Topological Classification of Families of Diffeomorphisms Without Small Divisors

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Abstract. We give a complete topological classification for germs of one-parameter families of one-dimensional diffeomorphisms without small divisors. In the non-trivial cases the topological invariants are given by some functions attached to the fixed points set plus the analytic class of the element of the family corresponding to the special parameter. The proof is based on the structure of the limits of orbits when we approach the special parameter.

Contents

Introduction

In this paper we give a complete topological classification for germs of one-parameter families of one-dimensional diffeomorphisms without small divisors. More precisely, we study germs of diffeomorphism in $(\mathbb{C}^2, 0)$ of the form

$$\varphi(x, y) = (x \circ \varphi, y)$$

The curve $Fix\varphi \subset \mathbb{C}^2$ of fixed points of φ is given by $x \circ \varphi - x = 0$. We associate $\varphi_{(x_0,y_0)} \in \text{Diff}\,(\mathbb{C}, 0)$ to every point $(x_0, y_0) \in Fix\varphi$; it is the germ defined by $\varphi_{|y=y_0}$ in a neighborhood of $x = x_0$. There are two kind of phenomena which can produce a complicated dynamical behavior for a diffeomorphism φ.

Presence of small divisors. We say that φ has small divisors if there exist $j \in \mathbb{Z}$ and $P \in Fix\varphi^{(j)}$ such that $(\partial\varphi_P^{(j)}/\partial x)(P) \in \mathbb{S}^1$ and $(\partial\varphi_P^{(j)}/\partial x)(P)$ is not a Bruno number [**Bru71**], [**Bru72**]. Then the dynamics of $\varphi_P^{(j)}$ is very chaotic if $\varphi_P^{(j)}$ is not linearizable [**Yoc95**], [**PM97**].

Evolution of the dynamics. In absence of small divisors the dynamics of $\varphi_{|y=s}$ admits a simple description. It depends in some sense continuously on s for $s \neq 0$, but it can change dramatically for different values of the parameter s.

There are some works identifying regular zones in the parameter space, i.e. zones where the dynamics of $\varphi_{|y=s}$ converges regularly to the dynamics of $\varphi_{|y=0}$ when $s \to 0$ (see [**Ris99**] for the case where $j^1\varphi_{(0,0)}$ is an irrational rotation or [**DES**] for the case $j^1\varphi_{(0,0)} \equiv Id$). But so far there was no description of the zones in the parameter space where the dynamical behavior does not commute with the limit. There was also no information about the dependence of the dynamics of $\varphi_{|y=s}$ with respect to s ($s \neq 0$) except in the topologically trivial case. Here we provide a description of these phenomena in the absence of small divisors.

A diffeomorphism φ without small divisors will be called (NSD) diffeomorphism. The (NSD) character implies that we are in one of the following cases:

- φ is analytically conjugated to $(\lambda(y)x, y)$ for some $\lambda \in \mathbb{C}\{y\}$.
- $j^1\varphi = (\lambda x, y)$ for a root $\lambda \in \mathbb{S}^1$ of the unit.
- $j^1\varphi = (x + \mu y, y)$ for some $\mu \in \mathbb{C}$.

We will deal with the latter scenarios since the first one is trivial. For $j^1\varphi = (\lambda x, y)$ and $\lambda^p = 1$ we can relate the dynamics of φ with the dynamics of $\varphi^{(p)}$. Then we can suppose $j^1\varphi = (x+\mu y, y)$ for some $\mu \in \mathbb{C}$ up to replace φ with an iterate. Thus, from now on (NSD) will mean (NSD)+unipotent. In the one-variable case the topological [**Lea97**], [**Cam78**], [**Shc82**] formal and analytical classifications [**Eca82**], [**Vor81**], [**MR83**] of unipotent diffeomorphisms are well-known (see [**Lor99**] for an excellent survey on these topics).

We are interested on giving a complete characterization of whether or not two (NSD) diffeomorphisms have the same dynamical behavior, or in other words when they are conjugated by a homeomorphism defined in a neighborhood of 0 in $\mathbb{C}^2$. Such a conjugating homeomorphism can be wild; for instance in general it is not of the form $(\sigma_1(x,y), \sigma_2(y))$. Since we want to describe the evolution of the dynamics of $\varphi_{|y=s}$ we impose two natural conditions. Let φ_1, φ_2 be (NSD) diffeomorphisms conjugated by a germ of homeomorphism σ; we say that σ is special if

- $y \circ \sigma \equiv y$.
- $\sigma_{|Fix\varphi_1 \setminus (y=0)} \equiv Id$.

If such a special conjugation exists we denote $\varphi_1 \overset{sp}{\sim} \varphi_2$. We denote the topological and the analytic conjugations by $\overset{top}{\sim}$ and $\overset{ana}{\sim}$ respectively.

If we have $\varphi_1 \overset{sp}{\sim} \varphi_2$ for (NSD) diffeomorphisms φ_1 and φ_2 then $Fix\varphi_1 = Fix\varphi_2$. This equation has two be understood as a relation between analytic sets with not necessarily reduced structure; for instance we have $Fix(x+x^2, y) \neq Fix(x+x^3, y)$.

Let φ be a (NSD) diffeomorphism. We denote by $m(\varphi)$ the unique non-negative number such that y^m divides $x \circ \varphi - x$ but y^{m+1} does not divide $x \circ \varphi - x$. Consider the decomposition $x \circ \varphi - x = y^m f_1^{n_1} \ldots f_p^{n_p}$ in irreducible factors. We define $N(\varphi) = \sum_{j=1}^{p} \nu(f_j(x,0))$. Then for every sufficiently small neighborhood U of $(0,0)$ and $y_0 \neq 0$ in a neighborhood of 0 we obtain $N = \sharp(Fix\varphi \cap U \cap [y = y_0])$. The couple (N, m) is a topological invariant.

Let φ be a (NSD) diffeomorphism. Consider an irreducible component $\gamma \neq [y = 0]$ of $Fix\varphi$. We define $Res_\varphi^\gamma : \gamma \setminus \{(0,0)\} \to \mathbb{C}$ as the function associating to P the residue of the diffeomorphism φ_P. The function Res_φ^γ is holomorphic. Our main theorem in this paper is:

MAIN THEOREM. *Let φ_1, φ_2 be two (NSD) diffeomorphisms with same invariant (N, m). We have*

- *If $N = 0$ or $(N, m) = (1, 0)$ then $\varphi_1 \overset{sp}{\sim} \varphi_2 \Leftrightarrow Fix\varphi_1 = Fix\varphi_2$.*
- *For the remaining cases $\varphi_1 \overset{sp}{\sim} \varphi_2$ if and only if*
  - *$Fix\varphi_1 = Fix\varphi_2$.*
  - *$y^m(Res_{\varphi_1}^\gamma - Res_{\varphi_2}^\gamma)$ extends continuously by 0 to $(0,0)$ for all irreducible component $\gamma \neq [y = 0]$ of $Fix\varphi_1$.*
  - *$\varphi_{1,(0,0)} \overset{ana}{\sim} \varphi_{2,(0,0)}$.*

Moreover if $(N, m) \neq (1, 0)$ then $\sigma_{|y=0}$ is complex analytic for every special germ of homeomorphism σ conjugating φ_1 and φ_2.

Suppose $m = 0$ throughout this paragraph. The condition $\varphi_{1,(0,0)} \overset{ana}{\sim} \varphi_{2,(0,0)}$ is much stronger than $\varphi_{1,(0,0)} \overset{top}{\sim} \varphi_{2,(0,0)}$ for $N > 1$ since the analytic classes contained in a topological class are parameterized by a functional invariant. Suppose $\varphi_1 \overset{sp}{\sim} \varphi_2$; we have

(N, m)	situation in $y = 0$	existence of irregular zones
$N = 1$, $m = 0$	$\varphi_{1,(0,0)} \overset{top}{\sim} \varphi_{2,(0,0)}$	NO
$N > 1$, $m = 0$	$\varphi_{1,(0,0)} \overset{ana}{\sim} \varphi_{2,(0,0)}$	YES

The rigidity provided by the main theorem is attached to the existence of irregular zones in the parameter space. Our work unveils a new phenomenon whose existence is based on the structure of the limits of orbits in the irregular zones.

Let us say a word about the proof of the main theorem. We study at first the real flow of a vector field $X = f\partial/\partial x$ such that $\exp(X)$ is a convergent normal form of a (NSD) diffeomorphism φ. We use techniques analogous to those in [**DES**] to study $Re(X)$. In fact we classify topologically all the vector fields $Re(X)$ where $X \in \mathcal{H}(\mathbb{C}^2, 0)$ and $\exp(X)$ is a (NSD) diffeomorphism. The same techniques can be used to classify the real flows of all the vector fields of the form $X = f\partial/\partial x$ for any $f \in \mathbb{C}\{x, y\}$. Anyway, we do not do it for simplicity and because it is of no utility to study the (NSD) diffeomorphisms.

CHAPTER 1

Outline of the Paper

A germ of diffeomorphism $\varphi = (x + \mu y + h.o.t., y) \in \text{Diff}\,(\mathbb{C}^2, 0)$ has no small divisors if and only if $\partial(x \circ \varphi)/\partial x \equiv 1$ by restriction to $Fix\varphi$. This condition has an algebraic translation. Let $y^m f_1^{n_1} \ldots f_p^{n_p}$ $(m \geq 0)$ be the decomposition of $x \circ \varphi - x$ in irreducible factors. Then φ is (NSD) if and only if $n_j \geq 2$ for all $1 \leq j \leq p$. This condition can be checked out on any $f = y^m f_1^{n_1} \ldots f_p^{n_p} \in \mathbb{C}\{x, y\}$ such that $f(0, 0) = 0$. Therefore, we can speak of germs of (NSD) functions. A germ $X \in \mathcal{H}(\mathbb{C}^2, 0)$ is a (NSD) vector field if $\exp(X)$ is a (NSD) diffeomorphism or in a equivalent way if X can be expressed in the form $f\partial/\partial x$ for some (NSD) germ of function.

Every germ of (NSD) diffeomorphism φ is the exponential $\exp(1\hat{X})$ of a unique formal vector field $\hat{X} = \hat{f}\partial/\partial x$ where $\hat{f} \in \mathbb{C}[[x, y]]$ and

$$\exp(t\hat{X}) = \left(\sum_{n=0}^{\infty} t^n \frac{\hat{X}^n(x)}{n!}, \sum_{n=0}^{\infty} t^n \frac{\hat{X}^n(y)}{n!} \right)$$

for $t \in \mathbb{C}$. By definition $\hat{X}^0(g) = g$ and $\hat{X}^{j+1}(g) = \hat{X}(\hat{X}^j(g))$ for $j \geq 0$. We just wrote down the Taylor formula for the formal vector field $t\hat{X}$. We have that $\hat{X}$ is of the form $\hat{u}f\partial/\partial x$ where $\hat{u} \in \mathbb{C}[[x, y]]$ is a unit and $f = x \circ \varphi - x$. The vector field $\hat{X}$ is transversally formal along $f = 0$.

Proposition 1.1. *Let $\varphi = \exp(\hat{u}f\partial/\partial x)$ be a (NSD) diffeomorphism. For all $k \in \mathbb{N}$ there exists $u_k \in \mathbb{C}\{x, y\}$ such that $\hat{u} - u_k \in (f^k)$.*

We say that $X = uf\partial/\partial x \in \mathcal{H}(\mathbb{C}^2, 0)$ is a convergent normal form of φ if $\hat{u} - u \in (f^2)$. The diffeomorphism φ is formally conjugated to $\exp(X)$. Our approach consists in comparing the dynamics of φ and $\exp(X)$. The first step of this program is describing the dynamical behavior of $Re(X)$ for a (NSD) vector field X. That is the purpose of chapters 2 through 5.

We fix domains $U_\epsilon = [|x| < \epsilon]$ and $U_{\epsilon,\delta} = B(0, \epsilon) \times B(0, \delta)$. We will always suppose that $SingX \cap (\epsilon\mathbb{S}^1 \times B(0, \delta)) \subset [y = 0]$. We want to study the vector field $\xi(X, y_0, \epsilon) = Re(X)_{|B(0,\epsilon)\times\{y_0\}}$ for a specific y_0. Afterwards, we are interested on the evolution of the dynamics of $\xi(X, y_0, \epsilon)$ with respect to y_0. Let us focus on the first task.

