ON THE INTERSECTION OF HOMOCLINIC CLASSES ON SINGULAR-HYPERBOLIC SETS

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Abstract. We know that two different homoclinic classes contained in the same hyperbolic set are disjoint [12]. Moreover, a connected singular-hyperbolic attracting set with dense periodic orbits and a unique equilibrium is either transitive or the union of two different homoclinic classes [6]. These results motivate the questions if two different homoclinic classes contained in the same singular-hyperbolic set are disjoint or if the second alternative in [6] cannot occur. Here we give negative answer for both questions. Indeed we prove that every compact 3-manifold supports a vector field exhibiting a connected singular-hyperbolic attracting set which has dense periodic orbits, a unique singularity, is the union of two homoclinic classes but is not transitive.

1. Introduction

The *singular-hyperbolic sets* introduced in [8] play nowadays important role in the study of robust transitive dynamics in dimension three. In fact, the main result in [8] asserts that a C^1 robust transitive set with singularities of a three-dimensional vector field is a singular-hyperbolic attractor for either the flow or the time reversed flow (see also [10]). Since [8] the search of a general theory of singular-hyperbolic sets has been taking place using the classical hyperbolic theory as model.

For example, Newhouse observed in the early seventies the simple (but interesting) fact that two different homoclinic classes contained in the same hyperbolic set are disjoint. More recently, it was proved a connected singular-hyperbolic attracting set with dense periodic orbits *and a unique singularity* is either transitive (as the *geometric Lorenz attractor* [1], [2]) or else the union of two different homoclinic classes. See [6] or [7].

In light of these results it is natural to ask if two different homoclinic classes contained in the same singular-hyperbolic set are disjoint or if the second alternative in [6] cannot occur. The goal of this paper is to answer these questions in a negative way. For this we show that on every compact 3-manifold there is a vector field exhibiting a connected singular-hyperbolic attracting set which has dense periodic orbits, a unique singularity, is the union of two homoclinic classes but is not transitive. Let us state this result in a precise way.

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Hereafter M denotes a compact 3-manifold and X denotes a C^1 vector field with flow X_t defined in M. The orbit of X through $q \in M$ is the set $O_X(q) = \{X_t(q) : t \in I\!R\}$ and an orbit of X is a set $O = O_X(q)$ for some $q \in M$. The *omega-limit set* of a point p is the set $\omega(p)$ defined by

$$\omega(p) = \left\{x \in M : x = \lim_{n\to\infty} X_{t_n}(p) \text{ for some sequence } t_n \to \infty\right\}.$$

A *singularity* is a point $\sigma \in M$ such that $X(\sigma) = 0$ while a *periodic orbit* is an orbit $O = O_X(p)$ such that $X_T(p) = p$ for some minimal $T > 0$. A *closed orbit* of X is either a singularity or a periodic orbit of X.

Definition 1.1. *A compact subset $\Lambda \subset M$ is:*

- Invariant *if $X_t(\Lambda) = \Lambda$, $\forall t \in I\!R$;*
- Transitive *if $\Lambda = \omega(p)$ for some $p \in \Lambda$;*
- Attracting set *if there is a compact neighborhood U of Λ such that*
$$\Lambda = \bigcap_{t>0} X_t(U);$$
- Attractor *if it is a transitive attracting set.*
- Hyperbolic *if there are positive constants K, λ and a continuous invariant tangent bundle decomposition $T_\Lambda M = E^s_\Lambda \oplus E^X_\Lambda \oplus E^u_\Lambda$ such that*
  - E^s_Λ *is* contracting, *i.e.*
  $$\| DX_t(x)/E^s_x \| \le Ke^{-\lambda t}, \qquad \forall t > 0, \forall x \in \Lambda.$$
  - E^u_Λ *is* expanding, *i.e.*
  $$\| DX_{-t}(x)/E^u_x \| \le Ke^{-\lambda t}, \qquad \forall t > 0, \forall x \in \Lambda.$$
  - E^X_Λ *is is the one-dimensional subbundle generated by X in Λ.*

A *hyperbolic closed orbit* of X is a closed orbit of X which is hyperbolic as a compact invariant set. The Invariant Manifold Theorem [3] asserts that if x belongs to a hyperbolic set H of X, then the sets

$$W^{ss}(x) = \left\{y \in M : \lim_{t\to\infty} d(X_t(x), X_t(y)) = 0\right\}$$

and

$$W^{uu}(x) = \left\{y \in M : \lim_{t\to-\infty} d(X_t(x), X_t(y)) = 0\right\}$$

are C^1 submanifolds of M. These manifolds are called *strong stable* and *strong unstable* manifolds of X through x. Therefore if O is an orbit of X contained in H, then the sets

$$W^s(O) = \bigcup_{x\in O} W^{ss}(x) \quad \text{and} \quad W^u(O) = \bigcup_{x\in O} W^{uu}(x)$$

are also C^1 submanifolds of M usually called the *stable* and *unstable* manifold of X throught O. Note that $W^s(O)$ and $W^u(O)$ are tangent to the subbundles $E^s_H \oplus E^X_H$ and $E^X_H \oplus E^u_H$ respectively.

By a *homoclinic orbit* associated to a hyperbolic closed orbit O of X we mean an orbit $\gamma \subset W^s(O) \cap W^u(O)$. If in addition $T_qM = T_qW^s(O) + T_qW^u(O)$ for some (and hence for every) point $q \in \gamma$ then we say that γ is a *transverse homoclinic orbit* of O.

Definition 1.2. *The* homoclinic class *of a hyperbolic periodic orbit O of X is the closure $H(O)$ of the union of the transverse homoclinic orbits of O. An invariant set Λ is a homoclinic class if $\Lambda = H(O)$ for some hyperbolic periodic orbit O.*

Denote by

$$m(L) = \inf_{v \neq 0} \frac{\|L(v)\|}{\|v\|}$$

the minimum norm of a linear operator L.

