Right-Permutative Cellular Automata on Topological Markov Chains

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Abstract

In this paper we consider cellular automata $(\mathfrak{G}, \Phi)$ with algebraic local rules and such that $\mathfrak{G}$ is a topological Markov chain which has a structure compatible to this local rule. We characterize such cellular automata and study the convergence of the Cesàro mean distribution of the iterates of any probability measure with complete connections and summable decay.

1 Introduction

Let $G^{\mathbb{Z}}$ be the two sided full shift on the finite alphabet G, and $\sigma : G^{\mathbb{Z}} \to G^{\mathbb{Z}}$ be the shift map. Suppose $\mathfrak{G} \subseteq G^{\mathbb{Z}}$ is a topological Markov Chain which, without loss of generality, we can consider uses all alphabet G.

Consider the cellular automaton $(\mathfrak{G}, \Phi)$ which has a local rule defined from some algebraic operation on G. Motivated by their several applications in information theory, physics, and biological sciences, among others, the problem of to characterize and to analyze the dynamical behavior of such cellular automata has been widely investigated. More specifically, there are three important questions about $(\mathfrak{G}, \Phi)$: if it is possible to recode it in the way to understand and to classify its dynamics (see [7] and [15]); what σ-invariant probability measures are also Φ-invariant (see [7], [19] and [23]); and how σ-invariant probability measures evolve under the dynamics of Φ (see [7], [13] and [15]).

When $\mathfrak{G} = G^{\mathbb{Z}}$ and $(\mathfrak{G}, \Phi)$ is a right-permutative Ψ-associative or N-scaling cellular automaton, Host-Maass-Martínez [7] proved that it is topologically conjugate to an affine cellular automaton product a translation $(K^{\mathbb{Z}} \times B^{\mathbb{Z}}, \Phi_K \times \mathbf{G}_B)$. Moreover, they showed sufficient conditions under which the unique shift-affine invariant measure is the maximum entropy measure (property which is known as rigidity), and studied the convergence of the Cesàro mean distribution of σ-invariant probability measures under the action of Φ. The results of [7] about rigidity were generalized by Pivato [19] for the case of bipermutative endomorphic cellular automata. Is his work, Pivato also showed results about the characterization of the topological dynamics of bipermutative cellular automata. Later, the rigidity results of Pivato were generalized by Sablik [23] who also includes the case of $\mathfrak{G}$ being a proper subgroup shift of $G^{\mathbb{Z}}$.

Recently, Mass-Martínez-Sobottka [15] have showed that if $(\mathfrak{G}, +)$ is an Abelian subgroup shift and a p^s-torsion for some prime number p, and Φ is an affine cellular automaton given by $\Phi := \mathbf{a} \cdot id \, + \, \mathbf{b} \cdot \sigma \, + \, \mathbf{c}$, where $\mathbf{a}, \mathbf{b} \in \mathbb{N}$ are relatively prime to p, and $\mathbf{c} \in \mathfrak{G}$ is a constant sequence,

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then the Cesàro mean distribution of any measure with complete connections (compatible with $\mathfrak{G}$) and summable decay under the action of Φ converges to the maximum entropy measure on $\mathfrak{G}$. The proof of this result combines regeneration theory, combinatorics, and the recodification of $\mathfrak{G}$. As consequence of the convergence of the Cesàro mean distribution we get a rigidity property, namely: the unique (σ, Φ)-invariant measure with complete connections (compatible with $\mathfrak{G}$) and summable decay for that case is the maximum entropy measure.

This paper concentrates mainly on the first problem, characterizing the dynamical behavior of bipermutative and some right-permutative cellular automata defined on subshifts $\mathfrak{G}$ which are not necessarily subgroup shifts, but which have some algebraic structure. As a direct application of these results we recuperate several results about rigidity and about the evolution of σ-invariant measures under the action of Φ.

This paper is organized as follows. In §2 we develop the background. In §3, we define the class of structurally-compatible cellular automata and study the case of bipermutative cellular automata. In §4 we study the representation of right-permutative Ψ-associative or N-scaling cellular automata. In §5 we present some sufficient conditions under which a block code preserves the properties of complete connections and summable decay of a probability measure, and so in §6 we apply the results obtained in the previous sections to study the convergence of the Cesàro mean distribution. In §7 we gives some results about rigidity.

2 Background

Let $\mathfrak{G} \subseteq G^{\mathbb{Z}}$ be a subshift. Given $\mathbf{g} \in \mathfrak{G}$, and $m \leq n$, we denote by $\mathbf{g}_m^n = (g_m, g_{m+1}, \ldots, g_n)$. For $k \geq 1$, denote by $\mathfrak{G}_k$ the set of all allowed words with length k in $\mathfrak{G}$. Given $\mathbf{g} \in \mathfrak{G}_k$, $\mathbf{g} = (g_1 \ldots, g_k)$ we write $\mathcal{F}(\mathbf{g})$, as the *follower set* of $\mathbf{g}$ in $\mathfrak{G}$:

$$\mathcal{F}(\mathbf{g}) = \{h \in G : (g_1, \ldots, g_k, h) \in \mathfrak{G}_{k+1}\}.$$

In the same way, we define $\mathcal{P}(\mathbf{g})$ the set of predecessors of $\mathbf{g} \in \mathfrak{G}_k$ in $\mathfrak{G}$.

We say a subshift $\mathfrak{G}$ is a *topological Markov chain* if for any $k \geq 1$ and $\mathbf{g} = (g_1, \ldots, g_k) \in \mathfrak{G}_k$ we have $\mathcal{F}(\mathbf{g}) = \mathcal{F}(g_k)$, which means $\mathfrak{G}$ can be thought as generated by a bi-infinite walking on an oriented graph. A topological Markov chain $\mathfrak{G}$ is *irreducible* if and only if for any $u, w \in G$ there exist $k \geq 1$ and $(v_1, \ldots, v_k) \in \mathfrak{G}_k$ such that $(u, v_1, \ldots, v_k, w) \in \mathfrak{G}_{k+2}$, and it is *mixing* if there exists $q \geq 1$ such that for any $k \geq q$ and $u, w \in G$ we always can find $(v_1, \ldots, v_k) \in \mathfrak{G}_k$ such that $(u, v_1, \ldots, v_k, w) \in \mathfrak{G}_{k+2}$.

Denote by $\mathfrak{G}^-$ and $\mathfrak{G}^+$, the projections of $\mathfrak{G}$ on $G^{-\mathbb{N}^*}$ and $G^{\mathbb{N}}$ respectively. Given $w \in \mathfrak{G}^-$ denote by $\mathfrak{G}_w^+$ the projection on $\mathfrak{G}^+$ of the set of all sequences $(g_i)_{i \in \mathbb{Z}} \in \mathfrak{G}$, with $g_i = w_i$ for $i \leq -1$.

Let $\sigma : \mathfrak{G} \to \mathfrak{G}$ be the *shift map*, which is defined for every $\mathbf{g} \in \mathfrak{G}$ and $n \in \mathbb{Z}$ as $(\sigma(\mathbf{g}))_n = g_{n+1}$.

