Submitted exclusively to the London Mathematical Society
doi:10.1112/S0000000000000000

RIGIDITY OF BRILLOUIN ZONES

F.H. KWAKKEL, M. MARTENS AND M.M. PEIXOTO

Abstract

In general, topological characteristics of manifolds do not determine the geometry. However, starting in the 1960's, examples have been discovered for which such characteristics do determine the geometry. The manifolds can not be deformed without changing the characteristic. The Mostow rigidity theorems are examples of this phenomenon. The universality observed in one-dimensional dynamics also leads to rigidity results.

The classical *Brillouin zones* were introduced by Brillouin in the quantum study of wave propagation in crystals. Here we will not go into the physical meaning but interpret the *Brillouin zones* as characteristics of two-dimensional flat tori. The main results will be rigidity theorems related to Brillouin zones, focal decompositions and torus puzzles.

1. *Introduction and Definitions*

In general, topological characteristics of manifolds do not determine the geometry. However, starting in the 1960's, examples have been discovered for which such characteristics do determine the geometry. The manifolds can not be deformed without changing the characteristic. One speaks of rigidity. The prototype rigidity theorem is due to Mostow [**1**].

Theorem 1.1 (Mostow rigidity theorem). *Suppose M and N are closed manifolds of constant sectional curvature -1 with the dimension of M is at least 3. If $\pi_1(M) \cong \pi_1(N)$, then M and N are isometric.*

The classical *Brillouin zones* were introduced by Brillouin in the quantum study of wave propagation in crystals. Here we will not go into the physical meaning but interpret the *Brillouin zones* as characteristics of two-dimensional flat tori. The main results will be rigidity theorems in the context of flat tori.

For completeness we will recall the definitions concerning Brillouin zones. A lattice Λ is a discrete subgroup of $\mathbb{R}^2$ generated by two linearly independent vectors $\omega_1, \omega_2 \in \mathbb{R}^2$, i.e.

$$\Lambda = \{n\omega_1 + m\omega_2 \mid n, m \in \mathbb{Z}\} = \omega_1\mathbb{Z} \oplus \omega_2\mathbb{Z}. \tag{1.1}$$

We define two elements $x, y \in \mathbb{R}^2$ to be equivalent, $x \sim y$, if and only if $x - y \in \Lambda$. The flat 2-torus $\mathbb{T}_\Lambda = \mathbb{R}^2/\Lambda$ is the quotient space of $\mathbb{R}^2$ under the equivalence $\sim$. Let $\pi : \mathbb{R}^2 \longrightarrow \mathbb{T}_\Lambda$ be the canonical projection and let $d(x, y) = |x - y|$ be the standard Euclidean metric on $\mathbb{R}^2$. Locally π is an isometry and induces the covering metric $\tilde{d}$ on the torus.

2000 *Mathematics Subject Classification* 52C25 (primary), 51H05 (secondary).

Definition 1 (Brillouin line). A *Brillouin line* $L_g \subset \mathbb{R}^2$ is defined as the perpendicular bisector of the line connecting the origin 0 and $g \in \Lambda$, i.e.

$$L_g = \{x \mid g \in \Lambda \text{ and } |x| = |x - g|\}$$

For $0 \in \Lambda$, we define $L_0 = \{0\}$.

Definition 2. Let $M_\Lambda \subset \mathbb{R}^2$ be the set of all Brillouin lines relative to the lattice Λ, i.e.

$$M_\Lambda = \bigcup_{g \in \Lambda_*} L_g, \tag{1.2}$$

where $\Lambda_* = \Lambda - \{0\}$.

Let ℓ_x be the open line segment connecting the origin 0 and x and let $\bar{\ell}_x$ be the closure of ℓ_x also containing 0 and x.

Definition 3. Let $\iota, \chi, \mu : \mathbb{R}^2 \to \mathbb{N}$ be the *indices* defined by

$$\iota(x) = \#\{g \in \Lambda \mid L_g \cap \ell_x \neq \emptyset\} \tag{1.3}$$

$$\chi(x) = \#\left\{g \in \Lambda \mid L_g \cap \bar{\ell}_x \neq \emptyset\right\} \tag{1.4}$$

$$\mu(x) = \#\{g \in \Lambda \mid L_g \ni x\} \tag{1.5}$$

where $\#$ means the cardinality of the set. The index $\mu(x)$ is referred to as the *multiplicity* of x.

It follows that $\chi(x) = \iota(x) + \mu(x) + 1$.

Definition 4 (Brillouin zone). The n-th Brillouin zone relative to a lattice Λ is the set

$$B_n = \{x \in \mathbb{R}^2 \mid \iota(x) \leq n \text{ and } \chi(x) \geq n + 1\}. \tag{1.6}$$

Remark 1. In [**4**], G.A. Jones gives three different definitions of Brillouin zones, in $\mathbb{R}^n$. Our definition can be equivalently formulated in terms of positive definite quadratic forms. This point of view is adopted in [**2**] and [**7**].

In [**8**] we find an abstract approach to Brillouin zones defined on quite general metric spaces. For physicists the most important Brillouin zone is the first zone, called the *Wigner-Seitz cell*. The first zone also appears in other mathematical pursuits and then it is called *Dirichlet region* in the study of Fuchsian groups, or *Voronoi cell* in the study of packing.

Notation. Although the Brillouin zones B_n and the torus $\mathbb{T}$ are defined relative to a lattice Λ, we omit the subscripts referring to the lattice. The results will hold for any (but fixed) lattice Λ, unless explicitly stated otherwise.

The Brillouin lines and zones are related to arithmetic properties of the geodesic flow on the given flat torus. This relation is explored in [**2**], [**7**]. Let us briefly recall this discussion. Consider the tangent plane at $0 = \pi(0) \in \mathbb{T}$. The function $\sigma : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$\sigma(x) = \#\{y \in \mathbb{R}^2 \mid \pi(y) = \pi(x), |y| = |x|\} \tag{1.7}$$

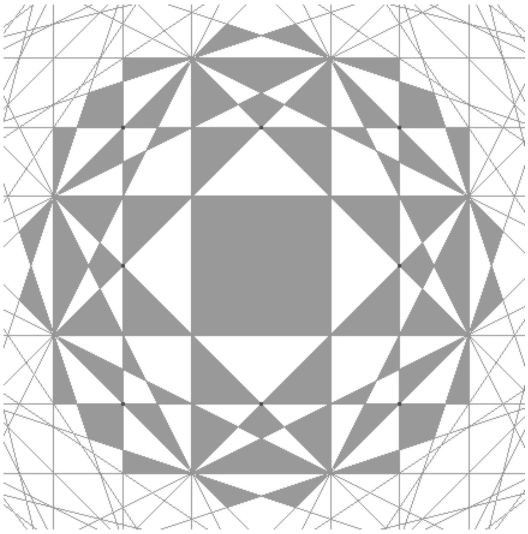

FIGURE 1. *The first 9 Brillouin zones relative to $\mathbb{Z}^2$. The consecutive zones are alternately shaded and unshaded.*

was introduced to count the number of closed geodesics of the same length starting at 0. The *focal decomposition* is the stratification of the tangent plane at $0 \in T$ defined by this function. The stratification consist of the following strata. The two-dimensional strata, also called *subzones*, are the connected components of $\sigma = 1$, the one-dimensional strata are the components of $\sigma = 2$ and the points are the connected components of $\sigma \geq 3$. Observe that, in the case of a flat torus, the focal decomposition is independent of the base point $0 \in \mathbb{T}$. According to the results in [**2**], the focal decomposition is characterized by the Brillouin lines. This is also expressed by our lemma 2.2 which states

$$\sigma(x) = \mu(x) + 1, \quad \text{for all } x \in \mathbb{R}^2. \tag{1.8}$$

We will identify M_Λ with the corresponding focal decomposition.

The topology of the focal decomposition M_Λ contains information about the geometry of the underlying torus. The first rigidity theorem we prove here is that it actually uniquely determines the geometry of the torus. We say that the focal decompositions associated to two flat tori are *equivalent* if there exists a homeomorphism between the corresponding tangent planes that maps the decomposition associated to the one torus onto the decomposition of the other.

THEOREM. *The focal decompositions of two tori are equivalent if and only if the tori are conformally equivalent.*

This result, proved in section 4, is inspired by Mostow's rigidity theorem. An important ingredient in the proof of this theorem is the asymptotic shape of Brillouin zones which, independent of the lattice, is a circle. This was shown by Jones in [**4**]. Using a result from analytic number theory [**5**], theorem 3.2 gives more precise bounds on the distance of B_n from the origin.

A classical result by Bieberbach [**3**], states that each zone B_n is a fundamental domain for the projection π. That is, each Brillouin zone gives rise to a tiling of the torus, which we call the corresponding *torus puzzle* of the n^{th} generation.

Again, the topology of the torus puzzles contains information on the geometry of the torus. We define an equivalence relation between torus puzzles, which in addition to requiring the puzzles to be homeomorphic, involves a fixed-point condition. The second rigidity result characterizes the torus in terms of these puzzles. The combinatorial properties of the torus puzzles obtained in section 2 play a crucial role in the proof of the following result, proved in section 5.

THEOREM. *Given two tori corresponding to lattices in general position. For every generation the two torus puzzles are equivalent if and only if the tori are conformally equivalent.*

The classical result of Bieberbach stating that Brillouin zones are fundamental domains can be extended in the context of higher dimensional flat tori. See for the definitions and results [**4**]. The method for proving this theorem does not have a strict two dimensional character. Although the proof of higher dimensional versions of for example lemma 4.2 and lemma 4.3 need some special care we state

CONJECTURE 1. The focal decomposition of two n-tori are equivalent if and only if the tori are isometric.

