FORMALITY IN GENERELIZED KÄHLER GEOMETRY

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Abstract. Using Gualtieri's $\partial\overline{\partial}$-lemma for generalized Kähler manifolds we prove that if one of the generalized complex structures forming a generalized Kähler pair has holomorphically trivial canonical bundle, then a certain differential graded algebra associated to it is formal. As an application we prove that no nilpotent Lie algebra has a generalized Kähler structure.

Introduction

The requirement that a compact manifold admits a Kähler structure has many topological implications, in particular the manifold is formal in the sense of Sullivan [9, 4]. We prove a formality result for their recently introduced relatives, generalized Kähler manifolds [5].

In order to state our result we recall some of the theory on generalized complex structures, as introduced by Hitchin [7] and developed by Gualtieri [5]. A generalized complex structure is an orthogonal complex structure $\mathcal{J}$ on $T \oplus T^*$, the sum of tangent and cotangent bundles of a manifold, integrable with respect to the Courant bracket. The Courant bracket always restricts to a Lie bracket on L, the i-eigenspace of $\mathcal{J}$, endowing it with the structure of a Lie algebroid. As a consequence $\Omega^\bullet(L^*)$, the space of sections of $\wedge^\bullet L^*$, has a differential making it a differential graded algebra. This DGA is also related to a decomposition of forms on the manifold.

A generalized Kähler structure is a pair of commuting generalized complex structures which together determine a metric. Since each generalized complex structure is related to a Lie algebroid, we have associated to a generalized Kähler structure two differential graded algebras. These DGAs are also related to a bigrading of differential forms and cohomology on the manifold. Developing a Hodge theory for generalized Kähler manifolds and the bigrading of forms, Gualtieri proved the equality of a number of Laplacians on differential forms associated to a generalized Kähler structure [6]. As a consequence, he obtained $\partial\overline{\partial}$-lemma-like results for the associated operators.

Using Gualtieri's results and the relationship between the different differential complexes related to a generalized Kähler manifold we prove that if one of the generalized complex structures making up a generalized Kähler structure has holomorphically trivial canonical bundle, then the respective Lie algebroid is associated to a formal DGA in the sense of Sullivan. As an application of this result we prove that no generalized complex structure on a nilpotent Lie algebra can be completed to a generalized Kähler pair.

I would like to thank Marco Gualtieri for his suggestions and inspired advice as well as Marisa Fernández, Nigel Hitchin and Simon Salamon for suggestions on the first manuscript. I also thank IMPA for their hospitality while writing this paper. This research is supported by EPSRC.

1. Differential graded algebras

In this section we give a lightening review formality for differential graded algebras and recover the well known fact that the DGA associated to a nontrivial nilpotent Lie algebra is not formal.

Definition. A *differential graded algebra*, or *DGA* for short, is an $\mathbb{N}$ graded vector space $\mathcal{A}^\bullet$, endowed with a product and a differential d satisfying:

(1) The product maps $\mathcal{A}^i \times \mathcal{A}^j$ to $\mathcal{A}^{i+j}$ and is graded commutative:
$$a \cdot b = (-1)^{ij} b \cdot a;$$
(2) The differential has degree 1, $d : \mathcal{A}^k \longrightarrow \mathcal{A}^{k+1}$, and squares to zero;
(3) The differential is a derivation: for $a \in \mathcal{A}^i$ and $b \in \mathcal{A}^j$
$$d(a \cdot b) = da \cdot b + (-1)^i a \cdot db.$$

The cohomology of a DGA is defined in the standard way and naturally inherits a grading and a product, making it into a DGA on its own with $d = 0$. A *morphism* of differential graded algebras is a map preserving the structure above, i.e., degree, products and differentials. Any morphism of DGAs $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ gives rise to a morphism of cohomology $\varphi^* : H^\bullet(\mathcal{A}) \longrightarrow H^\bullet(\mathcal{B})$. A morphism φ is a *quasi-isomorphism* if the induced map in cohomology is an isomorphism.

Given a DGA, $\mathcal{A}$, for which $H^k(\mathcal{A})$ is finite dimensional for every k, one can construct another differential graded algebra that captures all the information about the differential and which is minimal in the following sense.

Definition. A DGA $(\mathcal{M}, d)$ is *minimal* if it is free as a DGA (i.e. polynomial in even degree and skew symmetric in odd degree) and has generators $e_1, e_2 \ldots, e_n, \ldots$ such that

(1) The degree of the generators form a weakly increasing sequence of positive numbers;
(2) There are finitely many generators in each degree;
(3) The differential satisfies $de_i \in \wedge\mathrm{span}\{e_1, \ldots, e_{i-1}\}$.