We say a map $\Theta : \Lambda \to \Lambda'$, between two topological Markov chains is a $(\ell + r + 1)$-block code if it has a local rule $\theta : \Lambda_{\ell+r+1} \to \Lambda'_1$ such that for any $\mathbf{x} = (x_i)_{i \in \mathbb{Z}} \in \Lambda$ and $j \in \mathbb{Z}$ follows that $\big(\Theta(\mathbf{x})\big)_j = \theta(x_{j-\ell}, \ldots, x_{j+r})$. Under these notations, we say Θ has memory ℓ and anticipation r. We recall a map Θ is a block code if and only if it is continuous and commutes with the shift map.

$$\begin{array}{cccccc}
\mathbf{x} & = & (\ldots, & \boxed{x_{j-\ell}, \ldots, x_j \ldots, x_{j+r}} & , \ldots) \\
 & & & \Big\downarrow \theta & \\
\Theta(\mathbf{x}) & = & (\ldots, & \big(\Theta(\mathbf{x})\big)_j & , \ldots)
\end{array}$$

A *cellular automaton (c.a.)* is a pair $(\mathfrak{G},\Phi)$, where $\Phi:\mathfrak{G}\to\mathfrak{G}$ is $(\ell+r+1)$-block code. Without loss of generality we always can consider $\ell=0$ and so to say the c.a. has radio r.

A c.a. with radio r is said *right permutative*, if its local rule ϕ verifies for any fixed word $(w_0,\dots,w_{r-1})\in\mathfrak{G}_r$ that the map $g\mapsto\phi(w_0,\dots,w_{r-1},g)$ is a permutation on G. In the analogous way we define left permutativity. When a c.a. is right and left permutative, we will say it is *bipermutative*. From now on, we will consider that $\Phi:\mathfrak{G}\to\mathfrak{G}$ is a restriction on $\mathfrak{G}$ of some c.a. $\tilde{\Phi}:G^{\mathbb{Z}}\to G^{\mathbb{Z}}$. It is equivalent to say that there exists a map $\tilde{\phi}:G^{r+1}\to G$ such that the local rule of Φ is $\phi=\tilde{\phi}|_{\mathfrak{G}_{r+1}}$.

Let us to define three types of cellular automata which are fundamental in this work: **translations:** $(\mathfrak{G},\mathbf{g})$ is a translation if $\mathbf{g}=\mathbf{s}\circ\sigma$, where $\mathbf{s}:G^{\mathbb{Z}}\to G^{\mathbb{Z}}$ is a 1-block code with local rule $s:G\to G$ which is a permutation on G; **affine c.a.:** $(\mathfrak{G},\Phi)$ is an affine c.a. if its local rule is given by $\phi(a,b)=\eta(a)+\rho(b)+c$, where $+$ is an Abelian group operation on G, $\eta:G\to G$ and $\rho:G\to G$ are two commuting automorphisms (that is, $\eta\circ\rho=\rho\circ\eta$), and $c\in G$; and **group c.a.:** $(\mathfrak{G},\Phi)$ is a group c.a. if its local rule is given by $\phi(a,b)=a+b$, where $+$ is an Abelian group operation on G.

We say a binary operation $*$ on $\mathfrak{G}$ is $(\ell+r+1)$-block if the map $(\mathbf{x},\mathbf{y})\in\mathfrak{G}\times\mathfrak{G}\mapsto\mathbf{x}*\mathbf{y}\in\mathfrak{G}$ is a $(\ell+r+1)$-block code. When $*$ is a (quasi) group operation, then we say $(\mathfrak{G},*)$ is a (quasi) group shift.

Let μ be any σ-invariant probability measure on $\mathfrak{G}$. For a past $w\in\mathfrak{G}^-$, $w=(\dots,w_{-2},w_{-1})$, let μ_w be the probability measure on $\mathfrak{G}_w^+$ obtained for μ conditioning to the past w.

We say μ has *complete connections* (compatible with $\mathfrak{G}$) if given $a\in G$, for all $w\in\mathfrak{G}^-$ such that $a\in\mathcal{F}(w_{-1})$, one has $\mu_w(a)>0$.

If μ is a probability measure with complete connections, we define the quantities γ_m, for $m\geq 1$, by

$$\gamma_m:=\sup\left\{\left|\frac{\mu_v(a)}{\mu_w(a)}-1\right|:\quad\begin{array}{l}v,w\in\mathfrak{G}^-;\quad v_{-i}=w_{-i},\ 1\leq i\leq m;\\ a\in\mathcal{F}(v_{-1})=\mathcal{F}(w_{-1})\end{array}\right\}.$$

When $\sum_{m\geq 1}\gamma_m<\infty$, we say μ has *summable decay.*

3 Cellular automata with algebraic local rules

In this section we shall define the class of STRUCTURALLY-COMPATIBLE cellular automata, which is the subject of this work. Moreover, we will study the case of structurally-compatible bipermutative c.a..

Definition 3.1. *We say a cellular automaton $(\mathfrak{G},\Phi)$ with radio 1 is structurally compatible (SC) if it verifies the following property:*

$$(x_i)_{i\in\mathbb{Z}},(y_i)_{i\in\mathbb{Z}}\in\mathfrak{G}\Longrightarrow\big(\phi(x_i,y_i)\big)_{i\in\mathbb{Z}}\in\mathfrak{G},\tag{3.1}$$

where ϕ denotes the local rule of Φ.

Define $\bullet$ as the binary operation on G giving for all $a,b\in G$ by $a\bullet b:=\phi(a,b)$. The structural compatibility implies we can consider the componentwise operation $*$ on $\mathfrak{G}$:

$$\forall(x_i)_{i\in\mathbb{Z}},(y_i)_{i\in\mathbb{Z}}\in\mathfrak{G},(x_i)_{i\in\mathbb{Z}}*(y_i)_{i\in\mathbb{Z}}:=(x_i\bullet y_i)_{i\in\mathbb{Z}},$$

Notice neither $\bullet$ nor $*$ are necessarily algebraic operations on G and $\mathfrak{G}$ respectively. However, the c.a. is left permutative (as well right permutative or bipermutative) if and only if $\bullet$ (and so $*$) is a left cancellable operation (as well right cancellable or a quasi-group operation respectively). We recall that an operation which is left-right cancellable is called a quasi-group operation.

In terms of $*$, the map Φ can be written as

$$\Phi = id * \sigma.$$

Example 3.2. *Let $\bullet$ be the quasi-group operation on $G = \{a_i, b_i, c_i, d_i : \ i = 1, 2, 3\}$, giving by the following Latin square:*

$\bullet$	a_1	b_1	c_1	d_1	a_2	b_2	c_2	d_2	a_3	b_3	c_3	d_3
a_1	a_3	b_3	c_3	d_3	a_2	b_2	c_2	d_2	a_1	b_1	c_1	d_1
b_1	b_3	a_3	d_3	c_3	b_2	a_2	d_2	c_2	b_1	a_1	d_1	c_1
c_1	c_3	d_3	a_3	b_3	c_2	d_2	a_2	b_2	c_1	d_1	a_1	b_1
d_1	d_3	c_3	b_3	a_3	d_2	c_2	b_2	a_2	d_1	c_1	b_1	a_1
a_2	a_2	b_2	c_2	d_2	a_1	b_1	c_1	d_1	a_3	b_3	c_3	d_3
b_2	b_2	a_2	d_2	c_2	b_1	a_1	d_1	c_1	b_3	a_3	d_3	c_3
c_2	c_2	d_2	a_2	b_2	c_1	d_1	a_1	b_1	c_3	d_3	a_3	b_3
d_2	d_2	c_2	b_2	a_2	d_1	c_1	b_1	a_1	d_3	c_3	b_3	a_3
a_3	a_1	b_1	c_1	d_1	a_3	b_3	c_3	d_3	a_2	b_2	c_2	d_2
b_3	b_1	a_1	d_1	c_1	b_3	a_3	d_3	c_3	b_2	a_2	d_2	c_2
c_3	c_1	d_1	a_1	b_1	c_3	d_3	a_3	b_3	c_2	d_2	a_2	b_2
d_3	d_1	c_1	b_1	a_1	d_3	c_3	b_3	a_3	d_2	c_2	b_2	a_2

Denote as $*$ the 1-block operation induced by $\bullet$ on $G^{\mathbb{Z}}$. Let $\mathfrak{G} \subset G^{\mathbb{Z}}$ be the topological Markov chain defined by the oriented graph of Figure 1. We have that $(\mathfrak{G}, *)$ is an irreducible quasi-group shift.