The study on the boundary of Brillouin zones presented here relies strongly on two dimensional arguments. The boundaries in higher dimensions might be much more complex. Nevertheless, we state

CONJECTURE 2. Two n-tori are isometric if and only if for every generation the two torus puzzles are topologically equivalent.

2. *Torus Puzzles*

We start with studying the topological properties of B_n and we show that the projection of every B_n tiles the torus. Such a tiling of the torus is called a *torus puzzle*. Our special interest lies in determining what combinatorial information about the set M_Λ is encoded in these torus puzzles.

LEMMA 2.1. *Let* $x \in \mathbb{R}^2$, *then*

$$\iota(x) = \#\left\{y \in \mathbb{R}^2 \mid \pi(y) = \pi(x) \text{ and } |y| < |x|\right\}. \tag{2.1}$$

Proof. Let $x \in \mathbb{R}^2$ with index $\iota(x)$. We let $D_1 = D(0, |x|)$, be the open disc with center 0 and radius $|x|$ and similarly $D_2 = D(x, |x|)$, see figure 2.
We show that for $g \in \Lambda$ the following are equivalent:

1) $\ell_x \cap L_g \neq \emptyset$,
2) $g \in \Lambda \cap D_2$,
3) $y_g = x - g \in D_1$.

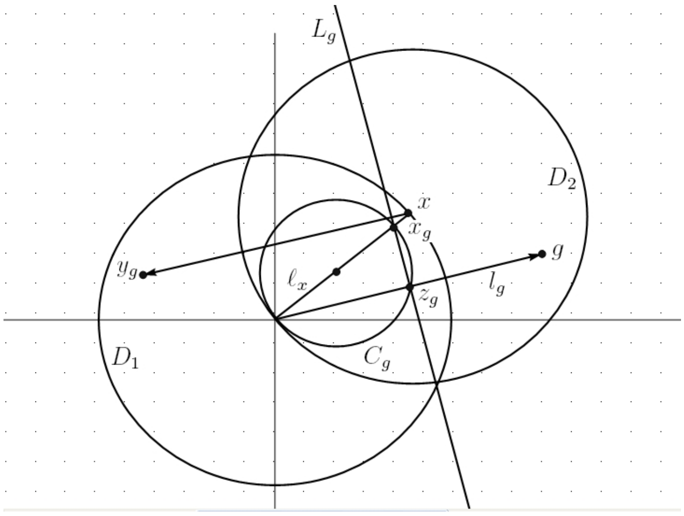

FIGURE 2. *Proof of lemma 2.1*

1) $\Leftrightarrow$ 2). Suppose $\ell_x \cap L_g \neq \emptyset$ for some $g \in \Lambda$. Let $\ell_x \cap L_g = \{x_g\}$, then $|x_g| < |x|$. Let C_g be the circle centered at $\frac{1}{2}x_g$ and radius $\rho_g = \frac{1}{2}|x_g|$. Let l_g be the line segment connecting the origin O and g and let $z_g = \frac{1}{2}g$. Since L_g is perpendicular to l_g, $z_g \in l_g \cap C_g$. Since $|x_g| < |x|$ and $\rho_g < \frac{1}{2}|x|$, by congruence, $g \in D_2 \cap \Lambda$. Reading the previous arguments backwards yields the other direction.
2) $\Leftrightarrow$ 3). By symmetry, $\alpha \in D_2$ if and only if $x - \alpha \in D_1$.

Hence, there is a one-to-one correspondence between the set of points

$$\left\{y \in \mathbb{R}^2 \mid \pi(y) = \pi(x) \text{ and } |y| < |x|\right\}$$

and the set of Brillouin lines L_g such that $\ell_x \cap L_g \neq \emptyset$ and this proves the lemma. □

DEFINITION 5. For $x \in \mathbb{R}^2$, let

$$\mathcal{O}(x) = \left\{y \in \mathbb{R}^2 \mid \pi(x) = \pi(y), \quad |x| = |y|\right\}. \tag{2.2}$$

and $\sigma : \mathbb{R}^2 \to \mathbb{N}$, $\sigma(x) = \#\mathcal{O}(x)$.

LEMMA 2.2. *Let $x \in B_n$ and $v = \pi(x)$. Then $\sigma(x) = \mu(x) + 1$ and ι, σ, χ and μ are constant on $\mathcal{O}(x)$. Moreover,*

$$\pi^{-1}(v) \cap B_n = \mathcal{O}(x). \tag{2.3}$$

Proof. In the notation of the proof of lemma 2.1, let $C_i = \partial D_i$ for $i = 1, 2$. If $x \in L_g$, then $g \in C_2$ and $y_g = x - g \in C_1$. Hence $y_g \in \mathcal{O}(x)$. Moreover, if $L_g \neq L_{g'}$, i.e. $g \neq g'$, then $y_g \neq y_{g'}$. Conversely, every $y \in \mathcal{O}(x)$ gives rise to a L_g such that $x \in L_g$; because $\pi(x) = \pi(y)$, $x - y = g$ for some $g \in \Lambda$ and it is easily seen that $x \in L_g$. So $\sigma(x)$ equals the number of points y_g plus x itself, hence $\sigma(x) = \mu(x) + 1$.

Since $\sigma(x) = \sigma(y)$ for all $y \in \mathcal{O}(x)$, μ is constant on $\mathcal{O}(x)$. From lemma 2.1 it is easy to see that ι (and hence χ) is constant on $\mathcal{O}(x)$.

To prove (2.3), first note that $|x'| = |x|$ for all $x' \in \pi^{-1}(v) \cap B_n$. Suppose otherwise that $x, x' \in \pi^{-1}(v) \cap B_n$ but $|x| \neq |x'|$, say $|x| > |x'|$. Then, by lemma 2.1 and the fact that $\sigma(x') = \mu(x') + 1$ we get $\iota(x) \geq \mu(x') + 1 + \iota(x') = \chi(x') \geq n + 1$, a contradiction. Hence, $\pi^{-1}(v) \cap B_n \subset \mathcal{O}(x)$. Furthermore, the indices ι and χ are constant on $\mathcal{O}(x)$, it follows that $y \in B_n$ for all $y \in \mathcal{O}(x)$. We finished the proof of equation (2.3). $\square$

REMARK 2. In general there are no explicit formulas for $\sigma(x)$. However, in the case when $\Lambda = \mathbb{Z}^2$ Gauss obtained an explicit formula for $\sigma(g)$ with $g \in \Lambda$ in terms of the prime decomposition of $|g|^2$. This result is as follows. Let $N \in \mathbb{N}$ with prime factorization

$$N = 2^\alpha \prod_{i=1}^{k} p_i^{\beta_i} \prod_{j=1}^{l} q_j^{\gamma_j},$$

where $p_i \equiv 1 \mod 4$ and $q_j \equiv 3 \mod 4$. Denote $R(N)$ be the number of solutions in $\mathbb{Z}^2$ of $n^2 + m^2 = N$. If all γ_j are even, which is the case for $N = |g|^2$ for $g \in \mathbb{Z}^2$, then $R(N) = 4\prod_{i=1}^{k}(1 + \beta_i)$. See for instance [**6**, p.166] for this result. Thus we have $\sigma(g) = R(|g|^2)$.

LEMMA 2.3. *B_n is closed.*

Proof. Let $x \in B_n^c$, the complement of B_n. Then either $\iota(x) \geq n+1$ or $\chi(x) \leq n$. The latter is equivalent to $\iota(x) + \sigma(x) = \iota(x) + \mu(x) + 1 = \chi(x) \leq n$. In both cases, because $\pi^{-1}(v)$ with $v = \pi(x)$ is discrete, there exists an open neighborhood around x for which $\iota(x) \geq n + 1$ or $\iota(x) + \sigma(x) \leq n$ respectively, which shows that the complement of B_n is open and hence B_n is closed. $\square$

LEMMA 2.4. *Let $x \in B_n$, then $x \in \mathrm{Int}(B_n)$ if and only if $\mu(x) = 0$. Consequently,*

$$M_\Lambda = \bigcup_{g \in \Lambda} L_g = \bigcup_{n \in \mathbb{N}} \partial B_n. \tag{2.4}$$

Proof. If $\mu(x) = 0$, then $\iota(x) = n$. Therefore, there exists a small neighbourhood around x such that $\iota(y) = n$ and $\mu(y) = 0$. Thus $y \in B_n$ for all y in this neighbourhood, so $x \in \mathrm{Int}(B_n)$.

Conversely, let $\mu(x) \geq 1$. If $\iota(x) \leq n - 1$ then consider the point $y = (1 - \epsilon)x$ with $0 < \epsilon < 1$. Observe, $\chi(y) \leq \iota(x) + 1 \leq n$. Hence, $y \in B_n^c$ and thus $x \in \partial B_n$. If $\iota(x) = n$ then consider the point $y = (1 + \epsilon)x$ with $\epsilon > 0$. Observe, $\iota(y) \leq \iota(x) + \mu(x) \geq n + 1$. Hence, $y \in B_n^c$ and thus $x \in \partial B_n$. $\square$

If $x \in \mathrm{Int}(B_n)$, then $\iota(x) = n$ and $\chi(x) = n + 1$. This yields that the zones tile $\mathbb{R}^2$ in the sense that

$$\bigcup_{n \in \mathbb{N}} B_n = \mathbb{R}^2 \quad \text{and} \quad \mathrm{Int}(B_n) \cap \mathrm{Int}(B_m) = \emptyset \quad \text{if} \quad n \neq m. \tag{2.5}$$

DEFINITION 6. Define

$$\partial_n^- = B_n \cap B_{n-1} \quad \text{and} \quad \partial_n^+ = B_n \cap B_{n+1}. \tag{2.6}$$

If $x \in \partial B_n$, then either $\iota(x) \leq n-1$ and $\chi(x) \geq n+1$ or $\iota(x) = n$ and $\chi(x) \geq n+2$, corresponding to $x \in B_n \cap B_{n-1}$ and $x \in B_n \cap B_{n+1}$ respectively. It follows that

$$\partial B_n = \partial_n^- \cup \partial_n^+ \tag{2.7}$$

We denote $\partial_n = \partial_n^- \cup \partial_n^+$ and (2.4) rewrites as

$$M_\Lambda = \bigcup_{n \in \mathbb{N}} \partial_n = \bigcup_{n \in \mathbb{N}} \partial_n^+, \tag{2.8}$$

since $\partial_n^+ = \partial_{n+1}^-$ and $\partial_0^- = \emptyset$.