For $P \in SingX$ we can define $X_P \in \mathcal{H}(\mathbb{C}, 0)$; the definition is analogous to the definition of φ_P for $P \in Fix\varphi$. The (NSD) character implies that X_P is nilpotent for all $P \in SingX$. The dynamics of $Re(Y)$ and $\exp(Y)$ for a nilpotent $Y = a(z)\partial/\partial z$ is well-known. There exists a fundamental system $\{V_n\}_{n\in\mathbb{N}}$ of open neighborhoods of 0 such that $V_n \setminus \{0\}$ is the union of $\nu(a(z)) - 1$ basins of attraction of $z = 0$ for $Re(Y)$ and $\nu(a(z)) - 1$ basins of attraction of $z = 0$ for $Re(-Y)$ [**Lea97**], [**Cam78**]. As a consequence the real parts of nilpotent vector fields in $\mathcal{H}(\mathbb{C}, 0)$ have an open character since the set of points whose α limit is $z = 0$ is an open set (ditto for the ω limit). The nilpotent character of the singular points also implies

Proposition 1.2. *Let X be a (NSD) vector field. For all $y_0 \in B(0, \delta)$ the vector field $\xi(X, y_0, \epsilon)$ satisfies the Rolle property.*

In other words a trajectory of $\xi(X, y_0, \epsilon)$ never intersects a connected transversal for two different times. In particular for any positive trajectory $\gamma : [0, c) \to U_{\epsilon,\delta} \cap [y = y_0]$ of $Re(X)$ the following dichotomy holds:

- $c \in \mathbb{R}^+$ and $\lim_{t\to c} \gamma(t) \in \partial U_{\epsilon,\delta}$.
- $c = \infty$ and $\omega(\gamma) \in SingX \cap [y = y_0]$.

Roughly speaking the trajectories of $Re(X)$ are attracted either by the boundary of $U_{\epsilon,\delta}$ or by the singular points.

The dynamics of $Re(X)_{|y=y_0}$ in the neighborhood of every point $(x_0, y_0) \in \partial U_{\epsilon,\delta}$ where $Re(X)_{|y=y_0}$ is transversal to $\epsilon\mathbb{S}^1 \times \{y_0\}$ is locally a product. Since nilpotent singular points have an open character then the unstable trajectories of $\xi(X, y_0, \epsilon)$ are contained in trajectories of $Re(X)_{|\overline{B}(0,\epsilon)\times\{y_0\}}$ passing through points where $Re(X)$ and $\partial U_{\epsilon,\delta}$ are tangent. The unstable trajectories are also called *critical trajectories.*

Proposition 1.3. *Let X be a (NSD) vector field. For all $y_0 \in B(0, \delta)$ the critical trajectories of $\xi(X, y_0, \epsilon)$ determine $\xi(X, y_0, \epsilon)$ up to topological equivalence.*

Next we focus on the evolution of the dynamics of $Re(X)_{|y=y_0}$ with respect to $y = y_0$. In chapter 3 we divide $U_{\epsilon,\delta}$ in a union of "basic" sets. There are two kind of basic sets, namely "exterior" and "compact-like" ones. Let $y_0 \in B(0, \delta)$; the dynamics of $\xi(X, y, \epsilon)$ restricted to an exterior set is locally a product in the neighborhood of y_0. Such a property is no longer true for a "compact-like" basic set; anyway since it is somehow compact the dynamics of the restriction of $Re(X)$ to a "compact-like" basic set is bound to be non-chaotic. The decomposition in basic sets is used throughout this paper to find uniform patterns of regularity for the orbits of $Re(X)$ (or φ for (NSD) diffeomorphisms) in $U_{\epsilon,\delta} \setminus [y = 0]$.

We are interested in the evolution of the dynamics of $\xi(X, y, \epsilon)$ with respect to y. In chapter 4 we study the set UN_X^ϵ of instability of the dynamics. By definition $y_0 \in B(0, \delta) \setminus UN_X^\epsilon$ if there exists a neighborhood V of y_0 in $\mathbb{C}$ and a homeomorphism $\sigma : \overline{B}(0, \epsilon) \times V \to \overline{B}(0, \epsilon) \times V$ such that

- $\sigma_{|y=y_0} \equiv Id$.
- $\sigma_{|y=s}$ is a topological equivalence between $\xi(X, y_0, \epsilon)$ and $\xi(X, s, \epsilon)$ for all $s \in V$.

We denote by $T_X^\epsilon \subset \partial U_{\epsilon,\delta}$ the set of points where $Re(X)$ is tangent to $\partial U_{\epsilon,\delta}$. The unstable trajectories of $\xi(X, y_0, \epsilon)$ are the ones contained in trajectories of $Re(X)_{|\overline{B}(0,\epsilon)\times\{y_0\}}$ passing through points of T_X^ϵ. Thus, the following proposition is natural.

Proposition 1.4. *Let X be a (NSD) vector field. Then $y_0 \in UN_X^\epsilon$ if and only if there exists a trajectory γ of $Re(X)_{|\overline{B}(0,\epsilon)\times\{y_0\}}$ such that $\sharp(\gamma \cap T_X^\epsilon) > 1$.*

The connected components of UN_X^ϵ are called T-sets since they connect tangent points. We describe the nature of UN_X^ϵ.

Proposition 1.5. *Let X be a germ of (NSD) vector field. There are finitely many T-sets. Moreover, every T-set is a semi-analytic curve.*

Chapters 2 through 4 allow to describe the behavior of $Re(X)$ restricted to $U_{\epsilon,\delta}$. The downside is that the information that we obtain depends not only on the germ $X \in \mathcal{H}(\mathbb{C}^2, 0)$ but also on the domain U_ϵ. The sets UN_X^ϵ and $UN_X^{\epsilon'}$ are different if $\epsilon \neq \epsilon'$. We would like to have a domain independent tool to study the dynamics. We accomplish this goal by studying the L-limits. In the remainder of the introduction we suppose $m = 0$, i.e. $[y = 0] \not\subset SingX$ since the notations and definitions are simpler. It is the generic case among (NSD) objects. Anyway, the propositions are enounced in complete generality.

We denote by $\Gamma^U_{\xi(X),+}[P]$ the positive trajectory of $Re(X)_{|U}$ passing through P. Analogously we define $\Gamma^U_{\xi(X),-}[P]$ and finally we define $\Gamma^U_{\xi(X)}[P] = \Gamma^U_{\xi(X),+}[P] \cup \Gamma^U_{\xi(X),+}[P]$. The positive L-limit $L^{\epsilon,+}_{\beta,x_0}$ of a point $x_0 \in B(0,\epsilon)$ along a semi-analytic curve β is the subset of $\overline{B}(0,\epsilon) \setminus \{0\}$ such that $x_1 \in L^{\epsilon,+}_{\beta,x_0}$ if there exists $(x_n, y_n) \to (x_1, 0)$ such that

- $y_n \in \beta$ for all $n \in \mathbb{N}$.
- For all $\eta > 0$ we have $(x_n, y_n) \in \Gamma^{|x|<\epsilon+\eta}_{\xi(X),+}[x_0, y_n]$ for all $n >> 0$.
- $(x_1, 0) \notin \Gamma^{|x|\leq\epsilon}_{\xi(X)}[x_0, 0]$.

In other words, the L-limit $L^{\epsilon,+}_{\beta,x_0}$ is the accumulation set of the positive trajectories $\Gamma_{\xi(X),+}(x_0, y)$ when $y \in \beta$ and $y \to 0$ deprived of the trajectory passing through $(x_0, 0)$.

Proposition 1.6. *A L-limit is a limit.*

We prove this by finding a continuous $S : \beta \cup \{0\} \to \mathbb{C}^2$ satisfying that for all $\eta > 0$ there exists $k(\eta) > 0$ such that $S(s) \in \Gamma^{|x|<\epsilon+\eta}_{\xi(X),+}(x_0, s)$ for all $s \in B(0, k(\eta)) \cap \beta$. We also require $S(0) = (x_1, 0)$. The L-limit would behave like an accumulation set and not like a limit if we would drop the hypothesis on the semi-analyticity of β.

The connected components of $L^{\epsilon,+}_{\beta,x_0}$ are naturally ordered by the time of the flow $Re(X)$; moreover, there are only finitely many. We claimed that the L-limit does not depend on the domain of definition (and then on ϵ) and that is not exactly true. The L-limit depends on ϵ but

Proposition 1.7. *Let X be a (NSD) vector field. Consider a L-limit $L^{\epsilon,+}_{\beta,x_0} \neq \emptyset$. Then, the first component of $L^{\epsilon,+}_{\beta,x_0}$ does not depend on the domain of definition of X.*

For $\epsilon > 0$ and $\delta(\epsilon) > 0$ small enough we define

$$N = N(X) = \sharp(SingX \cap [y = y_0])$$

for $y_0 \in B(0, \delta) \setminus \{0\}$. The number N does not depend on y_0 since $SingX \cap \partial U_{\epsilon,\delta} \subset [y = 0]$. We have

Proposition 1.8. *Let X be a germ of (NSD) vector field. Then there exists a non-empty L-limit if and only if $N > 1$.*

The existence of a non-empty L-limit $L^{\epsilon,+}_{\beta,x_0}$ implies that the limit of the positive trajectories of $Re(X)$ passing through (x_0, y) $(y \in \beta)$ is not the positive trajectory of $Re(X)$ passing through $(x_0, 0)$. Somehow "$\lim_{y_0 \to 0} Re(X)_{|y=y_0}$" is richer than $Re(X)_{|y=0}$. Let $m = \nu_y(X(x))$; we have

PROPOSITION 1.9. *Let X be a germ of (NSD) vector field. Then*

$$\lim_{y_0 \to 0} Re(X)_{|y=y_0} = X_{|y=0}$$

for $(N, m) \neq (1, 0)$. Otherwise $\lim_{y_0 \to 0} Re(X)_{|y=y_0} = Re(X)_{|y=0}$.

The formula $\lim_{y_0 \to 0} Re(X)_{|y=y_0} = X_{|y=0}$ means that the complex flow of X at $y = 0$ is generated by the real flow of $X_{|y=y_0}$ when $y_0 \to 0$. Proposition 1.9 is based in the following result:

PROPOSITION 1.10. *Let X be a (NSD) vector field with a non-empty $L^{\epsilon,+}_{\beta,x_0}$. There exist $x_1 \in L^{\epsilon,+}_{\beta,x_0}$, a neighborhood V of 0 in $\mathbb{R}$ and a continuous family of semi-analytic curves $\{\beta(s)\}_{s \in V}$ such that $\beta(0) = \beta$ and $\cup_{s \in V} L^{\epsilon,+}_{\beta(s),x_0}$ is a neighborhood of $(x_1, 0)$.*

In particular the previous proposition implies that for a germ of homeomorphism σ conjugating two (NSD) vector fields and defined in $U_{\epsilon,\delta}$ the value of $\sigma(x_0, 0)$ determines the value of $\sigma(x, 0)$ for x in the neighborhood of x_1. The proof of this kind of results relies in the fact that we can calculate the time $T(y)$ spent by $Re(X)$ to go from (x_0, y) $(y \in \beta)$ to the neighborhood of (x_1, y) for $x_1 \in L^{\epsilon,+}_{\beta,x_0}$. Roughly speaking T is the restriction of a meromorphic function

$$s \to -2\pi i \sum_{P \in E(s)} Res_X(P)$$

where $E(s) \subset SingX \cap [y = s]$ is a set depending on the connected component of $L^{\epsilon,+}_{\beta,x_0}$ containing x_1. Moreover $E(s)$ depends continuously on s. The functions Res are the usual residue functions. More precisely, for a nilpotent $Y \in \mathcal{H}(\mathbb{C}, 0)$ there exists a unique form $\omega \in \Omega(\mathbb{C}, 0)$ such that $\omega(Y) = 1$; we define $Res_Y(0)$ as the residue at 0 of ω and then $Res_X(P) = Res_{X_P}(P)$ for all $P \in SingX \setminus [y = 0]$.

We are interested on determining whether or not the real flows of germs of (NSD) vector fields X_1, X_2 are topologically conjugated. Our approach is based on studying the evolution of the dynamical behavior of $Re(X)_{|y=y_0}$ with respect to y_0 and in particular the evolution of the dynamics of $Re(X_P)$ with respect to $P \in SingX$. Then, it is natural to assume that the topological conjugations satisfy:

- $y \circ \sigma \equiv y$.
- $\sigma_{|SingX \setminus [y=0]} \equiv Id$.