A *minimal model* for a differential graded algebra $\mathcal{A}$ is a minimal DGA, $\mathcal{M}$, together with a quasi-isomorphism $\psi : \mathcal{M} \longrightarrow \mathcal{A}$.

Since the cohomology of a DGA is also a DGA we can also construct its minimal model. The minimal models for $\mathcal{A}$ and $H^\bullet(\mathcal{A})$ are not the same in general.

Definition. A DGA $\mathcal{A}$ is *formal* if it has the same minimal model as its cohomology, or equivalently, there is a quasi-isomorphism $\psi : \mathcal{M} \longrightarrow H^\bullet(\mathcal{A})$, where $\mathcal{M}$ is the minimal model of $\mathcal{A}$. A manifold M is *formal* if $(\Omega^\bullet(M), d)$ is formal.

Example 1.1. (Nilpotent Lie algebras) A typical example of nonformal DGA can be obtained from a finite dimensional nilpotent Lie algebra $\mathfrak{g}$ with nontrivial bracket. The Lie bracket induces a differential d on $\wedge^\bullet\mathfrak{g}^*$ making it into a DGA. Furthermore, $\mathfrak{g}^*$ is filtered by $\mathfrak{g}^*_1 = \ker d$ and
$$\mathfrak{g}^*_i = \{v \in \mathfrak{g}^* : dv \in \wedge^2\mathfrak{g}^*_{i-1}\}.$$

Nilpotency implies that $\mathfrak{g}_s^* = \mathfrak{g}^*$ for some s. Let $\{e^1, \cdots, e^n\}$ be a basis for $\mathfrak{g}^*$ compatible with this filtration. Then

$$de^i \in \wedge^2 \mathrm{span}\{e^1, \cdots, e^{i-1}\}.$$

showing that $(\wedge^\bullet \mathfrak{g}^*, d)$ is minimal.

Since the bracket is nontrivial, $de^n \neq 0$ and hence one can see that $e^1 \wedge \cdots \wedge e^{n-1}$ is exact and $e^1 \wedge \cdots \wedge e^n$ is a volume element and therefore represents a nontrivial cohomology class. If $(\wedge \mathfrak{g}^*, d)$ was formal, there would be a map $\psi : (\wedge^\bullet \mathfrak{g}^*, d) \longrightarrow H^\bullet(\mathfrak{g})$, but

$$0 \neq \psi(e^1 \wedge \cdots \wedge e^n) = \psi(e^1 \wedge \cdots \wedge e^{n-1}) \cdot \psi(e^n) = 0 \cdot \psi(e^n) = 0.$$

So there is no such ψ and $\wedge^\bullet \mathfrak{g}^*$ is not formal.

2. Generalized complex structures and Lie algebroids

In this section we recall the definition of generalized complex structures and their relation to Lie algebroids, following [5].

Given a closed 3-form H on a manifold M, we define the *Courant bracket* of sections of the sum $T \oplus T^*$ of the tangent and cotangent bundles by

$$[\![X + \xi, Y + \eta]\!] = [X, Y] + \mathcal{L}_X \eta - \mathcal{L}_Y \xi - \tfrac{1}{2} d(\eta(X) - \xi(Y)) + i_Y i_X H.$$

The bundle $T \oplus T^*$ is also endowed with a natural symmetric pairing of signature (n, n):

$$\langle X + \xi, Y + \eta \rangle = \frac{1}{2}(\eta(X) + \xi(Y)).$$

Definition. A *generalized complex structure* on a manifold with closed 3-form (M, H) is a complex structure on the bundle $T \oplus T^*$ which preserves the natural pairing and whose i-eigenspace is closed under the Courant bracket.

A generalized complex structure can be fully described in terms of its i-eigenspace L, which is a maximal isotropic subspace of $T_{\mathbb{C}} \oplus T_{\mathbb{C}}^*$ satisfying $L \cap \overline{L} = \{0\}$.

Two extreme examples of generalized complex structures, with $H = 0$, are given by complex and symplectic structures: in a complex manifold we let $L = T^{0,1} \oplus T^{*1,0}$ and in a symplectic manifold we let $L = \{X - i\omega(X) : X \in T_{\mathbb{C}}\}$, where ω is the symplectic form. What distinguishes these structures is their *type* which is the dimension of the kernel of $\pi : L \longrightarrow T_{\mathbb{C}}$. So, a complex structure on M^n has type n at all points and symplectic structures have type zero at all points.