Topological properties of Brillouin zones are given in the next proposition, which was proved in [**4**].

PROPOSITION 2.5 (Jones). *For every Brillouin zone B_n, the following holds:*
(i) *B_n is compact,*
(ii) *B_n is path-connected,*
(iii) *$\partial_n^\pm$ is homeomorphic to the circle $\mathbb{S}^1$.*

Although B_n is connected, the interior of B_n is in general not connected. Let $\{b_n^j\}_{j \in J_n}$ be the set of connected components of $\mathrm{Int}(B_n)$, then

$$\bigcup_{j \in J_n} b_n^j = \mathrm{Int}(B_n). \tag{2.9}$$

The set $B_n^j = b_n^j \cup \partial b_n^j$ is called a *subzone*† and we have

$$B_n = \bigcup_{j \in J_n} B_n^j. \tag{2.10}$$

LEMMA 2.6. *B_n is a finite union of convex polygons.*

Proof. Because Λ is discrete only finitely many Brillouin lines can meet B_n because B_n is bounded. This yields that every B_n consists of finitely many subzones and that the boundary of a subzone is comprised of finitely many edges, so every subzone is a polygon.

To prove convexity, notice that every Brillouin line L_g divides $\mathbb{R}^2$ into two half planes H_g^i, $i = 1, 2$. Since $M_\Lambda = \bigcup_{g \in \Lambda_*} L_g = \bigcup_{n \in \mathbb{N}} \partial_n$, every subzone is the intersection of finitely many convex half planes and thus convex. □

A point $x \in M_\Lambda$ is called a *vertex* if $\mu(x) \geq 2$. The connected components of $\{x \in M_\Lambda \mid \mu(x) = 1\}$ are the *edges* of M_Λ.

Let $\mathcal{P}_n = \bigcup_{j \in J_n} \mathcal{P}_n^j$ with $\mathcal{P}_n^j = \pi(B_n^j)$. Moreover, let $\partial^\pm \mathcal{P}_n = \pi(\partial_n^\pm)$ and $\partial \mathcal{P}_n = \pi(\partial_n)$.

---

†Subzones are also referred to as *Landsberg subzones*.

We define $\tilde{e} \subset \mathbb{T}$ to be an *edge* if $e \subset M_\Lambda$ is an edge and $\tilde{e} = \pi(e)$. Similarly, we say a region $\tilde{P} \subset \mathbb{T}$ is a *convex polygon* on the torus, if $P \subset \mathbb{R}^2$ is a convex polygon and $\pi(P) = \tilde{P}$ and π injective on $\mathrm{Int}(P)$.

Let $\{P_i\}_{i\in I}$ be a finite family of polygons on $\mathbb{R}^2$ and $v \in \mathbb{T}$ such that $v \in \tilde{P}_i = \pi(P_i)$ for all $i \in I$. Then v is called a *vertex* if $\pi^{-1}(v) \cap P_i$ is a vertex of P_i for every $i \in I$. We call an edge $\tilde{e} \subset \partial\mathcal{P}_n$ a *plus edge* if $\tilde{e} \subset \partial^+\mathcal{P}_n$ and a *minus edge* if $\tilde{e} \subset \partial^-\mathcal{P}_n$.

Definition 7 (Torus Puzzle). A *torus puzzle* is a finite family of convex polygons, $\{P^j\}_{j\in J_n}$ with $P^j \subset \mathbb{T}$, such that

(i) the union of the polygons covers the torus,
(ii) if $i \neq j$, then the intersection $P^i \cap P^j$ is either empty, or a single vertex of both P^i and P^j or a single edge of both.

When the polygons are all triangles, the notion of a torus puzzle coincides with that of a triangulation.

Theorem 2.7. *Every $\mathcal{P}_n$ is a torus puzzle.*

Proof. By lemma 2.1, $\{B_n^j\}_{j\in J_n}$ is a finite family of convex polygons on $\mathbb{R}^2$, hence $\{\mathcal{P}_n^j\}_{j\in J_n}$ is a finite family of convex polygons on $\mathbb{T}$. To show that $\pi : B_n \to \mathbb{T}$ is surjective, let $v \in \mathbb{T}$ and consider $\pi^{-1}(v)$. Because Λ is discrete, $\pi^{-1}(v)$ is discrete. Hence, there exists an $x \in \pi^{-1}(v)$ such that

$$\#\left\{y \in \mathbb{R}^2 \mid \pi(y) = \pi(x) \text{ and } |y| < |x|\right\} \leq n$$

and

$$\#\left\{y \in \mathbb{R}^2 \mid \pi(y) = \pi(x) \text{ and } |y| \leq |x|\right\} \geq n+1.$$

We have shown this to be equivalent to $\iota(x) \leq n$ and $\chi(x) \geq n+1$, hence $x \in B_n$. This shows that $\{\mathcal{P}_n^j\}_{j\in J_n}$ satisfies property (i) of definition 7.

To prove $\mathcal{P}_n$ satisfies part (ii), $\pi : \mathrm{Int}(B_n) \to \mathbb{T}$ is injective since $\pi^{-1}(v) \cap B_n = \mathcal{O}(x)$ and $\sigma(x) = 1$ if and only if $x \in \mathrm{Int}(B_n)$. This shows that $\mathrm{Int}(\mathcal{P}_n^i) \cap \mathrm{Int}(\mathcal{P}_n^j) = \emptyset$ if $i \neq j$ for $i, j \in J_n$. Let $x \in \partial_n$ and $\pi(x) = v$. A point $x \in \partial_n$ is a vertex if and only if $\mu(x) \geq 2$. By lemma 2.2, for all points $y \in \mathcal{O}(x)$, $\mu(y) = \mu(x)$ and these are exactly all the points in ∂_n that are mapped to v, hence v is a vertex. Conversely, if x is not a vertex then $\mu(x) = 1$ and $\mu(y) = 1$ for the other $y \in \mathcal{O}(x)$ so y is not a vertex and this proves $\{\mathcal{P}_n^j\}_{j\in J_n}$ satisfies part (ii) of definition 7. □

It particular, this shows that B_n is a *fundamental domain* for Λ. That is, B_n is closed by proposition 2.5 (i) and, moreover,

$$\bigcup_{g\in\Lambda} gB_n = \pi^{-1}(\pi(B_n)) = \pi^{-1}(\mathcal{P}_n) = \mathbb{R}^2$$

and

$$\mathrm{Int}(gB_n) \cap \mathrm{Int}(g'B_n) = \emptyset \qquad \text{if } g \neq g'.$$

The first equality follows from surjectivity of $\pi : B_n \to \mathbb{T}$ and the second by injectivity of $\pi : \mathrm{Int}(B_n) \to \mathbb{T}$. This result was first shown by Bieberbach in [**3**] and later by Jones in [**4**]. It also follows that

COROLLARY 2.8. *The measure of B_n, $n \in \mathbb{N}$, equals the area of the torus $\mathbb{T}$.*

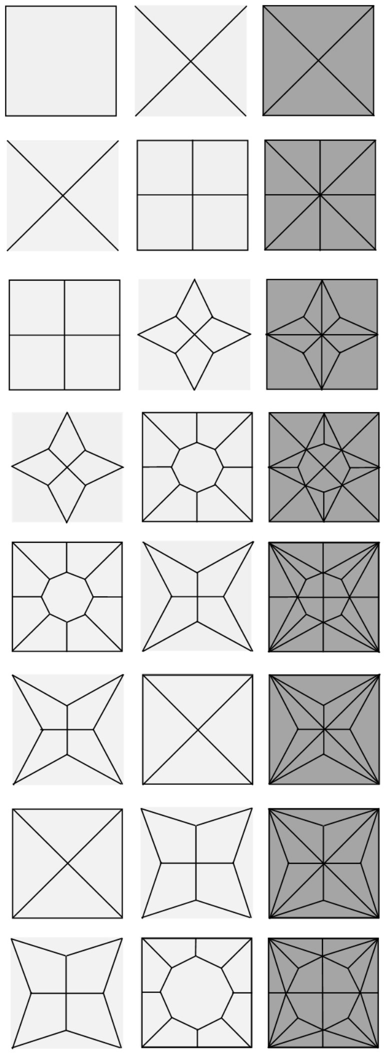

FIGURE 3. *The first 8 puzzles $\mathcal{P}_n$ relative to $\mathbb{Z}^2$. The left and middle pictures are the minus and plus boundaries $\partial^-\mathcal{P}_n$ and $\partial^+\mathcal{P}_n$ respectively and the right pictures the puzzles $\mathcal{P}_n$.*

Let $x \in \partial_n^- \cap \partial_n^+$, or equivalently, $x \in B_{n-1} \cap B_n \cap B_{n+1}$. Then $\iota(x) \leq n-1$ and $\chi(x) \geq n+2$. Hence, $\mu(x) \geq 2$. Every $x \in \partial_n^- \cap \partial_n^+$ is a vertex.