Such mappings will be called *special*. A special mapping has a certain degree of regularity, that is not always the case for conjugations. For instance, a general germ of homeomorphism conjugating real (NSD) flows does not preserve the fibration $y = cte$.

Let X_1, X_2 be (NSD) vector fields. If $Re(X_1)$ and $Re(X_2)$ are conjugated by a special germ of homeomorphism then they both belong to some set

$$\mathcal{H}_f = \{uf\partial/\partial x : u \in C\{x, y\} \text{ is a unit}\}$$

where f satisfies the (NSD) conditions. As a consequence we restrict our study to the sets $\mathcal{H}_f$.

Let $x_1 \in L^{\epsilon,+}_{\beta,x_0}$ and suppose that $Re(X_1)$ and $Re(X_2)$ are topologically conjugated by a special σ. We already pointed out the existence of a real function $T(y) \sim -2\pi i \sum_{P \in E(y)} Res_{X_1}(P)$ such that

$$\lim_{y \in \beta, y \to 0} \exp(T(y)X_1)(x_0, y) = (x_1, 0).$$

Moreover, we have

$$\lim_{y\in\beta, y\to 0} \exp(T(y)X_2)(\sigma(x_0,y)) = \sigma(x_1,0)$$

since σ conjugates $Re(X_1)$ and $Re(X_2)$. Because of this last equation we will see that $T(y) \sim -2\pi i \sum_{P\in E(y)} Res_{X_2}(P)$. Therefore, we obtain $\sum_{P\in E(y)} Res_{X_1}(P) \sim \sum_{P\in E(y)} Res_{X_2}(P)$, i.e. the residue functions attached to X_1 and X_2 are related. The ideas in this discussion will lead us to prove the sufficient condition in the next theorem:

THEOREM 1.1. *Let X_1, X_2 be elements of $\mathcal{H}_f$ for some $f \in \mathbb{C}\{x,y\}$ satisfying the (NSD) conditions. Suppose $(N,m) \neq (1,0)$. Then $Re(X_1)$ and $Re(X_2)$ are topologically conjugated by a special mapping if and only if*

$$\lim_{y\to 0} y^m (Res_{X_1}(S(y)) - Res_{X_2}(S(y))) = 0$$

for all continuous section $S : (0,\delta)\times\mathbb{R} \to [f=0]$ such that $S(r,\theta)$ belongs to $SingX \cap [y = re^{i\theta}]$ for all $(r,\theta)\in(0,\delta)\times\mathbb{R}$. Moreover, every special conjugation is analytic by restriction to $y=0$.

The analyticity of the special topological conjugation by restriction to $y=0$ is a consequence of proposition 1.9. For the dynamically simple case $(N,m)=(1,0)$ we have

PROPOSITION 1.11. *Let X_1, X_2 be elements of $\mathcal{H}_f$ for some $f \in \mathbb{C}\{x,y\}$ satisfying the (NSD) conditions. Suppose $(N,m)=(1,0)$. Then $Re(X_1)$ and $Re(X_2)$ are topologically conjugated by a special mapping.*

We explain briefly how we can prove proposition 1.11 and the necessary condition in theorem 1.1. To conjugate $Re(X_1)$ and $Re(X_2)$ we replace $Img(X_1)$ with $h(x,y)Img(X_1)$ where $h : U_{\epsilon,\delta}\setminus[f=0]\to\mathbb{R}^+$ is a continuous function such that $(Re(X_1))(h)=0$ and $c_0 < |h(x,y)| < C_0$ for some $c_0, C_0 > 0$ and all $(x,y)\in U_{\epsilon,\delta}\setminus[f=0]$.

Let $y_0 \in B(0,\delta)$. Consider a loop $\gamma : [0,1]\to[y=y_0]$ such that $\gamma \sim 1 \in \mathbb{Z} \sim \pi_1([y=y_0]\setminus\{P\})$ for some $P\in[f=0]\cap[y=y_0]$ and $\gamma\sim 0\in\mathbb{Z}\sim\pi_1([y=y_0]\setminus\{Q\})$ for all $Q\in([f=0]\cap[y=y_0])\setminus\{P\}$. Let ψ_1 be a complex function in the neighborhood of $\gamma(0)$ in $y=y_0$ such that

$$Re(X_1)(\psi_1) = 1 \text{ and } (hImg(X_1))(\psi_1) = i.$$

Such a function ψ_1 exists since $[Re(X_1), hImg(X_1)]=0$; moreover we can extend it continuously along γ. If $p_\gamma\psi_1$ is germ of the extension of ψ_1 at $\gamma(1)=\gamma(0)$ then $p_\gamma\psi_1 - \psi_1$ is a constant function. We denote by X_1' the complex vector field such that $Re(X_1') = Re(X_1)$ and $Img(X_1') = hImg(X_1)$. We have

$$Res_{X_1'}(P) = Res_{X_{1,P}'}(P) = \frac{1}{2\pi i}\left(\psi_1\circ\gamma(1) - \psi_1\circ\gamma(0)\right).$$

We can choose h to obtain $Res_{X_1'} \equiv Res_{X_1}$ in $SingX\setminus[y=0]$. Now, we can apply the method of the path to conjugate the complex vector fields X_1' and X_2. We obtain a special germ of homeomorphism σ such that

$$\sigma\circ\exp(tX_1') = \exp(tX_2)\circ\sigma$$

for $t\in\mathbb{C}$ and then

$$\sigma\circ\exp(tX_1) = \exp(tX_2)\circ\sigma$$

for $t \in \mathbb{R}$. The choice of h and X_1' is based on the dynamical properties of $Re(X_1)$.

The real goal of this work is classifying the dynamics of germs of (NSD) diffeomorphisms. We define

$$\mathcal{D}_f = \{(x + uf, y) : u \in C\{x, y\} \text{ is a unit}\};$$

this is the analogue of $\mathcal{H}_f$ for (NSD) diffeomorphisms. We have

THEOREM 1.2. *Let φ be a (NSD) diffeomorphism and let X be one of its convergent normal forms. For all $\mu > 0$ there exists $U_{\epsilon,\delta}$ such that*

$$\varphi^{(j)}(P) \in \exp(\overline{B}(0, \mu)X)(\exp(jX)(P))$$

for all $j \in \mathbb{Z}$ and P such that $\{\exp(0X)(P), \ldots, \exp(jX)(P)\} \subset U_{\epsilon,\delta}$.

As a consequence of last theorem the dynamics of a (NSD) diffeomorphism is a slight deformation of the dynamics of the exponential of its normal form. The main ingredient of the proof of theorem 1.2 is the division of $U_{\epsilon,\delta}$ in exterior and compact-like sets that we develop in chapter 3.

The similarity between a (NSD) diffeomorphism φ and a normal form X implies that there is an analogue of the L-limit phenomenon for (NSD) diffeomorphisms and $N > 1$. We obtain points $x_0 \in B(0, \epsilon) \setminus \{0\}$, semi-analytic curves β and sequences $\{y_n\} \subset \beta$ and $\{T(y_n)\} \subset \mathbb{Z}$ such that

- $\lim_{n\to\infty} y_n = 0$ and $\lim_{n\to\infty} T(y_n) = \infty$
- $\exists \lim_{n\to\infty} \exp(T(y_n)X)(x_0, y_n)$ and $\exists \lim_{n\to\infty} \varphi^{(T(y_n))}(x_0, y_n)$
- $\lim_{n\to\infty} \exp(T(y_n)X)(x_0, y_n)$ is in the first component of $L^{\epsilon,+}_{\beta,x_0}$.

Moreover, in this context we have

PROPOSITION 1.12. *There exists a neighborhood V of 0 in $\mathbb{R}$ and a continuous family of semi-analytic curves $\{\beta(s)\}_{s\in V}$ ($\beta(0) = \beta$) such that for all $(x_1, 0)$ in a neighborhood of $\lim_{n\to\infty} \varphi^{(T(y_n))}(x_0, y_n)$ there exist $s_0 \in V$ and sequences $\{y_n^0\} \subset \beta(s_0)$ and $\{T(y_n^0)\} \subset \mathbb{Z}$ satisfying*

$$\lim_{n\to\infty} y_n^0 = 0 \quad \text{and} \quad \lim_{n\to\infty} \varphi^{(T(y_n^0))}(x_0, y_n^0) = (x_1, 0).$$

The value of a topological conjugation σ at $(x_0, 0)$ determines $\sigma_{|y=0}$ in the neighborhood of $\lim_{n\to\infty} \varphi^{(T(y_n))}(x_0, y_n)$. We obtain

PROPOSITION 1.13. *Let $\varphi_1, \varphi_2 \in \mathbb{D}_f$ be (NSD) diffeomorphisms. Suppose $(N, m) \neq (1, 0)$. Let σ be a germ of special homeomorphism conjugating φ_1 and φ_2. Then $\sigma_{|y=0}$ is complex analytic.*

We take profit of the previous proposition and the similarity between (NSD) diffeomorphisms and normal forms to obtain the sufficient condition in next theorem

THEOREM 1.3. *Let $\varphi_1, \varphi_2 \in \mathbb{D}_f$ be (NSD) diffeomorphisms. Let X_j be a convergent normal form for φ_j ($j \in \{1, 2\}$). Suppose $(N, m) \neq (1, 0)$. Then*

- *φ_1 and φ_2 are conjugated by a special homeomorphism*

if and only if both following conditions are satisfied

- *$Re(X_1)$ is conjugated to $Re(X_2)$ by a special homeomorphism.*
- *$\varphi_{1|y=0}$ is analytically conjugated to $\varphi_{2|y=0}$.*

We also have

PROPOSITION 1.14. *Let $\varphi_1, \varphi_2 \in \mathcal{D}_f$ be (NSD) diffeomorphisms. Suppose $(N, m) = (1, 0)$. Then φ_1 and φ_2 are topologically conjugated by a special mapping.*

Theorem 1.3 and proposition 1.14 are equivalent to the Main Theorem for $(N, m) \neq (1, 0)$ and $(N, m) = (1, 0)$ respectively. To prove the necessary condition in theorem 1.3 and proposition 1.14 we embed φ in a complex flow which is not in general analytic. That is equivalent to exhibit a special homeomorphism conjugating the exponential $\exp(X)$ of the normal form and φ. Then, we just define

$$\varphi^{(t)}(P) = \sigma(\exp(tX)(\sigma^{(-1)}(P)))$$

for $t \in \mathbb{C}$. Unfortunately, theorem 1.3 implies that such a σ does not exist if $\varphi_{|y=0}$ is not the exponential of a nilpotent element in $\mathcal{H}(\mathbb{C}, 0)$. As a consequence instead of germs of homeomorphism we will consider tg-sp (tangential-special) mappings σ. By definition σ is a tg-sp mapping if there exist V and V' neighborhoods of $(0, 0)$ such that

- σ is a homeomorphism defined in $(V \setminus [y = 0]) \cup \{(0, 0)\}$.
- $\sigma^{(-1)}$ is a homeomorphism defined in $(V' \setminus [y = 0]) \cup \{(0, 0)\}$.
- $\sigma(0, 0) = (0, 0)$ and $y \circ \sigma \equiv y$ and $\sigma_{|[f=0] \setminus [y=0]} \equiv Id$.