Define the bipermutative cellular automaton $(\mathfrak{G}, \Phi)$, where $\Phi := id * \sigma$. It follows Φ verifies the property (3.1) and so it is structurally compatible. Moreover, since $\bullet$ has the medial property:

$$\forall a, b, c, d \in G, \qquad (a \bullet b) \bullet (c \bullet d) = (a \bullet c) \bullet (b \bullet d),$$

it follows, from ([3], Theorem 2.2.2, p.70), that there exist an Abelian group operation on G, η and ρ commuting automorphisms, and $c \in G$, such that $a \bullet b = \eta(a) + \rho(b) + c$. Therefore, $(\mathfrak{G}, \Phi)$ is an affine c.a..

The next proposition gives a characterization of SC bipermutative cellular automata.

Proposition 3.3. *Let $(\mathfrak{G}, \Phi)$ be a SC bipermutative c.a.. Then,*

(i) *$(\mathfrak{G}, \Phi)$ is topologically conjugate to $(\mathfrak{H}, \Phi_{\mathfrak{H}})$ through a 1-block code, where $\mathfrak{H} = \mathbb{F} \times \Sigma_n$, $\mathbb{F}$ is finite, Σ_n is a* full n shift, *and $\Phi_{\mathfrak{H}} = id_{\mathfrak{H}} \otimes \sigma_{\mathfrak{H}}$ where $\otimes$ is a k-block quasi-group operation on $\mathfrak{H}$.*

(ii) $\mathbf{h}(\mathfrak{G}) = 0$ *(the topological entropy of the shift is zero) if and only if $\Sigma_n = \{(\ldots, a, a, a, \ldots)\}$ (that is, the full shift is trivial).*

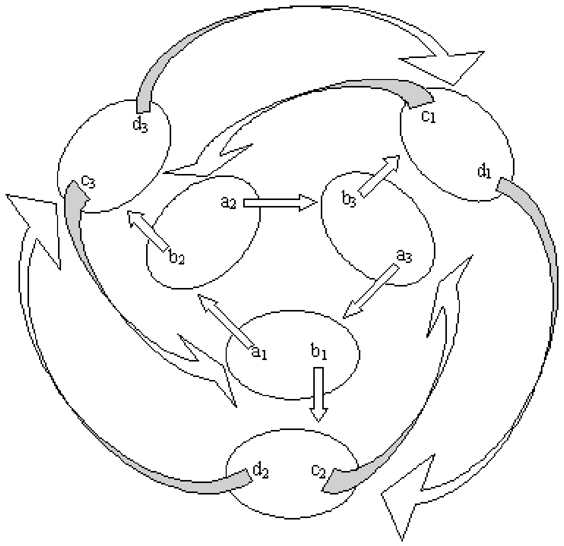

Figure 1: Graph generates $\mathfrak{G}$.

(iii) $\mathfrak{G}$ *is irreducible and has constant sequence if and only if* $\mathbb{F} = \{e\}$ *(that is,* $\mathbb{F}$ *is unitary).*

Proof.

(i) Let $(\mathfrak{G}, \Phi)$ be a bipermutative c.a. with radio 1 which verifies (3.1). As before, for $a, b \in G$, denote $a \bullet b = \phi(a, b)$ which is a quasi-group operation on G. Thus, $\Phi = id * \sigma$, where $*$ is the componentwise quasi-group operation on $\mathfrak{G}$ induced from $\bullet$.

From Theorem 4.25 and Remark 4.28 in [26], the quasi group $(\mathfrak{G}, *)$ is isomorphic to a quasi group $(\mathbb{F} \times \Sigma_n, \otimes)$, where $\otimes$ is a k-block operation, with anticipation $k-1$. We denote $\mathfrak{H} := \mathbb{F} \times \Sigma_n$, as $\varphi : \mathfrak{G} \to \mathfrak{H}$ the isomorphism between the quasi groups, and $\Phi_{\mathfrak{H}} := id_{\mathfrak{H}} \otimes \sigma_{\mathfrak{H}}$. It follows that

$$\begin{aligned} \varphi \circ \Phi \quad &= \quad \varphi \circ (id_{\mathfrak{G}} * \sigma_{\mathfrak{G}}) =_{(a)} (\varphi \circ id_{\mathfrak{G}}) \otimes (\varphi \circ \sigma_{\mathfrak{G}}) \\ &=_{(b)} \quad (id_{\mathfrak{H}} \circ \varphi) \otimes (\sigma_{\mathfrak{H}} \circ \varphi) = (id_{\mathfrak{H}} \otimes \sigma_{\mathfrak{H}}) \circ \varphi = \Phi_{\mathfrak{H}} \circ \varphi, \end{aligned}$$

where $=_{(a)}$ comes from the fact that φ is an isomorphism between $(\mathfrak{G}, *)$ and $(\mathfrak{H}, \otimes)$, and $=_{(b)}$ is due the fact that φ is a 1-block code (see Theorem 4.25 in [26]) and so it commutes with the shift map.

Since $\otimes$ is a k-block quasi-group operation (with memory 0 and anticipation $k-1$), we have that $\Phi_{\mathfrak{H}}$ has radio k.

(ii) and (iii) They follow straightforward from Theorem 4.25 of [26].

$\square$

Remark 3.4. *From Theorem 4.25 in [26] we could get an analogous result, but with* $\otimes$ *being an operation with memory* $k-1$ *and anticipation* 0*. Therefore,* $\Phi_{\mathfrak{H}}$ *would have memory* k *and anticipation* 0*.*

We notice $(\mathfrak{H}, \Phi_{\mathfrak{H}})$ in the previous theorem is not necessarily a bipermutative c.a.. For instance, if $(\mathfrak{G}, \Phi)$ is a group c.a. (which means $(G, \bullet)$ is a group) such that (3.1) holds, then it verifies all hypothesis of Theorem 3.3, but $\otimes$ can be a k-block group operation with memory 0 and anticipation $k-1$, for some $k > 1$. Thus, $(\mathfrak{H}, \Phi_{\mathfrak{H}})$ can not be right permutative. In fact, since $\rho : \mathfrak{H}_k \times \mathfrak{H}_k \to \mathfrak{H}_1$ the local rule of $\otimes$, then since $\sigma_{\mathfrak{H}}$ is an automorphism to the group $(\mathfrak{H}, \otimes)$, it follows the identity element $\mathbf{e}$ of the group is such that $\sigma_{\mathfrak{H}}(\mathbf{e}) = \mathbf{e}$, which implies $\mathbf{e} = (\dots e, e, e, \dots)$. Therefore, taking $w \in \mathfrak{H}_k$, $w = (\overbrace{e, e, \dots, e}^{k})$, we have for all $a \in \mathfrak{H}_1$:

$$\phi_{\mathfrak{H}}(wa) = \phi_{\mathfrak{H}}\big((\overbrace{e, e, \dots, e, a}^{k+1})\big) = \rho\big((\overbrace{e, e, \dots, e}^{k}), (\overbrace{e, e, \dots, a}^{k})\big) = e.$$

4 Right-permutative cellular automata

In this section we shall study two types of cellular automata: *N-scaling*; and *Ψ-associative*.