The Courant bracket does not satisfy the Jacobi identity. Instead we have the relation for the Jacobiator

$$\mathrm{Jac}\,(A, B, C) := [\![[\![A, B]\!], C]\!] + c.p. = \tfrac{1}{3} d(\langle [\![A, B]\!], C \rangle + c.p.),$$

where *c.p.* stands for cyclic permutations. However, the identity above also shows that the Courant bracket induces a Lie bracket when restricted to sections of any involutive isotropic space L. This Lie bracket together with the projection $\pi_T : L \longrightarrow TM$, makes L into a Lie algebroid and allows us to define a differential d_L on $\Omega^\bullet(L^*) = \mathbb{C}^\infty(\wedge^\bullet L^*)$ making it into a DGA. If L is the i-eigenspace of a generalized complex structure, then the natural pairing gives an isomorphism $L^* \cong \overline{L}$ and with this identification $(\Omega^\bullet(\overline{L}), d_L)$ is a DGA.

If a generalized complex structure has type zero over M, i.e., is of symplectic type, then $\pi : L \xrightarrow{\cong} T_{\mathbb{C}}$ is an isomorphism and the Courant bracket on $C^\infty(L)$

is mapped to the Lie bracket on $C^\infty(T_\mathbb{C})$. Therefore, in this particular case, $(\Omega^\bullet(\overline{L}), d_L)$ and $(\Omega^\bullet_\mathbb{C}(M), d)$ are isomorphic DGAs.

2.1. **Decomposition of forms.** A generalized complex structure can also be described using differential forms. Recall that the exterior algebra $\wedge^\bullet T^*$ carries a natural spin representation for the metric bundle $T \oplus T^*$; the Clifford action of $X + \xi \in T \oplus T^*$ on $\rho \in \wedge^\bullet T^*$ is

$$(X + \xi) \cdot \rho = i_X \rho + \xi \wedge \rho.$$

The subspace $L \subset T_\mathbb{C} \oplus T^*_\mathbb{C}$ annihilating a spinor $\rho \in \wedge^\bullet T^*_\mathbb{C}$ is always isotropic. If L is maximal isotropic, then ρ is called a *pure spinor* and it must have the following algebraic form at every point:

$$\rho = e^{B+i\omega} \wedge \Omega, \tag{2.1}$$

where B and ω are real 2-forms and Ω is a decomposable complex form. Pure spinors annihilating the same space must be equal up to rescaling, hence a maximal isotropic $L \subset T_\mathbb{C} \oplus T^*_\mathbb{C}$ may be uniquely described by a line subbundle $U \subset \wedge^\bullet T^*_\mathbb{C}$.

For a complex manifold $U = \wedge^{n,0} T^*$ and for a symplectic manifold U is generated by the globally defined closed form $e^{i\omega}$. In general we have the following definition.

Definition. Given a generalized complex structure $\mathcal{J}$, the line subbundle $U \subset \wedge^\bullet T^*_\mathbb{C}$ annihilating its i-eigenspace is the *canonical bundle* of $\mathcal{J}$.

Note that the condition $L \cap \overline{L} = \{0\}$ at the fiber of E over $p \in M$ is equivalent to the requirement that

$$\Omega \wedge \overline{\Omega} \wedge \omega^{n-k} \neq 0 \tag{2.2}$$

for a generator $\rho = e^{B+i\omega} \wedge \Omega$ of U at p, where $k = \deg(\Omega)$ and $2n = \dim(M)$.

By letting $\wedge^\bullet \overline{L} \subset \mathrm{Cliff}\,(L \oplus \overline{L})$ act on the canonical line bundle we obtain a decomposition of the differential forms on M^{2n}:

$$\wedge^\bullet T^*_\mathbb{C}(M) = \oplus_{k=-n}^{n} U^k, \qquad \text{where } U^k = \wedge^{n-k}\overline{L} \cdot \rho.$$

one can also describe the spaces U^k as the ik-eigenspaces of $\mathcal{J}$ acting on forms.

Letting $\mathcal{U}^k = C^\infty(U^k)$ and $d_H = d + H\wedge$, Courant integrability of the generalized complex structure is equivalent to

$$d_H : \mathcal{U}^k \longrightarrow \mathcal{U}^{k+1} \oplus \mathcal{U}^{k-1},$$

which allows us to define operators $\partial : \mathcal{U}^k \longrightarrow \mathcal{U}^{k+1}$ and $\overline{\partial} : \mathcal{U}^k \longrightarrow \mathcal{U}^{k-1}$ by composing d_H with the appropriate projections.