We explain now how to build a tg-sp mapping conjugating the normal form $\exp(X)$ and a (NSD) φ. A possible approach to embed φ in a complex flow is by using transversals. Let Tr be a 3-dimensional transversal to $Re(X)$. We suppose that $Tr \cap [y = y_0]$ when non-empty is contained in a trajectory of $Img(X)$ for all $y_0 \in B(0, \delta)$. We define the function Δ such that

$$\varphi(P) = \exp((1 + \Delta(P))X)(P)$$

for all P in a neighborhood of $(0, 0)$. Now we can define

$$\varphi^{(a+ib)}(P) = \exp(a[1 + \Delta(\exp(ibX)(P))]X)(\exp(ibX)(P))$$

for $a \in [0, 1]$ and $\exp(ibX)(P) \in Tr$. To define $\varphi^{(a+ib)}$ for $a \in \mathbb{R}$ we consider $c \in [0, 1]$ such that $a - c \in \mathbb{Z}$; we define

$$\varphi^{(a+ib)} = \varphi^{(a-c)} \circ \varphi^{(c+ib)}.$$

Now we build a mapping σ_{Tr} conjugating $\exp(X)$ and φ; we define $\sigma_{Tr}(\exp(aX)(P)) = \varphi^{(a)}(P)$ for $a \in \mathbb{R}$ and $P \in Tr$. This mapping is not C^∞ because the complex flow $\varphi^{(t)}$ is not C^∞ for $Re(t) \in \mathbb{Z}$ but only continuous. Anyway we can change slightly the definition to obtain a C^∞ flow. We have to face another problem; let $y_0 \in B(0, \delta)$, there is no in general a connected 1-dimensional transversal to $Re(X)_{|y=y_0}$ intersecting all the trajectories of $Re(X)$. Therefore, we have to interpolate conjugations obtained by considering different transversals. For both the construction of σ_{Tr} and the interpolation of different σ_{Tr} and $\sigma_{Tr'}$ we use dynamical properties of $Re(X)$. Then, to make this construction to depend continuously on y we have to work in the neighborhood of parameters y_0 such that $Re(X)_{|y=s}$ is topologically equivalent to a product in the neighborhood of $s = y_0$. We are in that situation for $y_0 \notin UN_X^\epsilon$. If $y_0 \in UN_X^\epsilon \setminus \{0\}$ we change slightly $U_{\epsilon,\delta}$ in order to have $y_0 \notin UN_X$ with respect to the new domain. Hence, for all $y_0 \notin UN_X^\epsilon \cap \{0\}$ there exists a neighborhood V_{y_0} such that we can build a C^∞ mapping σ_{y_0} defined in $(U_{\epsilon,\delta} \cap [y \in V_{y_0}]) \setminus [f = 0]$ and conjugating $\exp(X)$ and φ. The mapping σ_{y_0} is obtained by interpolating conjugations σ_{Tr}. Moreover, we can extend σ_{y_0} continuously to $f=0$ by defining $\sigma_{y_0,|f=0} \equiv Id$. For $(N,m)=(1,0)$ we have $0 \notin UN_X^\epsilon$ and then σ_0 is a special germ of homeomorphism conjugating $\exp(X)$ and φ. Otherwise we have to interpolate some conjugations σ_{y_0} to obtain a conjugation σ defined in $U_{\epsilon,\delta} \setminus [y=0]$. Again, we can extend σ continuously to $f=0$ by defining $\sigma_{|f=0} \equiv Id$. The mapping σ turns out to be tangential-special. We obtain

PROPOSITION 1.15. *Let φ be a (NSD) diffeomorphism with normal form X. There exists a tg-sp mapping σ conjugating* $\exp(X)$ *and φ. Moreover σ can be chosen to be a germ of homeomorphism if $N \leq 1$ or $m > 0$.*

Now proposition 1.11 implies proposition 1.14. Analogously theorem 1.1 implies the necessary condition in theorem 1.3 for $N \leq 1$ or $m > 0$.

The remaining case in theorem 1.3 is $N > 1$ and $m = 0$. Since $\varphi_{1|y=0}$ is analytically conjugated to $\varphi_{2|y=0}$ we can suppose $\varphi_{1|y=0} \equiv \varphi_{2|y=0}$ up to replace φ_2 with $h^{(-1)} \circ \varphi_2 \circ h$ for some special $h \in \mathrm{Diff}\,(\mathbb{C}^2,0)$. Hence, we can choose the convergent normal forms to satisfy $X_{1|y=0} \equiv X_{2|y=0}$ too. As a consequence there exists a special homeomorphism σ_X conjugating $Re(X_1)$ and $Re(X_2)$ such that $\sigma_{X,|y=0} \equiv Id$. Consider a tg-sp mapping σ_j conjugating $\exp(X_j)$ and φ_j for $j \in \{1,2\}$. The mapping

$$\sigma = \sigma_2 \circ \sigma_X \circ \sigma_1^{(-1)}$$

is a tg-sp mapping conjugating φ_1 and φ_2. The last part of the paper is devoted to prove that there is a choice of σ_1 and σ_2 such that σ is a special germ of homeomorphism. We define the function Δ_j^k such that

$$\varphi_k^{(j)}(P) = \exp((j+\Delta_j^k(P))X_k)(P)$$

for $(j,k) \in \mathbb{Z} \times \{1,2\}$ and $\{\exp(0X_k)(P), \ldots, \exp(jX_k)(P)\} \subset U_{\epsilon,\delta}$.

LEMMA 1.1. *We have $|\Delta_j^1 - \Delta_j^2| \leq L(y)$ for all $j \in \mathbb{Z}$ where $L = o(1)$ is independent of $j \in \mathbb{Z}$.*

The lemma claims that the orbits of φ_1 and φ_2 are very similar, even outside of $y=0$, since the "distance" tends to 0 uniformly on the orbits. This fact allows to choose σ_1 and σ_2 in a way such that $\sigma_{|y=0} \equiv Id$ and $\sigma^{(-1)}_{|y=0} \equiv Id$ are continuous extensions of σ and $\sigma^{(-1)}$ respectively.

CHAPTER 2

Flower Type Vector Fields

2.1. Definition and basic properties

Consider a real analytic vector field ξ defined over an open subset V of $\mathbb{R}^2$. Let $P \in V$ be a singular point of ξ; there is a "flower type" singularity at P if for all neighborhood U of P there exist two non-empty open sets $U_+, U_- \subset U$ such that

- $U_+ \cup U_- \cup \{P\}$ is a neighborhood of P.
- U_+ is positively invariant by ξ and the ω limit $\omega(Q)$ of any $Q \in U_+$ is equal to $\{P\}$.
- U_- is negatively invariant by ξ and $\alpha(U_-) = \{P\}$.

Throughout this section we will consider a real analytic vector field ξ defined in a neighborhood of $\overline{\mathbb{D}}$. Such a vector field is of flower type if

(1) $Sing\xi \cap \partial\mathbb{D} = \emptyset$
(2) There are only flower type singularities.

Remark 2.1.1. *The only relevant property is the second one; property (1) can be skipped by enlarging the domain of definition.*

Let V be a set where ξ is defined. We define $\Gamma_\xi^V[Q]$ the trajectory of ξ in V passing through Q. On top of that we define the positive and negative trajectories $\Gamma_{\xi,+}^V[Q]$ and $\Gamma_{\xi,-}^V[Q]$ obtained by restraining $\Gamma_\xi^V[Q]$ for positive and negative times respectively. We can define the mapping ω_V associating to each $Q \in V$ the ω limit of the trajectory $\Gamma_\xi^V[Q]$ of ξ passing through Q in V. We can define the mapping α_V in an analogous way.

We say that a set $S \subset \overline{\mathbb{D}}$ is *positively invariant* if for every open neighborhood B of $\overline{\mathbb{D}}$ we have

$$\cup_{Q \in S} \Gamma_{\xi,+}^B[Q] \subset S.$$

We can define a negatively invariant domain in an analogous way.

Remark 2.1.2. *Let U be any neighborhood of a singular point $P \in \mathbb{D}$ and consider a point $Q \in \mathbb{D} \setminus U$. Since the singularity at P is of flower type we have*

$$\Gamma_{\xi,+}^{\mathbb{D}}[Q] \cap (U_+ \cup U_- \cup \{P\}) = \Gamma_{\xi,+}^{\mathbb{D}}[Q] \cap U_+.$$

As a consequence we have

- *If $\omega_{\mathbb{D}}(Q)$ contains a singular point P then $\omega_{\mathbb{D}}(Q) = \{P\}$.*
- *$\omega_{\mathbb{D}}^{-1}(P)$ is an open set for all $P \in Sing\xi$.*
- *By Poincaré-Bendixon's theorem the only values for $\omega_{\mathbb{D}}(Q)$ are*
  (1) *$\omega_{\mathbb{D}}(Q) = \infty$; by definition this happens when $\Gamma_{\xi,+}^{\mathbb{D}}[Q]$ reaches $\partial\mathbb{D}$ for a finite time.*
  (2) *$\omega_{\mathbb{D}}(Q) = \{P\}$ for some $P \in Sing\xi$.*
  (3) *$\omega_{\mathbb{D}}(Q)$ is a cycle.*

Next property on the α and ω limits is not restricted to the flower type setting.

REMARK 2.1.3. *Let γ be a cycle of a C^1 vector field X defined in a neighborhood of γ in $\mathbb{R}^2$. There is an open set U containing γ such that either $\omega(Q)$ or $\alpha(Q)$ is a cycle for all $Q \in U$. This property is based on Poincaré-Bendixon's arguments.*

2.1.1. The Rolle property. We say that vector field ξ satisfies the dynamical Rolle property if there is no connected transversal I such that $\Gamma_\xi^{\mathbb{D}}[Q]$ cuts I for two different values of time. Our definition implies that any vector field having cycles can not hold the Rolle condition. Anyway, the definition coincides with the usual one if all the cycles are isolated.

LEMMA 2.1.1. *Let ξ be a flower type vector field and let $P \in Sing\xi$. Then $(\alpha, \omega)^{-1}(P, P) \setminus \{P\} \neq \emptyset$.*

PROOF. Let $U = \mathbb{D}$. Consider an open connected neighborhood V of P contained in $U_+ \cup U_- \cup \{P\}$. Since $\omega_{\mathbb{D}}(U_+) = \{P\}$ then $U_+ \cap V \neq \emptyset$; in an analogous way we have $U_- \cap V \neq \emptyset$. We obtain

$$V \setminus \{P\} = ([U_+ \cap V] \setminus \{P\}) \cup ([U_- \cap V] \setminus \{P\}).$$

The set $V \setminus \{P\}$ is connected; as a consequence there exists a point

$$Q \in (U_+ \cap U_-) \setminus \{P\} \subset (\alpha, \omega)^{-1}(P, P) \setminus \{P\}.$$

□

PROPOSITION 2.1.1. *Let ξ be a flower type vector field. Then ξ satisfies the dynamical Rolle property.*

PROOF. Suppose the proposition is not true. Let $\gamma(t) \subset \mathbb{D}$ be a trajectory of ξ and let $I \subset \mathbb{D}$ be a connected transversal to ξ such that $\gamma(t_0), \gamma(t_1) \in T$ for different t_0 and t_1. We can suppose without lack of generality that $\gamma(t_0, t_1)$ does not intersect I.

We denote by ST the segment of transversal in between $\gamma(t_0)$ and $\gamma(t_1)$. The union of the sets $\{\gamma(t)/t_0 \leq t \leq t_1\}$ and ST is a simple curve β. We denote by U the bounded region limited by β and contained in $\mathbb{D}$. By replacing ξ with $-\xi$ if necessary we suppose that ξ points towards U at the points in ST. If γ is a cycle then $ST = \emptyset$ and the last condition is empty.

The region U is positively invariant. We claim that $U \cap Sing\xi$ is not empty. Let Q be any point in U, then $\omega_{\mathbb{D}}(Q)$ is either a singular point $P \in \mathbb{D}$ or a cycle C. In the latter case the cycle is in the boundary of a bounded region containing a singular point P. We consider the set

$$A_{+,-} = \{Q \in \overline{U} \setminus Sing\xi \ / \ \alpha_{\mathbb{D}}(Q) \subset U\}.$$

The set $(\overline{U} \setminus Sing\xi) \setminus A_{+,-}$ is equal to $\cup_{Q \in ST} \Gamma_{\xi,+}^{\mathbb{D}}[Q]$, hence it is an open set. We also define

$$A_p = \{Q \in A_{+,-} \ \text{ s.t. } \alpha_{\mathbb{D}}(Q) \text{ and } \omega_{\mathbb{D}}(Q) \text{ are points}\} \ ,$$

$$B_p = \{Q \in A_{+,-} \ \text{ s.t. either } \alpha_{\mathbb{D}}(Q) \text{ or } \omega_{\mathbb{D}}(Q) \text{ is a cycle}\}.$$

The set A_p is open in $\overline{U}$ because of the flower nature of the equilibrium points. The set B_p is also open in $\overline{U} \setminus Sing\xi$, it is a consequence of the remark 2.1.3. Therefore, we can express $\overline{U} \setminus Sing\xi$ as a disjoint union of open sets, more precisely

$$\overline{U} \setminus Sing\xi = (A_p \cup B_p) \cup [(\overline{U} \setminus Sing\xi) \setminus A_{+,-}].$$

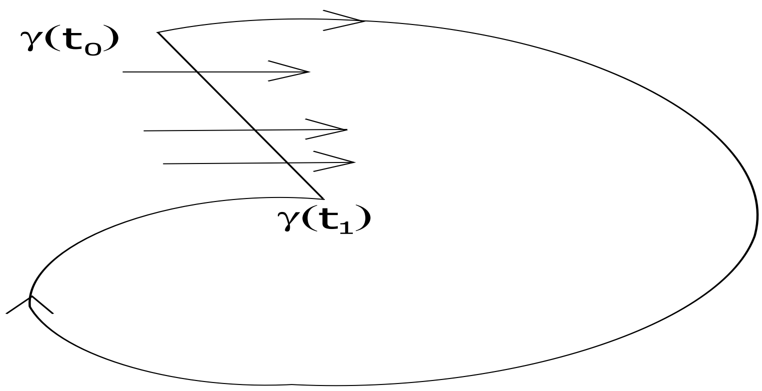

FIGURE 1.

