadratic tangency q^+ unfolding generically with respect to $\{\psi_{\mu,0}\}$. Let $\tilde{\mathcal{N}}_{\mu,\nu} = \tilde{\mathcal{N}}_{\mu,\nu}(\tilde{x},\tilde{y})$ be a C^∞ coordinate of a neighborhood of q^+ C^∞ depending on (μ,ν) such that $q^+ = (0,0)$ in $\tilde{\mathcal{N}}_{0,0}$ and the $\tilde{x}$-axis in $\tilde{\mathcal{N}}_{\mu,\nu}$ is contained in $W^s(p^+_{\mu,\nu})$ for any (μ,ν) near $(0,0)$. Note that we have not assumed that $p^+_{\mu,\nu}$ satisfies the Sternberg-Takens condition, and hence $W^u_{\mathrm{loc}}(p^+_{\mu,\nu})$ is not necessarily represented as a straight segment in $\tilde{\mathcal{N}}_{\mu,\nu}$, possibly $W^u_{\rm loc}(p^+_{\mu,\nu}) \cap \tilde{\mathcal{N}}_{\mu,\nu} = \emptyset$. Since the y-axis in $\mathcal{N}_{\mu,\nu}$ is contained in $W^u(p^-_{\mu,\nu})$, $\gamma_{\mu,\nu}$ is a regular curve parametrized by $\psi^N_{\mu,\nu}(0,y)$ for y near 0. Denote the $\tilde{x}$ and $\tilde{y}$-entries of the coordinate of $\psi^N_{\mu,\nu}(x,y)$ by $\tilde{x}(x,y,\mu,\nu)$ and $\tilde{y}(x,y,\mu,\nu)$ respectively. Since $\tilde{y}(0,y,0,0)$ is a C^8 function with $\tilde{y}(0,0,0,0) = \partial_y\tilde{y}(0,0,0,0) = 0$ and $\partial_{yy}\tilde{y}(0,0,0,0) \neq 0$, there exists a C^6 function $k(y)$ satisfying $y^2k(y) = \tilde{y}(0,y,0,0)$ and $k(y) \neq 0$ for any y near 0. The correspondence $y \mapsto \tilde{x}(0,y,0,0)$ has the C^8 inverse $\tilde{x} \mapsto y = y(\tilde{x})$ if $|y|$ is sufficiently small. For any (μ,ν) near $(0,0)$, we consider the C^6 coordinate change of a neighborhood of the origin in $\mathbb{R}^2$ defined by

$$(\tilde{x},\tilde{y}) \longmapsto \left(\tilde{x}, \frac{\tilde{y}}{k(y(\tilde{x}))}\right).$$

Note that this coordinate change is independent of (μ,ν). For saving symbols, we denote the new C^6 coordinate again by $(\tilde{x},\tilde{y})$. In particular, when $(\mu,\nu) = (0,0)$,

$$\tilde{y}(0,y,0,0) = y^2 \tag{2.17}$$

with respect to the new coordinate.

Now, we will consider the Taylor expansion of $\tilde{y}(x,y,\mu,\nu)$ given as

$$\tilde{y}(x,y,\mu,\nu) = \mu + y^2 + \tilde{a}x + \tilde{h}_1(\mu,\nu) + \tilde{h}_2(y,\mu,\nu) + \tilde{h}_3(x,y,\mu,\nu). \tag{2.18}$$

Since q^+ unfolds generically with respect to $\{\psi_{\mu,0}\}$, the coefficient of the μ-term is non-zero. Here, we adjusted the μ-parameter linearly so that the coefficient is one. The absence of a ν-term with non-zero constant coefficient is derived from the condition (iii) of Theorem A. Let $\tilde{h}_1$, $\tilde{h}_2$, $\tilde{h}_3$ be the C^6 functions defined as

$$\begin{aligned}\tilde{h}_1(\mu,\nu) &= \tilde{h}(0,0,\mu,\nu),\\ \tilde{h}_2(y,\mu,\nu) &= \tilde{h}(0,y,\mu,\nu) - \tilde{h}_1(\mu,\nu),\\ \tilde{h}_3(x,y,\mu,\nu) &= \tilde{h}(x,y,\mu,\nu) - \tilde{h}_1(\mu,\nu) - \tilde{h}_2(y,\mu,\nu),\end{aligned}$$

where $\tilde{h}(x,y,\mu,\nu) = \tilde{y}(x,y,\mu,\nu) - (\mu + y^2 + \tilde{a}x)$. These functions satisfy the following conditions.

(c.i) $\tilde{h}_1(0,0) = \partial_\mu\tilde{h}_1(0,0) = \partial_\nu\tilde{h}_1(0,0) = 0$.
(c.ii) $\tilde{h}_2(y,0,0) = 0$ for any y near 0, and $\tilde{h}_2(0,\mu,\nu) = 0$ for any (μ,ν) near $(0,0)$.
(c.iii) $\partial_x\tilde{h}_3(0,0,0,0) = 0$, and $\tilde{h}_3(0,y,\mu,\nu) = 0$ for any (y,μ,ν) near $(0,0,0)$.

Since $\tilde{h}_1$ does not contain any constant term or μ, ν-terms with non-zero constant coefficient, the condition (c.i) holds. The first condition of (c.ii) is derived from (2.17). The first condition of (c.iii) is derived from the fact that $\tilde{h}_3$ does not contain the x-term with non-zero constant coefficient.

By (2.18) and (c.ii), (c.iii), $\partial_x\tilde{y}(0,0,0,0) = \tilde{a}$ and $\partial_y\tilde{y}(0,0,0,0) = 0$. Since $\psi^N_{0,0}$ is a diffeomorphism, $\tilde{a}$ must be non-zero. If necessary reflecting the coordinates on $\tilde{\mathcal{N}}_{\mu,\nu}$ along the $\tilde{y}$-axis, we may assume that

$$\tilde{a} > 0.$$

We need to estimate the t-derivatives of $\tilde{y}(x,y,\mu,\nu)$ with $x = x_n(t,\mu,\nu), y = y_n(t,\mu,\nu)$, $\mu = \sigma^{-n}\bar{\mu}$, $\nu = \sigma^{-n}\bar{\nu}$ for $(\bar{\mu},\bar{\nu}) \in \bar{R}$ up to third order. Since $x_n(t,\mu,\nu)$, $y_n(t,\mu,\nu)$ are C^5 functions, the $\tilde{y}$ is also a C^5 function of t,μ,ν.

By the first condition of (c.ii), we have $\partial_y\tilde{h}_2(y,0,0) = 0$ for any y near 0. This fact together with (2.5) shows that $\partial_y\tilde{h}_2(y,\mu,\nu) = O(\sigma^{-n})$. Since $\dot{y}|_{t=0} = 0$ by

(2.13), $d(\partial_y \tilde{h}_2)/dt|_{t=0} = (\partial_{yy}\tilde{h}_2)\dot{y}|_{t=0} = 0$ and hence $\partial_y \tilde{h}_2$ does not have a t-term with non-zero constant coefficient. This shows that $\partial_y \tilde{h}_2 = \hat{O}(\sigma^{-n}) + O(\sigma^{-n}t^2)$. Since $\dot{\tilde{h}}_2(y,\mu,\nu) = \partial_y \tilde{h}_2(y,\mu,\nu)\dot{y}_n$, we have again by (2.13)

$$\begin{aligned} \dot{\tilde{h}}_2(y,\mu,\nu) &= \hat{O}(\sigma^{-n})t + O(\sigma^{-n}t^3) \\ &= \hat{O}(\sigma^{-n})t + \hat{O}(\sigma^{-n})t^3 + O(\sigma^{-n}t^4). \end{aligned} \tag{2.19}$$

Differentiating $\tilde{h}_3$ by t, we have

$$\dot{\tilde{h}}_3(x,y,\mu,\nu) = \partial_x \tilde{h}_3(x,y,\mu,\nu)\dot{x}_n + \partial_y \tilde{h}_3(x,y,\mu,\nu)\dot{y}_n.$$

From the first condition of (c.iii) together with (2.13),

$$\begin{aligned} \partial_x \tilde{h}_3(x,y,\mu,\nu)\dot{x}_n &= (O(x) + O(y) + O(\mu) + O(\nu))\dot{x}_n \\ &= (O(\lambda^n) + O(\varepsilon) + O(\sigma^{-n}) + O(\sigma^{-n})) \cdot O(\tau^n) \\ &= O(\varepsilon\tau^n). \end{aligned}$$