Given a local section ρ of the canonical bundle the operator $\overline{\partial}$ is related to d_L by

$$\overline{\partial}(\alpha \cdot \rho) = (d_L \alpha) \cdot \rho + (-1)^{|\alpha|} d_H \rho,$$

where $\alpha \in \Omega^\bullet(\overline{L})$ and $|\alpha|$ is the degree of α. In the particular case when $(M, \mathcal{J})$ has *holomorphically trivial canonical bundle*, i.e., there is a nonvanishing d_H-closed global section ρ of the canonical bundle, the above becomes

$$\overline{\partial}(\alpha \cdot \rho) = (d_L \alpha) \cdot \rho \tag{2.3}$$

and hence ρ furnishes an isomorphism of differential complexes.

3. Generalized Kähler manifolds

In this section we introduce generalized Kähler manifolds. For these manifolds both $\Omega^\bullet_{\mathbb{C}}(M)$ and $(\Omega^\bullet(\overline{L}), d_L)$ admit a bigrading and, in certain conditions, some differential operators $\Omega^\bullet_{\mathbb{C}}(M)$ correspond to differential operators on $\Omega^\bullet(\overline{L})$. This correspondence was also used by Yi Li to study the moduli space of a generalized Kähler structure [8] and is the key ingredient for our formality theorem.

Definition. A *generalized Kähler structure* on a manifold M^{2n} is a pair of commuting generalized complex structures $\mathcal{J}_1$, $\mathcal{J}_2$ on M such that

$$\langle \mathcal{J}_1\mathcal{J}_2 v, v\rangle > 0 \qquad \text{for } v \in T\oplus T^*\backslash\{0\}.$$

Let L_i be the i-eingenspace of $\mathcal{J}_i$. Since $\mathcal{J}_1$ and $\mathcal{J}_2$ communte, $\mathcal{J}_2$ furnishes a complex structure on L_1 with i-eigenspace $L_1\cap L_2$. Using the fact that the natural pairing has signature (n,n) and that $\langle \mathcal{J}_1\mathcal{J}_2\cdot,\cdot\rangle$ is positive definite one can show $\dim(L_1) = 2\dim(L_1\cap L_2)$. Since L_2 is closed under the Courant bracket, we see that $L_1\cap L_2$ is closed under the bracket in the Lie algebroid L_1, and hence $\mathcal{J}_2|_{L_1}$ is an integrable complex structure on L_1. Using this complex structure we can decompose

$$\wedge^\bullet\overline{L_1} = \oplus_{p,q}\wedge^{p,q}\overline{L_1} \qquad \text{and} \qquad d_{L_1} = \partial_{L_1} + \overline{\partial}_{L_1},$$

As in a complex manifold, the operators ∂_{L_1} and $\overline{\partial}_{L_1}$ are derivations, in the sense that they satisfy the Leibniz rule.

A generalized Kähler structure also gives a refinement of the deposition of forms into the spaces U^k. Since $\mathcal{J}_1$ and $\mathcal{J}_2$ commute one immediately obtains that the space of differential forms can be decomposed in terms of the eigenspaces of $\mathcal{J}_1$ and $\mathcal{J}_2$: $U^{p,q} = U^p_{\mathcal{J}_1}\cap U^q_{\mathcal{J}_2}$. This allows us to decompose d_H further in 4 components

$$d_H : \mathcal{U}^{p,q} \longrightarrow \mathcal{U}^{p+1,q+1} + \mathcal{U}^{p+1,q-1} + \mathcal{U}^{p-1,q+1} + \mathcal{U}^{p-1,q-1}.$$

In this case, the operator $\overline{\partial}$ for the generalized complex structure $\mathcal{J}_1$ corresponds to the sum of the last two terms:

$$\overline{\partial}_1 : \mathcal{U}^{p,q} \longrightarrow \mathcal{U}^{p-1,q+1} + \mathcal{U}^{p-1,q-1},$$

and we can define δ_+ and δ_- as the projections of $\overline{\partial}_1$ into each of the components

$$\delta_+ : \mathcal{U}^{p,q} \longrightarrow \mathcal{U}^{p-1,q+1} \qquad \delta_- : \mathcal{U}^{p,q} \longrightarrow \mathcal{U}^{p-1,q-1}.$$

By studying the Hodge theory of a generalized Kähler manifold, Gualtieri proved the following