Since $(\alpha,\omega)_{\mathbb{D}}^{-1}(P,P)$ is contained in A_p then $A_p \cup B_p$ is not empty (lemma 2.1.1). Moreover $[(\overline{U} \setminus Sing\xi) \setminus A_{+,-}]$ contains the curve β and then it is not empty. But $\overline{U} \setminus Sing\xi$ is connected, we obtain a contradiction. □

COROLLARY 2.1.1. *There are no cycles.*

REMARK 2.1.4. *The curve $\partial\mathbb{D}$ is not invariant by ξ. This result can be obtained by applying the corollary 2.1.1 to $\xi' = (x/(1+\eta), y/(1+\eta))_*\xi$ for some $\eta > 0$ small enough.*

2.1.2. Critical trajectories. Let $Q \in \partial\mathbb{D}$ be a point where ξ is tangent to $\partial\mathbb{D}$. The point Q is a *convex tangent* point if for some $\eta > 0$ and every open neighborhood U of $\overline{\mathbb{D}}$ we have

$$\Gamma_\xi^U[Q](-\eta,\eta) \cap (U \setminus \overline{\mathbb{D}}) = \emptyset.$$

In other words $\Gamma_{\xi,-}^{\overline{\mathbb{D}}}[Q] \neq \{Q\}$ and $\Gamma_{\xi,+}^{\overline{\mathbb{D}}}[Q] \neq \{Q\}$. The behavior of all the trajectories in a neighborhood of a point M of $\partial\mathbb{D}$ is the same except if M is a convex tangent point (see picture 2). The point Q is a *concave tangent* point if there exist an open neighborhood U of $\overline{\mathbb{D}}$ and some $\eta > 0$ such that

$$\Gamma_\xi^U[Q](-\eta,\eta) \cap \mathbb{D} = \emptyset.$$

If Q is neither convex nor concave then it is by definition an *inflexion tangent* point. We define the set $T_{\xi,+}^{\mathbb{D}} \subset \partial\mathbb{D}$ as the set of tangent convex points whereas $T_{\xi,-}^{\mathbb{D}} \subset \partial\mathbb{D}$ is the set of tangent concave points. We define the set of tangent points $T_\xi^{\mathbb{D}} = T_{\xi,+}^{\mathbb{D}} \cup T_{\xi,-}^{\mathbb{D}}$; we dismiss the inflexion points.

For any convex tangent point Q we can define the critical trajectories passing through Q. The *positive critical trajectory* passing through a convex tangent point P is the set

$$\overline{\Gamma_{\xi,+}^{\mathbb{D}\cup\{Q\}}[Q]}.$$

It is equal to $\Gamma_{\xi,+}^{\mathbb{D}\cup\{Q\}}[Q] \cup \omega_{\mathbb{D}\cup\{Q\}}(Q)$ if $\omega_{\mathbb{D}\cup\{Q\}}(Q) \in Sing\xi$, otherwise it is a curve joining Q and a point in $\partial\mathbb{D}$ whose interior is contained in $\mathbb{D}$. To define the negative critical trajectories just replace $+$ with $-$. We denote by $\mathcal{H}_\xi^{\mathbb{D}}$ the union of the critical trajectories; it is a closed set.

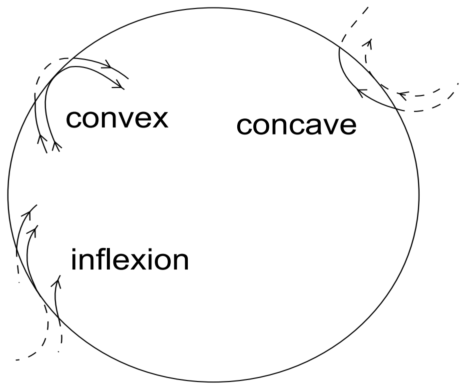

FIGURE 2. Convex, concave, inflexion points

LEMMA 2.1.2. *The mapping*

$$(\alpha, \omega)_{\mathbb{D}} : \mathbb{D} \setminus [\mathcal{H}_\xi^{\mathbb{D}} \cup Sing\xi] \to (Sing\xi \cup \{\infty\}) \times (Sing\xi \cup \{\infty\})$$

is locally constant. In particular, it is constant by restriction to any connected component of $\mathbb{D} \setminus [\mathcal{H}_\xi^{\mathbb{D}} \cup Sing\xi]$.

PROOF. We will prove that $\omega_{\mathbb{D}}$ is locally constant; the proof for $\alpha_{\mathbb{D}}$ is analogous. Let $Q \in \mathbb{D} \setminus [\mathcal{H}_\xi^{\mathbb{D}} \cup Sing\xi]$. If $\omega_{\mathbb{D}}(Q) \in Sing\xi$ then $\omega_{\mathbb{D}}^{-1}(Q)$ is a neighborhood of Q. If $\omega_{\mathbb{D}}(Q) = \infty$ then the closure of $\Gamma_{\xi,+}^{\mathbb{D}}[Q]$ contains a unique point Q' such that $Q' \in \partial\mathbb{D}$. Since $Q \notin \mathcal{H}_\xi^{\mathbb{D}}$ then ξ is either transversal to $\partial\mathbb{D}$ at Q' or Q' is an inflexion point. As a consequence $\Gamma_{\xi,-}^{\overline{\mathbb{D}}}[\partial\mathbb{D}]$ is a neighborhood of Q. Since $\Gamma_{\xi,-}^{\overline{\mathbb{D}}}[\partial\mathbb{D}]$ is contained in $\omega_{\mathbb{D}}^{-1}(\infty)$ then $\omega_{\mathbb{D}}$ is locally constant. $\square$

Let C be a connected component of $\mathbb{D} \setminus [\mathcal{H}_\xi^{\mathbb{D}} \cup Sing\xi]$ such that $\omega_{\mathbb{D}}(C) = \infty$. Consider the mapping

$$\begin{array}{rccc} end_\xi^+ : & C & \to & \partial\mathbb{D} \\ & Q & \mapsto & \overline{\Gamma_{\xi,+}^{\mathbb{D}}[Q]} \cap \partial\mathbb{D}. \end{array}$$

The mapping end_ξ^+ is continuous. Hence, the set $end_\xi^+(C)$ is connected and then it is an open arc. Moreover $end_\xi^+(C)$ does not contain neither tangent convex points nor concave tangent points. If $\omega_{\mathbb{D}}(C) \neq \infty$ then we define $end_\xi^+(C) = \emptyset$. In an analogous way we can define end_ξ^- for the components contained in $\alpha_{\mathbb{D}}^{-1}(\infty)$.

LEMMA 2.1.3. *Let C be a connected component of $\mathbb{D} \setminus [\mathcal{H}_\xi^{\mathbb{D}} \cup Sing\xi]$ contained in $(\alpha, \omega)_{\mathbb{D}}^{-1}(\infty, \infty)$. Then*

$$\partial C \setminus [end_\xi^+(C) \cup end_\xi^-(C)]$$

has two connected components.

PROOF. We consider the boundary points A_1 and A_2 of $end_\xi^+(C)$. We define

$$\gamma_j = \Gamma_{\xi,-}^{\overline{\mathbb{D}}}[A_j] \cap \overline{C}$$

for $j \in \{0,1\}$. The sets γ_1 and γ_2 are connected. We have

$$\partial C \setminus [end^+_\xi(C) \cup end^-_\xi(C) \cup Sing\xi] = \gamma_1 \cup \gamma_2.$$

We choose $Q \in end^+_\xi(C)$. We have $\gamma_1 \neq \gamma_2$ because they are in different connected components of $\overline{\mathbb{D}} \setminus \Gamma^{\overline{\mathbb{D}}}_{\xi,-}[Q]$. $\square$

The previous lemma characterizes the dynamics for the components in $(\alpha,\omega)^{-1}_{\mathbb{D}}(\infty,\infty)$ (see picture 3). Next we focus on the components in $\alpha^{-1}_{\mathbb{D}}(Sing\xi) \cup \omega^{-1}_{\mathbb{D}}(Sing\xi)$.

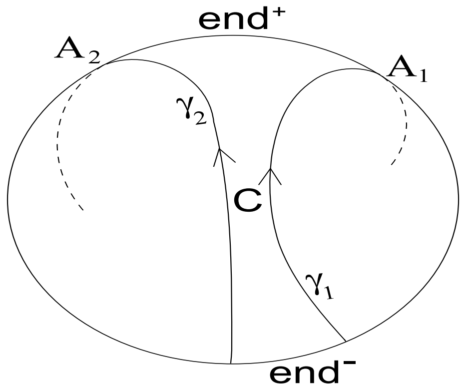

FIGURE 3. Component of $(\alpha,\omega)^{-1}_{\mathbb{D}}(\infty,\infty)$

LEMMA 2.1.4. *Let $P \in Sing\xi$. For every neighborhood U of P we have that $\partial((\alpha,\omega)^{-1}_{\mathbb{D}}(P,P)) \cap (U \setminus \{P\}) \neq \emptyset$. Moreover $\omega^{-1}_{\mathbb{D}}(P)$ does not contain a neighborhood of P.*

PROOF. Let B any open neighborhood of $\overline{\mathbb{D}}$ contained in the domain of definition of ξ. We define $D = (\alpha,\omega)^{-1}_{\mathbb{D}}(P,P)$. By lemma 2.1.1 we obtain that $D \setminus Sing\xi \neq \emptyset$. We have

$$D \setminus Sing\xi \neq \mathbb{D} \setminus Sing\xi$$

because $\partial\mathbb{D}$ is contained in the closure of $\alpha^{-1}_{\mathbb{D}}(\infty) \cup \omega^{-1}_{\mathbb{D}}(\infty)$. We choose a point Q in $\partial D \cap (\mathbb{D} \setminus Sing\xi)$. Since $Q \in \partial D$ the trajectory $\Gamma^B_\xi[Q]$ is contained in $\overline{\mathbb{D}}$ and there exists $Q' \in \Gamma^B_\xi[Q] \cap \partial\mathbb{D}$. We have that $\alpha_B(\Gamma^B_{\xi,-}[Q']) = P$; hence there exists $M_U \in \Gamma^B_{\xi,-}[Q'] \cap (U \setminus \{P\})$. Then we have $M_U \in \partial D \cap (U \setminus \{P\}) \neq \emptyset$. Moreover, for all neighborhood U of P the set $\omega^{-1}_{\mathbb{D}}(P)$ does not contain U because $\omega_{\mathbb{D}}(M_U) = \infty$. $\square$

COROLLARY 2.1.2.

$$Sing\xi \subset \mathcal{H}^{\mathbb{D}}_\xi$$

PROOF. Let $P \in Sing\xi$. Suppose $P \notin \mathcal{H}^{\mathbb{D}}_\xi$. We deduce that (α,ω) is constant in some pointed neighborhood of P. But then $\omega^{-1}_{\mathbb{D}}(P)$ contains a neighborhood of P, that is a contradiction. $\square$

LEMMA 2.1.5. *The mapping ω is constant over any positively invariant domain $D \subset \mathbb{D}$ and $\omega(D)$ is a singleton contained in ∂D. In particular D does not contain any equilibrium point.*

PROOF. The mapping $\omega_{\mathbb{D}} : D \setminus Sing\xi \to \overline{D} \cap Sing\xi$ is locally constant since the singular set is composed by flower points. As a consequence $\omega_{\mathbb{D}}(D \setminus Sing\xi)$ contains a unique point $P \in \overline{D}$. If the point P belongs to D then $D \subset \omega_{\mathbb{D}}^{-1}(P)$, that contradicts lemma 2.1.4. $\square$

LEMMA 2.1.6. *Let C be a connected component of $\mathbb{D} \setminus \mathcal{H}_\xi^{\mathbb{D}}$ contained in the set $(\alpha,\omega)_{\mathbb{D}}^{-1}(Sing\xi \times \{\infty\})$. Then $\partial C \setminus (end_\xi^+(C) \cup Sing\xi)$ has two connected components.*

