

Instanton sheaves on complex projective spaces

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Abstract

We study a class of torsion-free sheaves on complex projective spaces which generalize the much studied mathematical instanton bundles. Instanton sheaves can be obtained as cohomologies of linear monads and are shown to be semistable if its rank is not too large, while semistable torsion-free sheaves satisfying certain cohomological conditions are instanton. We also study a few examples of moduli spaces of instanton sheaves.

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Introduction

The study of vector bundles and more general sheaves on complex projective spaces has been a topic of great interest for algebraic geometers for many years, see for instance the excellent book by Okonek, Schneider and Spindler [20] and Hartshorne's problem list [11]. In this paper we concentrate on a particular class of sheaves defined as follows, generalizing the concept of admissible sheaves on \mathbb{P}^3 due to Manin [17], see also [10, 15].

Definition. *An instanton sheaf on \mathbb{P}^n ($n \geq 2$) is a torsion-free coherent sheaf E on \mathbb{P}^n with $c_1(E) = 0$ satisfying the following cohomological conditions:*

1. for $n \geq 2$, $H^0(E(-1)) = H^n(E(-n)) = 0$;
2. for $n \geq 3$, $H^1(E(-2)) = H^{n-1}(E(1-n)) = 0$;
3. for $n \geq 4$, $H^p(E(k)) = 0$, $2 \leq p \leq n-2$ and $\forall k$;

The integer $c = h^1(E(-1)) = -\chi(E(-1))$ is called the charge of E .

If E is a rank $2m$ locally-free instanton sheaf on \mathbb{P}^{2m+1} of trivial splitting type (i.e. there exist a line $\ell \subset \mathbb{P}^n$ such that $E|_\ell \simeq \mathcal{O}_\ell^{2m}$), then E is a *mathematical instanton bundle* as originally defined by Okonek and Spindler [21]. There is an extensive literature on such objects, see for instance [1, 24]. The nomenclature is motivated by gauge theory: mathematical instanton bundles on \mathbb{P}^{2n+1} correspond to quaternionic instantons on $\mathbb{P}\mathbb{H}^n$ [22].

The goal of this paper is to extend the discussion in two directions: the inclusion of even-dimensional projective spaces and the analysis of more general sheaves, allowing for non-locally-free sheaves of arbitrary rank. Such extension is motivated by the concept that in order to better understand moduli spaces of stable vector bundles over a variety one must also consider semistable torsion-free sheaves [11]. It turns out that many of the well-known results regarding mathematical instanton bundles on \mathbb{P}^{2n+1} generalize in sometimes surprising ways to more general instanton sheaves.

The paper is organized as follows. We start by studying linear monads and their cohomologies in Section 1, spelling out criteria to decide whether the cohomology of a given monad is torsion-free, reflexive or locally-free. We then shown that every instanton sheaf is the cohomology of a linear monad, and that rank r instanton sheaves on \mathbb{P}^n exist if and only if $r \geq n - 1$. Further properties of instanton sheaves are also studied in Section 2.

The bulk of the paper lies in Section 3, where we analyze the semistability (in the sense of Mumford-Takemoto) of instanton sheaves. It is shown, for instance, that every rank $r \leq 2n - 1$ locally-free instanton sheaf on \mathbb{P}^n is semistable, while every rank $r \leq n$ reflexive instanton sheaf on \mathbb{P}^n is semistable. We also determine when a semistable torsion-free sheaf on \mathbb{P}^n is instanton, showing for instance that every semistable torsion-free sheaf on \mathbb{P}^2 is instanton.

In Section 4 it is shown that every rank $n - 1$ instanton sheaf is simple, generalizing a result of Ancona and Ottaviani for mathematical instanton bundles. We then conclude in Section 5 with a few results concerning the moduli spaces of instanton sheaves.

It is also worth noting that Buchdahl has studied monads over arbitrary blow-ups of \mathbb{P}^2 [3] while Costa and Miró-Roig have initiated the study of locally-free instanton sheaves over smooth quadric hypersurfaces within \mathbb{P}^n [6], obtaining some results similar to ours. Many of the results here obtained are also valid for instanton sheaves suitably defined over projective varieties with cyclic Picard group [16].

Notation. We work over an algebraically closed field of characteristic zero. It might be interesting from the algebraic point of view to study how the results here obtained generalize to finite fields.

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1 Monads

Let X be a smooth projective variety. A *monad* on X is a sequence \mathcal{V}_\bullet of the following form:

$$\mathcal{V}_\bullet : 0 \rightarrow V_{-1} \xrightarrow{\alpha} V_0 \xrightarrow{\beta} V_1 \rightarrow 0 \quad (1)$$

which is exact on the first and last terms. Here, V_k are locally free sheaves on X . The sheaf $E = \ker \beta / \text{Im } \alpha$ is called the cohomology of the monad \mathcal{V}_\bullet , also denoted by $H^1(\mathcal{V}_\bullet)$.

Monads were first introduced by Horrocks, who has shown that every rank 2 locally free sheaf on \mathbb{P}^3 can be obtained as the cohomology of a monad where V_k are sums of line bundles [14].

In this paper, we will focus on the so-called *linear monads* on \mathbb{P}^n , which are of the form:

$$0 \rightarrow V \otimes \mathcal{O}_{\mathbb{P}^n}(-1) \xrightarrow{\alpha} W \otimes \mathcal{O}_{\mathbb{P}^n} \xrightarrow{\beta} U \otimes \mathcal{O}_{\mathbb{P}^n}(1) \rightarrow 0, \quad (2)$$

where $\alpha \in \text{Hom}(V, W) \otimes H^0(\mathcal{O}_{\mathbb{P}^n}(1))$ is injective (as a sheaf map) and $\beta \in \text{Hom}(W, U) \otimes H^0(\mathcal{O}_{\mathbb{P}^n}(1))$ is surjective. The *degeneration locus* Σ of the monad (2) consists of the set of points $x \in \mathbb{P}^n$ such that the localized map $\alpha_x \in \text{Hom}(V, W)$ is not injective; in other words $\Sigma = \text{supp } \{\text{coker } \alpha^*\}$.

Linear monads have appeared in a wide variety of contexts within algebraic geometry, like the construction of locally free sheaves on complex projective spaces and the study of curves in \mathbb{P}^3 and surfaces in \mathbb{P}^4 , see for instance [9] and the references therein. Our main motivation to study them comes from gauge theory; as it is well-known, linear monads on \mathbb{P}^2 and \mathbb{P}^3 are closely related to instantons on \mathbb{R}^4 [7].

The existence of linear monads on \mathbb{P}^n has been completely classified by Fløystad in [9]; let $v = \dim V$, $w = \dim W$ and $u = \dim U$.

Theorem 1. [9] *Let $n \geq 1$. There exists a linear monad on \mathbb{P}^n as above if and only if at least one of the following conditions hold:*

1. $w \geq 2u + n - 1$ and $w \geq v + u$;
2. $w \geq v + u + n$.

If the conditions hold, there exists a linear monad whose degeneration locus is a codimension $w - v - u + 1$ subvariety.

In particular, if v, w, u satisfy condition (2) above, then the degeneration locus is empty.

Remark. A similar classification result for linear monads over n -dimensional quadric hypersurfaces within \mathbb{P}^{n+1} has been proved by Costa and Miró-Roig in [6].