We say a cellular automaton $(\mathfrak{G}, \Phi)$ with radio 1 is a N-scaling c.a. for some $N \geq 2$ if its local rule $\phi : G \times G \to G$ is such that for any $\mathbf{x} = (x_i)_{i \in \mathbb{Z}} \in \mathfrak{G}$,

$$\big(\Phi^N(\mathbf{x})\big)_0 = x_0 \bullet x_N.$$

On the other hand $(\mathfrak{G}, \Phi)$ is said Ψ-associative, if there exists a permutation $\Psi : G \to G$ such that for any $a, b, c \in G$, we have

$$(a \bullet b) \bullet c = \Psi\big(a \bullet (b \bullet c)\big)$$

When $\mathfrak{G} = G^{\mathbb{Z}}$, Host-Maass-Martínez [7] have proved that every right-permutative N-scaling c.a. $(\mathfrak{G}, \Phi)$ is topologically conjugate to the product of an affine c.a. with a translation, while every right-permutative Ψ-associative is topologically conjugate to the product of a group c.a. with a translation.

Theorems 4.2 and 4.3 below reproduce those results for the general case of cellular automata defined on topological Markov chains. To proof these theorems we shall remark some basics on these types of cellular automata:

Remark 4.1.

- *If $(G^{\mathbb{Z}}, \Phi)$ is a right-permutative ψ-associative c.a., from Theorem 6 in [7], we get that there exists a* 1*-block code $\mathbf{u} : G^{\mathbb{Z}} \to K^{\mathbb{Z}} \times B^{\mathbb{Z}}$, which is a topological conjugacy between $(G^{\mathbb{Z}}, \Phi)$ and $(K^{\mathbb{Z}} \times B^{\mathbb{Z}}, \Phi_K \times \mathbf{g}_B)$, where $B \subseteq G$ and K are two finite alphabets, ϕ_K is a* group c.a. *and $\mathbf{g}_B$ is a translation.*

  We recall $\mathbf{g}_B = \mathbf{s}_B \circ \sigma_B$, where $\mathbf{s}_B : B^{\mathbb{Z}} \to B^{\mathbb{Z}}$ is a 1-block code with local rule $s_B : B \to B$ which is a permutation on B. Moreover, [7] gives $s_B : B \to B$ is defined for all $e' \in B$ by $s_B(e') = e'' \bullet e'$, where $e'' \in B$ is any element.

  Furthermore, $\mathbf{u}$ has local rule $u : G \to K \times B$ which is a bijection and is given for any $a \in G$ by

  $$u(a) = (\tilde{a}, e_a),$$

  where $\tilde{a}$ is the equivalent class of $a \in G$ to the equivalence relation,

  $$a \sim b \iff \forall c \in G,\ a \bullet c = b \bullet c,$$

  and e_a is the unique element of G for which $a \bullet e_a = a$. We notice for all $a \in G$ and $e \in B$, we have $a \bullet e \sim a$. Moreover, the following property holds: $e_{a \bullet b} = e_a \bullet e_b = s_B(e_b)$.

Finally, since Φ_K is a group, its local rule define a group operation on K:

$$\forall \tilde{a}, \tilde{b} \in K, \tilde{a} \tilde{\bullet} \tilde{b} := \phi_K(\tilde{a}, \tilde{b}).$$

- *From Theorem 8 in [7], if $(G^{\mathbb{Z}}, \Phi)$ is a N-scaling c.a, then the above statements hold, but Φ_K will be an affine c.a. and in the code $u(a) = (\tilde{a}, e_a)$, e_a will be defined as the unique element of B for which the equation $e_a = x \bullet a$ has solution.*

Theorem 4.2. *Let $(\mathfrak{G}, \Phi)$ be a SC right-permutative Ψ-associative c.a.. Then, $(\mathfrak{G}, \Phi)$ is topologically conjugate through a 1-block code to $(\mathfrak{K} \times \mathfrak{B}, \Phi_{\mathfrak{K}} \times \mathbf{g}_{\mathfrak{B}})$, where $\mathfrak{K}$ and $\mathfrak{B}$ are topological Markov chains, $(\mathfrak{K}, \Phi_{\mathfrak{K}})$ is a group c.a., and $(\mathfrak{B}, \mathbf{g}_{\mathfrak{B}})$ is a translation.*

Proof.

Step 1 Since $(\mathfrak{G}, \Phi)$ has radio 1 and verifies (3.1) we can consider that $\Phi : \mathfrak{G} \to \mathfrak{G}$ is a restriction on $\mathfrak{G}$ of some right-permutative Ψ-associative c.a. $(G^{\mathbb{Z}}, \tilde{\Phi})$ which has the same local rule $\phi : G \times G \to G$.

Let $(K^{\mathbb{Z}} \times B^{\mathbb{Z}}, \Phi_K \times \mathbf{g}_B)$ be the group-translation and $\mathbf{u} : G^{\mathbb{Z}} \to K^{\mathbb{Z}} \times B^{\mathbb{Z}}$ the topological conjugacy presented in Remark 4.1.

We consider on $K \times B$ the right-permutative operation also denoted as $\bullet$ and induced from the local rule of $\Phi_K \times \mathbf{g}_B$: given $(\tilde{a}_1, e_1), (\tilde{a}_2, e_2) \in K \times B$ define

$$(\tilde{a}_1, e_1) \bullet (\tilde{a}_2, e_2) = \big(\tilde{a}_1 \tilde{\bullet} \tilde{a}_2, s_B(e_2)\big).$$

Notice that $u : G \to K \times B$ is an isomorphism between $(G, \bullet)$ and $(K \times B, \bullet)$. In fact, u is bijective and

$$u(a \bullet c) = (\widetilde{a \bullet c}, e_{a \bullet c}) =_{(a)} (\tilde{a} \tilde{\bullet} \tilde{c}, e_a \bullet e_c)$$
$$= \big(\tilde{a} \tilde{\bullet} \tilde{c}, s_B(e_c)\big) = (\tilde{a}, e_a) \bullet (\tilde{c}, e_c) = u(a) \bullet u(c),$$

where $=_{(a)}$ comes from Theorem 6 of [7].

The operation $\bullet$ on $K \times B$ induces the componentwise operation also denoted as $*$ on $K^{\mathbb{Z}} \times B^{\mathbb{Z}}$. Thus, $\mathbf{u} : G^{\mathbb{Z}} \to K^{\mathbb{Z}} \times B^{\mathbb{Z}}$ is an isomorphism between $(G^{\mathbb{Z}}, *)$ and $(K^{\mathbb{Z}} \times B^{\mathbb{Z}}, *)$.