PROOF. Consider the same notations than in lemma 2.1.3. Let $P = \alpha_{\mathbb{D}}(C)$. We have

$$\partial C \setminus (end_\xi^+(C) \cup Sing\xi) = \gamma_1 \cup \gamma_2.$$

Since γ_1 and γ_2 are connected it is enough to prove that $\gamma_1 \neq \gamma_2$. Suppose $\gamma_1 = \gamma_2$; we choose an open neighborhood $V \subset \mathbb{D}$ of P such that $V \setminus (\gamma_1 \cup \{P\})$ and V are connected. Since $[V \setminus (\gamma_1 \cup \{P\})] \cap C \neq \emptyset$ and $[V \setminus (\gamma_1 \cup \{P\})] \cap \partial C = \emptyset$ then $V \setminus (\gamma_1 \cup \{P\}) \subset C$. Therefore, we have $(\alpha_{\mathbb{D}},\omega_{\mathbb{D}})[V \setminus (\gamma_1 \cup \{P\})] = (P,\infty)$. If V is a small neighborhood of P we also obtain that $\alpha_{\mathbb{D}}^{-1}(P)$ contains $V \cap (\gamma_1 \cup \{P\})$ and then the whole V; that contradicts lemma 2.1.4. $\square$

For $C \subset (\alpha,\omega)_{\mathbb{D}}^{-1}(Sing\xi \times \{\infty\})$ the picture 4 is a faithful representation of the dynamics. We describe next the dynamics in the connected components of $\mathbb{D} \setminus \mathcal{H}_\xi^{\mathbb{D}}$

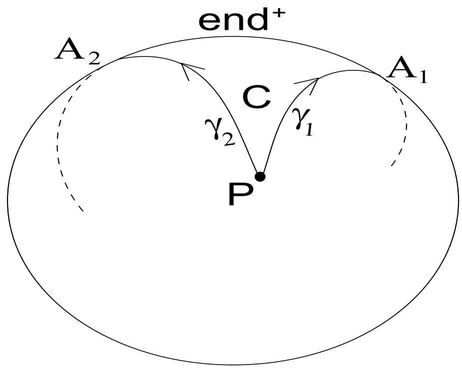

FIGURE 4. Component of $(\alpha,\omega)_{\mathbb{D}}^{-1}(P,\infty)$

contained in $(\alpha,\omega)_{\mathbb{D}}^{-1}(Sing\xi \times Sing\xi)$.

LEMMA 2.1.7. *Let $P,Q \in Sing\xi$. Suppose that $(\alpha,\omega)_{\mathbb{D}}^{-1}(P,Q) \neq \emptyset$ and $P \neq Q$. Then $\partial((\alpha,\omega)_{\mathbb{D}}^{-1}(P,Q))$ is a closed simple curve of the form $\gamma = \gamma_1 \cup \gamma_2 \cup \{P\} \cup \{Q\}$ where γ_1 and γ_2 are different trajectories of ξ in $\overline{\mathbb{D}}$. Moreover $(\alpha,\omega)_{\mathbb{D}}^{-1}(P,Q)$ is the bounded component of $\mathbb{R}^2 \setminus \gamma$.*

PROOF. Let $D = (\alpha,\omega)_{\mathbb{D}}^{-1}(P,Q)$. Since $(\alpha,\omega)_{\mathbb{D}}^{-1}(P,P) \neq \emptyset$ by lemma 2.1.4 then there exists A_1 in $[\partial D \cap \mathbb{D}] \setminus Sing\xi$. We define $\gamma_1 = \Gamma_\xi^{\overline{\mathbb{D}}}[A_1]$. Since $\gamma_1 \subset \partial D$ there exists a convex tangent point $Q_1 \in \gamma_1 \cap \partial\mathbb{D}$.

We claim that $\partial D \neq \gamma_1 \cup \{P\} \cup \{Q\}$. Otherwise we proceed as in lemma 2.1.6 to obtain that $\alpha_{\mathbb{D}}^{-1}(P)$ is a neighborhood of P; that is impossible by lemma 2.1.4.

There exists A_2 in $(\partial D \cap \mathbb{D}) \setminus (\gamma_1 \cup \{P\} \cup \{Q\})$. We define $\gamma_2 = \Gamma_\xi^{\overline{\mathbb{D}}}[A_2]$. There exists at least a convex tangent point $Q_2 \in \gamma_2 \cap \partial\mathbb{D}$.

The curve $\gamma = \gamma_1 \cup \gamma_2 \cup \{P\} \cup \{Q\}$ is a simple closed curve defining a bounded region B. The region B is invariant by ξ, hence α and ω are constant on B. Since $\alpha_{\overline{\mathbb{D}}}(\gamma_1 \cup \gamma_2) = \{P\}$ and $\omega_{\overline{\mathbb{D}}}(\gamma_1 \cup \gamma_2) = \{Q\}$ then $B \subset D$. We have that $\overline{B} \sim \overline{\mathbb{D}}$ because of Jordan's curve theorem. We can choose a curve $I[0,1] \subset \overline{D}$ such that $I[0,1]$ is transversal to ξ, $I(0) = Q_1$ and $I(1) = Q_2$. Since P and Q are in different connected components of $\overline{\mathbb{D}} \setminus I[0,1]$ then $D = B$. $\square$

The dynamics in $(\alpha,\omega)_{\mathbb{D}}^{-1}(P,Q)$ $(P \neq Q)$ is represented in picture 5.

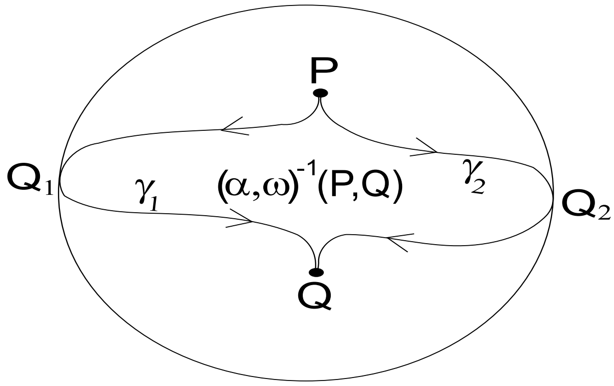

FIGURE 5. $(\alpha,\omega)_{\mathbb{D}}^{-1}(P,Q)$ for $P \neq Q$

LEMMA 2.1.8. *Let $P \in Sing\xi$ and let C be a connected component of the set $(\alpha,\omega)_{\mathbb{D}}^{-1}(P,P) \setminus \{P\}$. Then ∂C is a simple closed curve $\{P\} \cup \gamma'$ where γ' is a trajectory of ξ in $\overline{\mathbb{D}}$. Moreover, C is the bounded component of $\mathbb{R}^2 \setminus (\{P\} \cup \gamma')$.*

PROOF. By lemma 2.1.4 there exists $Q \in (\partial C \cap \mathbb{D}) \setminus Sing\xi$. Let $\gamma' = \Gamma_\xi^{\overline{\mathbb{D}}}[Q]$. We have $(\alpha,\omega)_{\overline{\mathbb{D}}}(\gamma') = (P,P)$, as a consequence $\gamma = \gamma' \cup \{P\}$ is a simple closed curve. Let B be the bounded component of $\mathbb{R}^2 \setminus \gamma$. By lemma 2.1.5 we have $(\alpha,\omega)(B) \in \partial B \times \partial B$ and then $(\alpha,\omega)(B) = (P,P)$. Since $\gamma' \cap \partial\mathbb{D} \neq \emptyset$ then γ is a union of critical trajectories. Therefore B is a connected component of $\mathbb{D} \setminus [\mathcal{H}_\xi^{\mathbb{D}} \cup Sing\xi]$. For a small neighborhood V of Q the set $V \setminus C$ is contained in $\alpha_{\mathbb{D}}^{-1}(\infty) \cup \omega_{\mathbb{D}}^{-1}(\infty)$; we obtain that $C = B$. $\square$

Last lemma is not enough to describe the dynamics in C. We need a little bit more.

LEMMA 2.1.9. *In the setting of the previous lemma let $M, Q \in \overline{C} \setminus \{P\}$. There exists a continuous mapping $F : [0,1] \times [0,1] \to \overline{C}$ such that*

- $F(\{0\} \times [0,1]) = F(\{1\} \times [0,1]) = P$
- $F((0,1) \times [0,1]) \subset \overline{C} \setminus \{P\}$ *and* $F_{|(0,1)\times[0,1]}$ *is injective*
- $F((0,1) \times \{t\})$ *is a trajectory of* ξ *in* $\overline{\mathbb{D}}$ *for all* $t \in [0,1]$
- $F((0,1) \times \{0\}) = \Gamma_\xi^{\overline{\mathbb{D}}}[M]$ *and* $F((0,1) \times \{1\}) = \Gamma_\xi^{\overline{\mathbb{D}}}[Q]$

PROOF. It is enough to prove the lemma for Q in a small neighborhood of M since $\overline{C} \setminus \{P\}$ is connected. Let $I(t) \subset \overline{C}$ ($t \in [0,1]$) be a transversal to ξ passing through M. We define $F(s,t) = \Gamma_\xi^{\overline{\mathbb{D}}}[I(t)](s)$. We claim that F is continuous at the points of type (∞, t) and $(-\infty, t)$. For instance, for a point (∞, t_0) we consider any neighborhood U of P such that $F(0,t_0) \notin U$. By remark 2.1.2 there exists $s_0 > 0$ such that $F(s_0,t_0) \in U_+$. Therefore $F(s,t) \in U_+$ for all $s \geq s_0$ and all t in a neighborhood of t_0. We deduce that F is continuous. We parameterize $[-\infty, \infty]$ by the interval $[0,1]$; in this way we can consider F as defined over $[0,1] \times [0,1]$. □

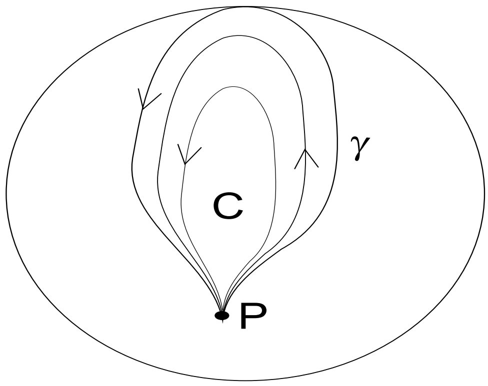

FIGURE 6. Dynamics in a component C of $(\alpha,\omega)_{\mathbb{D}}^{-1}(P,P)$

Because of the last lemma the picture 6 represents the dynamics in $\overline{C}$ for $C \subset (\alpha,\omega)_{\mathbb{D}}^{-1}(P,P)$.

2.1.3. Tangent singular diagram. Let ξ and ξ' be flower type vector fields; we say that $\mathcal{H}_\xi^{\mathcal{D}} \sim \mathcal{H}_{\xi'}^{\mathcal{D}}$ if there exists an oriented homeomorphism $h : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ such that $h(\mathcal{H}_\xi^{\mathcal{D}}) = \mathcal{H}_{\xi'}^{\mathcal{D}}$.

We enumerate the points T_ξ^0, T_ξ^1, …, $T_\xi^{N_T(\xi)-1}$, $T_\xi^{N_T(\xi)} = T_0$ contained in $T_\xi^{\mathbb{D}}$. The set of indexes is $\mathbb{Z}/(N_T\mathbb{Z})$. The order is induced by turning in $\mathbb{S}^1$ in counter clock wise sense. The enumeration is unique up to a translation $j \mapsto j + C$ for some $C \in \mathbb{Z}/(N_T\mathbb{Z})$. We also enumerate the points S_ξ^1, …, S_ξ^l in $Sing\xi$. We consider a list $L_\xi^{\mathbb{D}}$ of sets of types

$$\{S_\xi^a, T_\xi^b\},\ \{T_\xi^a, T_\xi^b\},\ \{T_\xi^{a,a+1}, T_\xi^b\}.$$

The set $\{C_\xi^a, D_\xi^b\}$ ($C, D \in \{T, S\}$) belongs to $L_\xi^{\mathbb{D}}$ if there is a critical trajectory either from C_ξ^a to D_ξ^b or from D_ξ^b to C_ξ^a. The set $\{T_\xi^{a,a+1}, T_\xi^b\}$ belongs to $L_\xi^{\mathbb{D}}$ if either the negative or the positive critical trajectory passing through T_ξ^b contains a point in the open arc $(T_\xi^a, T_\xi^{a+1}) \subset \partial\mathbb{D}$. It is clear that every point T_ξ^b belongs to at least one couple in $L_\xi^{\mathbb{D}}$; we also have that every S_ξ^a is contained in a couple of $L_\xi^{\mathbb{D}}$ because of corollary 2.1.2.