Definition. *A coherent sheaf on \mathbb{P}^n is said to be a linear sheaf if it can be represented as the cohomology of a linear monad.*

The goal of this section is to study linear sheaves, with their characterization in mind. First, notice that E is the cohomology of (2) then

$$\text{rank}(E) = w - v - u \quad , \quad c_1(E) = v - u \quad \text{and}$$

$$c(E) = \left(\frac{1}{1-H} \right)^v \left(\frac{1}{1+H} \right)^u \quad .$$

Proposition 2. *If E is a linear sheaf on \mathbb{P}^n , then:*

1. for $n \geq 2$, $H^0(E(k)) = H^0(E^*(k)) = 0$, $\forall k \leq -1$;
2. for $n \geq 3$, $H^1(E(k)) = 0$, $\forall k \leq -2$;
3. for $n \geq 4$, $H^p(E(k)) = 0$, $2 \leq p \leq n - 2$ and $\forall k$;

4. for $n \geq 3$, $H^{n-1}(E(k)) = 0$, $\forall k \geq -n + 1$;
5. for $n \geq 2$, $H^n(E(k)) = 0$ for $k \geq -n$;
6. for $n \geq 2$, $\mathcal{E}xt^1(E, \mathcal{O}_{\mathbb{P}^n}) = \text{coker} \alpha^*$ and $\mathcal{E}xt^p(E(k), \mathcal{O}_{\mathbb{P}^n}) = 0$ for $p \geq 2$ and all k .

If E is locally-free, then also $H^n(E^*(k)) = 0$ for $k \geq -n$.

In particular, note that linear sheaves have natural cohomology in the range $-n \leq k \leq -1$, i.e. for the values of k in the range at most one of the cohomology groups $H^q(E(k))$ is nontrivial. However, not all sheaves on \mathbb{P}^n having natural cohomology in the range $-n \leq k \leq -1$ are linear, the simplest example being $\Omega_{\mathbb{P}^n}^1(n)$.

Proof. Assume that E is the cohomology of the monad (2). The kernel sheaf $\mathcal{K} = \ker \beta$ is locally-free, and one has the sequences $\forall k$:

$$0 \rightarrow \mathcal{K}(k) \rightarrow W \otimes \mathcal{O}_{\mathbb{P}^n}(k) \xrightarrow{\beta} U \otimes \mathcal{O}_{\mathbb{P}^n}(k+1) \rightarrow 0 \quad \text{and} \quad (3)$$

$$0 \rightarrow V \otimes \mathcal{O}_{\mathbb{P}^n}(k-1) \xrightarrow{\alpha} \mathcal{K}(k) \rightarrow E(k) \rightarrow 0 \quad . \quad (4)$$

From the first sequence, we see that $H^p(\mathcal{K}(k)) = 0$ for $p = 0, 1, n$ and $p+k \leq -1$ and for $2 \leq p \leq n-1$ and all k . The second sequence tell us that $H^p(\mathcal{K}(k)) \xrightarrow{\sim} H^p(E(k))$ for $p = 0$ and $k \leq 0$, for $1 \leq p \leq n-2$ and all k , and for $p \geq n-1$ and $k \geq -n$. Putting these together, we obtain the conditions 1 through 5 above.

Dualizing sequence (3), we obtain that $H^0(\mathcal{K}^*(k)) = 0$ for $k \leq -1$. Now dualizing sequence (4), we get, since \mathcal{K} is locally-free:

$$0 \rightarrow E^*(-k) \rightarrow \mathcal{K}^*(-k) \xrightarrow{\alpha^*} V \otimes \mathcal{O}_{\mathbb{P}^n}(-k+1) \rightarrow \mathcal{E}xt^1(E(k), \mathcal{O}_{\mathbb{P}^n}) \rightarrow 0 \quad . \quad (5)$$

Condition 6 and the second part of condition 1 follow easily. \square

Conversely, linear sheaves can be characterized by their cohomology:

Theorem 3. *If E is a torsion-free sheaf on \mathbb{P}^n satisfying:*

1. *for $n \geq 2$, $H^0(E(-1)) = H^n(E(-n)) = 0$;*
2. *for $n \geq 3$, $H^1(E(-2)) = H^{n-1}(E(1-n)) = 0$;*
3. *for $n \geq 4$, $H^p(E(k)) = 0$, $2 \leq p \leq n-2$ and $\forall k$;*

then E is linear, and can be represented as the cohomology of the monad:

$$\begin{aligned} 0 &\rightarrow H^1(E \otimes \Omega_{\mathbb{P}^n}^2(1)) \otimes \mathcal{O}_{\mathbb{P}^n}(-1) \rightarrow & (6) \\ &\rightarrow H^1(E \otimes \Omega_{\mathbb{P}^n}^1) \otimes \mathcal{O}_{\mathbb{P}^n} \rightarrow H^1(E(-1)) \otimes \mathcal{O}_{\mathbb{P}^n}(1) \rightarrow 0 \end{aligned}$$

Proof. Given a hyperplane $\varphi \subset \mathbb{P}^n$, consider the restriction sequence:

$$0 \rightarrow E(k-1) \rightarrow E(k) \rightarrow E(k)|_{\varphi} \rightarrow 0 .$$

Clearly, $H^0(E(-1)) = 0$ implies that $H^0(E(k)) = 0$ for $k \leq -1$, while $H^n(E(-n)) = 0$ forces $H^0(E(k)) = 0$ for $k \geq -n$.

Since $H^0(E(-1)) = H^1(E(-2)) = 0$, it follows that $H^0(E(-1)|_{\varphi}) = 0$, hence by further restriction $H^0(E(k)|_{\varphi}) = 0$ for $k \leq -1$. So we have the sequence:

$$0 \rightarrow H^1(E(k-1)) \rightarrow H^1(E(k)) \quad , \quad \text{for } k \leq -2$$

thus by induction $H^1(E(k)) = 0$ for $k \leq -2$.

Since $H^n(E(-n)) = H^{n-1}(E(1-n)) = 0$, it follows that $H^{n-1}(E(1-n)|_{\varphi}) = 0$, hence by further restriction $H^{n-1}(E(k)|_{\varphi}) = 0$ for $k \geq 1-n$. So we have the sequence:

$$H^{n-1}(E(k-1)) \rightarrow H^{n-1}(E(k)) \rightarrow 0 \quad , \quad \text{for } k \geq 1-n$$

thus by induction $H^{n-1}(E(k)) = 0$ for $k \geq 1-n$.

The key ingredient of the proof is the Beilinson spectral sequence [20]: for any coherent sheaf E on \mathbb{P}^n there exists a spectral sequence $\{E_r^{p,q}\}$ whose E_1 -term is given by ($q = 0, \dots, n$ and $p = 0, -1, \dots, -n$):

$$E_1^{p,q} = H^q(E \otimes \Omega_{\mathbb{P}^n}^{-p}(-p)) \otimes \mathcal{O}_{\mathbb{P}^n}(p)$$

which converges to

$$E^i = \begin{cases} E, & \text{if } p + q = 0 \\ 0 & \text{otherwise} \end{cases} .$$

Applying the Beilinson spectral sequence to $E(-1)$, we must show that

$$H^q(E(-1) \otimes \Omega_{\mathbb{P}^n}^{-p}(-p)) = 0 \text{ for } q \neq 1 \text{ and for } q = 1, p \leq -3 . \quad (7)$$

It then follows that the Beilinson spectral sequence degenerates at the E_2 -term and the monad

$$\begin{aligned} 0 &\rightarrow H^1(E(-1) \otimes \Omega_{\mathbb{P}^n}^2(2)) \otimes \mathcal{O}_{\mathbb{P}^n}(-2) \rightarrow \\ &\rightarrow H^1(E(-1) \otimes \Omega_{\mathbb{P}^n}^1(1)) \otimes \mathcal{O}_{\mathbb{P}^n}(-1) \rightarrow H^1(E(-1)) \otimes \mathcal{O}_{\mathbb{P}^n} \rightarrow 0 \end{aligned} \quad (8)$$

has $E(-1)$ as its cohomology. Tensoring (8) by $\mathcal{O}_{\mathbb{P}^n}(1)$, we conclude that E is the cohomology of (6), as desired.