Definition 1.3. *Let Λ be a compact invariant set of X. A continuous invariant splitting $T_\Lambda M = E_\Lambda \oplus F_\Lambda$ over Λ satisfying $E_x \neq 0$ and $F_x \neq 0$ for every $x \in \Lambda$ is* dominated *if there are positive constants K, λ such that*

$$\frac{\|DX_t(x)/_{E_x}\|}{m(DX_t(x)/_{F_x})} \leq Ke^{-\lambda t}, \qquad \forall t > 0, \forall x \in \Lambda.$$

A compact invariant set Λ is partially hyperbolic *if it exhibits a dominated splitting*

$$T_\Lambda M = F^s_\Lambda \oplus F^c_\Lambda$$

such that F^s_Λ is contracting*, i.e.*

$$\|DX_t(x)/_{F^s_x}\| \leq Ke^{-\lambda t}, \qquad \forall t > 0, \forall x \in \Lambda.$$

Definition 1.4. *A* singular-hyperbolic set *Λ of X is a partially hyperbolic set with hyperbolic singularities and* volume expanding central subbundle *F^c_Λ, i.e.,*

$$\left|\det(DX_t(x)/_{F^c_x})\right| \geq K^{-1}e^{\lambda t}, \qquad \forall t > 0, \forall x \in \Lambda.$$

A singular-hyperbolic attracting set *is a singular-hyperbolic set which is also an attracting set.*

Now we can state our main result.

Theorem A. *On every compact 3-manifold there is a vector field X exhibiting a connected singular-hyperbolic attracting set A with the following properties:*

1. *X has a unique equilibrium in A.*
2. *The periodic orbits of X in A are dense in A.*
3. *A is the union of two homoclinic classes.*
4. *A is not transitive.*

To obtain the vector field X in the theorem we shall realize the map G described in Section 3 as the return map of X. The main difficulty behind this realization is the fact that G is a singular map, and so, the standard suspension procedure cannot be applied. Instead we make a suspension of G via the plugs 1 and 2 defined in Section 4. The properties in the theorem will be obtained basically from the one-dimensional results in Section 2. This will be done in Section 5.

2. One-dimensional map f

We introduce a one-dimensional map f to be used in the construction of the vector field X in Theorem A.

2.1. **Domain of f.** Consider the branched 1-manifold $\mathcal{B}$ consisting of a compact interval and a circle with branch point b described in Figure 1.

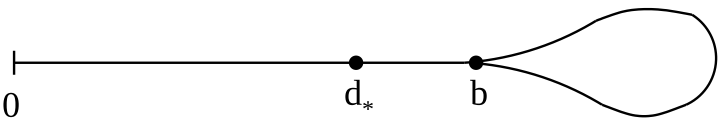

Figure 1. Branched manifold $\mathcal{B}$.

We cut $\mathcal{B}$ open along b to obtain a compact interval which we assume to be $[0,1]$ for simplicity. In $[0,1]$ we consider three points

$$0 < d_1 < d_* < d_2 < 1,$$

where d_* is depicted also in the figure. These will be the discontinuity points of f as a map of $[0,1]$. The set $\mathcal{B} \setminus \{d^*\}$ will be the domain of f.

2.2. **Definition of f.** We define $f : \mathcal{B} \setminus \{d^*\} \to \mathcal{B}$ in a way that its graph in $[0,1]$ is the one in Figure 2.

More precisely, f satisfies the following hypotheses:

(H1): $Dom(f) = [0,1] \setminus \{d_*\}$.
(H2): $f(0) = 0$; $f(d_1) = f(d_2) = 1$; $f(1) = f(b) \in (0, d_1)$.
(H3): $f(d_1+) = f(d_2+) = b$; $f(d_1-) = f(d_2-) = 1$; $f(d_*+) = f(d_*-) = 0$.
(H4): $f([0,d_1]) = [0,1]$; $f((d_1,d_*)) = (0,b)$; $f((d_*,d_2]) = (0,1]$; $f((d_2,1]) = [b_*, b)$.
(H5): f is `expanding`, i.e. f is C^1 in $Dom(f)$ and there is $\lambda > 1$ such that

$$\mid f'(x) \mid \geq \lambda, \qquad \forall x \in Dom(f).$$

2.3. **Properties of f.** The main one is given in Theorem 2.4. We start with some notations and facts. Let $\mid I \mid$ denote the length of an interval I. Consider the points $d_2', d_2'' \in f^{-1}(d_2)$ and $d_*', d_*'' \in f^{-1}(d_*)$ as in Figure 2. We define the following intervals:

- $I_1 = [d_2', d_1]$, $I_2 = (d_1, d_*']$;

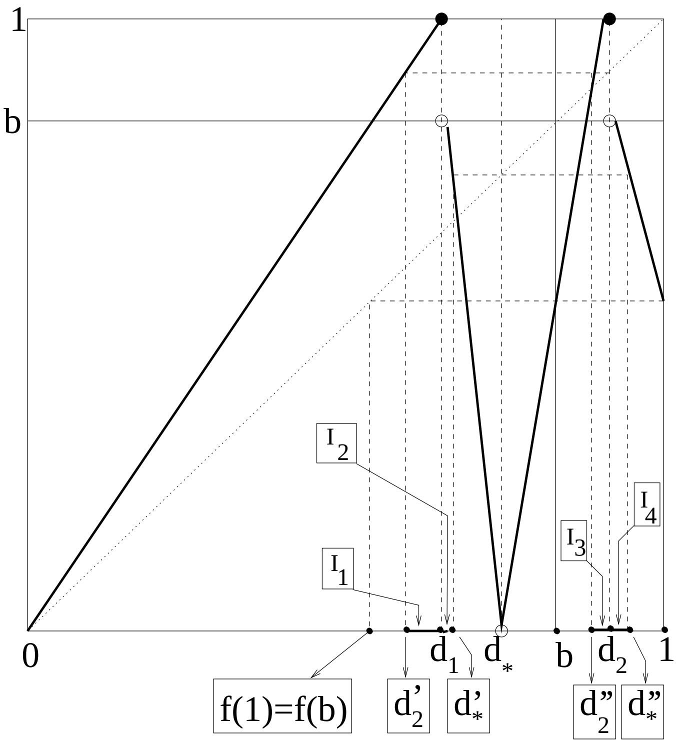

FIGURE 2. One-dimensional map.