From the second condition of (c.iii), $\partial_y \tilde{h}_3(0,y,\mu,\nu) = 0$. Hence, $\partial_y \tilde{h}_3(x,y,\mu,\nu) = O(x_n) = O(\lambda^n)$. Since moreover

$$\left.\frac{d(\partial_y \tilde{h}_3)}{dt}\right|_{t=0} = (\partial_{yx}\tilde{h}_3)\dot{x}|_{t=0} + (\partial_{yy}\tilde{h}_3)\dot{y}|_{t=0} = (\partial_{yx}\tilde{h}_3)\dot{x}|_{t=0} = \hat{O}(\tau^n),$$

it follows that $\partial_y \tilde{h}_3(x,y,\mu,\nu) = \hat{O}(\lambda^n) + \hat{O}(\tau^n)t + O(\lambda^n t^2)$. Thus, we have

$$\partial_y \tilde{h}_3(x,y,\mu,\nu)\dot{y}_n = \hat{O}(\lambda^n)t + \hat{O}(\tau^n)t^2 + O(\lambda^n t^3).$$

Since $\hat{O}(\tau^n)t^2 = O(\varepsilon\tau^n)$,

$$\begin{aligned} \dot{\tilde{h}}_3(x,y,\mu,\nu) &= \hat{O}(\lambda^n)t + O(\lambda^n t^3) + O(\varepsilon\tau^n) \\ &= \hat{O}(\lambda^n)t + \hat{O}(\lambda^n)t^3 + O(\lambda^n t^4) + O(\varepsilon\tau^n). \end{aligned} \tag{2.20}$$

Then, by (2.13), (2.18), (2.19), (2.20) together with $d\tilde{h}_1(\mu,\nu)/dt = 0$,

$$\begin{aligned} \dot{\tilde{y}} &= \tilde{a}\tau^n \ddot{u}_n + K_1 t + Lt^3 + O(\sigma^{-n}t^4) + O(\varepsilon\tau^n), \\ \ddot{\tilde{y}} &= K_1 + 3Lt^2 + O(\sigma^{-n}t^3) + O(\tau^n), \\ \dddot{\tilde{y}} &= 6Lt + O(\sigma^{-n}t^2) + O(\tau^n), \end{aligned} \tag{2.21}$$

where

$$\begin{aligned} K_1 &= 4b_1 K + \hat{O}(\sigma^{-n}) = 4b(\bar{\nu} - 1) + \hat{O}(\sigma^{-n}), \\ L &= 4b_1^2 + \hat{O}(\lambda^n) + \hat{O}(\sigma^{-n}) = 4b^2 + \hat{O}(\sigma^{-n}). \end{aligned} \tag{2.22}$$

Here, we used the facts that $\dot{K}_1 = \dot{L} = 0$ and $\dot{O(\varepsilon\tau^n)} = O(\tau^n)$. Note that (2.3) implies $L > 0$ for all sufficiently large n. Since $\dot{\tilde{y}}$ is a C^4 function, K_1 and L are C^3 and C^1 functions of μ, ν respectively.

2.5. **Proof of Theorem A.** We will give the proof of Theorem A under the notation and estimates as in Subsection 2.4. Define the C^5 function $\tilde{y}_{n;\bar{\mu},\bar{\nu}}(t)$ by

$$\tilde{y}_{n;\bar{\mu},\bar{\nu}}(t) = \tilde{y}\big(x_n(t,\bar{\mu},\bar{\nu}), y_n(t,\bar{\mu},\bar{\nu}), \sigma^{-n}\bar{\mu}, \sigma^{-n}\bar{\nu}\big).$$

We often write $\tilde{y}_{n;\bar{\mu},\bar{\nu}}(t) = \tilde{y}_n(t)$ for short if the parameters $\bar{\mu},\bar{\nu}$ are understood explicitly.

Throughout the remainder of this section, for non-zero sequences a_n, b_n, we denote the property $\lim_{n\to\infty} a_n/b_n = 1$ by $a_n \approx b_n$. When $a_n = a_n(t), b_n = b_n(t)$ are functions, $a_n \approx b_n$ means that $a_n(t)/b_n(t)$ converges uniformly to 1 as $n \to \infty$. We note that, for smooth functions $a_n(t), b_n(t)$, $a_n \approx b_n$ does not necessarily imply $da_n/dt \approx db_n/dt$.

By Lemma 1.3, the following assertion implies Theorem A.

Assertion 2.1. *For any sufficiently large $n > 0$, the following* (i)-(iii) *hold.*

(i) *For any $\bar{\mu} \in [-1,1]$, there exists an open interval $J_{n,\bar{\mu}}$ in the $\bar{\nu}$-parameter space such that $\mathcal{J}_n = \bigcup_{\bar{\mu}\in[-1,1]}\{\bar{\mu}\} \times J_{n,\bar{\mu}}$ is an open subset of $\bar{R}$, and there exists a C^3 function $t_{\bar{\mu},\bar{\nu}}$ (or more strictly $t_{n;\bar{\mu},\bar{\nu}}$) on $\mathcal{J}_n$ with $\ddot{\tilde{y}}_n(t_{\bar{\mu},\bar{\nu}}) = 0$.*

(ii) *There is a C^3 function $\bar{\nu}_n : [-1,1] \longrightarrow \mathbb{R}$ of $\bar{\mu}$ with $(\bar{\mu}, \bar{\nu}_n(\bar{\mu})) \in \bar{R}$ and satisfying $\dot{\tilde{y}}_n(t_{\bar{\mu},\bar{\nu}_n(\bar{\mu})}) = 0$, $\dddot{\tilde{y}}_n(t_{\bar{\mu},\bar{\nu}_n(\bar{\mu})}) \neq 0$ and $(\partial_{\bar{\nu}}\dot{\tilde{y}}_n)(t_{\bar{\mu},\bar{\nu}_n(\bar{\mu})}) \neq 0$ for any $\bar{\mu} \in [-1,1]$.*

(iii) *$\tilde{y}_n(t_{\bar{\mu}_0,\bar{\nu}_n(\bar{\mu}_0)}) = 0$ for some $\bar{\mu}_0 \in [-1,1]$, and $(\partial_{\bar{\mu}}\tilde{y}_{n;\bar{\mu},\bar{\nu}_n(\bar{\mu})})(t_{\bar{\mu},\bar{\nu}_n(\bar{\mu})}) \neq 0$ for any $\bar{\mu} \in [-1,1]$.*

In particular, (ii) and (iii) imply that

$$(\partial_{\bar{\nu}}\dot{\tilde{y}}_{n;\bar{\mu}_0,\bar{\nu}})(t_{\bar{\mu}_0,\bar{\nu}_0})|_{\bar{\nu}=\bar{\nu}_0} \neq 0 \quad \text{and} \quad (\partial_{\bar{\mu}}\tilde{y}_{n;\bar{\mu},\bar{\nu}_n(\bar{\mu})})(t_{\bar{\mu}_0,\bar{\nu}_0})|_{\bar{\mu}=\bar{\mu}_0} \neq 0, \tag{2.23}$$

where $\bar{\nu}_0 = \bar{\nu}_n(\bar{\mu}_0)$. We reparametrize $(\bar{\mu},\bar{\nu})$ (resp. t) by the parallel translation $(\bar{\mu},\bar{\nu}) \mapsto (\bar{\mu}-\bar{\mu}_0, \bar{\nu}-\bar{\nu}_0)$ (resp. $t \mapsto t - t_{\bar{\mu}_0,\bar{\mu}_0}$) and apply Lemma 1.3, where (2.23) corresponds to the condition (iv) of Lemma 1.3.