Define $\Lambda := \mathbf{u}(\mathfrak{G}) \subseteq K^{\mathbb{Z}} \times B^{\mathbb{Z}}$. Since $\mathbf{u}$ is topological conjugacy between Φ and $\Phi_K \times \mathbf{g}_B$, it follows $\Phi_K \times \mathbf{g}_B(\Lambda) = \Lambda$. Therefore, we have the cellular automaton $(\Lambda, \Phi_K \times \mathbf{g}_B)$ is well defined and $*$ is closed on Λ. Moreover, $\mathbf{u}|_{\mathfrak{G}}$ is a topological conjugacy between $(\mathfrak{G}, \Phi)$ and $(\Lambda, \Phi_K \times \mathbf{g}_B)$, and an isomorphism between $(\mathfrak{G}, *)$ and $(\Lambda, *)$.

Step 2 We will show that there exists $M \geq 1$ such that for all $e \in B$ we have $s_B^M(e) = e$.

Since s_B is a permutation on B, it follows for all $e \in B$ there exists $M_e \geq 1$ such that $s_B^{M_e}(e) = e$. Because B is a finite alphabet, we can take M a multiple of all periods of each element of B. Then, the result follows.

Step 3 Let us to prove that $\Lambda = \mathfrak{K} \times \mathfrak{B}$, where $\mathfrak{K} \subseteq K^{\mathbb{Z}}$ and $\mathfrak{B} \subseteq B^{\mathbb{Z}}$ are both topological Markov chains.

First, notice that, because $\tilde{\bullet}$ is a quasi-group operation, there exists $L \in \mathbb{N}$ such that for all $\tilde{a}, \tilde{c} \in K$:

$$\underbrace{(\ (\ (\tilde{c}\tilde{\bullet}\tilde{a})\dots\tilde{\bullet}\tilde{a})\tilde{\bullet}\tilde{a})\tilde{\bullet}\tilde{a}}_{\tilde{c} \text{ multiplied } L \text{ times by } \tilde{a} \text{ for the right side}} = \tilde{c}.$$

Denote $\pi_K : \Lambda \to K^{\mathbb{Z}}$ and $\pi_B : \Lambda \to B^{\mathbb{Z}}$ the canonical projections on the first and second coordinates respectively.

It is straightforward that $\Lambda \subseteq \pi_K(\Lambda) \times \pi_B(\Lambda)$. So, we only need to show that $\pi_K(\Lambda) \times \pi_B(\Lambda) \subseteq \Lambda$.

In fact, given $(\tilde{c}_i)_{i\in\mathbb{Z}} \in \pi_K(\Lambda)$ and $(e_i)_{i\in\mathbb{Z}} \in \pi_B(\Lambda)$ must there exist $(\tilde{c}_i, e'_i)_{i\in\mathbb{Z}}, (\tilde{a}_i, e_i)_{i\in\mathbb{Z}} \in \Lambda$, and so

$$\underbrace{\Big(\Big(\Big((\tilde{c}_i, e'_i)_{i\in\mathbb{Z}} * (\tilde{a}_i, e_i)_{i\in\mathbb{Z}}\Big) * (\tilde{a}_i, e_i)_{i\in\mathbb{Z}}\Big) \dots * (\tilde{a}_i, e_i)_{i\in\mathbb{Z}}\Big) * (\tilde{a}_i, e_i)_{i\in\mathbb{Z}}}_{\text{multiplying } (\tilde{c}_i, e'_i)_{i\in\mathbb{Z}} \ L \text{ times by } (\tilde{a}_i, e_i)_{i\in\mathbb{Z}} \text{ for the right side}}$$

$$= \Big(\underbrace{(((\tilde{c}_i\tilde{\bullet}\tilde{a}_i)\tilde{\bullet}\tilde{a}_i)\dots\tilde{\bullet}\tilde{a}_i)\tilde{\bullet}\tilde{a}_i}_{\text{multiplying } \tilde{c}_i \ L \text{ times by } \tilde{a}_i \text{ for the right side}}, s_B(e_i)\Big)_{i\in\mathbb{Z}}$$

$$= \big(\tilde{c}_i, s_B(e_i)\big)_{i\in\mathbb{Z}} \in \Lambda.$$

Now, we repeat the above procedure, but multiplying $\big(\tilde{c}_i, s_B(e_i)\big)_{i\in\mathbb{Z}}$ L times by itself for the right side, and so we get $\big(\tilde{c}_i, s_B^2(e_i)\big)_{i\in\mathbb{Z}} \in \Lambda$. By induction, we can obtain that for all $m \geq 0$, $\big(\tilde{c}_i, s_B^m(e_i)\big)_{i\in\mathbb{Z}} \in \Lambda$. From Step 2 there exists $M \geq 1$ such that for all $i \in \mathbb{Z}$ we have $s_B^M(e_i) = e_i$. Therefore, we get $\big(\tilde{c}_i, e_i\big)_{i\in\mathbb{Z}} \in \Lambda$, which allows us to deduce that $\Lambda = \pi_K(\Lambda) \times \pi_B(\Lambda)$.

Notice that $\mathbf{u}|_{\mathfrak{G}}$ is a 1-block code from $\mathfrak{G}$ to Λ such that its inverse is also a 1-block code. Thus, since $\mathfrak{G}$ is a topological Markov chain, it follows that Λ is also a topological Markov chain. Finally, since $\Lambda = \pi_K(\Lambda) \times \pi_B(\Lambda)$ we have that $\pi_K(\Lambda)$ and $\pi_B(\Lambda)$ are also both topological Markov chains, and denoting $\mathfrak{K} := \pi_K(\Lambda)$, $\mathfrak{B} := \pi_B(\Lambda)$, $\Phi_{\mathfrak{K}} := \Phi_K|_{\mathfrak{K}}$ and $\Phi_{\mathfrak{B}} := \Phi_B|_{\mathfrak{B}}$ we finish the proof.

$\square$

Theorem 4.3. *Let $(\mathfrak{G}, \Phi)$ be a SC right-permutative N-scaling c.a.. If its extension $(G^{\mathbb{Z}}, \tilde{\Phi})$ is also a N-scaling c.a., then $(\mathfrak{G}, \Phi)$ is topologically conjugate through a 1-block code to $(\mathfrak{K} \times \mathfrak{B}, \Phi_{\mathfrak{K}} \times \mathbf{g}_{\mathfrak{B}})$, where $\mathfrak{K}$ and $\mathfrak{B}$ are topological Markov chains, $(\mathfrak{K}, \Phi_{\mathfrak{K}})$ is an affine c.a., and $(\mathfrak{B}, \mathbf{g}_{\mathfrak{B}})$ is a translation.*

Proof. Since $(\mathfrak{G}, \Phi)$ is the restriction on $\mathfrak{G}$ of a N-scaling c.a. $(G^{\mathbb{Z}}, \tilde{\Phi})$, we can apply a similar reasoning than Theorem 4.2.