- $I_3 = [d_2'', d_2]$, $I_4 = (d_2, d_*']$;
- $H = [d_2', d_*'] = I_1 \cup I_2$, $H' = [d_2'', d_*''] = I_3 \cup I_4$.

Lemma 2.1. *The intervals $H, H', I_1, \cdots, I_4$ defined above satisfy the following properties:*

1. *If $J \subset H \cup H'$ is an open interval, then $f^2(J)$ is an interval and*
$$| f^2(J) | > \lambda^2 | J | .$$
2. $d_* \in f^2(I_1) \cap f(I_2) \cap f^2(I_3) \cap f(I_4)$.

Proof. Observe that $f(I_1) = f(I_3) = [d_2, 1]$ and $f(I_2) = f(I_4) = [d_*, b)$. In particular, $d_* \in f(I_2) \cap f(I_4)$. Since $f((d_2, 1]) = [f(1), b)$ and $f(1) < d_*$ by **(H2)** we have that $d_* \in f^2(I_1) \cap f^2(I_3)$ proving Item (2). Now we prove Item (1). Let $J \subset H \cup H'$ be an open interval. Note that H and H' are disjoint and so $J \subset H$ or $J \subset H'$. We assume $J \subset H$ for the other case is similar. If $d_1 \notin J$ then $f(J)$ is an interval and $f(J) \cap \{d_1, d_*, d_2\} = \emptyset$. If $d_1 \in J$ then $f(J)$ is not an interval. However, $f(J)$ consists of two intervals $J_1 \subset (d_2, 1]$ and $J_2 \subset (d_*, b)$ having 1 and b as boundary points respectively. In particular, $(J_1 \cup J_2) \cap \{d_1, d_*, d_2\} = \emptyset$. Since $f(b) = f(1)$ by **(H2)** we have that $f^2(J) = f(J_1) \cup f(J_2)$ is an interval. The inequality follows because f is expanding (see **(H5)**). $\square$

Lemma 2.2. *If $I \subset [0,1]$ is an open interval containing d_*, then there are a subinterval $J \subset I$ and $r \in \mathbb{N}$ such that $f^r(J) = [0,1]$.*

Proof. As $d_* \in I$ we can choose an interval $J \subset I$ with boundary point d_* such that $f(J)$ is a closed interval with 0 as a boundary point. Indeed, it suffices to use $f(d_*+) = f(d_*-) = 0$ in **(H3)**. As $f(0) = 0$ (c.f. **(H2)**) there is a sequence of preimage points of d_1 converging to 0^+. Hence we can assume that there is $n_1 \in \mathbb{N}$ such that if b_{**} is the remainder boundary point of J then $f^{n_1}(b_{**}) = d_1$. Using $f([0,d_1]) = [0,1]$ in **(H4)** one sees that $f^{n_1+2}(J) = [0,1]$. Then, we are done by choosing $r = n_1 + 2$. □

Definition 2.3. *We say that f is locally eventually onto (*`leo` *for short) if given any open interval $I \subset [0,1]$ there is $m \geq 0$ such that $f^m(I) = [0,1]$.*

Theorem 2.4. *f is leo.*

Proof. By **(H5)** if $K \subset [0,1] \setminus \{d_1, d_*, d_2\}$ is an interval, then $f(K)$ is an interval and

$$| f(K) |> \lambda \mid K \mid . \tag{1}$$

Fix an interval $I \subset [0,1] \setminus \{d_*\}$. By Lemma 2.2 it suffices to show the existence of $n \in \mathbb{N}$ such that

$$d_* \in f^n(I). \tag{2}$$

Indeed, to obtain the conclusion of Definition 2.3 for I it suffices to choose $m = n + r$ where r comes from Lemma 2.2. Note that $I \subset Dom(f)$ by **(H1)**. Hence, applying (1), we can find a first integer $n_1 \geq 0$ such that $f^i(I)$ is an interval for all $0 \leq i \leq n_1$ and

$$f^{n_1}(I) \cap \{d_1, d_*, d_2\} \neq \emptyset.$$

If $d_* \in f^{n_1}(I)$ then we prove (2) with $n = n_1$. Assume therefore that either $d_1 \in f^{n_1}(I)$ or $d_2 \in f^{n_1}(I)$.

Case I: $d_1 \in f^{n_1}(I)$. We have two subcases, namely $f^{n_1}(I) \subset H$ or not. If not then either $f^{n_1}(I) \supset I_1$ or $f^{n_1}(I) \supset I_2$ since $f^{n_1}(I)$ is an interval containing d_1. Then Lemma 2.1-(2) implies $d_* \in f^{n_1+2}(I)$ or $d_* \in f^{n_1+1}(I)$ and we are done with $n = n_1 + 2$ or $n = n_1 + 1$ respectively. If $f^{n_1}(I) \subset H$ then Lemma 2.1-(1) implies that $f^{n_1+2}(I)$ is an interval and

$$| f^{n_1+2}(I) |> \lambda^{n_1+2} \mid I \mid .$$

In this subcase we replace I by $f^{n_1+2}(I)$ and restart.

Case II: $d_2 \in f^{n_1}(I)$. This case is analogous to Case I. Again we have two subcases, namely, $f^{n_1}(I) \subset H'$ or not. If not then either $f^{n_1}(I) \supset I_3$ or $f^{n_1}(I) \supset I_4$ since $f^{n_1}(I)$ is an interval containing d_2. Then Lemma 2.1-(2) implies $d_* \in f^{n_1+2}(I)$ or $d_* \in f^{n_1+1}(I)$ and we are done with $n = n_1 + 2$ or

$n = n_1 + 1$ respectively. If $f^{n_1}(I) \subset H'$ then Lemma 2.1-(2) implies that $f^{n_1+2}(I)$ is an interval and

$$| f^{n_1+2}(I) |> \lambda^{n_1+2} | I | .$$

In this subcase we replace I by $f^{n_1+2}(I)$ and restart.