Proof. **(i)** By (2.22), for any $\bar{\mu} \in [-1,1]$, one can choose $\bar{\nu}$ with $(\bar{\mu},\bar{\nu}) \in \bar{R}$ so that K_1 takes an arbitrarily given value in $[-2|b|\varepsilon, 2|b|\varepsilon]$ for all sufficiently large n. Take positive constants η_1, η_2 independent of n with $\eta_1 < \eta_2$, which will be chosen suitably later. When $\eta_2\tau^{2n/3} \leq 2|b|\varepsilon$, for any $s \in [\eta_1,\eta_2]$ and $\bar{\mu} \in [-1,1]$, there exists $\bar{\nu}$ with $(\bar{\mu},\bar{\nu}) \in \bar{R}$ satisfying

$$K_1 = -s\tau^{2n/3}.$$

Since $\partial_{\bar{\nu}}K_1 \approx 4b \neq 0$ by (2.22) and $\partial_{\bar{\nu}}\tau^{2n/3} = O(\tau^{2n/3})$ by (2.16), $\bar{\nu}$ is a uniquely determined from and C^3 depending on $\bar{\mu}, s$. In fact, for any fixed $\bar{\mu} \in [-1,1]$, $\bar{\nu}$ and s are mutually related as

$$4b(\bar{\nu}-1) + \hat{O}(\sigma^{-n}) = -s\tau^{2n/3}. \tag{2.24}$$

The second equation of (2.21) implies that the graph of the C^3 function $u = \ddot{\tilde{y}}_n(t)$ is an 'almost quadratic' C^3 curve in the tu-plane meeting the t-axis transversely in two points one of which has a positive t-coordinate and the other negative. Thus, the there exists a unique $t_{\bar{\mu};s} > 0$ C^3 depending on $\bar{\mu}, s$ (and hence on $\bar{\mu},\bar{\nu}$) with $\ddot{\tilde{y}}_n(t_{\bar{\mu};s}) = 0$ and satisfying

$$t_{\bar{\mu},\bar{\nu}} := t_{\bar{\mu};s} \approx \Big(\frac{s}{3L}\Big)^{1/2}\tau^{n/3}. \tag{2.25}$$

Let $J_{n,\bar\mu}$ be the open interval in the $\bar\nu$-space corresponding to (η_1, η_2). Since $\bar\nu - 1$ satisfying (2.24) converges uniformly to zero as $n \to \infty$ for any $s \in (\eta_1, \eta_2)$, the union $\mathcal{J}_n = \bigcup_{\bar\mu \in [-1,1]} \{\bar\mu\} \times J_{n,\bar\mu}$ is an open subset of $\bar R$ for all sufficiently large n. This shows (i).

(ii) Since $\dot{\tilde u}_n \approx 1$ uniformly, we may assume that $\dot{\tilde u}_n(t_{\bar\mu;s}) - 1 = O(\varepsilon)$ and hence

$$\begin{aligned}\dot{\tilde y}_n(t_{\bar\mu;s}) &= \tilde a \tau^n \dot{\tilde u}_n(t_{\bar\mu;s}) + K_1 t_{\bar\mu;s} + L t_{\bar\mu;s}^3 + O(\sigma^{-n} t_{\bar\mu;s}^4) + O(\varepsilon\tau^n)\\ &\approx (\tilde a + O(\varepsilon))\tau^n - s\Big(\frac{s}{3L}\Big)^{1/2}\tau^n + L\Big(\frac{s}{3L}\Big)^{3/2}\tau^n\\ &= \Big(\tilde a + O(\varepsilon) - \frac{2}{3\sqrt{3L}} s^{3/2}\Big)\tau^n.\end{aligned}$$

Since $\tilde a$ is a positive constant, one can choose the constants $\varepsilon, \eta_1, \eta_2$ so that the above function $O(\varepsilon)$ of $(t, \bar\mu, \bar\nu) \in [-\varepsilon, \varepsilon] \times \bar R$ satisfies $\sup\{|O(\varepsilon)|\} \le \tilde a/10$ and, for any $\bar\mu \in [-1, 1]$, there exists an $s = s_n(\bar\mu) \in (\eta_1, \eta_2)$ with $\dot{\tilde y}_n(t_{\bar\mu;s_n(\bar\mu)}) = 0$. Let $\bar\nu_n(\bar\mu) \in J_{n,\bar\mu}$ be the element of the $\bar\nu$-space corresponding to $s_n(\bar\mu)$.

We need to prove that the $\bar\nu_n(\bar\mu)$ is a uniquely determined smooth function. By (2.15) and (2.22),

$$\partial_{\bar\mu} K_1 = \hat O(\sigma^{-n}),\ \partial_{\bar\nu} K_1 = 4b + \hat O(\sigma^{-n}),\ \partial_{\bar\mu} L = \hat O(\sigma^{-n}),\ \partial_{\bar\nu} L = \hat O(\sigma^{-n}).$$

Here, we used the fact that L is a C^1 function, which was shown above under the assumption that p^- satisfies the Sternberg-Takens C^8 condition. The authors do not know whether L is differentiable under the condition for p^- weaker than C^8.

It follows from (2.21) and (2.16)

$$\begin{aligned}\partial_{\bar\mu}\dot{\tilde y}_n(t) &= O(\tau^n) + \hat O(\sigma^{-n})t + \hat O(\sigma^{-n})t^3 + O(\sigma^{-n}t^4) + O(\tau^n)\\ &= O(\sigma^{-n})t + O(\tau^n),\\ \partial_{\bar\nu}\dot{\tilde y}_n(t) &= (4b + O(\sigma^{-n}))t + O(\tau^n).\end{aligned}$$

Then, for all sufficiently large n and any $(\bar\mu, \bar\nu) \in \mathcal{J}_n$,

$$\begin{aligned}\partial_{\bar\nu}(\dot{\tilde y}_n(t_{\bar\mu,\bar\nu})) &= (\partial_{\bar\nu}\dot{\tilde y}_n)(t_{\bar\mu,\bar\nu}) + \ddot{\tilde y}_n(t_{\bar\mu,\bar\nu}) \cdot \partial_{\bar\nu} t_{\bar\mu,\bar\nu}\\ &= (\partial_{\bar\nu}\dot{\tilde y}_n)(t_{\bar\mu,\bar\nu}) \approx 4b t_{\bar\mu,\bar\nu} \ne 0,\end{aligned}$$

where the last approximation is induced from $t_{\bar\mu,\bar\nu} = O(\tau^{n/3})$ in (2.25). This shows that the value $\bar\nu_n(\bar\mu) \in J_{n,\bar\mu}$ with $\dot{\tilde y}_n(t_{\bar\mu,\bar\nu_n(\bar\mu)}) = 0$ is uniquely determined and C^3 depends on $\bar\mu \in [-1, 1]$, and moreover $(\partial_{\bar\nu}\dot{\tilde y}_n)(t_{\bar\mu,\bar\nu}) \ne 0$. Similarly,

$$\partial_{\bar\mu}(\dot{\tilde y}_n(t_{\bar\mu,\bar\nu})) = (\partial_{\bar\mu}\dot{\tilde y}_n)(t_{\bar\mu,\bar\nu}) = O(\sigma^{-n})t_{\bar\mu,\bar\nu} + O(\tau^n).$$

Again by $t_{\bar\mu,\bar\nu} = O(\tau^{n/3})$,

$$\frac{d\bar\nu_n}{d\bar\mu}(\bar\mu) = -\frac{\partial_{\bar\mu}(\dot{\tilde y}_n(t_{\bar\mu,\bar\nu}))}{\partial_{\bar\nu}(\dot{\tilde y}_n(t_{\bar\mu,\bar\nu}))} = O(\sigma^{-n}) + O(\tau^{2n/3}) = O(\sigma^{-n}), \tag{2.26}$$

where we used the fact that $\tau^{2/3} = \lambda^{2/3}\sigma^{-1/3} < \sigma^{-2/3}\sigma^{-1/3} = \sigma^{-1}$. By using the third equality of (2.21), it is not hard to show that $\dddot{\tilde y}_n(t_{\bar\mu,\bar\nu}) \ne 0$ for all sufficiently large n and any $(\bar\mu, \bar\nu) \in \mathcal{J}_n$. This completes the proof of (ii).

(iii) By the property (c.i) of $\tilde h_1$, $\tilde h_1(\bar\mu, \bar\nu) = O(\sigma^{-2n})$. The property (c.ii) implies that $\tilde h_2$ does not contain any y^m $(m = 1, 2, \dots)$, μ or ν-terms with non-zero constant coefficients. It follows that $\tilde h_2(y_n(t), \bar\mu, \bar\nu_n(\bar\mu)) = O(\sigma^{-n})y_n(t) + O(\sigma^{-2n})$. From (c.iii) and the first equation of (2.12), we have $\tilde{h}_3(x_n, y_n, \mu, \nu) = O(x_n) = O(\lambda^n)$. By (2.18),

$$\tilde{y}_{n;\bar{\mu},\bar{\nu}_n(\bar{\mu})}(t) = \sigma^{-n}\bar{\mu} + y_n^2(t) + O(\sigma^{-n})y_n(t) + O(\sigma^{-2n}) + O(\lambda^n).$$

By (2.22), $K = K_1/(4b_1) + \hat{O}(\sigma^{-n}) = \hat{O}(\tau^{2n/3}) + \hat{O}(\sigma^{-n}) = \hat{O}(\sigma^{-n})$. Then, from (2.12), we have $y_{n;\bar{\mu},\bar{\nu}_n(\bar{\mu})}(t_{\bar{\mu},\bar{\nu}_n(\bar{\mu})}) = O(\sigma^{-n})$. It follows that

$$\tilde{y}_{n;\bar{\mu},\bar{\nu}_n(\bar{\mu})}(t_{\bar{\mu},\bar{\nu}_n(\bar{\mu})}) = \sigma^{-n}\bar{\mu} + O(\sigma^{-2n}) + O(\lambda^n) = \sigma^{-n}\bar{\mu} + o(\sigma^{-n}).$$

By applying the Intermediate Value Theorem to the continuous function $\tilde{y}_n(t_{\bar{\mu},\bar{\nu}_n(\bar{\mu})})$ for $\bar{\mu} \in [-1, 1]$, there exists $\bar{\mu}_0 \in [-1, 1]$ such that $\tilde{y}_n(t_{\bar{\mu}_0,\bar{\nu}_n(\bar{\mu}_0)}) = 0$.