Theorem 3.1. (Gualtieri [6]) $\delta_+\delta_-$**-lemma**. *In a compact generalized Kähler manifold*

$$\operatorname{Im}\delta_+\cap\operatorname{Ker}\delta_- = \operatorname{Im}\delta_-\cap\operatorname{Ker}\delta_+ = \operatorname{Im}(\delta_+\delta_-)$$

If $\mathcal{J}_1$ has holomorphically trivial canonical bundle, then the correspondence between $\overline{\partial}_1$ and d_{L_1} given in equation (2.3) also furnishes a correspondence between the operators ∂_{L_1} and $\overline{\partial}_{L_1}$ in $\wedge^\bullet\overline{L}$ and the operators δ_+ and δ_- in $\wedge^\bullet T^*_{\mathbb{C}}$. So, as a consequence of Theorem 3.1, the operators ∂_{L_1} and $\overline{\partial}_{L_1}$ satisfy the $\partial_{L_1}\overline{\partial}_{L_1}$-lemma and since they are derivations the same argument from [4] gives:

Theorem 3.2. *If $(M,\mathcal{J}_1,\mathcal{J}_2)$ is a compact generalized Kähler manifold and $\mathcal{J}_1$ has holomophically trivial canonical bundle, then the DGA $(\Omega^\bullet(\overline{L_1}), d_{L_1})$ is formal.*

In the case when $\mathcal{J}_1$ is a symplectic structure, then not only does it have a holomorphically trivial canonical bundle, but $(\Omega^\bullet(\overline{L_1}), d_{L_1})$ is isomorphic to $(\Omega^\bullet_{\mathbb{C}}(M), d)$. Therefore we have:

Corollary 1. If $(M, \mathcal{J}_1, \mathcal{J}_2)$ is a compact generalized Kähler manifold and $\mathcal{J}_1$ is a symplectic structure, then M is formal.

This corollary generalizes the original theorem of formality of Kähler manifolds [4].

4. Nilpotent Lie algebras

In this section we use Theorem 3.2 to prove that no nilpotent Lie algebra admits a generalized Kähler structure. Before we state the theorem we should stress that a generalized complex structure on a Lie algebra $\mathfrak{g}$ with closed 3-form $H \in \wedge^3\mathfrak{g}^*$ is just an integrable linear complex structure on $(\mathfrak{g} \oplus \mathfrak{g}^*, [\![\ ,\ ]\!])$, orthogonal with respect to the natural pairing, where the Courant bracket is defined by

$$[\![X + \xi, Y + \eta]\!] = [X, Y] + \mathcal{L}_X\eta - \mathcal{L}_Y\xi + i_Y i_X H,$$

and is a Lie bracket in this situation.

We also recall that a complex structure on a Lie algebra $\mathfrak{g}$ is called *abelian* if its i-eigenspace, $\mathfrak{g}^{1,0}$, is an abelian subalgebra of $\mathfrak{g} \otimes \mathbb{C}$ [1, 3]. By analogy, we say that a generalized complex structure on $\mathfrak{g}$ is *abelian* if the corresponding complex structure on $\mathfrak{g} \oplus \mathfrak{g}^*$ is abelian. Before we state our theorem on generalized Kähler structures on nilpotent Lie algebras we need a little lemma:

Lemma 4.1. If a Lie algebra $\mathfrak{g}$ admits an abelian generalized complex structure, then $\mathfrak{g}$ is abelian.

Proof. Let L be the i-eigenspce of an abelian generalized complex structure on $\mathfrak{g}$. Since L is abelian, so is its projection over $\mathfrak{g} \otimes \mathbb{C}$. Further, if $v \in \pi(L) \cap \pi(\overline{L})$ then v is a central element in $\mathfrak{g}_{\mathbb{C}}$. Indeed for such a v there is a $\xi \in \mathfrak{g}^*_{\mathbb{C}}$ such that $\mathcal{J}(v + \xi) \in \mathfrak{g}^*_{\mathbb{C}}$ so, for $w \in \mathfrak{g}_{\mathbb{C}}$