Note that in the subcases above we replace I by $f^{n_1+2}(I)$ and restart with an interval with length $> \lambda^{n_1+2} | I |$. This implies the existence of n satisfying (2). The theorem is proved. □

Corollary 2.5. *The periodic points of f are dense in $\mathcal{B}$. If $x \in \mathcal{B}$, then*

$$\mathcal{B} = Cl\left(\bigcup_{n\geq 0} f^{-n}(x)\right).$$

Proof. This is a direct consequence of Theorem 2.4. □

3. Two-dimensional map G

In this section we define a two-dimensional map G through a construction closely related to the Plykin attractor in p. 304 of [13]. See also [5] for a similar construction.

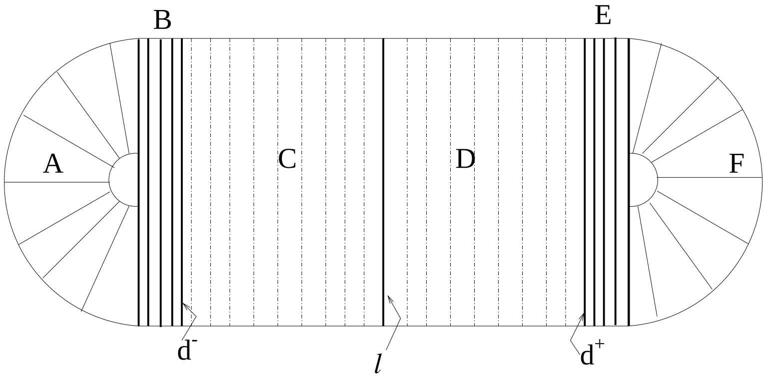

FIGURE 3. The region R.

3.1. **Domain of G.** Consider the twice punctured planar region R depicted in Figure 3. It is formed by: two half-annuli A, F; and four rectangles B, C, D, E. There is a middle vertical line denoted by l. Note that l defines a plane reflexion throughout denote by θ. We assume $\theta(D) = C$, $\theta(E) = B$ and $\theta(F) = A$. In particular, $\theta(R) = R$ and $\theta(d^+) = d^-$ where the vertical segments d^-, d^+ correspond to the right-hand and left-hand boundary curves of B and D respectively. We define

$$H^- = A \cup B \cup C \quad \text{and} \quad H^+ = D \cup E \cup F.$$

Let $\mathcal{F}$ denote the foliation of R formed by line segments depicted in Figure 3. Note that the middle leaf l is a leaf of $\mathcal{F}$. In addition, the segments forming $\mathcal{F}$ are vertical in the rectangular components B, C, D, E and radial in the annuli components A, F.

We use the standard notation $\mathcal{F}_x$ to denote the leaf of $\mathcal{F}$ containing $x \in R$.

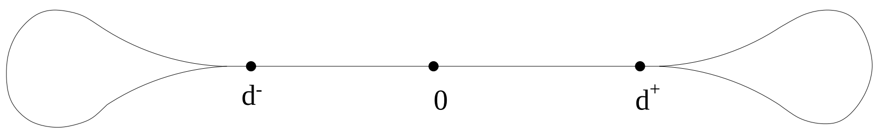

FIGURE 4. The quotient space.

Identifying points in the same leaf of $\mathcal{F}$ we get the quotient space K indicated in Figure 4. Note that $0 \in K$ represent the leaf l in the quotient space.

Remark 3.1. *The manifold K is obtained by gluing two copies $\mathcal{B}^+, \mathcal{B}^-$ of the branched manifold $\mathcal{B}$ depicted in Figure 1 of Section 2 along the boundary point 0. The set $R \setminus \{d^-, d^+\}$ will be the domain of G.*

3.2. **Definition of G.** We define the C^∞ map $G : R \setminus \{d^-, d^+\} \to Int(R)$ in a way that its image is as indicated in Figure 5. As usual we employ the notation $Int(R)$ to denote the interior of R. We require the following hypotheses:

- **(G1):** G and θ commute, i.e., $G \circ \theta = \theta \circ G$.
- **(G2):** G preserves and contracts the foliation $\mathcal{F}$.
- **(G3):** Let $g : K \setminus \{d^-, d^+\} \to K$ be the map induced by G in the leaf space K. Then, the map f^+ defined by $f^+ = g/\mathcal{B}^+$ satisfies the hypotheses **(H1)-(H5)** in Section 2 with $f = f^+$, $\mathcal{B} = \mathcal{B}^+$ and $d_* = d^+$ (remember $\mathcal{B}^\pm$ in Remark 3.1).

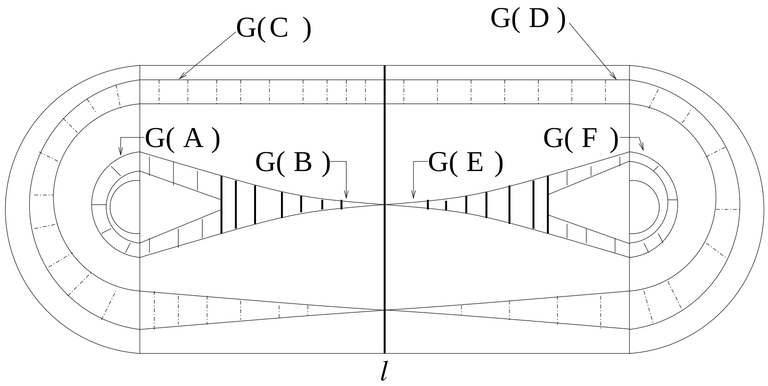

FIGURE 5. The map G.