It remains to show that $(\partial_{\bar{\mu}}\tilde{y}_{n;\bar{\mu},\bar{\nu}_n(\bar{\mu})})(t_{\bar{\mu},\bar{\nu}_n(\bar{\mu})}) \neq 0$. Here, we should remark that $t_{\bar{\mu},\bar{\nu}} = O(\tau^{n/3})$ does not necessarily imply that $\partial_{\bar{\mu}} t_{\bar{\mu},\bar{\nu}} = O(\tau^{n/3})$ since $t_{\bar{\mu},\bar{\nu}}$ is not in general a function with $\tau^{n/3}$ as a factor. So, we will give the proof without invoking the estimate of $\partial_{\bar{\mu}} t_{\bar{\mu},\bar{\nu}}$. By (2.15) and (2.16),

$$\begin{aligned}
\partial_{\bar{\mu}}\tilde{y}_{n;\bar{\mu},\bar{\nu}}(t) &= \sigma^{-n} + O(n\sigma^{-2n})\bar{\mu} + 2y_n(t)\partial_{\bar{\mu}} y_n(t) \\
&\qquad + \partial_{\bar{\mu}}\big[O(\sigma^{-n})y_n(t) + O(\sigma^{-2n}) + O(\lambda^n)\big], \\
\partial_{\bar{\nu}}\tilde{y}_{n;\bar{\mu},\bar{\nu}}(t) &= O(n\sigma^{-2n})\bar{\mu} + 2y_n(t)\partial_{\bar{\nu}} y_n(t) \\
&\qquad + \partial_{\bar{\nu}}\big[O(\sigma^{-n})y_n(t) + O(\sigma^{-2n}) + O(\lambda^n)\big].
\end{aligned}$$

Since $y_n(t) = O(\varepsilon)$ by (2.13) and since $\partial_{\bar{\mu}} y_n(t) = \hat{O}(\sigma^{-n}) + O(\sigma^{-n}t^4) = O(\sigma^{-n})$ and $\partial_{\bar{\nu}} y_n(t) = O(1)$ by (2.10) and (2.12), it follows that

$$\begin{aligned}
\partial_{\bar{\mu}}\tilde{y}_{n;\bar{\mu},\bar{\nu}}(t) &= (1 + O(\varepsilon))\sigma^{-n} + O(n\sigma^{-2n})\bar{\mu} + O(\sigma^{-2n}) + O(\lambda^n) \\
&= (1 + O(\varepsilon))\sigma^{-n} + o(\sigma^{-n}), \\
\partial_{\bar{\nu}}\tilde{y}_{n;\bar{\mu},\bar{\nu}}(t) &= O(y_n(t)) + O(\sigma^{-n}).
\end{aligned}$$

By the last equality together with $y_n(t_{\bar{\mu},\bar{\nu}_n(\bar{\mu})}) = O(\sigma^{-n})$,

$$(\partial_{\bar{\nu}}\tilde{y}_{n;\bar{\mu},\bar{\nu}})(t_{\bar{\mu},\bar{\nu}_n(\bar{\mu})})|_{\bar{\nu}=\bar{\nu}_n(\bar{\mu})} = O(\sigma^{-n}).$$

Then, by (2.26),

$$\begin{aligned}
&(\partial_{\bar{\mu}}\tilde{y}_{n;\bar{\mu},\bar{\nu}_n(\bar{\mu})})(t_{\bar{\mu},\bar{\nu}_n(\bar{\mu})}) \\
&\quad = (\partial_{\bar{\mu}}\tilde{y}_{n;\bar{\mu},\bar{\nu}})(t_{\bar{\mu},\bar{\nu}_n(\bar{\mu})})|_{\bar{\nu}=\bar{\nu}_n(\bar{\mu})} + (\partial_{\bar{\nu}}\tilde{y}_{n;\bar{\mu},\bar{\nu}})(t_{\bar{\mu},\bar{\nu}_n(\bar{\mu})})|_{\bar{\nu}=\bar{\nu}_n(\bar{\mu})} \cdot \frac{d\bar{\nu}_n}{d\bar{\mu}}(\bar{\mu}) \\
&\quad = (1 + O(\varepsilon))\sigma^{-n} + o(\sigma^{-n}) + O(\sigma^{-n}) \cdot O(\sigma^{-n}) > 0
\end{aligned}$$

for all sufficiently large n and any $\bar{\mu} \in [-1, 1]$. This completes the proof of (iii) and hence that of Theorem A. □

3. Generic cubic tangencies in the Hénon family

In this section, we give the proof of Theorem B.

3.1. **Saddle fixed points of Hénon maps.** Let $\varphi_{a,b}$ be the Hénon map given in Introduction such that

$$\varphi_{a,b}(x, y) = (y, a - bx + y^2).$$

For any element (a, b) of a small neighborhood of $(-2, 0)$ in the parameter space, $\varphi_{a,b}$ has the two fixed points $p^{\pm}_{a,b}$ with

$$p^{\pm}_{a,b} = (y^{\pm}_{a,b}, y^{\pm}_{a,b}), \quad \text{where} \quad y^{\pm}_{a,b} = \frac{1 + b \pm \sqrt{(1+b)^2 - 4a}}{2}. \tag{3.1}$$

Then, the eigenvalues of the differential $(D\varphi_{a,b})_{p^{\pm}_{a,b}}$ are

$$\lambda^{\pm}_{a,b} = y^{\pm}_{a,b} \mp \sqrt{\left(y^{\pm}_{a,b}\right)^2 - b}, \ \sigma^{\pm}_{a,b} = y^{\pm}_{a,b} \pm \sqrt{\left(y^{\pm}_{a,b}\right)^2 - b}. \tag{3.2}$$

Since $(\lambda^{+}_{a,b}, \sigma^{+}_{a,b}) \to (0, 4)$ and $(\lambda^{-}_{a,b}, \sigma^{-}_{a,b}) \to (0, -2)$ as $(a, b) \to (-2, 0)$, for any (a, b) near $(-2, 0)$ with $b \neq 0$, the eigenvalues satisfy

$$0 < |\lambda^{\pm}_{a,b}| < 1 < |\sigma^{\pm}_{a,b}| \quad \text{and} \quad |\lambda^{\pm}_{a,b}\sigma^{\pm}_{a,b}| < 1.$$

Thus, both the saddle fixed points $p^{\pm}_{a,b}$ are dissipative.

3.2. **Outline of proof of Theorem B.** Throughout the remainder of this paper, $A \sim B$ (resp. $\boldsymbol{v} \sim \boldsymbol{w}$) for two real numbers (resp. vectors) means that one can suppose that $|A - B| < \varepsilon$ (resp. $\|\boldsymbol{v} - \boldsymbol{w}\| < \varepsilon$) for any given $\varepsilon > 0$. We note that it does not necessarily imply that A/B is close to 1, e.g. $A = \varepsilon/10$ and $B = \varepsilon/1000$.