DEFINITION 8. Let

$$\mathcal{I}_n = \{x \in \mathbb{R}^2 \mid x \in \partial_n^- \cap \partial_n^+\}, \tag{2.11}$$

the set of *intermediate* vertices of ∂_n and let $\gamma_n^{\pm} = \partial_n^{\pm} - \mathcal{I}_n$. The vertices of $\gamma_n^{\pm}$ are called *plus* and *minus* vertices respectively, see figure 4.

Since the union of $\mathcal{I}_n$ and $\gamma_n^{\pm}$ is ∂_n, every vertex of ∂_n is either a plus, minus or intermediate vertex.

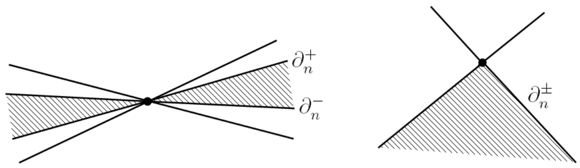

FIGURE 4. *An intermediate vertex (left) and a plus/minus vertex (right) of ∂_n.*

LEMMA 2.9. *Let $x \in \partial_n$ a vertex. If x is a plus, minus or intermediate vertex, then y is plus, minus or intermediate vertex respectively for all $y \in \mathcal{O}(x)$.*

Proof. For $x \in \mathcal{I}_n$ we have $\iota(x) \leq n-1$ and $\chi(x) \geq n+2$. If $x \in \gamma_n^+$, then $x \in B_n \cap B_{n+1}$ but $x \notin B_{n-1}$. Hence $\iota(x) = n$. For a vertex we must have $\mu(x) \geq 2$. Thus a vertex in γ_n^+ satisfies $\iota(x) = n$ and $\chi(x) \geq n+3$. Similarly, if $x \in \gamma_n^-$, then $\iota(x) \leq n-1$ and $\chi(x) = n+1$.

Since these conditions are mutually exclusive, and, by lemma 2.2, $\iota(x) = \iota(y)$ and $\chi(x) = \chi(y)$ for all $y \in \mathcal{O}(x)$, the result follows. □

DEFINITION 9. Let v be a vertex of $\mathcal{P}_n$, then v is a *vertex of type I* if all edges incident to v are plus edges and v is a *vertex of type II* if all edges incident to v are minus edges. Finally, a vertex v is a *vertex of type III*, if the edges incident to v are alternately plus and minus edges.

DEFINITION 10. Let $x \in \partial_n$ and $v = \pi(x) \in \partial\mathcal{P}_n$. We define $\tilde{\mu}(v)$ to be the number of edges that are *locally* incident to v. If x lies on the interior of an edge, we define $\tilde{\mu}(v) = 1$.

By *locally* in definition 10 we mean the number of edges incident to a vertex in a small neighborhood, since an edge can have its vertices identified on the torus, see for instance the puzzles $\mathcal{P}_1, \mathcal{P}_6$ and $\mathcal{P}_7$ relative to $\mathbb{Z}^2$ in Figure 3.

If we set $\tilde{\mathcal{I}}_n = \pi(\mathcal{I}_n)$, then

LEMMA 2.10.

$$\partial^-\mathcal{P}_n \cap \partial^+\mathcal{P}_n = \tilde{\mathcal{I}}_n \tag{2.12}$$

Proof. We need to show that $\pi(\gamma_n^-) \cap \pi(\gamma_n^+) = \emptyset$. Let $v \in \mathbb{T}$ and $x \in \gamma_n^+$ such that $\pi(x) = v$. Since $\pi^{-1}(v) \cap B_n = \mathcal{O}(x)$ by lemma 2.2 and $y \in \gamma_n^{\pm}$ for all $y \in \mathcal{O}(x)$ if $x \in \gamma_n^{\pm}$ by the proof of lemma 2.9, we have $\pi(\gamma_n^-) \cap \pi(\gamma_n^+) = \emptyset$ and hence (2.12). □

The following proposition relates the combinatorial properties of B_n to that of the torus puzzles $\mathcal{P}_n$ on the torus $\mathbb{T}$.

PROPOSITION 2.11. *Let $x \in \partial_n$ be a vertex and $v = \pi(x)$. If x is a plus, minus or intermediate vertex, then v is of type I, II or III respectively and*

$$\tilde{\mu}(v) = \mu(x) + 1 \qquad \text{if } v \text{ is of type I or II,} \tag{i}$$

$$\tilde{\mu}(v) = 2\mu(x) + 2 \qquad \text{if } v \text{ is of type III.} \tag{ii}$$

Proof. By lemma 2.9, if x is a plus or minus or intermediate vertex, then all vertices in $\mathcal{O}(x)$ are plus or minus vertices respectively. If x is a plus or minus vertex, it is clear that the corresponding vertex v is of type I or II respectively. So consider the case where $\mathcal{O}(x)$ consists of all intermediate vertices. For every subzone B_n^j sharing the intermediate vertex $y \in \mathcal{O}(x)$, the two edges contained in ∂B_n^j incident to y consists of one edge contained in ∂_n^- and one edge contained in ∂_n^+, cf. figure 4. Hence, for every $\mathcal{P}_n^j$ that shares the common vertex v, there is one minus edge and one plus edge incident to v. By lemma 2.10, $\partial^-\mathcal{P}_n \cap \partial^+\mathcal{P}_n = \tilde{\mathcal{I}}_n$, so the minus edge incident to v of one subzone $\mathcal{P}_n^i$ is identified to the minus edge incident to v of the neighbouring subzone $\mathcal{P}_n^j$ for certain $i, j \in J_n$. Similarly, plus edges are mapped to plus edges, thus the edges incident to v are alternately plus and minus edges, so v is of type III.

To prove the second statement, note that if v is of type I or II, then to every vertex $y \in \mathcal{O}(x)$ there are exactly two plus or minus edges of ∂B_n^j for of some $j \in J_n$ incident to y. Exactly $2\sigma(x) = 2(\mu(x) + 1)$ plus or minus edges are mapped to $\mathbb{T}$ and are incident to v. For any edge $\tilde{e} \subset \partial^-\mathcal{P}_n$, $\pi^{-1}(\tilde{e}) \cap \partial_n^- = e \cup e'$ for certain edges $e, e' \subset \partial_n^-$ by theorem 2.7 and this yields that $\tilde{\mu}(v) = \sigma(x) = \mu(x) + 1$ which proves (i). If v is of type III, then incident to every vertex $y \in \mathcal{O}(x)$ are exactly two plus edges and two minus edges of ∂_n. By similar reasoning, we have that $\tilde{\mu}(v) = 2\sigma(x) = 2\mu(x) + 2$ and proves (ii). □

The statement of the previous proposition is illustrated by figure 1 and figure 5, which shows the puzzle $\mathcal{P}_4$ relative to $\mathbb{Z}^2$.

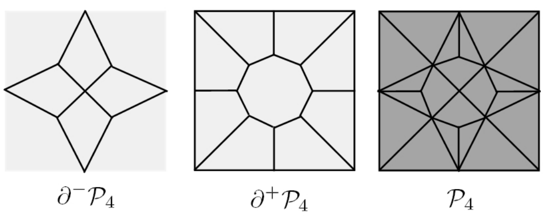

FIGURE 5.

DEFINITION 11. A lattice Λ is in *general position* if the Brillouin lines of M_Λ intersect at most pairwise: $\mu(x) \leq 2$ for all $x \in \mathbb{R}^2$.

Almost all lattices are in general position, in the sense that the set of lattices in general position has full measure in the set of all lattices. However, lattices not in general position are also dense in this set, see [**4**].

Example 1. Consider the following family of lattices

$$\Lambda(\rho,\theta) = (1,0)\mathbb{Z} \oplus \rho(\cos\theta, \sin\theta)\mathbb{Z},$$

with $\theta \in (0,\pi)$ and $\rho > 0$ rational. First consider the case where $\rho = 1$. It is clear that $L_{(2,0)}$ intersects $(1,0)$. An easy computation shows that both lines L_{g_1} en L_{g_2} intersect $(1,0)$, where $g_1, g_2 \in \Lambda(1,\theta)$,

$$g_1(\theta) = (1+\cos\theta, \sin\theta) \qquad \text{and} \qquad g_2(\theta) = (1-\cos\theta, -\sin\theta).$$

Hence, the lattices $\Lambda(1,\theta)$ with $\theta \in (0,\pi)$ are not in general position. Now write $\rho = \frac{p}{q}$ with $p, q \in \mathbb{N}$ coprime. Clearly, $q\Lambda(\rho,\theta) = (q,0)\mathbb{Z} \oplus p(\cos\theta, \sin\theta)\mathbb{Z} \subset \Lambda(\rho,\theta)$. Let τ be the least common multiple of p and q. Then $\tau g_1, \tau g_2 \in q\Lambda(\rho,\theta) \subset \Lambda(\rho,\theta)$ and $L_{(2\tau,0)}, L_{\tau g_1}$ and $L_{\tau g_2}$ intersect $(\tau, 0)$. Hence, the lattices $\Lambda(\rho,\theta)$ with $\theta \in (0,\pi)$ and $\rho > 0$ rational are not in general position. This is a dense and uncountable family in the set of all lattices.

We can write $\Lambda = B\mathbb{Z}^2$ with $B \in \mathrm{GL}(2,\mathbb{R})$. This matrix B gives rise to a (positive definite) quadratic form induced by the positive definite matrix B^tB. The Brillouin lines relative to the Euclidean metric and the lattice $\Lambda = B\mathbb{Z}^2$ are identical to the Brillouin lines relative to the lattice $\mathbb{Z}^2$ with the metric induced by the matrix B^tB.