3.3. **Properties of G.** Denote by $\pi : R \to B$ the projection along the leaves of $\mathcal{F}$. It follows that

$$H^- = \pi^{-1}(B^-), \qquad H^+ = \pi^{-1}(B^+).$$

(G1) implies that H^+, H^- are invariant under G, namely

$$G(H^- \setminus d^-) \subset H^- \quad \text{and} \quad G(H^+ \setminus d^+) \subset H^+. \tag{3}$$

Let $Per(G/_{H^+})$ denote the set of periodic points of $G/_{H^+}$. Let $W^s(x, G)$ denote the stable manifold of $x \in R$ with respect to G, i.e.

$$W^s(x, G) = \left\{ y \in R : \lim_{n\to\infty} d(G^n(y), G^n(x)) = 0 \right\}.$$

Since G contracts $\mathcal{F}$ (**(G2)**) we have that $W^s(x, G)$ is union of leaves of $\mathcal{F}$. It follows from **(G2)**, **(G3)** and the expansiveness in **(H5)** that all periodic points of G are hyperbolic saddles. Hence the unstable manifold

$$W^u(x, G) = \left\{ y \in R : \lim_{n\to-\infty} d(G^n(y), G^n(x)) = 0 \right\}$$

is well defined for all $x \in R$ periodic. By **(G1)** we have that $G(l) \subset l$ and so G has a fixed point P in l. Clearly one has $\pi(P) = 0$.

Given a set L we denote by $Cl(L)$ the closure of L. Define

$$A_G^- = Cl\left(\bigcap_{n\geq 1} G^n(H^-) \right), \; A_G^+ = Cl\left(\bigcap_{n\geq 1} G^n(H^+) \right). \tag{4}$$

Lemma 3.2. *If* $p \in Per(G/_{H^+}) \setminus \{P\}$, *then*

$$A_G^+ = Cl(W^u(p, G))$$

and the same holds replacing $+$ *by* $-$.

Proof. Clearly $Cl(W^u(p, G)) \subset A_G^+$. To prove the reversed inclusion we fix $a \in A_G^+$. It follows from the definition of A_G^+ that there is $a' \in H^+$ arbitrarily close to a such that $G^{-n}(a')$ exists and belongs to H^+ for every $n \in \mathbb{N}$. Consider the leaf $\mathcal{F}_{G^{-n}(a')}$ containing $G^{-n}(a')$. Pick $\epsilon > 0$ small. Since $\mathcal{F}$ is uniformly contracted by G (see **(G2)**) we have that there is $n_\epsilon \in \mathbb{N}$ such that

$$\sup_{n\geq n_\epsilon} \{ d(G^n(y), a') : y \in \mathcal{F}_{G^{-n}(a')} \} \leq \epsilon. \tag{5}$$

Now fix $n \geq n_\epsilon$ and consider $b_n = \pi\left(\mathcal{F}_{G^{-n}(x)}\right) \in B$. Fix an interval $I \subset W^u(p, G)$.

Let f^+ and $\mathcal{B}^+$ be the map and the branched manifold in **(G3)**. To simplify the notation we drop the exponent $+$.

Note that f is leo by Theorem 2.4. Therefore there is $n' \in \mathbb{N}$ such that $f^{n'}(\pi(I)) = \mathcal{B}$ and so $b_n \in f^{n'}(\pi(I))$. As $f^{n'}(\pi(I)) = \pi(G^{n'}(I))$ we get $\pi\left(\mathcal{F}_{G^{-n}(x)}\right) \in \pi(G^{n'}(I))$ and so

$$\mathcal{F}_{G^{-n}(a')} \cap G^{n'}(I) \neq \emptyset.$$

It follows that there is $x \in I$ such that

$$G^{n'}(x) \in \mathcal{F}_{G^{-n}(a')}.$$

As $n \geq n_\epsilon$ and $n' \in \mathbb{N}$ we get $n + n' \geq n_\epsilon$. As $y = G^{n'}(x) \in \mathcal{F}_{G^{-n}(a')}$ we obtain

$$d(G^{n+n'}(x), a') < \epsilon$$

from Equation (5) applied to y. As $W^u(p, G)$ is G-invariant and $x \in I \subset W^u(p, G)$ we have $G^{n+n'}(x) \in W^u(p, G)$. Consequently

$$d(W^u(p, G), a') < \epsilon.$$

As ϵ is arbitrary we conclude that

$$a' \in Cl(W^u(p, G)).$$

As a' is arbitrarily close to a we obtain

$$a \in Cl(W^u(p, G))$$

proving the result. Similar result holds replacing $+$ by $-$ because G and θ commute by **(G1)**. $\square$

Theorem 3.3. *A^-_G and A^+_G are homoclinic classes of G and $P \in A^+_G \cap A^-_G$*

Proof. Clearly $P \in H^+ \cap H^-$ and so $G^n(P) \in G^n(H^+) \cap G^n(H^-)$ for all $n \in \mathbb{N}$. As P is a fixed point of G we obtain $P \in G^n(H^+) \cap G^n(H^-)$ for all n. This proves $P \in A^+_G \cap A^-_G$.

Now we prove that A^+_G and A^-_G are homoclinic classes of G. We only prove that A^+_G is since if it does then A^-_G also does because G and θ commute by **(G1)**.

Let f^+ and $\mathcal{B}^+$ be the map and the branched manifold in **(G3)**. As before we drop the exponent $+$ for the sake of simplicity.

By Corollary 2.5 we have that the periodic orbits of f are dense in $\mathcal{B}$.

Hence we can fix $x \in Per(f) \setminus \{0\}$. It follows that the leaf $\pi^{-1}(x)$ of $\mathcal{F}$ is periodic. As $\mathcal{F}$ is a contracting foliation we conclude that there is $p \in Per(G/H^+) \setminus \{P\}$. Denote $H_G(p)$ the homoclinic class of p.

We claim that $A^+_G = H_G(p)$. Indeed, we have that $A^+_G = Cl(W^u(p, G))$ by Lemma 3.2. Since $H_G(p)$ is the closure of the transverse intersections points of $W^u(p, G)$ with $W^s(p, G)$ we obtain $H_G(p) \subset Cl(W^u(p, G))$ and so $H_G(p) \subset A^+_G$.