By using some results in Kiriki-Li-Soma [11, §2], we have a C^∞ function $h : I_\varepsilon = (-\varepsilon, \varepsilon) \longrightarrow \mathbb{R}$ with $h(0) = -2$ and such that $\varphi_{h(b),b}$ admits a heteroclinic quadratic tangency q^{+}_{b} associated with $p^{\pm}_{h(b),b}$ and contained in a small neighborhood $V(-2, 2)$ of $(-2, 2) \in \mathbb{R}^2$. One can also prove that the tangency q^{+}_{b} C^∞ varies with respect to b and unfolds generically with respect to the a-parameter family $\{\varphi_{a,b(\text{fixed})}\}$. Moreover, there exists a locally finite set B of $I_\varepsilon \setminus \{0\}$ such that, for any $b \in I_\varepsilon \setminus (B \cup \{0\})$, the fixed point $p^{-}_{h(b),b}$ (resp. $p^{+}_{h(b),b}$) satisfies the Sternberg-Takens C^8 (resp. C^4) condition. Then, there exists $b_0 > 0$ arbitrarily near 0 such that $\varphi_{h(b_0),b_0}$ admits a heteroclinic quadratic tangency $q^{-}_{b_0}$ in $V(-2, 2)$ associated to $p^{\pm}_{h(b_0),b_0}$ such that $q^{+}_{b_0}, q^{-}_{b_0}$ are cyclically associated with $p^{\pm}_{h(b_0),b_0}$. The situation in the present case is illustrated in Figure 3.1, which corresponds to the general situation in Figure 2.1. When $b_0 \notin B$, these tangencies are our desired ones. In the exceptional case of $b_0 \in B$, we will need some more arguments, see the proof of Lemma 3.1 for details. It remains now to show that $q^{-}_{b_0}$ unfolds generically with respect to the b-parameter family $\{\varphi_{h(b),b}\}$. The authors do not know effective approximations of the velocity vectors $\boldsymbol{v}^{u,\perp}_{b}(q^{-}_{b_0}), \boldsymbol{v}^{s,\perp}_{b}(q^{-}_{b_0})$ defined as in Subsection 1.1. However, as will be seen in Subsection 3.4, it is possible to approximate the difference so that

$$\boldsymbol{v}^{u,\perp}_{b}(q^{-}_{b_0}) - \boldsymbol{v}^{s,\perp}_{b}(q^{-}_{b_0}) \sim (0, -6\sqrt{2}).$$

From this fact together with (1.4), we know that $q^{-}_{b_0}$ unfolds generically. Then, the proof of Theorem B is completed by taking the new parameter (μ, ν) with $\mu = a - h(b), \nu = b - b_0$ and applying Theorem A.

3.3. **Existence of pairs of generic unfolding quadratic tangencies.** Now, we recall some arguments and results in [11, §2] which are needed to prove Theorem B.

When $b = 0$, $\varphi_{a,0}$ is not a diffeomorphism. Even in this case, one can define the stable and unstable manifolds associated with $p^{+}_{a,0}$ in a usual manner. The stable manifold $W^s(p^{+}_{a,0})$ of $\varphi_{a,0}$ contains the horizontal segment $S^{+}_{a,0} = \{(x, y_{a,0}); |x| \leq$

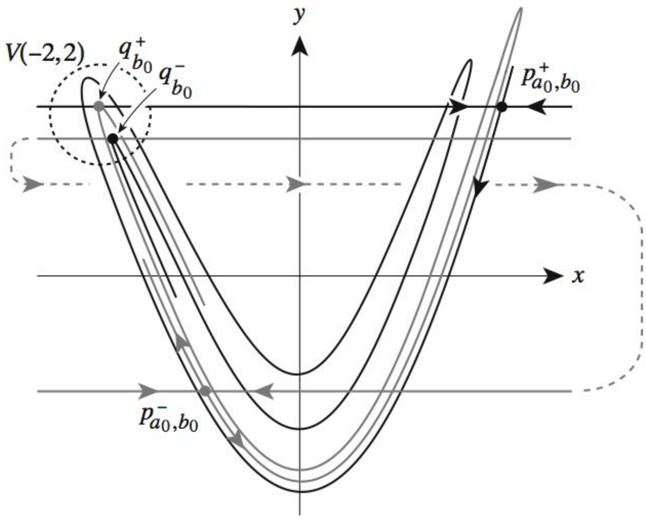

FIGURE 3.1

$5/2\}$ passing through $p^+_{a,0}$. By the Stable Manifold Theorem (see e.g. Robinson [16, Chapter 5, Theorem 10.1]), for any (a,b) near $(-2,0)$, there exists an almost horizontal segment $S^+_{a,b} \subset W^s(p^+_{a,b})$ containing $p^+_{a,b}$ which C^∞ depends on (a,b) and such that one of the end points of $S^+_{a,b}$ is in the vertical line $x=-5/2$ and the other in $x=5/2$. In particular, each $S^+_{a,b}$ has a holding function $\eta^+_{a,b}$, that is,

$$S^+_{a,b} = \{(x, \eta^+_{a,b}(x)); |x| \le 5/2\}.$$

Note that $\eta^+_{a,b}$ is a C^∞ function C^∞ depending on (a,b), and the family $\{\eta^+_{a,b}\}$ C^∞ converges to the constant function $\eta^+_{a_0,0}$ uniformly as $(a,b) \to (a_0,0)$.

From the definition, the unstable manifold $W^u(p^+_{a,0})$ consists of the points $q \in \mathbb{R}^2$ which admits a sequence $\{q_n\}_{n=0}^\infty$ in $\mathbb{R}^2$ with $q_0 = q$, $q_n \in \varphi^{-1}_{a,0}(q_{n-1})$ for $n = 1, 2, \dots$ and $\lim_{n\to\infty} q_n = p^+_{a,0}$. In particular, $W^u(p^+_{a,0})$ is contained in the parabolic curve $\mathrm{Im}(\varphi_{a,0}) = \{(x, x^2 + a); -\infty < x < \infty\}$. Then, it is not hard to show that

$$W^u(p^+_{a,0}) = \{(x, x^2 + a); a \le x < \infty\}$$

for any a near -2. Again by the Stable Manifold Theorem, for any (a,b) near $(-2,0)$ (possibly $b=0$), there exist short curves $T_{a,b}$ in $W^u_{\mathrm{loc}}(p^+_{a,b})$ with $\mathrm{Int}(T_{a,b}) \ni p^+_{a,b}$ and varying C^∞ with respect to (a,b). Thus, for any integer $m>0$, $T^{(m)}_{a,b} = \varphi^m_{a,b}(T_{a,b})$ C^∞ converges to $T^{(m)}_{a_0,0} = \varphi^m_{a_0,0}(T_{a_0,0})$ as $(a,b) \to (a_0,0)$. As was illustrated in [11, Fig. 2.1], $T^{(m)}_{a_0,0}$ is the folded curve when m is large enough.

Let $V(-2,2)$ be a fixed small neighborhood of $(-2,2)$ in the xy-plane. Since $S^+_{a,0}$ is the horizontal line $y_{a,0} = (1+\sqrt{1-4a})/2$, for any (a,b) near $(-2,0)$ and any point r in $S^+_{a,b}$, $\boldsymbol{v}^{s,\perp}_a(r)$ is arbitrarily and uniformly close to $\partial y_{a,0}/\partial a|_{a=-2} = -1/3$. Recall that $\boldsymbol{v}^{s,\perp}_a(r)$ is the velocity vector $dr^s(a)/da$ at r defined as in Subsection 1.1, where $r^s(a)$ is the intersection point of $S^+_{a,b(\mathrm{fixed})}$ and a short vertical segment passing through r. From now on, we denote such a closeness by

$$\boldsymbol{v}_a^{s,\perp}(r) \sim \Big(0, \frac{\partial y_{a,0}}{\partial a}\Big|_{a=-2}\Big) = \Big(0, -\frac{1}{3}\Big).$$

The fact that $S^+_{a,0}$ is horizontal does not necessarily imply that $\boldsymbol{v}_b^{s,\perp}(r)$ is constant on $r \in S^+_{a,0}$. However, if we take $V(-2,2)$ is sufficiently small and (a,b) is sufficiently near $(-2,0)$, then

$$\boldsymbol{v}_b^{s,\perp}(r') \sim \boldsymbol{v}_0 := \frac{dr^s}{db}(0)$$

for any $r' \in S^+_{a,b} \cap V(-2,2)$, where $r^s(b)$ is the intersection point of $S^+_{-2,b}$ and the vertical line $x = -2$.