An interesting result, proved by Kupka, Peixoto and Pugh in [**9**], is the following relation between the coefficients of a quadratic form and the notion of general position.

Theorem 2.12. *If the coefficients a, b, c of the positive definite quadratic form Q are rationally independent, then no three of its Brillouin lines meet at a common point.*

It is understood that the Brillouin lines in theorem 2.12 are the Brillouin lines relative to the metric induced by the quadratic form Q.

So if $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2,\mathbb{R})$, then M_Λ with $\Lambda = B\mathbb{Z}^2$ is in general position if the coefficients $a^2+c^2, ab+cd$ and b^2+d^2 are rationally independent. It is not known whether the converse of theorem 2.12 holds.

Definition 12. Two puzzles $\mathcal{P}_n$ and $\mathcal{P}'_n$ are homeomorphic if there exists a homeomorphism $h_n : \mathbb{T} \to \mathbb{T}'$ such that $h_n(\partial\mathcal{P}_n) = \partial\mathcal{P}'_n$.

Proposition 2.13. *Let Λ be in general position and Λ' not in general position. Then there exists an $n \in \mathbb{N}$ such that $\mathcal{P}_n$ and $\mathcal{P}'_n$ are not homeomorphic.*

Proof. By assumption, $\mu(x) = 2$ for every vertex $x \in M_\Lambda$, hence $\tilde{\mu}(u) = 2+1 = 3$ or $2(2+1) = 6$ for $u = \pi(x)$ of type I/II or III respectively, for every vertex u of every $\mathcal{P}_n$ by proposition 2.11. On the other hand, there exists at least two (antipodal) vertices $y \in M_{\Lambda'}$ for which $\mu(y) \geq 3$. For a certain n, $y \in \partial'_n$ is an intermediate vertex. Thus $\tilde{\mu}(v) \geq 2(3+1) = 8$ for $v = \pi'(y) \in \mathcal{P}_n'$. Hence, these puzzles can not be homeomorphic. $\square$

Hence, arbitrarily close to a given lattice, there exists a lattice such that the torus puzzles of the associated tori are not pairwise homeomorphic.

3. *Asymptotic Behavior of B_n*

We study the behavior of B_n for $n \to \infty$. More precisely, we derive bounds on the distance of B_n from the origin and show that B_n is contained in a circular annulus with decreasing modulus. Consequently, B_n always becomes circular shaped, independent of the underlying lattice.

If we let G be the set of all lattices in $\mathbb{R}^2$, then we define two lattices $\Lambda, \Lambda' \in G$ to be *conformally equivalent*, $\Lambda \sim \Lambda'$, if there exists a conformal matrix A,

$$A = \lambda \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \tag{3.1}$$

where $\lambda > 0$ and $\theta \in [0, \pi)$, such that $\Lambda' = A(\Lambda)$. We denote $\mathcal{G} = G/\sim$.

Remark 3. Note that A is orientation preserving and that $A(L_g) = L_{A(g)}$. Hence

$$A(M_\Lambda) = M_{A(\Lambda)}. \tag{3.2}$$

Every lattice $\Lambda \in \mathcal{G}$ can be represented as $\Lambda = B(\mathbb{Z}^2)$ where

$$B = \begin{pmatrix} 1 & \alpha \\ 0 & \beta \end{pmatrix} \tag{3.3}$$

with $(\alpha, \beta) \in \mathcal{H} = (-\infty, \infty) \times (0, \infty) \subset \mathbb{R}^2$, the upper half plane. In other words, a lattice in $\mathcal{G}$ has the form

$$\Lambda = (1,0)\mathbb{Z} \oplus (\alpha, \beta)\mathbb{Z}.$$

By *modular symmetry*, this representation is not unique. That is, if two lattices Λ, Λ' are generated by the vectors (ω_1, ω_2) and (ω_1', ω_2') then $\Lambda = \Lambda'$ if and only if

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix},$$

with $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})$. For $\Lambda, \Lambda' \in \mathcal{G}$, we have $\Lambda = \Lambda'$ if the associated matrix has the form $\begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}$ with $n \in \mathbb{Z}$. Hence, the points $(\alpha + n, \beta) \in \mathcal{H}$ for $n \in \mathbb{Z}$ all represent the same lattice.

If $x \in \mathrm{Int}(B_n)$, then $\iota(x) = n$ and by lemma 2.1,

$$\iota(x) = \#\left\{y \in \mathbb{R}^2 \mid \pi(y) = \pi(x) \text{ and } |y| < |x|\right\} = n,$$

which we proved to be equivalent to

$$\#\left\{g \mid g \in \Lambda \cap D(x, |x|)\right\} = n. \tag{3.4}$$

The following (classical) result is essential in this respect, the proof of which can be found in [**5**].

THEOREM 3.1 (Van der Corput). *Let D be a region bounded by a convex simple closed curve, piecewise twice differentiable, with radius of curvature bounded above by R. The discrepancy Δ of D, the difference between the number of integer points in D and the area of D, satisfies*

$$\Delta = O(R^{2/3}) \tag{3.5}$$

Theorem 3.1 gives rise to the following bounds on the distance of a point $x \in B_n$ from the origin.

THEOREM 3.2. *Let $\Lambda = B(\mathbb{Z}^2)$ where $B = \begin{pmatrix} 1 & \alpha \\ 0 & \beta \end{pmatrix}$, $(\alpha, \beta) \in \mathcal{H}$. Then there exists a constant $K_\Lambda > 0$ depending only on the lattice Λ such that for $x \in B_n$ and $n \geq 1$,*

$$|x| \in \left[\left(\frac{\beta n}{\pi}\right)^{1/2} - \frac{K_\Lambda}{n^{1/6}}, \ \left(\frac{\beta n}{\pi}\right)^{1/2} + \frac{K_\Lambda}{n^{1/6}} \right]. \tag{3.6}$$

Proof. First let $x \in \mathrm{Int}(B_n)$. Since $\det(B) = \beta \neq 0$, B is invertible. Let $C_x = \partial D(x, |x|)$, then $E_x := B^{-1}(C_x)$ is an ellipse and the region bounded by this ellipse satisfies the requirements of theorem 3.1. The radius of curvature of an ellipse with major and minor axes given by a and b respectively is bounded from above by $R = \frac{a^2}{b}$. Let R_x denote the upper bound on the radius of curvature of E_x and let $t_n(x) = (\frac{\pi}{\beta})^{1/2}|x|$. The (semi)axes of E_x are proportional to $|x|$ and hence to $t_n(x)$, thus R_x is proportional to $t_n(x)$ where the constant of proportionality depends only on the lattice Λ and

$$|B^{-1}(D(x, |x|))| = \det(B^{-1})\pi|x|^2 = \frac{\pi}{\beta}|x|^2 = t_n(x)^2. \tag{3.7}$$

From equation (3.4), it follows that

$$\#\left\{g \mid g \in \mathbb{Z}^2 \cap B^{-1}(D(x, |x|))\right\} = n, \tag{3.8}$$

so by theorem 3.1 and (3.7)

$$n = |B^{-1}(D(x, |x|))| + O(t_n(x)^{2/3}) = t_n(x)^2 + O(t_n(x)^{2/3}). \tag{3.9}$$

Put $t_n(x) = \sqrt{n}\,(1 + z_n(x))$, with $z_n(x)$ the error term. Since $t_n(x) > 0$, $1 + z_n(x) > 0$ and by (3.9),

$$|n - n\,(1 + z_n(x))^2\,| \leq C_\Lambda(\sqrt{n}\,(1 + z_n(x)))^{2/3}, \tag{3.10}$$

for some constant $C_\Lambda > 0$ depending only on the lattice Λ. Then (3.10) for $n \geq 1$, after some manipulation, reads

$$|z_n(x)| \leq \frac{C_\Lambda}{n^{2/3}} \frac{(1 + z_n(x))^{2/3}}{z_n(x) + 2}. \tag{3.11}$$

For $z_n \in (-1, \infty)$, $0 < \frac{(1+z_n)^{2/3}}{z_n+2} \leq \frac{2^{2/3}}{3}$, so (3.11) reduces to $|z_n(x)| \leq \frac{C'_\Lambda}{n^{2/3}}$, where $C'_\Lambda = \frac{2^{2/3}}{3}C_\Lambda$ yielding

$$|z_n(x)|\sqrt{n} \leq \frac{C'_\Lambda}{n^{2/3}}\sqrt{n} = \frac{C'_\Lambda}{n^{1/6}}. \tag{3.12}$$

Since $t_n(x) = (\frac{\pi}{\beta})^{1/2}|x|$ the result follows for all $x \in \mathrm{Int}(B_n)$ with $K_\Lambda = (\frac{\beta}{\pi})^{1/2}C'_\Lambda$. Letting x approach ∂_n, we see that these bounds are in fact valid for all $x \in B_n$. $\square$

Remark 4. Note that $\det(B) = \beta$ is independent of the representation of the lattice, so the statement of theorem 3.2 is well-defined.

4. *Rigidity of M_Λ*

Here we prove our main result that the focal decomposition M_Λ is *rigid* in the sense that M_Λ and $M_{\Lambda'}$ are homeomorphic if and only if Λ and Λ' are conformally equivalent.

Definition 13. We define $M_\Lambda \simeq M_{\Lambda'}$, if there exists an orientation preserving homeomorphism

$$\varphi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \quad \text{such that} \quad \varphi(M_\Lambda) = M_{\Lambda'}. \tag{4.1}$$

Notation. In order to distinguish between the Brillouin zones relative to Λ and Λ', we denote these B_n and B'_n respectively.

Theorem 4.1 (rigidity theorem). *$M_\Lambda \simeq M_{\Lambda'}$ if and only if Λ and Λ' are conformally equivalent.*

For the proof of theorem 4.1, we need the following lemmas.