Now we prove that $A^+_G \subset H_G(p)$. We only have to prove that $W^u(p, G) \subset H_G(p)$ because $W^u(p, G)$ is dense in A^+_G and $H_G(p)$ is closed. Fix $q \in W^u(p, G)$ and take an open interval $I \subset W^u(p, G)$ containing q. We have from by **(G3)** and

Theorem 2.4 as before that f is leo. Then, there is $n \geq 0$ such that $f^n(\pi(I)) = \mathcal{B}$. In particular, $\pi(\mathcal{F}_p) \in f^n(\pi(I))$ proving $\mathcal{F}_p \cap G^n(I) \neq \emptyset$. Hence

$$G^{-n}(\mathcal{F}_p) \cap I \neq \emptyset.$$

Now we have

$$G^{-n}(\mathcal{F}_p) \cap I \subset H_G(p) \cap I$$

because

$$G^{-n}(\mathcal{F}_p) \subset W^s(p, G)$$

and

$$I \subset W^u(p, G).$$

Consequently

$$H_G(p) \cap I \neq \emptyset.$$

Since the interval I is arbitrarily close to q and $H_G(p)$ is closed we conclude that $q \in H_G(p)$. This proves the claim. Theorem 3.3 follows from the claim. $\square$

Finally we define

$$A_G = Cl\left(\bigcap_{n\geq 1} G^n(R)\right).$$

Then we have the following property.

Lemma 3.4. $A_G = A_G^- \cup A_G^+$.

Proof. This follows from $R = H^- \cup H^+$ and the definition of A_G^-, A_G^+. $\square$

4. Three-dimensional vector field X

In this section we construct a vector field X which will satisfy the properties of Theorem A. The idea is to realize the two-dimensional map G in Section 3 as the return map associated to a singular-hyperbolic attracting set. Hereafter M will be a closed 3-manifold.

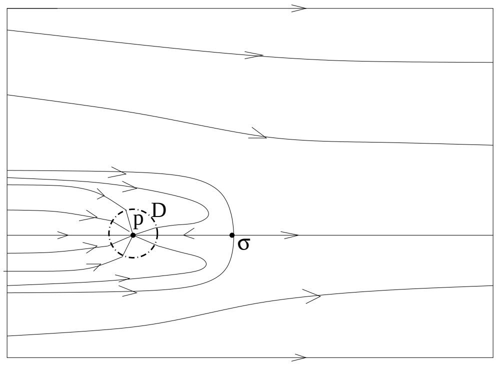

FIGURE 6.

4.1. **Plugs 1 and 2.** We shall introduce plugs 1 and 2. Plug 1 is constructed through three steps: First we consider a vector field in the square whose flow is described in Figure 6. Note that this vector field has two equilibria: a saddle σ and a sink p. We shall assume that the eigenvalues $\{\lambda_3, \lambda_1\}$ of σ satisfy the eigenvalue relation

$$-\lambda_3 < 0 < -\lambda_3 < \lambda_1.$$

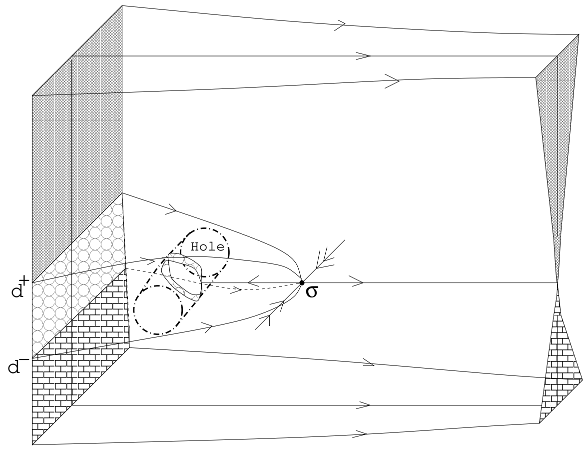

FIGURE 7.

Note in addition the similarity of this vector field with the Cherry flow described in [11]. We have depicted a small disk D centered at the attracting equilibrium p. This finishes the first step for the construction of Plug 1.

For the second step we multiply the above vector field by a strong contraction λ_2 in order to obtain the vector field described in Figure 7. The hole indicated in this figure is nothing but the disk D times a compact interval. Note also that if we choose λ_2 such that $-\lambda_2$ is large, then the resulting vector field has a unique singularity σ which is `Lorenz-like`, i. e., σ has real eigenvalues $\{-\lambda_2, -\lambda_3, \lambda_1\}$ that satisfy

$$-\lambda_2 < \lambda_3 < 0 < -\lambda_3 < \lambda_1.$$

The final step is to glue two handles in the 3-dimensional vector field above in order to obtain the vector field whose flow is given in Figure 8. The resulting vector field is what we shall call Plug 1.

Observe that by construction Plug 1 has the planar region R in Section 3 as a cross-section.

Plug 2 is a solid 3-ball and the flow has no singularities and acts as in Figure 9.

4.2. **Definition of X.** We construct a vector field X on a solid bitorus ST in a way that $X_t(ST) \subset Int(ST)$ for all $t > 0$ and X is transverse to the boundary

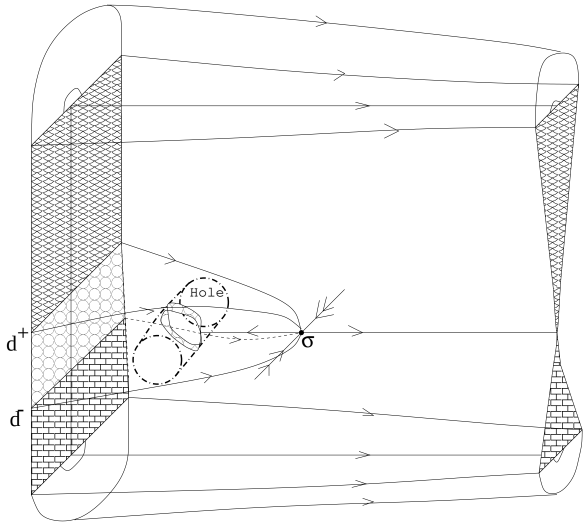

FIGURE 8. Plug 1

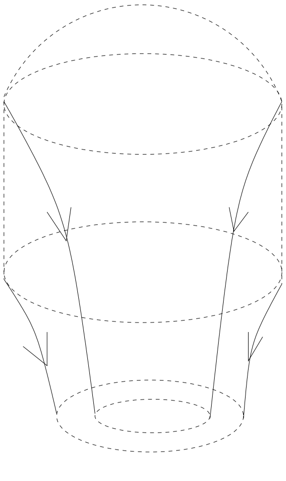

FIGURE 9. Plug 2

bitorus $\partial(ST)$ of ST. The flow is obtained gluing plugs 1 and 2 as indicated in Figure 10. We require the following hypotheses:

- **(X1):** There are two repelling periodic orbits $\mathcal{O}_1, \mathcal{O}_2$ in $Int(ST)$ crossing the holes of R as indicated in Figure 10.
- **(X2):** There are two solid tori neighborhoods $V_1, V_2 \subset Int(ST)$ of $\mathcal{O}_1, \mathcal{O}_2$ with boundaries transverse to X_t such that if $V = ST \setminus (V_1 \cup V_2)$, then V is a compact neighborhood with smooth boundary transverse to X_t and $X_t(V) \subset V$ for $t \geq 0$. As V is a solid bitorus with two solid tori removed we have that V is *connected*.

(X3): $R \subset V$ and the return map G induced by X in R satisfies the properties **(G1)-(G3)** in Section 3. Moreover,

$$\{q \in V : X_t(q) \notin R, \forall t \in \mathbb{R}\} = \{\sigma\}.$$

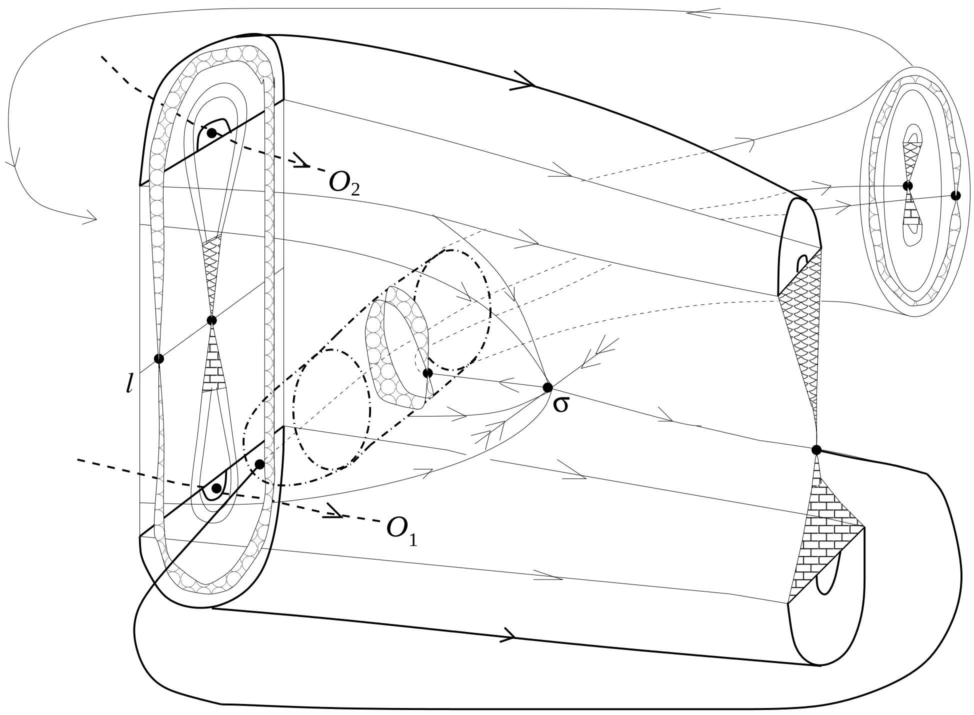

FIGURE 10. Gluing plugs 1 and 2.

4.3. **Properties of X_t.** Recall the definition of A_G in Section 3. Define

$$A = Cl\left(\bigcup_{t\in\mathbb{R}} X_t(A_G)\right) \tag{6}$$

As σ is Lorenz-like we have that it is hyperbolic. So, the stable and unstable manifolds $W^s_X(\sigma), W^u_X(\sigma)$ of σ are well defined.

Lemma 4.1. *$W^u_X(\sigma) \subset A$.*

Proof. First we claim that the full X-orbits of the periodic points of G are contained in A. Indeed, if $x \in R$ is a periodic point of G, then $G^n(x) \in R$ for all $n \leq 0$ and so $x \in A_G = Cl(\cap_{n\geq 1} G^n(R))$. Consequently $x \in A$ (for $A_G \subset A$) and so the full orbit of x is contained in A since A is X_t-invariant.

Second, the periodic points of f in **(G3)** are dense in $\mathcal{B}$ by Corollary 2.5. Then, the periodic points of G accumulate on d^+ in both connected components of $R \setminus d^+$ (analogously for d^-). Since d^+ (or d^-) is contained in $W^s_X(\sigma)$, the full

X_t-orbit of the periodic points of G accumulating d^+ (or d^-) also accumulate on $W^u_X(\sigma)$. Then the result follows from the claim and the fact that A is closed. $\square$

Lemma 4.2. *$A = \bigcap_{t\geq 0} X_t(V)$. In particular, A is a connected attracting set.*