The claim 7 follows from repeated use of the exact sequence

$$\begin{aligned} H^q(E(k)) \oplus^m &\rightarrow H^q(E(k+1) \otimes \Omega_{\mathbb{P}^n}^{-p-1}(-p-1)) \rightarrow \\ &\rightarrow H^{q+1}(E(k) \otimes \Omega_{\mathbb{P}^n}^{-p}(-p)) \rightarrow H^{q+1}(E(k)) \oplus^m \end{aligned} \quad (9)$$

associated with Euler sequence for p -forms on \mathbb{P}^n twisted by $E(k)$:

$$0 \rightarrow E(k) \otimes \Omega_{\mathbb{P}^n}^{-p}(-p) \rightarrow E(k) \oplus^m \rightarrow E(k) \otimes \Omega_{\mathbb{P}^n}^{-p-1}(-p) \rightarrow 0 , \quad (10)$$

where $q = 0, \dots, n$, $p = -1, \dots, -n$ and $m = \binom{n+1}{-p}$.

For instance, it is easy to see that:

$$H^0(E(k) \otimes \Omega_{\mathbb{P}^n}^{-p}(-p)) = 0 \text{ for all } p \text{ and } k \leq -1 ;$$

$$H^q(E(-1) \otimes \Omega_{\mathbb{P}^n}^n(n)) = H^q(E(-2)) = 0 \text{ for all } q ;$$

$$H^q(E(-1)) = 0 \text{ for all } q \neq 1 ;$$

$$H^n(E(k) \otimes \Omega_{\mathbb{P}^n}^{-p}(-p)) = 0 \text{ for all } p \text{ and } k \geq -n .$$

Setting $q = n - 1$, we also obtain:

$$H^{n-1}(E(k) \otimes \Omega_{\mathbb{P}^n}^{-p}(-p)) = 0 \text{ for } p \geq -n + 1 \text{ and } k \geq -n - 1 ,$$

and so on. □

Note that if E is assumed to be locally-free in the statement of Theorem 3, then it is enough to assume $H^{n-1}(E(1-n)) = 0$.

Clearly, the cohomology of a linear monad is always coherent, but more can be said if the codimension of the degeneration locus of α is known.

Proposition 4. *Let E be a monad sheaf.*

1. E is locally-free if and only if its degeneration locus is empty;
2. E is reflexive if and only if its degeneration locus is a subvariety of codimension at least 3;
3. E is torsion-free if and only if its degeneration locus is a subvariety of codimension at least 2.

Proof. Let Σ be the degeneration locus of the monad sheaf E . From Proposition 5, we know that $\mathcal{E}xt^p(E, \mathcal{O}_{\mathbb{P}^n}) = 0$ for $p \geq 2$ and

$$\Sigma = \text{supp } \mathcal{E}xt^1(E, \mathcal{O}_{\mathbb{P}^n}) = \{x \in \mathbb{P}^n \mid \alpha_x \text{ is not injective} \}$$

The first statement is clear; so it is now enough to argue that E is torsion-free if and only if Σ has codimension at least 2 and that E is reflexive if and only if Σ has codimension at least 3.

Recall that the m^{th} -singularity set of a coherent sheaf \mathcal{F} on \mathbb{P}^n is given by:

$$S_m(\mathcal{F}) = \{X \in \mathbb{P}^n \mid dh(\mathcal{F}_x) \geq n - m\}$$

where $dh(\mathcal{F}_x)$ stands for the homological dimension of \mathcal{F}_x as an \mathcal{O}_x -module:

$$dh(\mathcal{F}_x) = d \iff \begin{cases} \text{Ext}_{\mathcal{O}_x}^d(\mathcal{F}_x, \mathcal{O}_x) \neq 0 \\ \text{Ext}_{\mathcal{O}_x}^p(\mathcal{F}_x, \mathcal{O}_x) = 0 \quad \forall p > d \end{cases}$$

In the case at hand, we have that $dh(E_x) = 1$ if $x \in \Sigma$, and $dh(E_x) = 0$ if $x \notin \Sigma$. Therefore $S_0(E) = \cdots = S_{n-2}(E) = \emptyset$, while $S_{n-1}(E) = \Sigma$. It follows that [23, Proposition 1.20] :

- if $\text{codim } \Sigma \geq 2$, then $\dim S_m(E) \leq m - 1$ for all $m < n$, hence E is a locally 1st-syzygy sheaf;
- if $\text{codim } \Sigma \geq 3$, then $\dim S_m(E) \leq m - 2$ for all $m < n$, hence E is a locally 2nd-syzygy sheaf.

The desired statements follow from the observation that E is torsion-free if and only if it is a locally 1st-syzygy sheaf, while E is reflexive if and only if it is a locally 2nd-syzygy sheaf [20, p. 148-149]. \square

A splitting criterion for locally-free linear sheaves. Given a coherent sheaf E on \mathbb{P}^n , we define

$$H_*^p(E) = \bigoplus_{k \in \mathbb{Z}} H^p(E(k))$$

which has the structure of a graded module over $S^n = \bigoplus_{k \in \mathbb{Z}} H^p(\mathcal{O}(k))$. Kumar, Petersen & Rao prove the following result [18]:

Theorem 5. *Let E be a rank r locally-free sheaf on \mathbb{P}^n , $n \geq 4$.*

1. *If n is even and $r \leq n - 1$, then E splits as a sum of line bundles if and only if $H_*^p(E) = 0$ for $2 \leq p \leq n - 2$.*
2. *If n is odd and $r \leq n - 2$, then E splits as a sum of line bundles if and only if $H_*^p(E) = 0$ for $2 \leq p \leq n - 2$.*

Thus we obtain as an easy consequence of (3) in Proposition 2 and the previous theorem:

Corollary 6. *Let E be a rank r locally-free linear sheaf on \mathbb{P}^n .*

1. *If n is even and $r \leq n - 1$, then E splits as a sum of line bundles.*
2. *If n is odd and $r \leq n - 2$, then E splits as a sum of line bundles.*

This means that linear monads are not useful to produce locally free sheaves of low rank on \mathbb{P}^n , one of the problems suggested by Hartshorne in [11] and still a challenge in the subject.

Let us also point out that Kumar, Petersen & Rao's result is optimal, in the sense that there exist rank $2n$ locally-free sheaves on \mathbb{P}^{2n+1} and on \mathbb{P}^{2n} ($n \geq 2$) satisfying $H_*^p(E) = 0$ for $2 \leq p \leq 2n - 1$ which are stable, and hence do not split as a sum of line bundles. Moreover, as we will see in examples below, it does not generalize to reflexive or torsion-free sheaves either.

Thus linear monads will only produced interesting locally-free sheaves when $r = w - v - u \geq n$ if n is even and when $r = w - v - u \geq n - 1$ if n is odd. However, we can still expect to use monads to produce interesting *reflexive* sheaves of low rank, see Example 5 below.

2 Basic properties of instanton sheaves

It follows from Proposition 2 that the cohomology of a linear monad with $v = u$ and $w - 2v \geq n - 3$ is an instanton sheaf of rank $w - 2v$ and charge v . In particular, by Fløystad's theorem, there are rank r instanton sheaves on \mathbb{P}^n for each $r \geq n - 1$. Moreover, for every $r \geq 2n$, there are rank r locally-free instanton sheaves on \mathbb{P}^{2n+1} or \mathbb{P}^{2n} .

Moreover, by virtue of Theorem 3, every rank r instanton sheaf of charge c is the cohomology of a monad of the form:

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus c} \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^n}^{\oplus r+2c} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus c} \rightarrow 0 \quad (11)$$

for some injective map α degenerating in codimension at least 2 and some surjective map β , such that $\beta\alpha = 0$. It follows easily from Fløystad's theorem that there are no instanton sheaves on \mathbb{P}^n of rank $r \leq n - 2$.

Corollary 7. *If E is an instanton sheaf then:*

- $H^0(E^*(k)) = 0, \forall k \leq -1;$

- $\text{Ext}^p(E(k), \mathcal{O}_{\mathbb{P}^n}) = 0$ for $p \geq 2$ and all k ;
- $\text{Ext}^p(E, E) = 0$ for $p \geq 3$.