$\square$

Corollary 4.4. *Let $(\mathfrak{G},\Phi)$ be a SC right-permutative N-scaling c.a.. If $\mathfrak{G}$ is mixing, then $(\mathfrak{G},\Phi)$ is topologically conjugate through a 1-block code to $(\mathfrak{K}\times\mathfrak{B},\Phi_{\mathfrak{K}}\times\mathbf{g}_{\mathfrak{B}})$, where $\mathfrak{K}$ and $\mathfrak{B}$ are topological Markov chains, $(\mathfrak{K},\Phi_{\mathfrak{K}})$ is an affine c.a., and $(\mathfrak{B},\mathbf{g}_{\mathfrak{B}})$ is a translation.*

Proof. Since $\mathfrak{G}$ is mixing, there exists $q\geq 1$ such that for any $k\geq q$ and $u,w\in G$ we always can find $(v_1,\ldots,v_k)\in\mathfrak{G}_k$ such that $(u,v_1,\ldots,v_k,w)\in\mathfrak{G}_{k+2}$. Without loss of generality we can consider $N\geq q$, because if $(\mathfrak{G},\Phi)$ is N-scaling, then it is also N^m-scaling for any $m\geq 1$. We will show that $(G^{\mathbb{Z}},\tilde{\Phi})$ is also N-scaling:

Given a sequence $\mathbf{x}=(x_i)_{i\in\mathbb{Z}}\in G$, due the fact of $\mathfrak{G}$ is mixing and $N\geq q$, we can find a sequence $\mathbf{y}=(y_i)_{i\in\mathbb{Z}}\in\mathfrak{G}$ such that $y_{jN}=x_j$ for all $i\in\mathbb{Z}$. Thus,

$$\left(\tilde{\Phi}(\mathbf{x})\right)_j = x_j\bullet x_{j+1} = y_{jN}\bullet y_{(j+1)N} = \left(\Phi^N(\mathbf{y})\right)_{jN}\ ,$$

and by induction we get that for any $k\geq 1$, $\left(\tilde{\Phi}^k(\mathbf{x})\right)_j=\left(\Phi^{kN}(\mathbf{y})\right)_{jN}$. Therefore,

$$\left(\tilde{\Phi}^N(\mathbf{x})\right)_j = \left(\Phi^{N^2}(\mathbf{y})\right)_{jN} = y_{jN}\bullet y_{jN+N^2} = y_{jN}\bullet y_{(j+N)N} = x_j\bullet x_{j+N}\ .$$

Now, since $\tilde{\Phi}$ is a N-scaling c.a., we can apply Theorem 4.3 to conclude the proof.

□

Notice that $(\mathfrak{K},\Phi_K)$ obtained in the previous theorems is a group c.a. (or an affine c.a.) which is also structurally compatible. Thus, since $(\mathfrak{K},\Phi_K)$ is bipermutative, we can apply Proposition 3.3 to get it is topologically conjugate through a 1-block code to $(\mathfrak{H},\Phi_{\mathfrak{H}})$, where $G=\mathbb{F}\times\Sigma_n$ with $\mathbb{F}$ is finite and Σ_n is a full n shift, and Φ_G is a group c.a. (or an affine c.a.) with radio k.

5 Projections of measures with complete connections and summable decay

In this section we shall present sufficient conditions to reproduce results about the convergence of the Cesàro mean distribution ([7], [15]) to the more general case of $\mathfrak{G}$ being neither a full shift nor a groupshift, but $(\mathfrak{G},\Phi)$ being structurally compatible.

Lemma 5.1. *Let Λ and Λ' be two topological Markov chains, and $\Theta:\Lambda\to\Lambda'$ be an invertible 1-block code which action is constant on the predecessor sets. Suppose Θ^{-1} has memory 1 and anticipation 0. If μ is a σ-invariant probability measure on Λ with complete connections (compatible with Λ) and summable decay, then $\mu'=\mu\circ\Theta^{-1}$ also has complete connections (compatible with Λ') and summable decay.*

Proof. Let C' be a cylinder of Λ' defined by the coordinates $i=0,\ldots,m$ with $m\geq 1$, that is, $C'=[c'_0,\ldots,c'_m]$. We will show that $C:=\Theta^{-1}(C')$ is a cylinder of Λ defined by the coordinates $i=1,\ldots,m$, that is, $C=[c_1,\ldots,c_m]$.

Denote as θ the local rule of Θ. Notice that for all $1\leq i\leq m$, $c_i\in\Lambda_1$ is well defined by $c_i:=\theta^{-1}(c'_{i-1},c'_i)$. Therefore,

$$\Theta^{-1}(C') = \bigcup_{\substack{c_0\in\mathcal{P}(c_1)\\ \theta(c_0)=c'_0}} [c_0,c_1,\ldots,c_m] = \bigcup_{c_0\in\mathcal{P}(c_1)} [c_0,c_1,\ldots,c_m] = [c_1,\ldots,c_m]$$

Through the use of a similar reasoning and since Θ^{-1} has anticipation 0, we get that for any $v', w' \in \Lambda'^{-}$, we can define $v := \Theta^{-1}(v')$ and $w := \Theta^{-1}(w')$ which are both pasts belonging to Λ^{-}. In particular, if $v'_{-i} = w'_{-i}$ for $1 \le i \le m$, with $m \ge 2$, then $v_{-i} = w_{-i}$ for $1 \le i \le m-1$.

On the other hand, since μ has complete connections (compatible with Λ), given $w' \in \Lambda'^{-}$ and $a' \in \mathcal{F}(w'_{-1})$ there exist unique $w \in \Lambda$ and $a \in \mathcal{F}(w_{-1})$ such that $\mu'_{w'}(a') = \mu_w(a) > 0$. It means μ' also has complete connections (compatible with Λ'). Moreover, for $m \ge 2$, it follows

$$\gamma'_m = \sup\left\{ \left| \frac{\mu'_{v'}(a')}{\mu'_{w'}(a')} - 1 \right| : \begin{array}{l} v', w' \in \Lambda'^{-}; \quad v'_{-i} = w'_{-i},\ 1 \le i \le m; \\ a' \in \mathcal{F}(v'_{-1}) = \mathcal{F}(w'_{-1}) \end{array} \right\}$$

$$= \sup\left\{ \left| \frac{\mu_v(a)}{\mu_w(a)} - 1 \right| : \begin{array}{l} v, w \in \Lambda^{-}; \quad v_{-i} = w_{-i},\ 1 \le i \le m-1; \\ a \in \mathcal{F}(v_{-1}) = \mathcal{F}(w_{-1}) \end{array} \right\} = \gamma_{m-1},$$

which means μ' has summable decay.

$\square$

In an analogous way, we can prove the following Lemma:

Lemma 5.2. *Let Λ and Λ' be two topological Markov chains, and let $\varphi : \Lambda \times \Sigma \to \Lambda' \times \Sigma$ be block code defined by $\varphi := \Theta \times id$, where $\Theta : \Lambda \to \Lambda'$ is an invertible 1-block code which is constant on the predecessor sets. Suppose Θ^{-1} has memory 1 and anticipation 0. If μ is σ-invariant probability measure on Λ with complete connections (compatible with Λ) and summable decay, then $\mu' = \mu \circ \Theta^{-1}$ also has complete connections (compatible with Λ') and summable decay.*

$\square$

Now, consider $(\mathfrak{G}, \Phi)$ being a SC bipermutative c.a.. Let $\varphi : \mathfrak{G} \to G$ be the topological conjugacy between $(\mathfrak{G}, \Phi)$ and $(\mathfrak{H}, \Phi_{\mathfrak{H}})$, where $\mathfrak{H} = \mathbb{F} \times \Sigma_n$, given by Proposition 3.3. From Remark 3.4 we can suppose that φ has memoria k and anticipation 0. With this notations, we have that:

Proposition 5.3. *If $(\mathfrak{G}, \Phi)$ is a SC bipermutative c.a., and μ is a probability measure with complete connections (compatible with $\mathfrak{G}$) and summable decay, then $\mu \circ \varphi^{-1}$ is a probability measure on $\mathfrak{H} = \mathbb{F} \times \Sigma_n$ which also has complete connections and summable decay.*

Proof. From Theorem 4.25 of [26], φ is given by the following composition:

$$\varphi = \varphi_n \circ \eta_n \circ \varphi_{n-1} \circ \eta_{n-1} \circ \ldots \circ \varphi_1 \circ \eta_1,$$

where for all $i = 1, \ldots, n$, $\varphi_i = \Theta_i \times id$ is a block code as in Lemma 5.2, and η_i is an invertible 1-block code which inverse is also a 1-block code. Thus, for each $i \le n$ we have that η_i and φ_i preserve the properties of complete connections and summable decay of the measure, which conclude the proof.