$$\begin{aligned}
4[v, w] &= 4\pi([\![v + \xi, w]\!])\\
&= \pi([\![v + \xi + i\mathcal{J}(v + \xi) + v + \xi - i\mathcal{J}(v + \xi), w + i\mathcal{J}w + w - i\mathcal{J}w]\!])\\
&= \pi([\![v + \xi + i\mathcal{J}(v + \xi), w + i\mathcal{J}w]\!] + [\![v + \xi + i\mathcal{J}(v + \xi), w - i\mathcal{J}w]\!] +\\
&+ [\![v + \xi - i\mathcal{J}(v + \xi), w + i\mathcal{J}w]\!] + [\![v + \xi - i\mathcal{J}(v + \xi), w - i\mathcal{J}w]\!])\\
&= \pi([\![v + \xi + i\mathcal{J}(v + \xi), w - i\mathcal{J}w]\!] + [\![v + \xi - i\mathcal{J}(v + \xi), w + i\mathcal{J}w]\!])\\
&= \pi([\![v + \xi - i\mathcal{J}(v + \xi), w - i\mathcal{J}w]\!] + [\![v + \xi + i\mathcal{J}(v + \xi), w + i\mathcal{J}w]\!]) = 0,
\end{aligned}$$

where we have used that L and $\overline{L}$ are abelian in the fourth and in the last equalities and in the fifth equality we used that $\mathcal{J}(v + \xi) \in \mathfrak{g}^*_{\mathbb{C}}$, hence the change of signs does not affect the projection of the bracket onto $\mathfrak{g}_{\mathbb{C}}$.

If we let $e^{B+i\omega} \wedge \Omega$, with $\Omega = \theta^1 \wedge \cdots \wedge \theta^k$, be a generator for the canonical bundle of $\mathcal{J}$, then $\pi(L) \cap \pi(\overline{L})$ is the annihilator of $\Omega \wedge \overline{\Omega}$. Since $\theta^i \in L$ there are $\overline{\partial_j} \in \overline{L}$ such that $\langle \theta^i, \overline{\partial_j} \rangle = \delta^i_j$ and we can compute

$$\theta^i([\pi(\partial_j), \pi(\overline{\partial_k})]) = d\theta^i(\pi(\partial_j), \pi(\overline{\partial_k})) = \langle [\![\theta^i, \partial_j]\!], \overline{\partial_k} \rangle = 0$$

since $\theta^i, \partial_j \in L$. Analogously we see that $[\pi(\partial_j), \pi(\overline{\partial_k})]$ also annihilates $\overline{\theta^i}$ and hence $[\pi(\partial_j), \pi(\overline{\partial_k})] \in \pi(L) \cap \pi(\overline{L})$, hence $\mathfrak{g}$ is either abelian or 2-step nilpotent.

If $\mathfrak{g}$ was 2-step nilpotent there would be an element $\xi \in \mathfrak{g}^*$ with $d\xi \neq 0$. Since the only nonvanishing brackets are of the form $[\pi(\partial_i), \pi(\overline{\partial}_j)]$ and ξ is real, we see that there is a ∂_i for which $d\xi(\pi(\partial_i), \pi(\overline{\partial}_i)) = \xi([\pi(\partial_i), \pi(\overline{\partial}_i)]) \neq 0$. Since all the θ^i are closed, we can further assume that $\xi = \mathcal{J}(v - B(v))$, for some $v \in \mathfrak{g}$, therefore $v - B(v) - i\xi \in L$ and

$$0 = \langle [\![v - B(v) - i\xi, \partial_i]\!], \overline{\partial}_i \rangle = -id\xi(\pi(\partial_i), \pi(\overline{\partial}_i)) + (H + dB)(v, \pi(\partial_i), \pi(\overline{\partial}_i)).$$

Observe that the first term is real and nonzero while the second is purely imaginary, hence the equation above can never hold and $\mathfrak{g}$ is abelian. □

Theorem 4.1. *If a nilpotent Lie algebra* $\mathfrak{g}$ *admits a generalized Kähler structure, then* $\mathfrak{g}$ *is abelian.*

Proof. According to [2], Theorem 3.1, every generalized complex structure on a nilpotent Lie algebra $\mathfrak{g}$ has holomorphically trivial canonical bundle. Further, for any closed $H \in \wedge^3 \mathfrak{g}^*$, $\mathfrak{g} \oplus \mathfrak{g}^*$ with the Courant bracket is again a nilpotent Lie algebra, hence the i-eigenspace, L, of any generalized complex structures is a nilpotent Lie subalgebra of $(\mathfrak{g} \oplus \mathfrak{g}^*) \otimes \mathbb{C}$. According to Lemma 4.1, if $\mathfrak{g}$ has nontrivial bracket, then L has a nontrivial bracket. Then, Example 1.1 shows that $(\wedge^\bullet \overline{L}, d_L)$ is not formal and hence, by Theorem 3.2, can not be part of a generalized Kähler pair. □

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