Proof. The first two statements follow easily from Proposition 2 and the fact that every instanton sheaf is linear. For the last statement, let $K = \ker \beta$; taking the sequence:

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus c} \rightarrow K \rightarrow E \rightarrow 0 \quad ,$$

we obtain:

$$\text{Ext}^p(E, \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus c}) \rightarrow \text{Ext}^p(E, K) \rightarrow \text{Ext}^p(E, E) \rightarrow \text{Ext}^{p+1}(E, \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus c}) \quad (12)$$

Thus $\text{Ext}^p(E, K) \simeq \text{Ext}^p(E, E)$ for all $p \geq 2$. Now from the sequence

$$0 \rightarrow K \rightarrow \mathcal{O}_{\mathbb{P}^n}^{\oplus r+2c} \rightarrow \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus c} \rightarrow 0$$

we obtain:

$$\text{Ext}^p(E, \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus c}) \rightarrow \text{Ext}^{p+1}(E, K) \rightarrow \text{Ext}^{p+1}(E, \mathcal{O}_{\mathbb{P}^n}^{\oplus r+2c}) \quad . \quad (13)$$

Hence $\text{Ext}^p(E, K) = 0$ for $p \geq 3$, and the result follows. \square

Furthermore, $\text{Ext}^1(E, E)$ is completely determined by the map α , while $\text{Ext}^2(E, E)$ is completely determined by the map β . Indeed, first note that the maps α and β induce linear maps:

$$\text{Ext}^1 \alpha : \text{Ext}^1(E, \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus c}) \rightarrow \text{Ext}^1(E, K)$$

$$\text{Ext}^1 \beta : \text{Ext}^1(E, \mathcal{O}_{\mathbb{P}^n}^{\oplus r+2c}) \rightarrow \text{Ext}^1(E, \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus c})$$

Setting $p = 1$ on (12) and (13) we get:

$$\text{Ext}^1(E, E) = \text{coker} \{ \text{Ext}^1 \alpha \} \quad \text{and} \quad \text{Ext}^2(E, E) = \text{coker} \{ \text{Ext}^1 \beta \}$$

In particular, note that $\text{Ext}^2(E, E) = 0$ is an open condition on $\text{Hom}(\mathcal{O}_{\mathbb{P}^n}^{\oplus r+2c}, \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus c})$.

Given two instanton sheaves, one can produce a new instanton sheaf of higher rank using the following results:

Proposition 8. *An extension of instanton sheaves is also an instanton sheaf.*

Proof. Let E' and E'' be instanton sheaves, and consider the sequence

$$0 \rightarrow E' \rightarrow F \rightarrow E'' \rightarrow 0 .$$

The desired statement follows easily from the associated sequences of cohomology:

$$H^q(E'(k)) \rightarrow H^q(F(k)) \rightarrow H^q(E''(k)) , \quad \forall q = 0, \dots, n ,$$

so that $H^q(F(k))$ vanishes whenever $H^q(E'(k))$ and $H^q(E''(k))$ do. Note that F is classified by $\text{Ext}^1(E'', E')$. \square

Proposition 9. *Let E and F be instanton sheaves.*

- *If E is locally-free, then E^* is also instanton.*
- *If F is locally-free, then $E \otimes F$ is also instanton.*

Proof. The first statement is an easy consequence of Serre duality.

Let E be the cohomology of the monad:

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus c} \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^n}^{\oplus r+2c} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus c} \rightarrow 0 ,$$

and let $K_E = \ker \beta$. The second statement will follow from the analysis of the long exact cohomology sequences associated to the sheaf sequences:

$$0 \rightarrow F(k-1)^{\oplus c} \rightarrow F \otimes K_E(k) \rightarrow F \otimes E(k) \rightarrow 0$$

$$\text{and } 0 \rightarrow F \otimes K_E(k) \rightarrow F(k-1)^{\oplus 2r+c} \rightarrow F(k+1)^{\oplus c} \rightarrow 0 .$$

\square

In general, however, if E is not locally-free, its dual might not be an instanton sheaf, see example 2 below.

Proposition 10. *Let H be a hyperplane in \mathbb{P}^n . The restriction $E|_H$ of a locally-free instanton sheaf E on \mathbb{P}^n is also a locally-free instanton sheaf.*

Proof. Follows easily from the definition and the exact sequence

$$0 \rightarrow E(k-1) \xrightarrow{\sigma} E(k) \rightarrow E|_H(k) \rightarrow 0 \quad ,$$

and Theorem 3. □

Instanton sheaves and mathematical instanton bundles. Mathematical instanton bundles have been defined in [21] as a rank $2n$ locally-free sheaf on \mathbb{P}^{2n+1} satisfying the following conditions:

- $c(E) = \left(\frac{1}{1-H^2}\right)^c = (1 + H^2 + H^4 + \dots)^c$;
- E has natural cohomology in the range $-n \leq k \leq 0$;
- E is simple;
- E has trivial splitting type (i.e. there exist a line $\ell \subset \mathbb{P}^n$ such that $E|_\ell \simeq \mathcal{O}_\ell^{2m}$).

It was later shown by Ancona & Ottaviani have shown that the simplicity assumption is redundant [1, Proposition 2.11]: every rank $2n$ locally-free sheaf on \mathbb{P}^{2n+1} satisfying the first two conditions is simple (in fact, more is true, see Lemma 25 below). The last condition is also redundant for $n = 1$.

Every mathematical instanton bundle as above can be represented as the cohomology of a linear monad with $v = u = c$ [21], hence it is a rank $2n$ locally-free instanton sheaf on \mathbb{P}^{2n+1} of charge c .

Conversely, a rank $2n$ locally-free instanton sheaf E on \mathbb{P}^{2n+1} is a mathematical instanton bundle as above if $H^0(E) = 0$ and it is of trivial splitting type. We show that the vanishing of $H^0(E)$ is automatic.

Proposition 11. *If E is a rank $n-1$ instanton sheaf on \mathbb{P}^n , then $H^0(E) = 0$. If E is locally-free, then $H^0(E^*) = 0$.*

Proof. Let E be a rank $n - 1$ locally-free instanton sheaf on \mathbb{P}^n , and assume that $H^0(E) \neq 0$. So let $Q = E/\mathcal{O}_{\mathbb{P}^n}$ and consider the sequence:

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^n} \rightarrow E \rightarrow Q \rightarrow 0 \quad .$$

Note that $c_1(Q) = 0$, $H^0(Q(k)) = 0$ for $k \leq -1$, $H^n(Q(k)) = 0$ for $k \geq -n$ and $H^q(Q(k)) = H^q(E(k))$ for $1 \leq q \leq n - 1$. It follows that Q must be a rank $n - 2$ instanton sheaf, which cannot exist by Fløystad's theorem.

If E is locally-free, then E^* is also instanton and the vanishing of $H^0(E^*)$ follows by the same argument. \square

In particular, there are no rank $2n - 1$ locally-free instanton sheaves on \mathbb{P}^{2n} . Indeed, by Theorem 5, any such sheaf must split as a sum of line bundles, and this contradicts $c_1(E) = 0$ and $H^0(E) = 0$.

We also point out that there are rank $n - 1$ properly reflexive instanton sheaves E on \mathbb{P}^n for which $H^0(E^*) \neq 0$, see Example (4) below.

3 Semistability of monad sheaves

Recall that a torsion-free sheaf E on \mathbb{P}^2 is said to be *semistable* if for every coherent subsheaf $0 \neq F \hookrightarrow E$ we have

$$\mu(F) = \frac{c_1(F)}{\mathrm{rk}(F)} \leq \frac{c_1(E)}{\mathrm{rk}(E)} = \mu(E) \quad .$$

Furthermore, if for every coherent subsheaf $0 \neq F \hookrightarrow E$ with $0 < \mathrm{rk}(F) < \mathrm{rk}(E)$ we have

$$\frac{c_1(F)}{\mathrm{rk}(F)} < \frac{c_1(E)}{\mathrm{rk}(E)} \quad ,$$

then E is said to be *stable*. A sheaf is said to be properly semistable if it is semistable but not stable. It is also important to remember that E is (semi)stable if and only if E^* and $E(k)$ are.