$\square$

6 Cesàro mean convergence of measures with complete connections and summable decay

In this section we shall present some results about the convergence of the Cesàro mean distribution of probability measures under the action cellular automata, namely we study the following limit:

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \mu \circ \Phi^{-n}.$$

The essential tools that we will use to study the convergence of the Cesàro mean distribution are: propositions 3.3 and 5.3; and Corollary 29 of [7].

Definition 6.1. *Given a SC bipermutative c.a. $(\mathfrak{G}, \Phi)$, we say it is* regular *if the quasigroup $(\mathbb{F} \times \Sigma_n, \otimes)$ set in Theorem 3.3 is such that $\otimes = \otimes_{\mathbb{F}} \times \otimes_{\Sigma_n}$, where $(\mathbb{F}, \otimes_{\mathbb{F}})$ and $(\Sigma_n, \otimes_{\Sigma_n})$ are both quasigroups. Furthermore, if $\otimes_{\Sigma_n}$ is a 1-block operation, then we say $(\mathfrak{G}, \Phi)$ is* simple.

Example 6.2. *If $\mathfrak{G}$ is irreducible or $\mathbf{h}(\mathfrak{G}) = 0$, then $\mathfrak{G}$ is regular due Theorem 4.25(ii,iii) of [26]. If $\mathbf{h}(\mathfrak{G}) = p$, where p is a prime number, then $\mathfrak{G}$ is simple due Theorem 4.26 of [26].*

Theorem 6.3. *Let $(\mathfrak{G}, \Phi)$ be a SC cellular automaton, where $\mathfrak{G}$ is not necessarily irreducible. Denote as $(G^{\mathbb{Z}}, \tilde{\Phi})$ the extension of $(\mathfrak{G}, \Phi)$ to the full shift, and suppose μ is a probability measure on $\mathfrak{G}$ with complete connections (compatible with $\mathfrak{G}$) and summable decay. Then:*

(i) *If $(\mathfrak{G}, \Phi)$ is an affine c.a. which is regular and simple, then the Cesàro mean distribution of μ under the action of Φ converges;*

(ii) *If $(\mathfrak{G}, \Phi)$ is a right-permutative Ψ-associative c.a. and the group c.a. associate to it (see Theorem 4.2) is Abelian, regular and simple, then the Cesàro mean distribution of μ under the action of Φ converges.*

(iii) *If $(G^{\mathbb{Z}}, \tilde{\Phi})$ is right-permutative and N-scaling and the affine c.a. associate to it (see Theorem 4.3) is regular and simple, then the Cesàro mean distribution of μ under the action of Φ converges.*

Proof.

(i) Let $(\mathbb{F} \times \Sigma_n, \Phi_{\mathbb{F}\times\Sigma_n})$ and $\varphi : \mathfrak{G} \to \mathbb{F} \times \Sigma_n$ be the cellular automaton and the topological conjugacy given by Proposition 3.3. From Proposition 5.3, we have that $\mu' = \mu \circ \varphi^{-1}$ is a probability measure on $\mathbb{F} \times \Sigma_n$ with complete connections and summable decay. Moreover, since $(\mathfrak{G}, \Phi)$ is regular and simple, it follows that $\Phi_{\mathbb{F}\times\Sigma_n} = \Phi_{\mathbb{F}} \times \Phi_{\Sigma_n}$, where $(\Sigma_n, \Phi_{\Sigma_n})$ is an affine c.a.. In fact, $(\mathbb{F} \times \Sigma_n, \otimes_{\mathbb{F}\times\Sigma_n}) = (\mathbb{F} \times \Sigma_n, \otimes_{\mathbb{F}} \times \otimes_{\Sigma_n})$ has the medial property, thus $(\Sigma_n, \otimes_{\Sigma_n})$ also has the medial property and we can apply ([3], Theorem 2.2.2, p.70), in the same way as in Theorem 7.1, which allows us to deduce that Φ_{Σ_n} is an affine c.a..

Furthermore, since $\mathbb{F}$ is a finite set, we get $(\mathbb{F}, \Phi_{\mathbb{F}})$ is equicontinuous. Therefore, from Corollary 29 in [7], it follows that the Cesàro mean of μ' under the action of $\Phi_{\mathbb{F}\times\Sigma_n}$ converges to a probability measure $\mu'_{\mathbb{F}} \times \nu$, where $\mu'_{\mathbb{F}}$ is a $\Phi_{\mathbb{F}}$-invariant probability measure on $\mathbb{F}$, and ν is the Parry measure on Σ_n (that is, the uniform Bernoulli measure). Since $(\mathfrak{G}, \Phi)$ is topologically conjugate to $(\mathbb{F} \times \Sigma_n, \Phi_{\mathbb{F}\times\Sigma_n})$, we conclude that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \mu \circ \Phi^{-n} = (\mu'_{\mathbb{F}} \times \nu) \circ \varphi.$$

(ii) From Theorem 4.2 and Proposition 3.3, and using ([3], Theorem 2.2.2, p.70) in the same way as in the proof of Theorem 7.1, we deduce that $(\mathfrak{G}, \Phi)$ can be represented as $(\mathfrak{B} \times \mathbb{F} \times \Sigma_n, \mathbf{g}_B \times \Phi_{\mathbb{F}} \times \Phi_{\Sigma_n})$, that is: a translation on a topological Markov chain, product a group c.a. on a finite set, product a group c.a. on a full shift. Thus, by Corollary 29 of [7], we conclude

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \mu \circ \Phi^{-n} = \mu_{\mathfrak{B}} \times (\mu'_{\mathbb{F}} \times \nu) \circ \varphi,$$

where $\mu_{\mathfrak{B}}$ is a $\mathbf{g}_B$-invariant probability measure on $\mathfrak{B}$, $\mu'_{\mathbb{F}}$ is a $\Phi_{\mathbb{F}}$-invariant probability measure on $\mathbb{F}$, and ν is the Parry measure on Σ_n (that is, the uniform Bernoulli measure).

(iii) This proof is analogous to the part (ii), but uses Theorem 4.3 instead Theorem 4.2.

□

Example 6.4. *The affine c.a. of Example 3.2 is regular and simple, because applying the reasoning presented in the proof of Proposition 3.3 we deduce it is topologically conjugate to* $(\mathbb{F} \times \mathbb{Z}_2^{\mathbb{Z}}, \Phi_{\mathbb{F}} \times \Phi_{\mathbb{Z}_2^{\mathbb{Z}}})$*, where* $\mathbb{F} = \{(\ldots, 0, 1, 2, 0, 1, 2, \ldots)\}$, $\Phi_{\mathbb{F}} = \sigma_{\mathbb{F}}$ *and* $\Phi_{\mathbb{Z}_2^{\mathbb{Z}}} = id + \sigma$*. Therefore, the Cesàro mean of any probability measure on* $\mathfrak{G}$ *with complete connections summable decay converges under the action of* $(\mathfrak{G}, \Phi)$.