Since $W^s(p^-_{a,0})$ contains the horizontal line passing through $p^-_{a,0}$, $W^s(p^-_{a,b})$ and $W^u(p^+_{a,b})$ have a transverse point τ for any (a,b) near $(-2,0)$ as illustrated in Fig. 3.2. By the Inclination Lemma, there exists a sequence of curves in $W^s(p^-_{a,b})$ C^∞

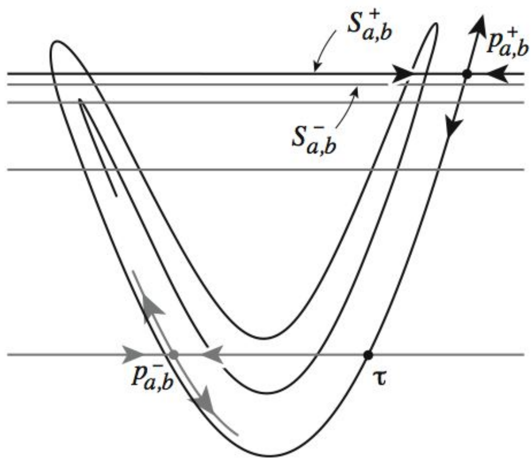

FIGURE 3.2

converges to $S^+_{a,b}$ for any (a,b) near $(-2,0)$. In particular, $W^s(p^-_{a,b})$ contains an almost horizontal curve $S^-_{a,b}$ well approximated by $S^+_{a,b}$. Thus, $S^-_{a,b}$ has a C^∞ holding function $\eta^-_{a,b}$ C^∞ depending on (a,b). Since $S^-_{a_0,0}$ is a horizontal segment, $\eta^-_{a,b}$ C^∞ converges to a constant map $\eta^-_{a_0,0}$ as $(a,b) \to (a_0,0)$. The Inclination Lemma also guarantees that one can choose the $S^-_{a,b}$ so that

$$\boldsymbol{v}_a^{s,\perp}(r) \sim \Big(0, -\frac{1}{3}\Big), \qquad \boldsymbol{v}_b^{s,\perp}(r') \sim \boldsymbol{v}_0 \tag{3.3}$$

for any r in $S^-_{a,b}$ and r' in $S^-_{a,b} \cap V(-2,2)$.

It is well known that there exists a continuation of horseshoe sets $\Lambda_{a,b}$ of $\varphi_{a,b}$ associated with a transverse homoclinic orbit of $p^+_{a,b}$ just as $\Lambda^{\mathrm{out}}_{\hat{a},\hat{b}}$ in [11, §2]. Let $W^s_{\mathrm{loc}}(\Lambda_{a,b})$ be a local stable manifold of $\Lambda_{a,b}$ consisting of almost horizontal leaves connecting the vertical lines $x = -5/2$ and $x = 5/2$. Consider an arc $I_{a,b}$ in $W^u(p^+_{a,b})$ with $p^+_{a,b}$ as an end, meeting each leaf of $W^s_{\mathrm{loc}}(\Lambda_{a,b})$ in a single point.

Moreover, $I_{a,b}$ is taken so that it is smallest among all such arcs. Then, $I_{a,b}$ is uniquely determined, see Fig. 3.3.

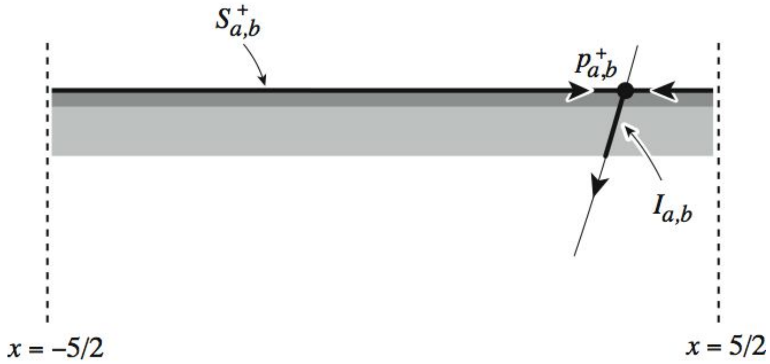

FIGURE 3.3. The union of the shaded regions represents the support of $\mathcal{F}^s_{a,b}$, and the darker part does that of $\mathcal{F}^{s(k)}_{a,b}$.

According to Lemma 4.1 in Kan-Koçak-Yorke [10] based on results in Franks [6], there exists a continuation of foliations $\mathcal{F}^s_{a,b}$ in the xy-plane satisfying the following conditions. Such foliations are said to be *compatible with* $W^s_{\rm loc}(\Lambda_{a,b})$.

(i) Each leaf of $W^s_{\rm loc}(\Lambda_{a,b})$ is a leaf of $\mathcal{F}^s_{a,b}$.
(ii) $I_{a,b}$ *crosses* $\mathcal{F}^s_{a,b}$ *exactly*. That is, if each leaf of $\mathcal{F}^u_{a,b}$ intersects $I_{a,b}$ transversely in a single point and any point of $I_{a,b}$ is passed through by a leaf of $\mathcal{F}^s_{a,b}$.
(iii) Leaves of $\mathcal{F}^s_{a,b}$ are C^3-curves connecting the vertical lines $x = \pm 5/2$ and such that themselves, their directions, and their curvatures vary C^1 with respect to any transverse direction and a, b.

For any sufficiently large integer $k > 0$, let $\mathcal{F}^{s(k)}_{a,b}$ be a foliation obtained by shortening all the leaves of $\varphi^{-k}_{a,b}(\mathcal{F}^s_{a,b})$ so that each leaf of $\mathcal{F}^{s(k)}_{a,b}$ is a C^3-curve connecting vertical lines $x = \pm 5/2$ and the *support* (the union of leaves of $\mathcal{F}^{s(k)}_{a,b}$) is contained in a small neighborhood of $S^+_{a,b}$. Then, it is not hard to show that

$$\boldsymbol{v}^{s,\perp}_a(r) \sim \Big(0, -\frac{1}{3}\Big), \qquad \boldsymbol{v}^{s,\perp}_b(r') \sim \boldsymbol{v}_0 \tag{3.4}$$

for any point r (resp. r') in a $p^+_{a,b}$-leaf l of $\mathcal{F}^{s(k)}_{a,b}$ (resp. $\mathcal{F}^{s(k)}_{a,b}|_{V(-2,2)}$), where l being a $p^+_{a,b}$-*leaf* means that $l \subset W^s(p^+_{a,b})$. This fact is a special case of Accompanying Lemma (Lemma 1.1 in [12]). Moreover, for any sufficiently large k, $\mathcal{F}^{s(k)}_{a,b}$ is disjoint from $S^-_{a,b}$.

Let $l^+_{a,b}$ (resp. $l^-_{a,b}$) be a short curve in $W^u(p^+_{a,b})$ (resp. $W^u(p^-_{a,b})$) as illustrated in Fig. 3.4 such that both $\mathrm{Int}(l^\pm_{a,b})$ meet the x-axis transversely. Set $\hat{l}^\pm_{a,b} = \varphi^2_{a,b}(l^\pm_{a,b})$. The curve $l^\pm_{a,b}$ is parametrized as $(x^\pm_{a,b}(t), t)$ for any t near 0, where $x^\pm_{a,b}$ is a C^∞ function converging uniformly to $x^\pm_{-2,0}(t) = \mp\sqrt{t+2}$ as $(a, b) \to (-2, 0)$. For

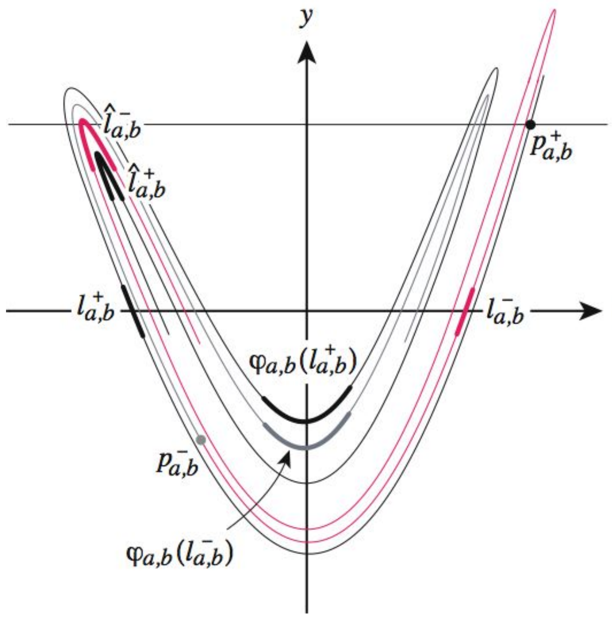

FIGURE 3.4. $\varphi_{a,b}|W^u(p_{a,b}^-)$ exchanges the two components of $W^u(p_{a,b}^-) \setminus \{p_{a,b}^-\}$.