Lemma 4.2. *Let φ be as in definition 13. Then φ induces a bijection $\psi : \Lambda_* \to \Lambda'_*$, defined by*

$$\varphi(L_g) = L_{\psi(g)} = L_{g'}. \tag{4.2}$$

Proof. Because φ is a homeomorphism, $\mu(x) = \mu(x')$ where $x, x' \in \mathbb{R}^2$, $\varphi(x) = x'$. In particular, φ maps vertices to vertices. Consider a vertex x that is the intersection point of m Brillouin lines L_{g_i}, $g_i \in \Lambda$ for $i = 1, ..., m$, so $\mu(x) = m$.

If $\varphi(x) = x'$, then x' is the intersection point of n Brillouin lines $L_{g'_j}$, $g'_j \in \Lambda'$, $j = 1, ..., m$. Let $g = g_i$ for some $i = 1, ..., m$. The plane minus L_g divides $\mathbb{R}^2$ into two connected half-planes H^1_g and H^2_g, i.e. $\mathbb{R}^2 \backslash L_g = H^1_g \cup H^2_g$. Locally, there are exactly $m-1$ edges e^1_k incident to x such that $e^1_k \subset H^1_g$ and $m-1$ edges e^2_k incident to x with $e^2_k \subset H^2_g$. Hence, $\varphi(H^1_g)$ contains $m-1$ edges $\tilde{e}^1_k = \varphi(e^1_k)$ incident to x' and $\varphi(H^2_g)$ contains $m-1$ edges $\tilde{e}^2_k = \varphi(e^2_k)$ incident to x'. So locally the image $\varphi(L_g)$ goes across x' as a straight line segment. Since this holds for every vertex, $\varphi(L_g) \subseteq L_{g'}$ for some $g' = g'_j \in \Lambda'_*$. The same arguments show that $\varphi^{-1}(L_{g'}) \subseteq L_g$, thus $\varphi(L_g) = L_{g'}$.

Since φ is a homeomorphism, it is seen that the map $\psi : \Lambda_* \to \Lambda'_*$ defined by (4.2) is a bijection and this concludes the proof. □

Lemma 4.3. *Given φ as in definition 13. There exists a uniform $N \in \mathbb{N}$ such that, if $x \in \mathrm{Int}(B_n)$, i.e. $\iota(x) = n$, then $n - N \leq \iota(x') \leq n + N$*

Proof. Let $x \in \mathrm{Int}(B_n)$, then $\iota(x) = n$, i.e. there are n lines L_g such that $L_g \cap \ell_x \neq \emptyset$. Let $x' = \varphi(x)$. By lemma 4.2, $\varphi(L_g) = L_{g'}$, with $g \in \Lambda_*$ and $g' \in \Lambda'_*$. Let $\gamma = \varphi(\ell_x)$, then γ is a continuous curve between $\varphi(0)$ and x'. Every L_{g_i}, $g_i \in \Lambda_*$, $i = 1, ..., n$ has exactly one point of intersection with ℓ_x and is transversal to ℓ_x.

Hence there are exactly n Brillouin lines $L_{g'_j}$, $g'_j \in \Lambda'_*$, $j = 1, ..., n$ which have exactly one point of intersection with γ. Moreover, because L_{g_i} are transversal to ℓ_x for all $i = 1, ..., n$, $L_{g'_j}$ are transversal to γ for all $j = 1, ..., n$.

Let D be the disc $D(O, R)$ with $R = |\varphi(0)| < \infty$. The curve γ can have (multiple) intersection points with $\ell_{x'}$ and γ meets $\ell_{x'}$ at the point x', see figure 6.

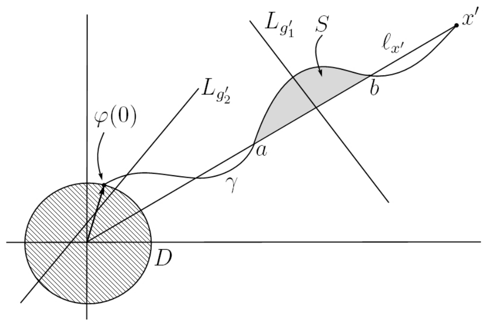

FIGURE 6. *Proof of lemma 4.3.*

Suppose γ has intersection points with $\ell_{x'}$ and let a, b be two consecutive intersection points. Let S be the Jordan domain enclosed by $\ell_{x'}$ and γ between a and b. If a Brillouin line $L_{g'_1}$, $g'_1 \in \Lambda'_*$, enters S by crossing γ, then it has to leave S through $\ell_{x'}$, since $L_{g'_1}$ has only one point of intersection with γ. Suppose however that a line L_{g_2} intersects ℓ_x, but that the image $L_{g'_2}$ does not intersect $\ell_{x'}$. In this case, the line $L_{g'_2}$ has to escape through the disc D. But because at most finitely many Brillouin lines can meet any bounded subset of $\mathbb{R}^2$, the number of Brillouin lines that can escape through the disc D is uniformly bounded by a certain $N \in \mathbb{N}$. Conversely, there are lines that could intersect with $\ell_{x'}$, but not with γ. Again, since $\iota(\varphi(0)) \leq N$, this number of lines is uniformly bounded by N. Hence $\iota(x') = n + \iota(\varphi(0)) \leq n + N$. Hence, if $\iota(x) = n$ and $\varphi(x) = x'$, then $n - N \leq \iota(x') \leq n + N$. □

Let $\mathcal{B}_g$ be the *bundle of Brillouin lines* consisting of all Brillouin lines parallel to L_g. The Brillouin lines in a bundle are parallel, so by lemma 4.2 and injectivity of φ, we see that bundles are mapped to bundles,

$$\varphi(\mathcal{B}_g) = \mathcal{B}_{g'} \quad \text{where} \quad g' = \psi(g). \tag{4.3}$$

We call an element g that is the generator of the subgroup formed by all lattice points on the line through 0 and g the *generator of the bundle* $\mathcal{B}_g$.

LEMMA 4.4. *Let φ be as in definition 13, then there exists a linear Q and a bounded $\delta : \mathbb{R}^2 \to \mathbb{R}^2$ such that*

$$\varphi(x) = Q(x) + \delta(x),$$

for all $x \in \mathbb{R}^2$.

Proof. We will show that there exists a tiling of the plane by a parallelogram which is mapped by φ to a tiling of the plane by a second parallelogram. This will give us an affine map $x \mapsto Q(x) + c$ which also preserves this tiling. Because the parallelogram in the image space of φ has a bounded diameter K_0 we get

$$|\varphi(x) - (Q(x) + c)| \leq K_0,$$

thus $|\delta(x)| \leq |c| + K_0 := K$.

Let $g \in \Lambda$ be the generator of the bundle

$$\mathcal{B}_g = \{L_{kg} \mid k \in \mathbb{Z} \setminus \{0\}\}$$

and $\varphi(\mathcal{B}_g) = \mathcal{B}_{g'}$ where $g' \in \Lambda'$ is the generator. Assume

$$\varphi(L_g) = L_{kg'},$$

say with $k > 0$. Define

$$\mathcal{B}_g^{\varphi} = \{L_{(1+j(k+1))g} \mid j \in \mathbb{Z}\} \subset \mathcal{B}_g$$

and

$$\mathcal{B}_{g'}^{\varphi} = \{L_{(k+j(k+1))g'} \mid j \in \mathbb{Z}\} \subset \mathcal{B}_{g'}.$$

Notice,

$$\varphi(L_{(1+j(k+1))g}) = L_{(k+j(k+1))g'},$$

for $j \in \mathbb{Z}$. Furthermore, the bundle $\mathcal{B}_g$ (and $\mathcal{B}_{g'}$) does not consist of equally spaced lines. Namely, the strip between L_g and L_{-g} is twice as wide as the other strips. However, the collections $\mathcal{B}_g^{\varphi}$ and $\mathcal{B}_{g'}^{\varphi}$ do consist of equally spaced lines.

Now choose $g_1, g_2 \in \Lambda$ independently. Then the collections $\mathcal{B}_{g_1}^{\varphi}$ and $\mathcal{B}_{g_2}^{\varphi}$ define a tiling of the plane by parallelograms. Even so the collections $\mathcal{B}_{g_1'}^{\varphi}$ and $\mathcal{B}_{g_2'}^{\varphi}$. This tiling is preserved by φ which now can be approximated by an affine map as described above. □

Proof of theorem 4.1. The *if* part easily follows, because if $\Lambda' = A(\Lambda)$ with A conformal, then $A(M_\Lambda) = M_{\Lambda'}$, hence $M_\Lambda \simeq M_{\Lambda'}$ with $\varphi = A$.

To prove the *only if* part, by lemma 4.4, the linear part $|Q(x)| \to \infty$ for $|x| \to \infty$. Since $\delta(x)$ is bounded, $\frac{|\delta(x)|}{|Q(x)|} \to 0$ for $|x| \to \infty$. Thus for $|x| \to \infty$, the behaviour of φ is completely determined by Q. By theorem 3.2, B_n converges to a large circle, for $n \to \infty$. This implies in turn, by lemma 4.3, that $\varphi(B_n)$ converges to a large circle for $n \to \infty$. Hence Q maps circles to circles. The only possible non-singular linear map that maps circles is to circles is a rotation or reflection combined with dilatation. A reflection reverses orientation, and φ is orientation preserving if and only if Q is orientation preserving. So Q cannot be a reflection. Hence, Q is a combination of a rotation and dilatation, that is, Q is conformal.