For any given torsion-free sheaf E of rank r , there is an uniquely determined integer k_E such that

$$c_1(E(k_E)) = c_1(E) + rk_E \in \{0, -1, \dots, -r + 1\} \quad ;$$

$E_\eta = E(k_E)$ is called the normalization of E . A sheaf E is said to be normalized if $-r + 1 \leq c_1(E) \leq 0$.

Lemma 12. ([20, p. 167]) *Let E be a normalized torsion-free sheaf on \mathbb{P}^n . If E is stable then $H^0(E) = 0$ and*

- $H^0(E^*) = 0$ if $c_1(E) = 0$;
- $H^0(E^*(-1)) = 0$ if $c_1(E) < 0$.

If E is semistable then $H^0(E(-1)) = H^0(E^(-1)) = 0$.*

For sheaves of rank 2 or 3, the above necessary criteria turns out to be also sufficient, as we recall in the next two lemmas. We fix $n \geq 2$.

Lemma 13. *Let E be a normalized rank 2 torsion-free sheaf E on \mathbb{P}^n . If $c_1(E) = 0$, then:*

- E is stable if and only if $H^0(E) = H^0(E^*) = 0$;
- E is semistable if and only if $H^0(E(-1)) = H^0(E^*(-1)) = 0$.

If $c_1(E) = -1$, then E is stable if and only if it is semistable if and only if $H^0(E) = H^0(E^(-1)) = 0$.*

Proof. For E being reflexive, this result is in [20, p. 166]. In general, simply note that E^* is reflexive and use the result just mentioned. \square

For rank 3 sheaves we have:

Lemma 14. ([20, p. 167]) *Let E be a normalized rank 3 reflexive sheaf on \mathbb{P}^n . E is stable if and only if:*

- $H^0(E) = H^0(E^*) = 0$ for $c_1(E) = 0$;
- $H^0(E) = H^0(E^*(-1)) = 0$ for $c_1(E) < 0$.

For $c_1(E) = 0$, E is semistable if and only if $H^0(E(-1)) = H^0(E^(-1)) = 0$.*

As an easy consequence of Proposition 2 and Lemmata 13 and 14, we have:

Proposition 15. *If E is a rank 2 instanton sheaf on \mathbb{P}^n ($n = 2, 3$), then E is semistable. It is stable if $H^0(E) = 0$.*

If E is a rank 3 reflexive instanton sheaf on \mathbb{P}^n ($n = 2, 3, 4$), then E is semistable. It is stable if $H^0(E) = H^0(E^) = 0$*

This result does not generalize for higher rank, even we restrict ourselves to locally-free instanton sheaves, as we show in the example below.

Example 1. For each $n \geq 2$, there are rank $2n$ locally-free instanton sheaves on \mathbb{P}^n which are not semistable. Indeed, by Fløystad's theorem, there is a linear monad:

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus a+1} \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^n}^{\oplus n+2a+1} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus a} \rightarrow 0 \quad (a \geq 1) \quad , \quad (14)$$

whose cohomology F is a locally-free sheaf of rank n on \mathbb{P}^n and $c_1(F) = 1$. The dual F^* is a locally-free sheaf of rank n on \mathbb{P}^n and $c_1(F) = -1$. Any extension of E of F^* by F :

$$0 \rightarrow F \rightarrow E \rightarrow F^* \rightarrow 0$$

is a rank $2n$ locally-free instanton sheaf which is clearly not semistable. Furthermore, one can adjust the parameter a depending on n to ensure the existence of nontrivial extensions.

For instanton sheaves of higher rank, the best statement one can have is the following:

Theorem 16. *Let E be a rank r instanton sheaf on \mathbb{P}^n .*

- *If E is reflexive and $r \leq n$, then E is semistable;*
- *if E is locally-free and $r \leq 2n - 1$, then E is semistable.*

The proof is based on Hoppe's criterion [13]: if E is a rank r reflexive sheaf on \mathbb{P}^n with $c_1(E) = 0$ satisfying

$$H^0(\Lambda^q E(-1)) = 0 \quad \text{for } 1 \leq q \leq r - 1$$

then E is semistable. Indeed, assume E is not semistable, and let F be a rank q destabilizing sheaf with $c_1(F) = d > 0$. Then $\Lambda^q F = \mathcal{O}_{\mathbb{P}^n}(d)$, and the induced map $\Lambda^q F \rightarrow \Lambda^q E$ yields a section in $H^0(\Lambda^q E(-d))$, which forces $h^0(\Lambda^q E(-1)) \neq 0$. Similarly, it is also easy to see that if

$$H^0(\Lambda^q E) = 0 \quad \text{for } 1 \leq q \leq r - 1$$

then E is stable.

Proof. Every rank r reflexive instanton sheaf on \mathbb{P}^n can be represented as the cohomology of the monad (11). Taking the sequence

$$0 \rightarrow K \rightarrow \mathcal{O}_{\mathbb{P}^n}^{\oplus(r+2c)} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus c} \rightarrow 0 \quad ,$$

we consider the associated long exact sequence of exterior powers, twisted by $\mathcal{O}_{\mathbb{P}^n}(-1)$:

$$0 \rightarrow \Lambda^q K(-1) \rightarrow \Lambda^q(\mathcal{O}_{\mathbb{P}^n}^{\oplus(r+2c)})(-1) \rightarrow \dots \quad .$$

Hence $H^0(\Lambda^q K(-1)) = 0$ for $1 \leq q \leq r + c - 1$. Now take the sequence:

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus c} \rightarrow K \rightarrow E \rightarrow 0 \quad ,$$

and consider the associated long exact sequence of symmetric powers, twisted by $\mathcal{O}_{\mathbb{P}^n}(-1)$:

$$\begin{aligned} 0 \rightarrow \mathcal{O}_{\mathbb{P}^n}(-q)^{\binom{c+q-1}{q}} \rightarrow K(-q+1)^{\binom{c+q-2}{q-1}} \rightarrow \dots \\ \rightarrow \Lambda^{q-1} K \otimes \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus c} \rightarrow \Lambda^q K \rightarrow \Lambda^q E \rightarrow 0 \quad . \end{aligned}$$

Cutting into short exact sequences and passing to cohomology, we have obtain

$$H^0(\Lambda^p E(-1)) = 0 \quad \text{for } 1 \leq p \leq n - 1 \quad , \quad (15)$$

and this proves the first statement.

Now if E is locally-free, the dual E^* is also an instanton sheaf on X , so

$$H^0(\Lambda^q(E^*)(-1)) = 0 \quad \text{for } 1 \leq q \leq n-1 \quad . \quad (16)$$

But $\Lambda^p(E^*) \simeq \Lambda^{r-p}(E^*)$, since $\det(E) = \mathcal{O}_{\mathbb{P}^n}$; it follows that:

$$\begin{aligned} H^0(\Lambda^p E(-1)) &= H^0(\Lambda^{r-p}(E^*)(-1)) = 0 \quad \text{for } 1 \leq r-p \leq n-1 \\ &\implies r-n+1 \leq p \leq r-1 \end{aligned} \quad (17)$$

Together, (16) and (17) imply that if E is a rank $r \leq 2n-1$ locally-free instanton sheaf, then:

$$H^0(\Lambda^p E(-1)) = 0 \quad \text{for } 1 \leq p \leq 2n-2$$

hence E is semistable by Hoppe's criterion. \square

On the other hand, we have:

Proposition 17. *For $r > (n-1)c$, there are no stable rank r instanton sheaf on \mathbb{P}^n of charge c .*

In other words, every stable rank r instanton sheaf on \mathbb{P}^n must be of charge $c \geq r/(n-1)$, and there are properly semistable rank r instanton sheaves for $n \leq r \leq 2n-1$.