By definition $L_\xi^{\mathbb{D}} \sim L_{\xi'}^{\mathbb{D}}$ if

- $N_T(\xi) = N_T(\xi')$ and $\sharp(Sing\xi) = \sharp(Sing\xi')$
- There exist $c \in \mathbb{Z}/(j\mathbb{Z})$ and $\sigma \in S_{\sharp Sing\xi}$ such that
  - $\{S_\xi^a, T_\xi^b\} \in L_\xi^{\mathbb{D}} \Leftrightarrow \{S_\xi^{\sigma(a)}, T_\xi^{b+c}\} \in L_{\xi'}^{\mathbb{D}}$
  - $\{T_\xi^a, T_\xi^b\} \in L_\xi^{\mathbb{D}} \Leftrightarrow \{T_\xi^{a+c}, T_\xi^{b+c}\} \in L_{\xi'}^{\mathbb{D}}$
  - $\{T_\xi^{a,a+1}, T_\xi^b\} \in L_\xi^{\mathbb{D}} \Leftrightarrow \{T_\xi^{a+c,a+c+1}, T_\xi^{b+c}\} \in L_{\xi'}^{\mathbb{D}}$

We define $IC_\xi^{\mathbb{D}} = [L_\xi^{\mathbb{D}}]$. We have

LEMMA 2.1.10.

$$\mathcal{H}_\xi^{\mathbb{D}} \sim \mathcal{H}_{\xi'}^{\mathbb{D}} \Leftrightarrow IC_\xi^{\mathbb{D}} = IC_{\xi'}^{\mathbb{D}}$$

PROOF. Implication ($\Rightarrow$). Suppose $h : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ is an oriented homeomorphism conjugating $\mathcal{H}_\xi^{\mathbb{D}}$ and $\mathcal{H}_{\xi'}^{\mathbb{D}}$. The homeomorphism h preserves the critical trajectories; as a consequence h also preserves the convex tangent points and the singular points (corollary 2.1.2).

We will denote $(\alpha_\xi, \omega_\xi)_{\mathbb{D}}$ and $(\alpha_{\xi'}, \omega_{\xi'})_{\mathbb{D}}$ the (α, ω) mappings for ξ and ξ' respectively. A concave tangent point Q is in the closure of a unique component C^Q of $\mathbb{D} \setminus \mathcal{H}^{\mathbb{D}}$ contained in $(\alpha, \omega)_{\mathbb{D}}^{-1}(\infty, \infty)$. Let C be a connected component of $\mathbb{D} \setminus \mathcal{H}_\xi^{\mathbb{D}}$ such that $(\alpha_\xi, \omega_\xi)_{\mathbb{D}}(C) = (\infty, \infty)$. The set of tangent concave points in $\overline{C}$ coincides with $\overline{end_\xi^+(C)} \cap \overline{end_\xi^-(C)}$. We define $l_\xi(C)$ as the number of connected components of $\partial C \cap \mathbb{D}$. The number of tangent concave points in $\overline{C}$ is equal to $2 - l_\xi(C)$. Since $l_\xi(C) = l_{\xi'}(h(C))$ then the number of tangent concave points in $\overline{C}$ and $\overline{h(C)}$ are the same. Therefore, there exists a bijection ι from $T_{\xi,-}^{\mathbb{D}}$ onto $T_{\xi',-}^{\mathbb{D}}$ such that $\iota(Q) \in \overline{h(C_\xi^Q)}$ for all $Q \in T_{\xi,-}^{\mathbb{D}}$. Consider the mapping $\theta : \mathcal{H}_\xi^{\mathbb{D}} \cup T_{\xi,-}^{\mathbb{D}} \to \mathcal{H}_{\xi'}^{\mathbb{D}} \cup T_{\xi',-}^{\mathbb{D}}$ such that $\theta_{|\mathcal{H}_\xi^{\mathbb{D}}} = h_{|\mathcal{H}_\xi^{\mathbb{D}}}$ and $\theta_{|T_{\xi,-}^{\mathbb{D}}} = \iota$. Thus θ conjugates $L_\xi^{\mathbb{D}}$ and $L_{\xi'}^{\mathbb{D}}$.

Implication ($\Leftarrow$). Let $j \to j + c$ and σ the permutations conjugating $L_\xi^{\mathbb{D}}$ and $L_{\xi'}^{\mathbb{D}}$. We define $h(T_\xi^a) = T_{\xi'}^{a+c}$ and $h(S_\xi^b) = S_\xi^{\sigma(b)}$ for all a in $\mathbb{Z}/(N_T\mathbb{Z})$ and $1 \leq b \leq \sharp Sing\xi$. We can extend h to the union of the critical trajectories. Consider a connected component C of $\mathbb{D} \setminus \mathcal{H}_\xi^{\mathbb{D}}$. We denote by $\lambda(C)$ the connected component of $\mathbb{D} \setminus \mathcal{H}_{\xi'}^{\mathbb{D}}$ such that

$$h(\partial C \setminus [end_\xi^+(C) \cup end_\xi^-(C)] = \partial\lambda(C) \setminus [end_{\xi'}^+(\lambda(C)) \cup end_{\xi'}^-(\lambda(C))].$$

The mapping λ induces a bijection from the connected components of $\mathbb{D} \setminus \mathcal{H}_\xi^{\mathbb{D}}$ onto the connected components of $\mathbb{D} \setminus \mathcal{H}_{\xi'}^{\mathbb{D}}$. It is enough to prove that we can extend h to $\overline{C}$ such that

$$h : \overline{C} \to \overline{\lambda(C)}$$

is a homeomorphism. It is straightforward since $C \sim \mathbb{D}$ and $\overline{C} \sim \overline{\mathbb{D}}$. □

By definition, two flower type vector fields ξ and ξ' are topologically equivalent if there exists an oriented homeomorphism $h : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ such that h maps orbits of ξ in orbits of ξ'.

PROPOSITION 2.1.2.

$$\xi \overset{\text{top}}{\sim} \xi' \Leftrightarrow \mathcal{H}_\xi^{\mathbb{D}} \sim \mathcal{H}_{\xi'}^{\mathbb{D}}$$

PROOF. The implication ($\Rightarrow$) is obvious.

Implication ($\Leftarrow$). We use again the notations in lemma 2.1.10. Let $h : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ an oriented homeomorphism such that $h(\mathcal{H}_\xi^{\mathbb{D}}) = \mathcal{H}_{\xi'}^{\mathbb{D}}$. By lemma 2.1.10 we can suppose that $h(T^{\mathbb{D}}_{\xi,-}) = T^{\mathbb{D}}_{\xi',-}$. Let θ be the mapping defined in $\mathcal{H}^{\mathbb{D}}_{\xi} \cup T^{\mathbb{D}}_{\xi,-} \cup \{\infty\}$ such that $\theta = h$ in $\mathcal{H}^{\mathbb{D}}_{\xi} \cup T^{\mathbb{D}}_{\xi,-}$ and $\theta(\infty) = \infty$.

Let C be any connected component of $\mathbb{D} \setminus \mathcal{H}^{\mathbb{D}}_{\xi}$. It is enough to prove that θ can be extended to a topological equivalence from $\overline{C}$ onto $\overline{\lambda(C)}$. We described the dynamics in both $\overline{C}$ and $\overline{\lambda(C)}$ and proved to be the same; that is a consequence of

$$\theta(\{\alpha_{\xi,\mathbb{D}}(C), \omega_{\xi,\mathbb{D}}(C)\}) = \{\alpha_{\xi',\mathbb{D}}(\lambda(C)), \omega_{\xi',\mathbb{D}}(\lambda(C))\}$$

and lemmas 2.1.3, 2.1.6, 2.1.7, 2.1.8 and 2.1.9. Therefore, it is straightforward to extend θ to $\overline{C}$. We obtain an oriented homeomorphism $\theta : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$, it is a topological equivalence by construction. □

2.1.4. The singular graph. We can associate an oriented graph $\mathcal{G}^{\mathbb{D}}_{\xi}$ to ξ. The vertexes of the graph are the points in $Sing\xi$. There is an edge $P \to Q$ going from $P \in Sing\xi$ to $Q \in Sing\xi$ $(P \neq Q)$ if $(\alpha,\omega)^{-1}_{\mathbb{D}}(P,Q) \neq \emptyset$.

For an oriented graph $\mathcal{G}$ we define $\mathcal{N}\mathcal{G}$ the non-oriented graph obtained from $\mathcal{G}$ by removing orientation of the edges.

LEMMA 2.1.11. *The graphs $\mathcal{G}^{\mathbb{D}}_{\xi}$ and $\mathcal{N}\mathcal{G}^{\mathbb{D}}_{\xi}$ are both acyclic.*

PROOF. Consider and edge $P \to Q$. The points P and Q belong to different connected components of $\mathbb{D} \setminus (\alpha,\omega)^{-1}_{\mathbb{D}}(P,Q)$ (proof of lemma 2.1.7). As a consequence the edge $P \to Q$ can not be contained in a cycle neither for $\mathcal{G}^{\mathbb{D}}_{\xi}$ nor for $\mathcal{N}\mathcal{G}^{\mathbb{D}}_{\xi}$. □

We will say that $P, Q \in Sing\xi$ are *separated* by ξ if there exists $M \in \mathbb{D} \setminus Sing\xi$ such that P and Q are in different connected components of $\mathbb{D} \setminus \Gamma^{\mathbb{D}}_{\xi}[M]$. Clearly P and Q can not be separated if they belong to the same connected component of $\mathcal{G}^{\mathbb{D}}_{\xi}$. It is a sharper idea to deal with separation of connected components of $\mathcal{G}^{\mathbb{D}}_{\xi}$ instead of separation of singular points.

We define the set of *critical tangent cords* $TC^{\mathbb{D}}_{\xi}$ as the union of all the critical trajectories in $L^{\mathbb{D}}_{\xi}$ not containing singular points.

PROPOSITION 2.1.3. *Two different connected components of $\mathcal{G}^{\mathbb{D}}_{\xi}$ are always separated by ξ. More precisely, they are separated by a critical tangent cord.*

PROOF. It is enough to prove that no connected component of $\mathbb{D} \setminus TC^{\mathbb{D}}_{\xi}$ contains more than one connected component of $\mathcal{G}^{\mathbb{D}}_{\xi}$. Suppose it is false. Let C be a connected component of $\mathbb{D} \setminus TC^{\mathbb{D}}_{\xi}$ containing $l > 1$ connected components $G_1, \ldots,$ G_l of the graph $\mathcal{G}^{\mathbb{D}}_{\xi}$. For $1 \leq j \leq l$ we denote by $Sing(G_j)$ the set of singular points (also vertexes) of G_j. We define

$$V_j = [\alpha^{-1}_{\mathbb{D}}(Sing(G_j)) \cup \omega^{-1}_{\mathbb{D}}(Sing(G_j))] \setminus Sing\xi$$

for all $1 \leq j \leq l$. The set $V_j \subset C$ $(1 \leq j \leq l)$ is open since ξ is a flower type vector field, moreover it is not empty by lemma 2.1.1. For all $1 \leq j < k \leq l$ we have $V_j \cap V_k = \emptyset$, otherwise the restriction of $\mathcal{G}^{\mathbb{D}}_{\xi}$ to $Sing(G_j) \cup Sing(G_k)$ is a connected graph. We define the set $ES \subset C \setminus Sing\xi$ such that $Q \in ES$ if $(\alpha,\omega)_{\mathbb{D}}(Q) = (\infty,\infty)$ and the two points in $\overline{\Gamma^{\mathbb{D}}_{\xi}[Q]} \cap \partial\mathbb{D}$ are not convex tangent points. The set ES is open and it satisfies $ES \cap V_j = \emptyset$ for all $1 \leq j \leq l$. Let $M \in (C \setminus Sing\xi) \setminus (V_1 \cup \ldots \cup V_l)$;

since $Sing\xi \cap C = \cup_{1\leq j\leq l} Sing(G_j)$ then $(\alpha,\omega)_{\mathbb{D}}(M) = (\infty,\infty)$. The point M belongs to ES because otherwise $M \in TC_{\xi}^{\mathbb{D}} \subset \mathbb{R}^2 \setminus C$. As a consequence

$$C \setminus Sing\xi = V_1 \cup (V_2 \cup \ldots \cup V_l \cup ES)$$

is a disjoint union of non-empty open sets. Since $C \setminus Sing\xi$ is connected we obtain a contradiction. □

We consider that two different critical tangent cords are equivalent if they induce the same partition in the singular points. Let $TC_{\xi,\sim}^{\mathbb{D}}$ a subset of $TC_{\xi}^{\mathbb{D}}$ containing one element for each equivalence class.