$\omega = \pm$, $\hat{l}_{a,b}^\omega$ is parametrized as $\hat{r}_{a,b}^\omega(t) = (\hat{x}_{a,b}^\omega(t), \hat{y}_{a,b}^\omega(t))$, where

$$\hat{x}_{a,b}^\omega(t) = t^2 - bx_{a,b}^\omega + a,$$
$$\hat{y}_{a,b}^\omega(t) = (\hat{x}_{a,b}^\omega)^2 - bt + a.$$

Since $\hat{y}_{a,b}^\omega(t)$ C^∞ converges to $\hat{y}_{a,0}^\omega(t) = (t^2 + a)^2 + a$ as $b \to 0$, $(\partial/\partial a)\hat{y}_{a,b}^\omega(t)$ C^∞ converges to $(\partial/\partial a)\hat{y}_{a,0}^\omega(t) = 2(t^2+a)+1$ as $b \to 0$. This implies that, for any (a, b) near $(-2, 0)$ and any t near 0, we have $(\partial/\partial a)\hat{y}_{a,b}^\omega(t) \sim -3$ and hence

$$\boldsymbol{v}_a^{u,\perp}(\hat{r}_{a,b}^\omega(t)) \sim (0, -3). \tag{3.5}$$

The following lemma is based on some results in [11].

Lemma 3.1. *There exists a C^∞ function $h(b)$ defined for b near 0, a continuation $\hat{S}_{a,b}^+$ of curves in $W^s(p_{a,b}^+)$ and a continuation $S_{a,b}^-$ of curves in $W^s(p_{a,b}^-)$ satisfying the following conditions.*

(i) *$h(0)$ is arbitrarily close to -2.*
(ii) *For any non-zero b near 0, there is a continuation q_b^+ of quadratic tangencies of $\hat{S}_{h(b),b}^+$ and $\hat{l}_{h(b),b}^-$ each of which unfolds generically with respect to the a-parameter family $\{\varphi_{a,b(\text{fixed})}\}$.*
(iii) *For any b_* with $0 < b_* < \varepsilon$, there is a b_0 with $0 < b_0 < b_*$ such that $S_{h(b_0),b_0}^-$ and $\hat{l}_{h(b_0),b_0}^+$ have a quadratic tangency $q_{b_0}^-$.*
(iv) *$p_{h(b_0),b_0}^-$, $p_{h(b_0),b_0}^+$ satisfy the Sternberg-Takens C^8 and C^4 conditions respectively.*

Proof. By an argument quite similar to the proofs of Propositions 2.2 and 3.1 in [11], there exists a C^∞ function $\tilde{h} : I_\varepsilon = (-\varepsilon, \varepsilon) \longrightarrow \mathbb{R}$ with $\tilde{h}(0) = -2$ and such that $S^+_{\tilde{h}(b),b}$ and $\hat{l}^-_{\tilde{h}(b),b}$ have a quadratic tangency $\tilde{q}^+_b$. Moreover, there exists a locally finite set B^- (resp. B^+) in $I_\varepsilon \setminus \{0\}$ such that, for any $b \in I_\varepsilon \setminus (B^- \cup \{0\})$ (resp. $b \in I_\varepsilon \setminus (B^+ \cup \{0\})$), $p^-_{h(b),b}$ (resp. $p^+_{h(b),b}$) satisfies the Sternberg-Takens C^8 (resp. C^4) condition. We will show that the $\tilde{h}$ is our desired function h except a special case. However, in the exceptional case, we need other two functions $\tilde{h}_1$ and h such that the deformation along $a = \tilde{h}_1(b)$ (resp. $a = h(b)$) keeps a tangency of $S^-_{a,b}$ and $\hat{l}^+_{a,b}$ (resp. $\hat{S}^+_{a,b}$ and $\hat{l}^-_{a,b}$).

Fix $b_* \in (0, \varepsilon)$ arbitrarily. If necessarily retaking $\mathcal{F}^{s(k)}_{\tilde{h}(b_*),b_*}$ and $S^-_{\tilde{h}(b_*),b_*}$ given as above, we may assume that

$$\big(|\mathcal{F}^{s(k)}_{\tilde{h}(b_*),b_*}| \cup S^-_{\tilde{h}(b_*),b_*}\big) \cap \hat{l}^+_{\tilde{h}(b_*),b_*} = \emptyset,$$

where $|\mathcal{F}^{s(k)}_{\tilde{h}(b_*),b_*}|$ denotes the support of $\mathcal{F}^{s(k)}_{\tilde{h}(b_*),b_*}$, see Fig. 3.5 (a). Note that $\hat{l}^+_{\tilde{h}(b),b}$

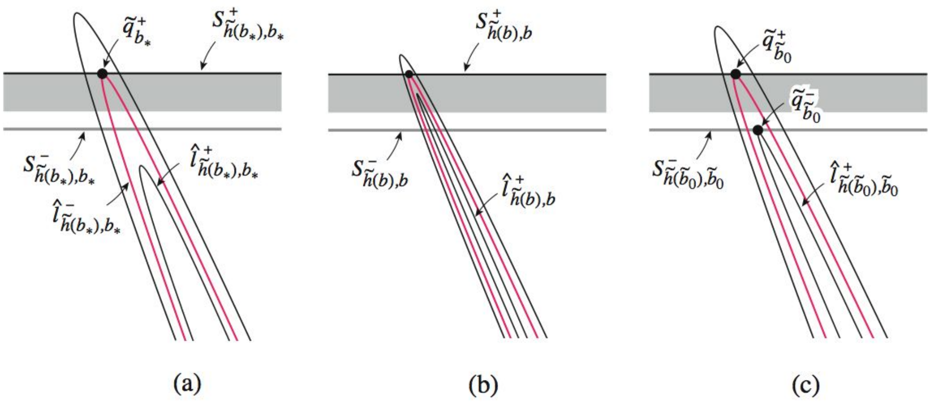

Figure 3.5. The shaded regions represent the support of $\mathcal{F}^{s(k)}_{\tilde{h}(b),b}$.

approaches to $S^+_{\tilde{h}(b),b}$ as $b \searrow 0$, see Fig. 3.5 (b). On the other hand, $S^-_{\tilde{h}(b),b}$ converges to $S^-_{-2,0}$, which is a horizontal segment disjoint from $S^+_{-2,0}$. By the Intermediate Value Theorem, there exists $\tilde{b}_0$ with $0 < \tilde{b}_0 < b_*$ such that $S^-_{\tilde{h}(\tilde{b}_0),\tilde{b}_0}$ and $\hat{l}_{\tilde{h}(\tilde{b}_0),\tilde{b}_0}$ have a quadratic tangency $\tilde{q}^-_{\tilde{b}_0}$, see Fig. 3.5 (c). If $\tilde{b}_0 \not\in B^+ \cup B^-$, we may set $\tilde{b}_0 = b_0$, $\tilde{h}(b) = h(b)$, $S^+_{\tilde{h}(b_0),b_0} = \hat{S}^+_{h(b_0),b_0}$ and $\tilde{q}^-_{\tilde{b}_0} = q^-_{b_0}$, see Fig. 3.6 (a).

Suppose that $\tilde{b}_0 \in B^+ \cup B^-$. Then, we have a C^∞ function $\tilde{h}_1(b)$ defined on an open neighborhood of $\tilde{b}_0$ with $\tilde{h}_1(\tilde{b}_0) = \tilde{h}(\tilde{b}_0)$ and such that there exists a continuation of quadratic tangencies of $S^-_{\tilde{h}_1(b),b}$ and $\hat{l}^+_{\tilde{h}_1(b),b}$ based at $\tilde{q}^-_{\tilde{b}_0}$, see Fig. 3.6 (b). Since $\Lambda_{\tilde{h}_1(b),b}$ is a horseshoe set, $\mathcal{F}^{s(k)}_{\tilde{h}_1(b),b}$ has a sequence of $p^+_{\tilde{h}_1(b),b}$-leaves $S^+_{n;\tilde{h}_1(b),b}$ converging to $S^+_{\tilde{h}_1(b),b}$, see Fig. 3.7 (a). Since $B^+ \cup B^-$ is locally finite in $I_\varepsilon \setminus \{0\}$, one can take n sufficiently large so that there exists $b_0 \in I_\varepsilon \setminus (B^+ \cup B^- \cup \{0\})$