Proof. Any rank r instanton sheaf of charge c is the cohomology of a monad of the form:

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus c} \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^n}^{\oplus r+2c} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus c} \rightarrow 0$$

and note that

$$H^0(E) \simeq H^0(\ker \beta) \simeq \ker \{ H^0 \beta : H^0(\mathcal{O}_{\mathbb{P}^n}^{\oplus r+2c}) \rightarrow H^0(\mathcal{O}_{\mathbb{P}^n}(1)^{\oplus c}) \}$$

thus if $r > (n-1)c$, then $h^0(\mathcal{O}_{\mathbb{P}^n}^{\oplus r+2c}) > h^0(\mathcal{O}_{\mathbb{P}^n}(1)^{\oplus c})$ and $H^0(E) \neq 0$, so that E is not stable by Lemma 12. \square

Let us now analyze the inverse question: are all semistable sheaves of degree zero on \mathbb{P}^n instanton? The answer is positive for $n = 2$, but there are cohomological restrictions for $n \geq 3$.

Theorem 18. *Let E be a torsion-free sheaf on \mathbb{P}^2 with $c_1(E) = 0$. If E is semistable, then E is instanton.*

Proof. The semistability of E and E^* immediately implies that $H^0(E(k)) = H^0(E^*(k)) = 0$ for $k \leq -1$. If E is a locally-free sheaf, then via Serre duality $H^2(E(k)) = 0$ for $k \geq -2$, thus E is instanton.

Now if E is properly torsion-free, we consider the sequence:

$$0 \rightarrow E \rightarrow E^{**} \rightarrow Q \rightarrow 0 \quad (18)$$

where $Q = E^{**}/E$ is supported on a zero dimensional subscheme. Clearly, E^{**} is a semistable locally-free sheaf with $c_1(E) = 0$, so it is instanton by the previous paragraph. It follows from (18) that:

$$H^0(E(k)) \hookrightarrow H^0(E^{**}(k)) = 0 \quad \text{for } k \leq -1, \text{ and}$$

$$H^2(E(k)) \xrightarrow{\cong} H^2(E^{**}(k)) = 0 \quad \text{for } k \geq -2,$$

so E is also instanton. □

For $n \geq 3$, we have:

Proposition 19. *If E is a semistable locally-free sheaf on \mathbb{P}^n with $c_1(E) = 0$ such that $H^1(E(-2)) = H^{n-1}(E(1-n)) = 0$ and, for $n \geq 4$, $H_*^p(E) = 0$ for $2 \leq p \leq n-2$, then E is instanton.*

Proof. If E is semistable, then $H^0(E(k)) = H^0(E^*(k)) = 0$ for $k \leq -1$, hence $H^n(E(k)) = 0$ for $k \geq -n$ by Serre duality. The statement then follows from Theorem 3 □

As simple consequence of Proposition 15 and Proposition 19 we have:

- A rank 2 torsion-free sheaf on \mathbb{P}^3 with $c_1(E) = 0$ is instanton if and only if it is semistable and $H^1(E(-2)) = H^2(E(-2)) = 0$.
- A rank 3 reflexive sheaf on \mathbb{P}^3 with $c_1(E) = 0$ is instanton if and only if it is semistable and $H^1(E(-2)) = H^2(E(-2)) = 0$.
- A rank 3 reflexive sheaf on \mathbb{P}^4 with $c_1(E) = 0$ is instanton if and only if it is semistable and $H^1(E(-2)) = H_*^2(E) = H^3(E(-2)) = 0$.
- A rank $4 \leq r \leq 2n - 1$ locally-free sheaf on \mathbb{P}^n ($n \geq 3$) with $c_1(E) = 0$ is instanton if and only if it is semistable and $H^1(E(-2)) = H^{n-1}(E(1 - n)) = 0$ and, for $n \geq 4$, $H_*^p(E) = 0$ for $2 \leq p \leq n - 2$.

Remark. Since every Gieseker semistable torsion-free sheaf on \mathbb{P}^n is semistable [20, p. 174], one can use the results above to decide when a Gieseker semistable torsion-free sheaf on \mathbb{P}^n is instanton. It is easy to see, however, that not all instanton sheaves are Gieseker semistable; indeed if E is an instanton sheaf satisfying $H^0(E) \neq 0$, then E is not Gieseker semistable. Thus, there are Gieseker unstable instanton sheaves of every rank.

A little more can be said about rank 2 reflexive instanton sheaves on \mathbb{P}^3 and locally-free instanton sheaves on \mathbb{P}^5 .

Proposition 20. *Every rank 2 reflexive instanton sheaf on \mathbb{P}^3 is locally-free and stable.*

Proof. Hartshorne has shown that if E is a rank 2 reflexive sheaf on \mathbb{P}^3 with $c_3(E) = 0$, then E is locally-free [12], thus every rank 2 reflexive instanton sheaf on \mathbb{P}^3 is locally-free. By Proposition 11, we have that $H^0(E) = 0$, hence E is stable by Proposition 15. \square

This result is sharp, in the sense that there are properly semistable rank 2 torsion-free sheaves on \mathbb{P}^3 and properly semistable rank 3 reflexive instanton sheaves on \mathbb{P}^3 :

Example 2. Consider the monad:

$$\mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^3}^{\oplus 4} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^3}(1) \quad (19)$$

$$\alpha = \begin{pmatrix} x_1 \\ x_2 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \beta = (-x_2 \ x_1 \ x_3 \ x_4) \ .$$

Since α is injective provided $x_1, x_2 \neq 0$, its cohomology is a rank 2 properly torsion-free instanton sheaf of charge 1. Moreover, E is not stable because it is a non-locally-free nullcorrelation sheaf [8, remark 1.2.1].

Finally, note that E^* is a properly semistable rank 2 properly reflexive sheaf on \mathbb{P}^3 with $c_1(E) = 0$; by Proposition 20, E^* cannot be instanton.

Example 3. Set $w = 5$ and $v = u = 1$ and consider the monad:

$$\mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^3}^{\oplus 5} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^3}(1) \quad (20)$$

$$\alpha = \begin{pmatrix} x_1 \\ x_2 \\ 0 \\ 0 \\ x_3 \end{pmatrix} \quad \text{and} \quad \beta = (-x_2 \ x_1 \ x_3 \ x_4 \ 0) \ .$$

It is easy to see that β is surjective for all $[x_1 : \cdots : x_4] \in \mathbb{P}^3$, while α is injective provided $x_1, x_2, x_3 \neq 0$. It follows that E is reflexive, but not locally-free; its singularity set is just the point $[0 : 0 : 0 : 1] \in \mathbb{P}^3$.

In summary, E is a rank 3 properly reflexive instanton sheaf of charge 1 on \mathbb{P}^3 . Note that E is properly semistable, by Proposition 17

Proposition 21. *Every rank 4 locally-free instanton sheaf E on \mathbb{P}^5 is stable.*

Proof. Noting that $H^0(E) = 0$ by Proposition 11, the claim follows from [1, Theorem 3.6]. \square

Again, this result is sharp, in the sense that there exists a properly semistable rank 4 properly reflexive instanton sheaf on \mathbb{P}^5 :

Example 4. Consider the cohomology E of the monad:

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^5}(-1) \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^5}^{\oplus 6} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^5}(1) \rightarrow 0$$

with the maps α and β given by:

$$\beta = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{pmatrix} \quad \alpha = \begin{pmatrix} -x_2 \\ x_1 \\ -x_4 \\ x_3 \\ 0 \\ 0 \end{pmatrix}.$$

Its degeneration locus is the line $\{x_1 = \dots = x_4 = 0\}$, so its cohomology is indeed properly reflexive.

Finally, E is properly semistable because it is a non-locally-free nullcorrelation sheaf [8, remark 1.2.1], and $H^0(E^*) \neq 0$.

Hartshorne's result could also suggest that every rank $2n$ reflexive instanton sheaf on \mathbb{P}^{2n+1} is locally-free. This is not true, though, as shown by the previous example.

Remark. It seems reasonable to conjecture that every rank $2n$ locally-free instanton sheaf on \mathbb{P}^{2n+1} is stable. In support of this conjecture, see Lemma 25 and Proposition 27 below. Results in this direction were also obtained in [1] for *symplectic* mathematical instanton bundles.