Proof. Suppose that $x \in \cap_{n\geq 1} G^n(R)$. Then $G^{-n}(x) \in R$ for all $n \leq 0$. Since G is the return map induced by X_t in R, we have that $X_t(x) \in V$ for all $t < 0$. This implies that $X_t(x) \in V$ for all $t \in \mathbb{R}$ since $x \in R \subset V$ (by **(X3)**) and V is positively invariant by X_t (by **(X2)**). This proves $\cap_{n\geq 1} G^n(R) \subset \cap_{t\in\mathbb{R}} X_t(V)$. As $\cap_{t\in\mathbb{R}} X_t(V)$ is closed (for V is compact) we conclude that $A_G = Cl(\cap_{n\geq 1} G^n(R)) \subset \cap_{t\in\mathbb{R}} X_t(V)$. As $\cap_{t\in\mathbb{R}} X_t(V)$ is X-invariant we conclude that $X_t(A_G) \subset \cap_{t\in\mathbb{R}} X_t(V)$ for all $t \in \mathbb{R}$ and so

$$A \subset \bigcap_{t\geq 0} X_t(V).$$

Now suppose $q \in \bigcap_{t\geq 0} X_t(V)$. Then $X_t(q) \in V$ for all $t \leq 0$. It follows that $X_t(q) \in V$ for all $t \in \mathbb{R}$ since V is positively invariant by **(X2)**. If $X_t(q) \notin R$ for all $t \in \mathbb{R}$, then $q = \sigma$ by the equality in **(X3)**. By Lemma 4.1 we have $W^u_X(\sigma) \subset A$. Hence $q \in A$ in this case. So, we can assume that $X_{t_0}(q) \in R$ for some t_0. Set $x = X_{t_0}(q)$. If $X_t(x) \in R$ for infinite many values of $t \leq 0$ then $x \in \cap_{n\geq 1} G^n(R)$. Otherwise $x \in W^u_X(\sigma)$ by **(X3)**. In any case we have $x \in A$ and so

$$\bigcap_{t\geq 0} X_t(V) \subset A.$$

The last sentence of the lemma follows from fact that V is a compact *connected* neighborhood of A. The lemma is proved $\square$

Next we define

$$A^+ = Cl\left(\bigcup_{t\in\mathbb{R}} X_t(A^+_G)\right) \qquad \text{and} \qquad A^- = Cl\left(\bigcup_{t\in\mathbb{R}} X_t(A^-_G)\right),$$

where A^+_G and A^-_G come from Section 3.

Lemma 4.3. *A^+ and A^- are homoclinic classes of X and*

$$A = A^+ \cup A^-.$$

Proof. Note that A^-_G and A^+_G are homoclinic classes of G by Theorem 3.3. Then, A^-, A^+ are homoclinic classes of X because G is the return map of X. On the other hand, Lemma 3.4 implies $A_G = A^-_G \cup A^+_G$. Then $X_t(A_G) = X_t(A^-_G) \cup X_t(A^+_G)$ for all $t \in \mathbb{R}$, and so, Equation (6) implies

$$A = Cl\left(\bigcup_{t\in\mathbb{R}} X_t(A_G)\right) = Cl\left(\bigcup_{t\in\mathbb{R}} X_t(A^-_G)\right) \cup Cl\left(\bigcup_{t\in\mathbb{R}} X_t(A^+_G)\right) = A^+ \cup A^-.$$

$\square$

Further properties of A are listed in the next lemma.

Lemma 4.4. *The following assertions hold:*

- *A has only one singularity.*
- *A is not transitive.*
- *A is a singular-hyperbolic set of X_t.*

Proof. We have that σ is the unique singularity of A by construction. In addition, A is not transitive by Equation (3) since the orbits of X starting in a connected component of $R \setminus l$ do not intersect the other component of $R \setminus l$. It remains to prove that A is a singular-hyperbolic set. Although this can be proved directly from the construction (e.g. [4], Section 3 in [9] or [13]) we shall use here the following quit argument based on [10]:

By Lemma 4.3 we have that $A = A^+ \cup A^-$ where $A^\pm$ are homoclinic classes. The Birkhoff-Smale Theorem then implies that the periodic orbits of X in A are dense in A. On the other hand, every periodic orbit of X contained in A is hyperbolic of saddle type. This follows from the fact that the foliation $\mathcal{F}$ preserved by G in **(G2)** is contracting; and the fact that the one-dimensional map g induced by G in the leaf space of $\mathcal{F}^s$ in **(G3)** is expanding. In particular, every periodic orbit of X contained in A is endowed with a hyperbolic splitting $T_O M = E^s_O \oplus E^X_O \oplus E^u_O$ (recall E_O denotes the one-dimensional subbundle over O induced by X). Observe that the above structures persist by small C^1 perturbations of X. Therefore, every periodic orbit in V of every vector field that is C^1 close to X is hyperbolic of saddle type as well.

Now denote by $Per(A)$ the union of the periodic orbits of X contained in A. By the discussion above we have that $Per(A)$ is a dense subset of A. Define the splitting

$$T_{Per(A)}M = F^s_{Per(A)} \oplus F^c_{Per(A)}$$

over $Per(A)$ by

$$F^s_x = E^s_x \qquad \text{and} \qquad F^c_x = E^X_x \oplus E^u_x,$$

for $x \in O$ where $O \subset Per(A)$ is a periodic orbit. As every periodic orbit in V of every vector field C^1 close to X is hyperbolic of saddle type we can use the arguments in [10] to prove that the splitting $T_{Per(A)}M = F^s_{Per(A)} \oplus F^c_{Per(A)}$ over $Per(A)$ extends to a singular-hyperbolic splitting $T_A M = F^s_A \oplus F^c_A$ over the whole $A = Cl(Per(A))$.

This completes the proof. ∎

5. Proof of Theorem A

Let M a closed 3-manifold and X be a vector field in the solid bitorus ST defined in Section 4. In particular, X satisfies the hypotheses **(X1)-(X3)**. Standard constructions in 3-manifold topology allow us to embed ST into M so that X extends to all of M. We shall prove that X satisfies the conclusion of Theorem A. For this we proceed as follows.

Let V the set in **(X2)** and A be defined in Equation (6). It follows from lemmas 4.2 and 4.4 that A is a connected singular-hyperbolic attracting set of X which has a unique singularity and is not transitive. On the other hand, A is the union of the homoclinic classes by Lemma 4.3, and so, the periodic orbits are dense in A by the Birkhoff-Smale Theorem. Finally these homoclinic classes must be different by the Birkhoff-Smale Theorem once more since A is not transitive. The proof is completed. □

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