We show that $\varphi(M_\Lambda) = Q(M_\Lambda)$ by showing that $\varphi(\mathcal{B}_g) = Q(\mathcal{B}_g)$ for every bundle. Given $\mathcal{B}_g$, there exists a conformal map $A = \lambda R(\theta)$ with $\lambda \neq 0$ and $R(\theta)$ a rotation, such that $\varphi(\mathcal{B}_g) = A(Q(\mathcal{B}_g))$. We claim that $\theta = 0 \pmod{2\pi}$ and $\lambda = 1$, i.e. $A = \mathrm{Id}$. First suppose that $\theta \neq 0$. Then $Q(L_g)$ and $\varphi(L_g)$ with $L_g \in \mathcal{B}_g$ are non-parallel lines in $\mathbb{R}^2$ and hence $|\varphi(x) - Q(x)|$ is unbounded for $x \in L_g$. This contradicts the fact that δ is bounded, see lemma 4.4.

To show that $\lambda = 1$, consider the points $x_k = \frac{1}{2}gk \in L_{gk}$, $k \in \mathbb{Z}^*$. By what we just showed we have that $\lambda Q(g) = g'$. Hence,

$$Q(x_k) = \frac{1}{2}\frac{k}{\lambda}g'.$$

The Brillouin lines in a bundle are mapped by φ in a bijective and order preserving manner to the image bundle $\mathcal{B}_{g'}$. We may assume that there exists $m \in \mathbb{Z}$ such that $\varphi(x_k) \in L_{g'(k+m)}$ for $k \geq 1$. Now the bound on δ we found in lemma 4.4 gives the following estimate for all $k \geq 1$.

$$\begin{aligned} K &\geq |\varphi(x_k) - Q(x_k)| \\ &\geq |\frac{1}{2}(k+m)g' - \frac{1}{2}\frac{k}{\lambda}g'| \\ &= \frac{1}{2}|k(1-\frac{1}{\lambda}) + m| \cdot |g'|. \end{aligned} \tag{4.4}$$

This is only possible if $\lambda = 1$.

Hence $A = \mathrm{Id}$ and it follows that

$$\varphi(M_\Lambda) = Q(M_\Lambda) = M_{Q(\Lambda)} = M_{\Lambda'}.$$

Thus $\Lambda' = Q(\Lambda)$ and this proves the theorem. $\square$

5. *Rigidity of Torus Puzzles*

Next we study the rigidity of torus puzzles. We define an equivalence relation on torus puzzles and show that, for almost all lattices, the torus puzzles relative to two lattices are pairwise equivalent if and only if the lattices are conformally equivalent. We use the rigidity of M_Λ to prove this result.

Let $\tau : \mathbb{R}^2 \to \mathbb{R}^2$, $\tau(x) = -x$ be the *antipodal map*. By symmetry, $\tau(M_\Lambda) = M_\Lambda$. Let $\tilde{\tau} : \mathbb{T} \to \mathbb{T}$ be the map that satisfies $\pi \circ \tau = \tilde{\tau} \circ \pi$. Let $\Lambda = (1,0)\mathbb{Z} \oplus (\alpha, \beta)\mathbb{Z} \in \mathcal{G}$. Denote symbolically the points $0, 1, 2, 3 \in \mathbb{T}$ defined by $i = \pi(x_i)$, $i = 0, 1, 2, 3$ with

$$x_0 = (0,0), \quad x_1 = \frac{1}{2}(1,0), \quad x_2 = \frac{1}{2}(\alpha+1, \beta), \quad x_3 = \frac{1}{2}(\alpha, \beta). \tag{5.1}$$

A straightforward computation shows that the points $0, 1, 2, 3$ are the only fixed points of $\tilde{\tau}$.

Example 2. Figure 7 depicts $\mathcal{P}_1$ for $\Lambda = (1,0)\mathbb{Z} \oplus (\frac{1}{4}, \frac{3}{4})\mathbb{Z}$. The associated fixed points $0, 1, 2, 3$ discussed above are indicated with dots.

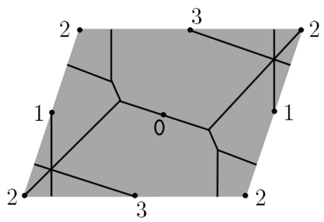

Figure 7. *Puzzle $\mathcal{P}_1$ of Example 2.*

Remark 5. The points 0 and 1 are independent of the representation of the lattice, but the the points 2 and 3 are not. If (α, β) represents Λ, then so does $(\alpha + n, \beta)$ with $n \in \mathbb{Z}$. The points 2 and 3 flip according to n being even or odd.

LEMMA 5.1. *Let Λ be in general position, then $0, 1 \in \partial\mathcal{P}_n$ but the points are not vertices, for all $n \geq 1$. In addition*

$$0 \in \partial^-\mathcal{P}_n \text{ and } 1 \in \partial^+\mathcal{P}_n \qquad \text{if } n \text{ is even,} \tag{i}$$

$$0 \in \partial^+\mathcal{P}_n \text{ and } 1 \in \partial^-\mathcal{P}_n \qquad \text{if } n \text{ is odd.} \tag{ii}$$

Proof. Let $x \in \pi^{-1}(0)$ or $\pi^{-1}(1)$. Because 0 and 1 are the fixed points of $\tilde{\tau}$, $\sigma(x)$ is always even. This yields that $\sigma(x) = 2$. Because if $\sigma(x) > 2$, i.e. $\sigma(x) \geq 4$, then $\mu(x) \geq 3$, contradicting the assumption that Λ is in general position. Hence, $\mu(x) = 1$, these points always lie on the interior of an edge of M_Λ. For each $n \geq 0$ we have $0, 1 \in \partial\mathcal{P}_n$.

We have that $\partial^-\mathcal{P}_0 = \{0\}$. Since $1 \neq 0$ and $1 \in \partial\mathcal{P}_0$, $1 \in \partial^+\mathcal{P}_0$. For $n = 1$, we have that $0 \in \partial^+\mathcal{P}_1$ and $1 \in \partial^-\mathcal{P}_1 = \partial^+\mathcal{P}_0$.

The properties (i) and (ii) for the point $0 \in \partial\mathcal{P}_n$ follows inductively. Namely, $0 \in \partial^+\mathcal{P}_1$ and the fact $\partial^+\mathcal{P}_n = \partial^-\mathcal{P}_{n+1}$ imply that 0 lies alternately on the plus and minus boundary. Similarly, one obtains both properties for the point $1 \in \partial\mathcal{P}_n$. □

DEFINITION 14 Equivalence of Puzzles. Let $\Lambda, \Lambda' \in \mathcal{G}$. Two puzzles $\mathcal{P}_n, \mathcal{P}'_n$ are equivalent, $\mathcal{P}_n \sim \mathcal{P}'_n$, if there exists an orientation preserving homeomorphism $h_n : \mathbb{T} \longrightarrow \mathbb{T}'$ such that

a) $h_n(\partial\mathcal{P}_n) = \partial\mathcal{P}'_n$ and
b) $h_n(0) = 0'$ and $h_n(1) = 1'$.

Comparing $\mathcal{P}_1$ relative to $\Lambda = (1,0)\mathbb{Z} \oplus (\frac{1}{4}, \frac{3}{4})\mathbb{Z}$ and $\mathcal{P}_1$ relative to $\Lambda = \mathbb{Z}^2$, cf. figure 3, it is clear that these two puzzles are not equivalent (or even homeomorphic).

THEOREM 5.2. *Let $\Lambda, \Lambda' \in \mathcal{G}$ in general position, then $\Lambda = \Lambda'$ if and only if $\mathcal{P}_n \sim \mathcal{P}'_n$ for all $n \in \mathbb{N}$.*

The proof of theorem 5.2 will be preceded by the following two lemmas.

NOTATION. In what follows, if a map on $\mathbb{R}^2$ or $\mathbb{T}$ has the property that it maps plus/minus or intermediate vertices (for a map on $\mathbb{R}^2$) or vertices of type I, II, or III (for a map on $\mathbb{T}$) to vertices of the same type, we say for short that the map *preserves the types of vertices.*

LEMMA 5.3. *Let Λ, Λ' in general position and $\mathcal{P}_n \sim \mathcal{P}'_n$, then*

$$h_n(\partial^\pm\mathcal{P}_n) = \partial^\pm\mathcal{P}'_n. \tag{5.2}$$

Moreover, h_n preserves the types of vertices.

Proof. We will only discuss the proof for $\partial^+\mathcal{P}_n$. The proof for the other boundary part is similar. Consider the following situation. Let $e_1, e_2 \subset \partial^+\mathcal{P}_n$ be plus edges with $\partial e_1 = \{u, v\}$ and $\partial e_2 = \{v, v_2\}$ and $h_n(e_1) \subset \partial^+\mathcal{P}'_n$. Then

$$h_n(e_2) \subset \partial^+\mathcal{P}'_n. \tag{5.3}$$

The proof of equation (5.3) will deal with two cases. The first when the vertex v of e_1 is of type I. Then $\tilde{\mu}(v) = 3$. Hence, $\tilde{\mu}(v') = 3$. Because, $v' \in \partial^+\mathcal{P}'_n$ we see that also v' is of type I (otherwise $\tilde{\mu}(v') = 6$). This means that every edge attached to v' is a plus edge: $h_n(e_2) \subset \partial^+\mathcal{P}'_n$.

The second case to consider is when v is of type III. Then $\tilde{\mu}(v) = 6$. Hence, $\tilde{\mu}(v') = 6$. Because, $v' \in \partial^+\mathcal{P}'_n$ we see that also v' is of type III (otherwise $\tilde{\mu}(v') = 3$). The edges attached to a type III vertex are alternately plus and minus. Because e_1 and $h_n(e_1)$ are plus edges we see that h_n maps all plus edges attached to v to plus edges attached to v' (and similarly, the minus edges are mapped to minus edges). This finishes the proof of equation (5.3).