Example 5. Indecomposable rank 2 locally-free sheaves on \mathbb{P}^n , $n \geq 4$, have been extremely difficult to construct, and linear monads do not help with this problem. However, stable rank 2 reflexive sheaves on \mathbb{P}^n are easy to construct. Indeed, Fløystad's theorem guarantees the existence of linear monad:

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus n+a-3} \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^n}^{\oplus n+2a-1} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus a} \rightarrow 0 \quad (a \geq 1)$$

whose cohomology is a rank 2 reflexive linear sheaf with $c_1(E) = n-3$. To see that it is also stable, note that $E_\eta = E(k)$ for some $k \leq -1$, thus $H^0(E_\eta) = 0$ and it follows from Lemma 13 that E must be stable.

This example implies that Kumar, Petersen & Rao's result (Theorem 5) is sharp, in the sense it cannot be extended to more general sheaves: there are rank 2 reflexive sheaves E on \mathbb{P}^n with $H_*^p(E) = 0$ for $2 \leq p \leq n-2$ which do not split as a sum of rank 1 sheaves.

For instance, with $n = 4$ and $a = 1$, we get the monad:

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^4}(-1)^{\oplus 2} \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^4}^{\oplus 5} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^4}(1) \rightarrow 0$$

with the maps α and β given by:

$$\beta = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \end{pmatrix} \quad \alpha = \begin{pmatrix} -x_2 & -x_5 \\ x_1 & x_3 \\ -x_4 & -x_2 \\ x_3 & 0 \\ 0 & x_1 \end{pmatrix},$$

where $[x_1 : \dots : x_5]$ are homogeneous coordinates in \mathbb{P}^4 . Note that the degeneration locus of this monad is given by the union of two lines:

$$\Sigma(E) = \{x_1 = x_2 = x_3 = x_4 = 0\} \cup \{x_1 = x_2 = x_3 = x_5 = 0\}.$$

One can thus hope to construct stable rank 2 locally-free sheaves on \mathbb{P}^n via some mechanism that turns reflexive into locally-free sheaves without introducing new global sections.

Lifting of instantons. It is known that for every locally-free instanton sheaf E of trivial splitting type on \mathbb{P}^2 , there exists a locally-free instanton sheaf \tilde{E} of trivial splitting type on \mathbb{P}^3 and a hyperplane $\wp \subset \mathbb{P}^3$ such that $\tilde{E}|_{\wp} \simeq E$ [7]. \tilde{E} is called a *lifting* of E .

It would be interesting to see whether this generalizes to higher dimensional projective spaces and/or to more general sheaves; more precisely, we propose the following conjecture:

Conjecture. *If E is a rank $r \geq 2n$ locally-free instanton sheaf on \mathbb{P}^{2n} of charge c , then there is a rank $r \geq 2n$ locally-free instanton sheaf \tilde{E} on \mathbb{P}^{2n+1} of charge c and a hyperplane $\wp \subset \mathbb{P}^3$ such that $\tilde{E}|_{\wp} \simeq E$.*

It is not difficult to see that if a locally-free instanton sheaf on \mathbb{P}^n can be lifted to \mathbb{P}^{n+k} , then the lifted sheaf must be instanton.

Proposition 22. *Let E be a locally-free instanton sheaf on \mathbb{P}^n of charge c , and assume there is a locally-free sheaf \tilde{E} on \mathbb{P}^{n+k} ($n \geq 2, k \geq 1$) and a n -dimensional linear subvariety $\wp_n \subset \mathbb{P}^{n+k}$ such that $\tilde{E}|_{\wp_n} \simeq E$. Then \tilde{E} is also instanton of charge c .*

Proof. It is enough to prove the statement for $k = 1$. So let E be a locally-free instanton sheaf on \mathbb{P}^n of charge c and assume that there is a locally-free sheaf \tilde{E} on \mathbb{P}^{n+1} and a hyperplane $\wp \subset \mathbb{P}^{n+1}$ such that $\tilde{E}|_{\wp} \simeq E$. We must show that

The desired result follows from the restriction sequence:

$$0 \rightarrow \tilde{E}(k-1) \rightarrow \tilde{E}(k) \rightarrow E(k) \rightarrow 0 \quad .$$

together with repeated use of Serre's duality and Serre's vanishing theorem.

For example, since $H^0(E(k)) = 0$ for all $k \leq -1$, we get by induction that:

$$H^0(\tilde{E}(-1)) \sim H^0(\tilde{E}(-2)) \sim H^0(\tilde{E}(-3)) \sim \dots$$

hence $H^0(\tilde{E}(-1)) = 0$. Similarly, since $H^n(E(k)) = 0$ for all $k \geq -n$, we get by induction that:

$$H^{n+1}(\tilde{E}(-n-1)) \sim H^{n+1}(\tilde{E}(-n)) \sim H^{n+1}(\tilde{E}(-n+1)) \sim \dots$$

hence $H^{n+1}(\tilde{E}(-n-1)) = 0$.

Finally, setting $k = -1$ on the restriction sequence, we get $h^1(\tilde{E}(-1)) = h^1(E(-1))$, showing that the charge is preserved. \square

As a simple consequence, we obtain the following interesting characterization of locally-free instanton sheaves on higher dimensional projective spaces; compare with [8, Theorem 1.6].

Proposition 23. *Let $r \geq 2$ locally-free sheaf on \mathbb{P}^n with $c_1(E) = 0$. If $E|_{\wp_2}$ is semistable for a generic 2-plane $\wp_2 \subset \mathbb{P}^n$, then E is instanton.*

Proof. Since $E|_\varphi$ is semistable, then $E|_\varphi$ is instanton. The conclusion follows from the previous proposition. \square

4 Simplicity of monad sheaves

Recall that a torsion-free sheaf E on a projective variety X is said to be *simple* if $\dim \text{Ext}^0(E, E) = 1$. Every stable torsion-free sheaf on \mathbb{P}^n is simple.

Theorem 24. *If E is the cohomology of the linear monad*

$$0 \rightarrow V \otimes \mathcal{O}_{\mathbb{P}^n}(-1) \xrightarrow{\alpha} W \otimes \mathcal{O}_{\mathbb{P}^n} \xrightarrow{\beta} U \otimes \mathcal{O}_{\mathbb{P}^n}(1) \rightarrow 0 ,$$

for which $\mathcal{K} = \ker \beta$ is simple, then E is simple.

Proof. Applying $\text{Ext}^*(\cdot, E)$ to the sequence

$$0 \rightarrow V \otimes \mathcal{O}_{\mathbb{P}^n}(-1) \rightarrow \mathcal{K} \rightarrow E \rightarrow 0 ,$$

we get

$$0 \rightarrow \text{Ext}^0(E, E) \rightarrow \text{Ext}^0(K, E) \rightarrow \dots . \quad (21)$$

Now applying $\text{Ext}^*(K, \cdot)$ we get:

$$V \otimes \text{Ext}^0(K, \mathcal{O}_{\mathbb{P}^n}(-1)) \rightarrow \text{Ext}^0(K, K) \rightarrow \text{Ext}^0(K, E) \rightarrow V \otimes \text{Ext}^1(K, \mathcal{O}_{\mathbb{P}^n}(-1))$$

But it follows from the dual of sequence (3) with $k = 1$ that $h^0(\mathcal{K}^*(-1)) = h^1(\mathcal{K}^*(-1)) = 0$, thus

$$\dim \text{Ext}^0(K, E) = \dim \text{Ext}^0(K, K) = 1$$

because K is simple. It then follows from (21) that E is also simple. \square

As a consequence of [1, Theorem 2.8(a)], we have in particular the following generalization of [1, Theorem 2.8(b)]:

Lemma 25. *Every rank $n - 1$ monad sheaf on \mathbb{P}^n is simple.*

Not all instanton sheaves are simple. Indeed, recall that a rank 2 locally-free sheaf on \mathbb{P}^n is simple if and only if it is stable; since there are properly semistable rank 2 locally-free instanton sheaves on \mathbb{P}^2 and \mathbb{P}^3 , we conclude that there are instanton sheaves that are not simple.