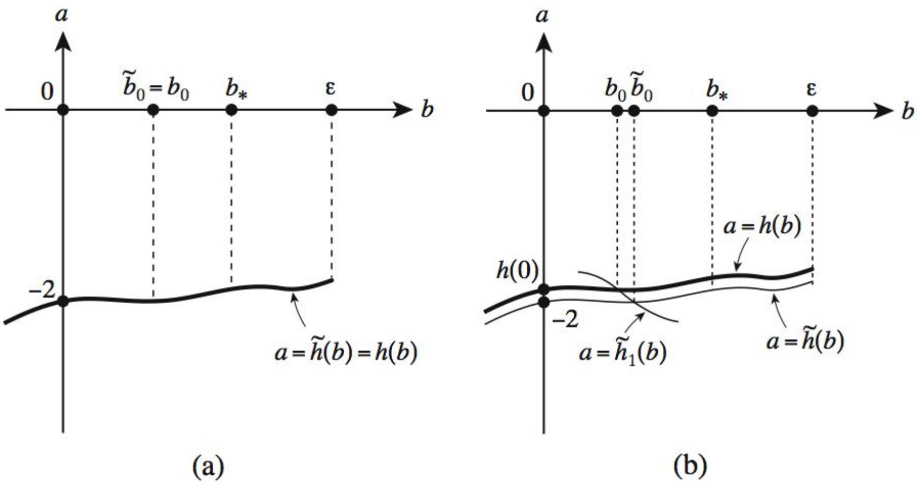

FIGURE 3.6

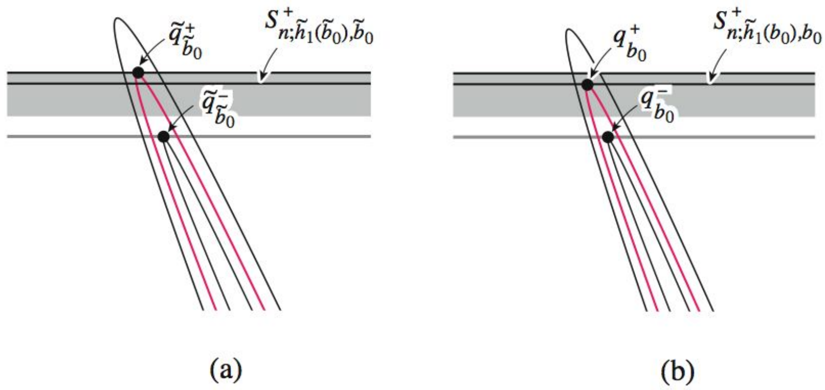

FIGURE 3.7

with $0 < b_0 < \tilde{b}_0$ arbitrarily close to $\tilde{b}_0$ and such that $S^+_{n;\tilde{h}_1(b_0),b_0}$ and $\hat{l}^-_{\tilde{h}_1(b_0),b_0}$ have a quadratic tangency $q^+_{b_0}$, see Fig. 3.7 (b). Let $\hat{S}^+_{a,b}$ be a continuation of $p^+_{a,b}$-leaves of $\mathcal{F}^{s(k)}_{a,b}$ based at $S^+_{n;\tilde{h}_1(b_0),b_0}$, that is, $\hat{S}^+_{\tilde{h}_1(b_0),b_0} = S^+_{n;\tilde{h}_1(b_0),b_0}$. Again by using results in [12, §2], one can define a C^∞ function $h : (-\varepsilon, \varepsilon) \longrightarrow \mathbb{R}$ with $h(b_0) = \tilde{h}_1(b_0)$ and such that there is a continuation q^+_b of quadratic tangencies of $\hat{S}^+_{h(b),b}$ and $\hat{l}^-_{h(b),b}$ based at $q^+_{b_0}$. If necessary replacing n by a larger integer, we may assume that $h(0)$ arbitrarily close to -2. In fact, since the horizontal segment $\hat{S}^+_{h(0),0}$ passes through the end point $q^+_0 = (h(0), h(0)^2 + h(0))$ of $W^u(p^+_{h(0),0})$, $h(0)^2 + h(0)$ coincides with the height of the horizontal segment $\hat{S}^+_{h(0),0} \sim 2$.

The tangency $q^+_{b_0}$ is equal to $\hat{r}^+_{h(b_0),b_0}(t_{b_0})$ for some t_{b_0} near 0. Then, the generic condition (1.4) and the approximations (3.3) and (3.5) show that q^+_b unfolds generically with respect to $\{\varphi_{a,b(\text{fixed})}\}$. This completes the proof. $\square$

3.4. **Proof of Theorem B.** For the completion of the proof of Theorem B, it suffices to show that the tangency $q^-_{b_0} = \hat{r}^-_{h(b_0),b_0}$ given in Lemma 3.1 unfolds generically with respect to the b-parameter family $\{\varphi_{h(b),b}\}$. Then, $\{\varphi_{a,b}\}$ satisfies the conditions (i)-(iv) of Theorem A with respect to the new parameter

$$\mu = a - h(b), \quad \nu = b - b_0.$$

From now on, we denote the holding function of $\hat{S}^+_{a,b}$ given in Lemma 3.1 newly by $\eta^+_{a,b}$ and the subscription pair '$h(b), b$' only by 'b', e.g. $\eta^\pm_{h(b),b} = \eta^\pm_b$. Recall that η^-_b is the holding function of $S^-_{h(b),b}$.

For $\omega = \pm$, consider the function $\theta^\omega_b(t)$ of t corresponding to (1.3) defined by

$$\theta^\omega_b(t) = (\hat{x}^\omega_b(t))^2 - bt + h(b) - \eta^{-\omega}_b(\hat{x}^\omega_b(t)), \tag{3.6}$$

where we set

$$\hat{x}^\omega_b(t) = t^2 - bx^\omega_b(t) + h(b)$$

and suppose $-\omega = \mp$ if $\omega = \pm$ respectively. Since $\theta^\omega_b(t)$ C^∞ converges to $\theta^\omega_0(t) = (t^2 + h(0))^2 + c^\omega$ as $b \to 0$ for some constant c^ω, $\dot{\theta}^\omega_b(t)$ and $\ddot{\theta}^\omega_b(t)$ C^∞ converge respectively to $\dot{\theta}^\omega_0(t) = 4t^3 + 4h(0)t$, $\ddot{\theta}^\omega_0(t) = 12t^2 + 4h(0) \sim -8 \neq 0$, where the 'dot' represents the derivative of a function by t. From this fact, we have a unique t^ω_b near 0 C^∞ depending on b and such that $\dot{\theta}^\omega_b(t^\omega_b) = 0$. In particular, $t^\omega_0 = 0$. Note that, since q^+_b is in $\hat{S}^+_b$ for any b near b_0, $\theta^-_b(t^-_b)$ is a zero function of b.

We set $x^\omega_b(t) = x^\omega(b,t)$ and $\eta^\omega_b(x) = \eta^\omega(b,x)$, and suppose that the 'prime' represents the derivative of a function by b. For example, $x^{\omega\prime}_b(t) = (\partial/\partial b)x^\omega(b,t)$ and $\eta^{\omega\prime}_b(x) = (\partial/\partial b)\eta^\omega(b,x)$.

Proof of Theorem B. First, we show that, for any sufficiently small $b > 0$,

$$\frac{d}{db}\theta^+_b(t^+_b) \sim -6\sqrt{2}. \tag{3.7}$$

Set $t^+_b = t_b$, $x^+_b = x_b$, $\hat{x}^+_b = \hat{x}_b$ and $\eta^-_b = \eta_b$ for short. By (3.6),

$$\begin{aligned}\frac{d\theta^+_b(t_b)}{db} &= 2\hat{x}_b(t_b)\big(2t_bt'_b - x_b(t_b) - bx'_b(t_b) - b\dot{x}_b(t_b)t'_b + h'(b)\big)\\ &\qquad - t_b - bt'_b + h'(b) - \eta'_b(\hat{x}_b(t_b)) - \frac{\partial\eta}{\partial x}(b,\hat{x}_b(t_b))(\hat{x}_b(t_b))'.\end{aligned}$$

Since η_0 is a constant function of x, $(\partial/\partial x)\eta_0 = 0$. Since moreover $t_0 = 0$, $x_0(0) = -\sqrt{2}$, $\hat{x}_0(0) = h(0) \sim -2$,

$$\frac{d\theta^+_b(t^+_b)}{db} \sim -3\sqrt{2} - 3h'(0) - \eta^{-\prime}(0,-2).$$

Similarly, we have

$$0 = \frac{d\theta^-_b(t^-_b)}{db} \sim 3\sqrt{2} - 3h'(0) - \eta^{+\prime}(0,-2).$$

By (3.3) and (3.4),

$$\big(0, \eta^{-\prime}(0,-2)\big) \sim \boldsymbol{v}_0, \quad \big(0, \eta^{+\prime}(0,-2)\big) \sim \boldsymbol{v}_0.$$

The above four approximations imply (3.7).

We have from (1.4) and (3.7)

$$\boldsymbol{v}_b^{u,\perp}(q_{b_0}^-) - \boldsymbol{v}_b^{s,\perp}(q_{b_0}^-) = \left(0, \frac{d\theta_b^+(t_b^+)}{db}(b_0)\right) \sim (0, -6\sqrt{2}).$$

This implies that $q_{b_0}^-$ unfolds generically with respect to the b-parameter family $\{\varphi_{h(b),b}\}$. This completes the proof of Theorem B. $\square$

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