The next step is to find a pair of plus edges e and $e' = h_n(e)$ to which we can apply the result stated in equation (5.3). If n is odd, then the edge e through 0 on $\mathbb{T}$ and e' through $0'$ on $\mathbb{T}'$ are plus edges. This follows from lemma 5.1. Since $h_n(0) = 0'$, we have $h_n(e) = e'$. If n is even we use the point $1 \in \partial^+\mathcal{P}_n$ and the fact that $h_n(1) = 1' \in \partial^+\mathcal{P}'_n$.

Because ∂_n^+ is path-connected (it is homeomorph to $\mathbb{S}^1$), $\partial^+\mathcal{P}_n$ is path-connected. Taking a path through $\partial^+\mathcal{P}_n$, traversing every plus edge at least once (possibly some edges more than once), the above arguments show that $h_n(\partial^+\mathcal{P}_n) \subseteq \partial^+\mathcal{P}'_n$. Similarly, we can apply these arguments to the inverse of h_n and obtain $h_n^{-1}(\partial^+\mathcal{P}'_n) \subseteq \partial^+\mathcal{P}_n$. We proved $h_n(\partial^+\mathcal{P}_n) = \partial^+\mathcal{P}'_n$. $\square$

LEMMA 5.4. *Let $\Lambda, \Lambda' \in \mathcal{G}$ in general position and $\mathcal{P}_n \sim \mathcal{P}'_n$. Then there exist an orientation preserving homeomorphism $\varphi_n : B_n \to B'_n$ such that*

$$\begin{tikzcd} B_n \arrow[r, "\varphi_n"] \arrow[d, "\pi"'] & B'_n \arrow[d, "\pi'"] \\ \mathcal{P}_n \arrow[r, "h_n"'] & \mathcal{P}'_n \end{tikzcd}$$

Moreover, $\varphi_n(\partial_n^\pm) = {\partial'}_n^\pm$ and φ_n preserves the types of vertices.

Proof. Write $B_n = \bigcup_{j \in J_n} B_n^j$ and $B'_n = \bigcup_{j' \in J'_n} B_n^{j'}$. Since $\mathcal{P}_n \sim \mathcal{P}'_n$, we have $|J_n| = |J'_n|$. Let $h_n^j = h_n|\mathrm{Int}(\mathcal{P}_n^j)$ with $h_n^j(\mathrm{Int}(\mathcal{P}_n^j)) = \mathrm{Int}({\mathcal{P}'}_n^{j'})$. Since the projections of the form $\pi : \mathrm{Int}(B_n) \to \mathbb{T} \setminus \partial\mathcal{P}_n$ are homeomoprhisms, the map

$$\varphi_n^j : \mathrm{Int}(B_n^j) \to \mathrm{Int}({B'}_n^{j'}), \quad x \mapsto {\pi'}^{-1}\left(h_n^j(\pi(x))\right) \cap \mathrm{Int}({B'}_n^{j'})$$

is also a homeomorphism. We can extend φ_n^j uniquely to a homeomorphism on the boundary ∂B_n^j, which we denote again φ_n^j. The extension $\varphi_n^j : B_n^j \to {B'}_n^{j'}$ is a homeomorphism for every $j \in J_n$. Moreover, it commutes: $\pi' \circ \varphi_n^j = h_n \circ \pi$.

Let $B_n^{j_1} \cap B_n^{j_2} = \{x\} \in \mathcal{I}_n$. We will prove that

$$\varphi_n^{j_1}(x) = \varphi_n^{j_2}(x).$$

Let $v = \pi(x)$. There are two lines through x, $\mu(x) = 2$. These lines determine two opposite sectors which locally coincide with $B_n^{j_1}$ and $B_n^{j_2}$ respectively. Furthermore,

$\pi^{-1}(v) \cap B_n$ consists of three points. Hence, there are three lines crossing v. Moreover, opposite sectors at v coincide locally with $\pi(B_n^{j_1})$ and $\pi(B_n^{j_2})$. To summarize, $B_n^{j_1} \cap B_n^{j_2} = \{x\}$ if and only if $\pi(B_n^{j_1})$ and $\pi(B_n^{j_2})$ coincide locally with opposite sectors at $v = \pi(x)$.

The map h_n is a homeomorphism. This means it will map opposite sectors at a type III vertex to opposite sectors of the image type III vertex. We proved that if $B_n^{j_1} \cap B_n^{j_2} = \{x\}$ then

$$\varphi_n^{j_1}(B_n^{j_1}) \cap \varphi_n^{j_2}(B_n^{j_2}) = \{\varphi_n^{j_1}(x)\} = \{\varphi_n^{j_2}(x)\}$$

This allows us to define a homeomorphism $\varphi_n : B_n \to B_n$ by $\varphi_n | B_n^j = \varphi_n^j$, which is orientation preserving since h_n is orientation preserving. □

Proof of theorem 5.2. If $\mathcal{P}_n \sim \mathcal{P}'_n$ for all $n \in \mathbb{N}$, then lemma 5.4 gives us a sequence of orientation preserving homeomorphisms $\{\varphi_n\}_{n\in\mathbb{N}}$, $\varphi_n : B_n \to B'_n$, satisfying the properties as stated in the lemma. Since $\partial_n^+ = \partial_{n+1}^-$,

$$\varphi_n(\partial_n^+) = \varphi_{n+1}(\partial_n^+). \tag{5.4}$$

We may assume that h_n is piecewise linear on $\partial\mathcal{P}_n$, i.e. linear on every edge of $\partial\mathcal{P}_n$, for all $n \in \mathbb{N}$. This makes the maps φ_n piecewise linear on ∂_n for all $n \in \mathbb{N}$. Assume that n is even, the case where n is odd is identical. Then $0 \in \partial^+\mathcal{P}_n$ and $\{\pm g\} = \pi^{-1}(0) \cap \partial_n$ and $\{\pm g'\} = \pi^{-1}(0') \cap \partial'_n$ for certain $g \in \Lambda_*$ and $g' \in \Lambda'_*$.

Since $\tau(B_n) = B_n$, $\tilde{\tau}(\mathcal{P}_n) = \mathcal{P}_n$. Hence, if h_n satisfies definition 14, then so does $\tilde{h}_n = h_n \circ \tilde{\tau}$. The map $\tilde{\varphi}_n = \varphi_n \circ \tau$ is the homeomorphism that commutes with the diagram of lemma 5.4 when one replaces h_n by $\tilde{h}_n$, so we may assume that if $\varphi_n(g) = g'$, then $\varphi_{n+1}(g) = g'$. Combining this with piecewise linearity of φ_n on ∂_n for all $n \in \mathbb{N}$ and (5.4) we have that

$$\varphi_n|\partial_n^+ = \varphi_{n+1}|\partial_n^+. \tag{5.5}$$

This holds for all $n \geq 1$. In fact, it also holds when $n = 0$, because $\partial_0^- = \emptyset$ thus $\partial_0 = \partial_0^+ = \partial_1^-$. Hence, the homeomorphisms $\{\varphi_n\}_{n\in\mathbb{N}}$ glue to a global homeomorphism $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$, with the property that $\varphi(M_\Lambda) = M_{\Lambda'}$, since $M_\Lambda = \bigcup_{n\in\mathbb{N}} \partial_n = \bigcup_{n\in\mathbb{N}} \partial_n^+$. Hence $M_\Lambda \simeq M_{\Lambda'}$. Since $\Lambda, \Lambda' \in \mathcal{G}$, $\Lambda = \Lambda'$ by theorem 4.1. □

References

1. G. D. Mostow, 'Quasi-Conformal Mappings in n-space and the Rigidity of the Hyperbolic Space Forms', *Publ. Math. IHES* 34, (1968), 53-104.
2. M.M.Peixoto and I. Kupka, 'On the Enumerative Geometry of Geodesics', *From Topology to Computation-Proceedings of the Smalefest*, Springer-Verlag, (1993), 243-253.
3. L. Bieberbach, 'Über die Inhaltsgleichheit der Brillouinschen Zonen', *Monatshefte für Mathematik und Physik* 48 (1939), 509-515.
4. G.A. Jones, 'Geometric and Asymptotic Properties of Brillouin Zones in Lattices', *Bulletin London Mathematical Society* 16 (1984), 241-263.
5. J.G. van der Corput, 'Über Gitterpunkte in der Ebene', *Mathematische Annalen* 81 (1920), 1-20.
6. I. Niven, H.S. Zuckerman and H.L Montgomery, 'An Introduction to the Theory of Numbers', 5th ed. John Wiley & Sons, (1991).
7. M.M. Peixoto, 'Focal Decompositions in Geometry, Arithmetic and Physics', *Geometry, Topology and Physics*, Eds. Apanosov / Bradlov / Rodrigues / Uhlembeck, Berlin, Walter de Gruyter, (1997), 213-232.

8. J.J.P. Veerman, M.M. Peixoto, A.C. Rocha and S. Sutherland, 'On Brillouin Zones', *Comm. Math. Phys.* 212/3 (2000), 725-744.

9. I. Kupka, M.M. Peixoto and C.C. Pugh, 'Focal Stability of Riemann Metrics', *Journal für die reine und angewandte Mathematik* 593 (2006), 31-72.

F.H. Kwakkel
Mathematics Institute
University of Warwick
Coventry CV4 7AL
United Kingdom

M. Martens
Department of Mathematics
SUNY at Stony Brook
Stony Brook NY 11794
USA

marco@math.sunysb.edu

M. M. Peixoto
Instituto Nacional Matemática Pura e Aplicada
Estrada Dona Castorina 110
Rio de Janeiro RJ 22460-320
Brasil

peixoto@impa.br