5 Moduli spaces of instanton sheaves

Let $\mathcal{I}_{\mathbb{P}^n}(r, c)$ denote the moduli space of equivalence classes of rank r instanton sheaves of charge c on \mathbb{P}^n . Let $\mathcal{I}_{\mathbb{P}^n}^{\text{lf}}(r, c)$ denote the open subset of $\mathcal{I}_{\mathbb{P}^n}(r, c)$ consisting of locally-free sheaves. Note that $\mathcal{I}_{\mathbb{P}^n}^{\text{lf}}(r, c)$ might be empty even though $\mathcal{I}_{\mathbb{P}^n}(r, c)$ is not.

Very little is known in general about $\mathcal{I}_{\mathbb{P}^n}(r, c)$; research so far has concentrated on $\mathcal{I}_{\mathbb{P}^{2n+1}}^{\text{lf}}(2n, c)$. Here is a summary of some of the known facts:

- $\mathcal{I}_{\mathbb{P}^{2n+1}}^{\text{lf}}(2n, c)$ is affine [5];
- $\mathcal{I}_{\mathbb{P}^{2n+1}}^{\text{lf}}(2n, 1)$ is an open subset of $\mathbb{P}^{n(2n+1)-1}$ [8];
- $\mathcal{I}_{\mathbb{P}^{2n+1}}^{\text{lf}}(2n, 2)$ is an irreducible, smooth variety of dimension $4n^2 + 12n - 3$ [1];
- $\mathcal{I}_{\mathbb{P}^3}^{\text{lf}}(2, c)$ is an irreducible, smooth variety of dimension $8c - 3$ for $1 \leq c \leq 5$, see [4] and the references therein;
- $\mathcal{I}_{\mathbb{P}^{2n+1}}^{\text{lf}}(2n, c)$ is singular for all $n \geq 2$ and $c \geq 3$ [19].

The smoothness of $\mathcal{I}_{\mathbb{P}^3}^{\text{lf}}(2, c)$ for arbitrary charge c is still an open problem; it is known however that its closure in the moduli space of semistable locally-free sheaves with Chern character $2 - cH^2$ is in general singular, see [2].

In this section, we will generalize the second statement, study the moduli spaces of instanton sheaves on \mathbb{P}^2 and conclude with a general conjecture that generalizes the first statement.

Instanton sheaves and nullcorrelation sheaves Recall that a nullcorrelation sheaf N on \mathbb{P}^n is a rank $n - 1$ torsion-free sheaf defined by the short exact sequence:

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^n}(-1) \xrightarrow{\sigma} \Omega_{\mathbb{P}^n}^1(1) \rightarrow N \rightarrow 0 \quad ,$$

where $\sigma \in H^0(\Omega_{\mathbb{P}^n}^1(2)) = \Lambda^2 H^0(\mathcal{O}_{\mathbb{P}^n}(1))$. If n is odd and σ is generic, then N is locally-free. If n is even, then N is never locally-free; however, the generic one is reflexive if $n \geq 4$.

It is easy to see that any nullcorrelation sheaf is instanton of charge 1. The converse is also true: every rank $n - 1$ torsion-free instanton sheaf of charge 1 is a nullcorrelation sheaf. Indeed, let E be a rank $n - 1$ instanton sheaf of charge 1, so that it is the cohomology of the sequence:

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^n}(-1) \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^n}^{\oplus(n+1)} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^n}(1) \rightarrow 0 \quad .$$

Comparing this with the Euler sequence, it follows that $\ker \beta$ coincides $\Omega_{\mathbb{P}^n}^1(1)$, up to an automorphism of \mathbb{P}^n . Thus E fits into the sequence:

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^n}(-1) \xrightarrow{\alpha} \Omega_{\mathbb{P}^n}^1(1) \rightarrow E \rightarrow 0 \quad ,$$

hence E is nullcorrelation.

Since a nullcorrelation sheaf is completely determined by a section $\sigma \in H^0(\Omega_{\mathbb{P}^n}^1(2))$, we have that the set of equivalence classes of nullcorrelation sheaves on \mathbb{P}^n is exactly $\mathbb{P}(H^0(\Omega_{\mathbb{P}^n}^1(2)))$. Therefore we conclude:

Theorem 26. $\mathcal{I}_{\mathbb{P}^n}(n - 1, 1) \simeq \mathbb{P}^{\frac{n(n+1)}{2} - 1}$.

It is known that every nullcorrelation locally-free sheaf is stable, while nullcorrelation sheaves which are not locally-free are not stable [8, Remark 1.2.1]; they are however properly semistable. In particular, we have:

Proposition 27. *Every rank $2n$ locally-free instanton sheaf of charge 1 on \mathbb{P}^{2n+1} is stable.*

Moduli spaces of instanton sheaves on \mathbb{P}^2 . The simplest possible instanton sheaves are the rank 1 instanton sheaves on \mathbb{P}^2 . It is not difficult to see that such sheaves are exactly the ideals of points in \mathbb{P}^2 .

Indeed, let Z be a closed zero dimensional subscheme in \mathbb{P}^2 , and let I_Z denote its ideal sheaf; it fits into the sequence:

$$0 \rightarrow I_Z \rightarrow \mathcal{O}_{\mathbb{P}^2} \rightarrow \mathcal{O}_Z \rightarrow 0 \quad . \tag{22}$$

After tensoring with $\mathcal{O}_{\mathbb{P}^2}(k)$, it follows that $H^0(\mathbb{P}^2, I_Z(k)) = 0$ for $k \leq -1$, and $H^2(\mathbb{P}^2, I_Z(k)) = 0$ for $k \geq -2$, so I_Z is indeed instanton. Moreover, the charge of I_Z is just the length of Z .

Conversely, let E be a rank 1 instanton sheaf of charge c on \mathbb{P}^2 . Then E^{**} is a rank 1 locally-free sheaf with $c_1(E^{**}) = c_1(E) = 0$, so $E^{**} = \mathcal{O}_{\mathbb{P}^2}$. Thus E is the ideal sheaf associated with the zero-dimensional scheme $\mathcal{O}_{\mathbb{P}^2}/E$, whose length is equal to the charge of E .

In other words, there is a 1-1 correspondence between rank 1 instanton sheaves of charge c on \mathbb{P}^2 and closed zero dimensional subschemes of length c in \mathbb{P}^2 . This gives us the following identity:

Theorem 28. $\mathcal{I}_{\mathbb{P}^2}(1, c) \simeq (\mathbb{P}^2)^{[c]}$.

Remark. Let E be a locally-free instanton sheaf of charge c ; by tensoring sequence (22) with $E(k)$, it is easy to see that $E_Z = E \otimes I_Z$ is a properly torsion-free instanton sheaf of charge $c + r \cdot \text{lenght}(Z)$. Moreover, if $H^0(\mathbb{P}^2, E) = 0$, then $H^0(\mathbb{P}^2, E_Z) = 0$. Is it true that every properly torsion-free instanton is a locally-free instanton sheaf tensored by an ideal of points?

As we have seen in Section 3, rank 2 and 3 instanton sheaves on \mathbb{P}^2 are in 1-1 correspondence with semistable torsion-free sheaves with zero first Chern class. So we have:

Corollary 29. *For $r = 2, 3$, $\mathcal{I}_{\mathbb{P}^2}(r, c) \simeq \mathcal{M}_{\mathbb{P}^2}(r, 0, c)$, the moduli space of rank r semistable torsion-free sheaves E on \mathbb{P}^2 with $c_1(E) = 0$ and $c_2(E) = c$.*

In particular, $\mathcal{I}_{\mathbb{P}^2}(2, c)$ and $\mathcal{I}_{\mathbb{P}^2}(3, c)$ are quasi-projective variety.

In general, we conjecture that $\mathcal{I}_{\mathbb{P}^n}(r, c)$ is always a quasi-projective variety.

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