Example 6.5. *Let* $\bullet$ *be a binary operation defined on the set* $G = \{a, b, c, d, e, f, g, h\}$ *by the following table:*

$\bullet$	a	b	c	d	e	f	g	h
a	b	a	d	c	f	e	h	g
b	b	a	d	c	f	e	h	g
c	d	c	b	a	h	g	f	e
d	d	c	b	a	h	g	f	e
e	f	e	h	g	b	a	d	c
f	f	e	h	g	b	a	d	c
g	h	g	f	e	d	c	b	a
h	h	g	f	e	d	c	b	a

Let $\Lambda \subset G^{\mathbb{Z}}$ *be the topological Markov chain defined by the oriented graph presented in Figure 2.*

We define the map $\phi : G \times G \to G$ *giving by* $\phi(u, v) = u \bullet v$*, and we consider the cellular automaton* $\Phi : \Lambda \to \Lambda$ *with radio 1, which the local rule is* ϕ.

It is easy to check that (Λ, Φ) *is a SC right-permutative and* Ψ*-associative c.a., where* Ψ *is given by* $\Psi(\cdot) := \phi(a, \cdot)$.

From the algorithm developed in the proof of Theorem 4.2, we get (Λ, Φ) *is topologically conjugate to* $\left(\Sigma \times \{0,1\}^{\mathbb{Z}}, \Phi_\Sigma \times \mathbf{g}\right)$*, where:* $\Sigma \subset (\mathbb{Z}_2 \oplus \mathbb{Z}_2)^{\mathbb{Z}}$*;* Φ_Σ *is a group c.a.; and* $\mathbf{g}$ *is a translation on* $\{0,1\}^{\mathbb{Z}}$ *giving by* $\mathbf{g}\big((x_i)_{i\in\mathbb{Z}}\big) = \big(\Psi(x_{i+1})\big)_{i\in\mathbb{Z}}$.

Using the algorithm developed in the proof of Proposition 3.3 we find (Σ, Φ_Σ) *is topologically conjugate to the group c.a.* $(\mathbb{Z}_2^{\mathbb{Z}}, \Phi_{\mathbb{Z}_2^{\mathbb{Z}}})$*, and so it is regular and simple. Therefore, Theorem 6.3 guarantees the convergence of the Cesàro mean of any probability measure on* Λ *with complete connections summable decay under the action of* (Λ, Φ).

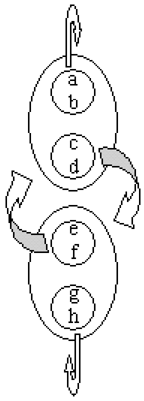

Figure 2: Graph which generates Λ.

7 Invariant measures for cellular automata on topological Markov chains

Let $(\mathfrak{G}, \Phi)$ be a SC cellular automaton and suppose that $\mathfrak{G}$ is irreducible.

An important problem is to characterize probability measures on $\mathfrak{G}$ which are invariant for the $\mathbb{Z}^2$-action defined on $\mathfrak{G}$ by (Φ, σ). Several works ([7], [19], [23]) have studied this problem and for many cases have showed that the Parry measure (the unique maximum entropy measure for $(\mathfrak{G}, \sigma)$) is the unique (σ, Φ)-invariant measure.

We can deduce results about (Φ, σ)-invariant measures through the use of the topological conjugacies presented previously. When $\mathfrak{G}$ is a group shift, then the following theorems are particular cases of the results presented by Sablik [23].

Theorem 7.1. *Let $(\mathfrak{G}, \Phi)$ be a SC affine c.a, with $\mathfrak{G}$ being irreducible and $\mathbf{h}(\mathfrak{G}) = \log p$, where p is a prime number. Let μ be a (Φ, σ)-invariant probability measure on $\mathfrak{G}$. If μ is ergodic to σ and has positive entropy to Φ, then μ is the Parry measure.*

Proof. Since $(\mathfrak{G}, \Phi)$ is an affine c.a. it follows that it is bipermutative, which implies the operation $\bullet$, defined by $a \bullet b := \phi(a, b)$ is a quasi-group operation on G.

On the other hand, from definition of affine c.a. there exists an Abelian group operation on G, η and ρ commuting automorphism and $k \in G$, such that $\phi(a, b) = \eta(a) + \rho(b) + k$. It implies that $\bullet$ has the medial property. Thus, the componentwise quasi-group operation $*$ induced from $\bullet$ on $\mathfrak{G}$, also has the medial property.

From Proposition 3.3, $(\mathfrak{G}, \Phi)$ is topologically conjugate to $(K^{\mathbb{Z}}, \Phi_K)$ through a 1-block code, where Φ_K is given by $\Phi_K = id \otimes \sigma$. Moreover the same code is an isomorphism between $(\mathfrak{G}, *)$ and $(K^{\mathbb{Z}}, \otimes)$. Therefore, $\otimes$ is also a quasi-group operation which has the medial property.

Since $\mathbf{h}(\mathfrak{G}) = \log p$, with p being a prime number, from Theorem 4.26 of [26] gives $|K| = p$ and $\otimes$ is a 1-block operation. Thus, there exists a quasi-group operation $\odot$ on K, which induces the operation $\otimes$. Notice that the local rule of Φ_K is given by $\phi_K(a', b') = a' \odot b'$.

Hence, $\odot$ also has the medial property, and so from ([3], Theorem 2.2.2, p.70) there exist an Abelian group operation $\oplus$ on K, two commuting automorphism η' and ρ', and $c' \in K$, such that $a' \odot b' = \eta'(a') \oplus \rho'(b') \oplus c'$. With other words, $(K^{\mathbb{Z}}, \Phi_K)$ is an affine c.a..

Now, defining $\mu' := \mu \circ \varphi^{-1}$, we have that $(K^{\mathbb{Z}}, \mathbf{g})$ and μ' verify all hypothesis of Theorem 12 in [7] which implies μ' is the uniform Bernoulli measure on $K^{\mathbb{Z}}$, i.e., the maximum entropy measure for the full shift. Therefore, we conclude that μ is the maximum entropy measure on $\mathfrak{G}$.

□

The following theorem has analogous proof than the previous one, but uses Theorem 13 instead Theorem 12 of [7].

Theorem 7.2. *Let $(\mathfrak{G}, \Phi)$ be a SC affine c.a., such that $\mathfrak{G}$ is irreducible and $\mathbf{h}(\mathfrak{G}) = \log p$, where p is a prime number. Let μ be (Φ, σ)-invariant probability measure on $\mathfrak{G}$. Suppose that*

(i) *μ is ergodic for the action (Φ, σ);*

(ii) *μ has positive entropy for Φ;*

(iii) *the sigma-algebra of the $\sigma^{(p-1)p}$-invariant sets coincides mod μ to the sigma-algebra of the σ-invariant sets.*

Then, μ is the Parry measure.

□

Remark 7.3. *Given a SC bipermutative c.a. $(\mathfrak{G}, \Phi)$, the c.a. $(K^{\mathbb{Z}}, \Phi_K)$ obtained from Proposition 3.3 would not be necessarily bipermutative. For the cases when $(K^{\mathbb{Z}}, \Phi_K)$ is bipermutative, we can extend the results about invariant measures set out by Pivato [19].*

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