Let G be a connected component of $\mathcal{G}_{\xi}^{\mathbb{D}}$. Then $Sing(G)$ is contained in a unique connected component C of $\mathbb{D} \setminus \cup_{\eta\in TC_{\xi,\sim}^{\mathbb{D}}} \eta$. We define $\Xi(G)$ the set of elements of $TC_{\xi,\sim}^{\mathbb{D}}$ contained in $\overline{C}$. We denote by $(E_{\eta}^{G,1}, E_{\eta}^{G,2})$ the partition of the singular points induced by a $\eta \in \Xi(G)$; we choose $E_{\eta}^{G,1}$ to satisfy $Sing(G) \subset E_{\eta}^{G,1}$.

LEMMA 2.1.12. *Let G be a connected component of $\mathcal{G}_{\xi}^{\mathbb{D}}$. For η,η' in $\Xi(G)$ such that $\eta \neq \eta'$ we have that $E_{\eta}^{G,2} \cap E_{\eta'}^{G,2} = \emptyset$.*

PROOF. Since $\eta' \cap \mathbb{D}$ and $Sing(G)$ are in the same connected component of $\mathbb{D} \setminus \eta$ then $E_{\eta'}^{G,2} \subset E_{\eta}^{G,1}$ and we are done. □

PROPOSITION 2.1.4. *$L_{\xi}^{\mathbb{D}}$ determines completely $\mathcal{NG}_{\xi}^{\mathbb{D}}$.*

PROOF. For $\{T_{\xi}^a, T_{\xi}^b\} \in L_{\xi}^{\mathbb{D}}$ we denote by $\beta^{a,b}$ the critical trajectory joining T_{ξ}^a and T_{ξ}^b. For $\{P, T_{\xi}^a\} \in L_{\xi}^{\mathbb{D}}$ and $P \in Sing\xi$ we denote by β_P^a the critical trajectory joining P and T_{ξ}^a. Let $P,Q \in Sing\xi$ such that $P \neq Q$. We claim that $P \leftrightarrow Q$ belongs to $\mathcal{NG}_{\xi}^{\mathbb{D}}$ if there exists $k \geq 1$ and a sequence

$$\{P, T_{\xi}^{a_1}\}\ ,\ \{T_{\xi}^{a_1}, T_{\xi}^{a_2}\}\ ,\ \ldots,\ \{T_{\xi}^{a_{k-1}}, T_{\xi}^{a_k}\}\ ,\ \{T_{\xi}^{a_k}, Q\}$$

contained in $L_{\xi}^{\mathbb{D}}$ such that P and Q are in the same connected component of $\mathbb{D} \setminus \beta^{a_j,a_{j+1}}$ for all $1 \leq j < k$.

Suppose $P \to Q$ belongs to $\mathcal{G}_{\xi}^{\mathbb{D}}$. Consider a trajectory γ_1 of ξ in $\overline{\mathbb{D}}$ contained in the boundary of $(\alpha,\omega)_{\mathbb{D}}^{-1}(P,Q)$. By the proof of lemma 2.1.7 the curve $\{P\} \cup \gamma_1 \cup \{Q\}$ is a union of critical trajectories. Therefore, there exist $k \geq 1$ and a sequence

$$\{P, T_{\xi}^{a_1}\}\ ,\ \{T_{\xi}^{a_1}, T_{\xi}^{a_2}\}\ ,\ \ldots,\ \{T_{\xi}^{a_{k-1}}, T_{\xi}^{a_k}\}\ ,\ \{T_{\xi}^{a_k}, Q\}$$

contained in $L_{\xi}^{\mathbb{D}}$. Moreover P and Q belong to the same connected component of $\mathbb{D} \setminus \beta^{a_j,a_{j+1}}$ for $1 \leq j < k$, otherwise $(\alpha,\omega)_{\mathbb{D}}^{-1}(P,Q) = \emptyset$. The proof for $Q \to P$ in $\mathcal{G}_{\xi}^{\mathbb{D}}$ is analogous.

Suppose we have a sequence satisfying the aforementioned properties but $(P \leftrightarrow Q) \notin \mathcal{NG}_{\xi}^{\mathbb{D}}$. Then P and Q are in different connected components of $\mathcal{G}_{\xi}^{\mathbb{D}}$, otherwise $\mathcal{NG}_{\xi'}^{\mathbb{D}}$ has a cycle for $\xi' = (x/(1+\eta), y/(1+\eta))_* \xi$ and $\eta > 0$ small enough. By proposition 2.1.3 there exists $M \in \mathbb{D} \setminus Sing\xi$ such that $\Gamma_{\xi}^{\mathbb{D}}[M]$ separates P and Q. We claim that

$$(\beta_P^{a_1} \setminus \{P\}) \cup_{1\leq j<k} \beta^{a_j,a_{j+1}} \cup (\beta_Q^{a_k} \setminus \{Q\}) = \Gamma_{\xi}^{\overline{\mathbb{D}}}[M].$$

The right hand side term separates P and Q and then the intersection of both terms is not empty; since both sides are trajectories then they are equal. Hence $\Gamma_{\xi}^{\mathbb{D}}[M]$

coincides with $\beta^{a_j,a_{j+1}}$ for some $1 \leq j < k$; that is a contradiction since $\Gamma_\xi^{\mathbb{D}}[M]$ separates P and Q. $\square$

2.2. Families of vector fields without small divisors

We will define throughout this section the objects that we are going to study. We will consider families of flower type vector fields. The flower singularities we will deal with are parabolic.

2.2.1. Parabolic germs of vector fields. Let $Y \in \mathcal{H}(\mathbb{C}, 0)$ such that $\nu_Y \geq 2$. The vector field Y can be expressed in the form

$$Y = (a_{\nu_Y} x^{\nu_Y} + a_{\nu_Y+1} x^{\nu_Y+1} + \ldots)\frac{\partial}{\partial x}$$

where $a_{\nu_Y} \neq 0$. We define the set $\Theta^-(Y) \subset \mathbb{S}^1$ of $\nu_Y - 1$ roots of $|a_{\nu_Y}|/a_{\nu_Y}$. We define $\Theta^+(Y) = e^{(\pi i)/(\nu_Y-1)}\Theta^-(Y)$. The set $\Theta^-(Y)$ is composed by the directions contained in $(a_{v_Y} x^{\nu_Y})/x \in \mathbb{R}^+$, in other words $[(a_{v_Y} x^{\nu_Y})/x \in \mathbb{R}^+] \equiv \Theta^-(Y)\mathbb{R}^+$. We expect a repulsive behavior in the neighborhood of the directions in $\Theta^-(Y)$ and an attractive one in the neighborhood of $[(a_{v_Y} x^{\nu_Y})/x \in \mathbb{R}^-] \equiv \Theta^+(Y)\mathbb{R}^+$. We define $\Theta(Y) = \Theta^+(Y) \cup \Theta^-(Y)$. The set $\Theta(Y)$ is ordered in a natural way; for every $l \in \Theta(Y)$ there exists a next one $NE(l) = le^{(\pi i)/(\nu_Y-1)}$ and a previous one $PR(l) = le^{-(\pi i)/(\nu_Y-1)}$. Moreover, if $l \in \Theta^+(Y)$ then $NE(l)$ and $PR(l)$ belong to $\Theta^-(Y)$ whereas if $l \in \Theta^-(Y)$ then $NE(l)$ and $PR(l)$ belong to $\Theta^+(Y)$.

Let $\pi : (\mathbb{R}^+ \cup \{0\}) \times \mathbb{S}^1 \to \mathbb{R}^2$ be the mapping defined by $\pi(r, \lambda) = r\lambda$. This is the real blow-up of the origin in $\mathbb{R}^2$. We say that a set $E \subset \mathbb{R}^2$ adheres to a direction λ if $(0, \lambda) \in \overline{\pi^{-1}(E \setminus \{0\})}$. Consider a vector field ξ and a point $Q \notin Sing\xi$ such that $\omega(Q)$ is a point. We define $l_{\xi,+}[Q]$ the set of directions at $\omega(Q)$ such that $\Gamma_{\xi,+}[Q]$ adheres at. In an analogous way we define $l_{\xi,-}[Q]$.

Proposition 2.2.1. *(Leau* [**Lea97**]*, see also* [**Cam78**]*) Let $Y \in \mathcal{H}(\mathbb{C}, 0)$. Suppose that $\nu_Y \geq 2$. For any neighborhood V of 0 there exists a family of open non-empty connected subsets $\{V_l\}_{l\in\Theta(Y)}$ of $V \setminus \{0\}$ such that*

1. *$W \stackrel{def}{=} (\cup_{l\in\Theta(Y)} V_l) \cup \{0\}$ is a neighborhood of 0.*
2. *For $l \in \Theta^+(Y)$ the domain V_l is positively invariant by $Re(Y)$, moreover $\omega_{Re(Y)}(V_l) = \{0\}$.*
3. *For $l \in \Theta^-(Y)$ the domain V_l is negatively invariant by $Re(Y)$, moreover $\alpha_{Re(Y)}(V_l) = \{0\}$.*
4. *For $l \in \Theta^+(Y)$ and $Q \in V_l$ we have $l_{Re(Y),+}^{V_l}[Q] = \{l\}$.*
5. *For $l \in \Theta^-(Y)$ and $Q \in V_l$ we have $l_{Re(Y),-}^{V_l}[Q] = \{l\}$.*
6. *Let $Q \in W \setminus \{0\}$; if $\omega_{Re(Y),W}(Q) = \{0\}$ then*
$$\Gamma_{Re(Y),+}^W[Q] \cap (\cup_{l\in\Theta^+(Y)} V_l) \neq \emptyset.$$
7. *Let $Q \in W \setminus \{0\}$; if $\alpha_{Re(Y),W}(Q) = \{0\}$ then*
$$\Gamma_{Re(Y),-}^W[Q] \cap (\cup_{l\in\Theta^-(Y)} V_l) \neq \emptyset.$$
8. *$V_l \cap V_k \neq \emptyset$ if and only if $k \in \{NE(l), PR(l)\}$.*

In particular 0 is a flower type singular point for $Re(Y)$.

REMARK 2.2.1. *Consider* $Y = (a_{\nu_Y}(y)x^{\nu_Y} + a_{\nu_Y+1}(y)x^{\nu_Y+1} + \ldots)\partial/\partial x$ *where* $a_{\nu_Y}(0) \neq 0$. *There exists a family of open connected sets* $\{V_l\}_{l\in\Theta}$ *such that* $\{V_l \cap [y = y_0]\}_{l\in\Theta}$ *satisfies the conditions in proposition 2.2.1 for all* y_0 *in some neighborhood of* 0.

2.2.2. Holomorphic families. We will consider germs of vector field of the form $X = f\partial/\partial x$. We will ask f for fulfilling the no small divisors (NSD) conditions:

- $f(0,0) = 0$ and $f \not\equiv 0$
- The decomposition $f_1^{n_1} \ldots f_p^{n_p} y^m$ of f in irreducible factors satisfies that $m \geq 0$ and $n_j \geq 2$ for all $1 \leq j \leq p$.

The first condition implies that $f = 0$ is an analytic curve whereas the second one guarantees the absence of small divisors.

We define the sets

$$U_\epsilon = \{(x,y) \ : \ |x| < \epsilon\} \text{ and } U_{\epsilon,\delta} = \{(x,y) \ : \ |x| < \epsilon \text{ and } |y| < \delta\}.$$

Our results will be valid for $\epsilon > 0$ and $\delta > 0$ small enough. Many times it will be implicit that the results are true up to shrink the domain.

Let $U_{\epsilon,\delta}$ a domain such that f is defined in a neighborhood of $\overline{U_{\epsilon,\delta}}$. We also request that $[f_j = 0] \setminus Sing(f_j = 0)$ is connected in $U_{\epsilon,\delta}$ and $[([x] = \epsilon) \times (|y| \leq \delta)] \cap [f_j = 0] = \emptyset$ for all $1 \leq j \leq p$. We define $\xi(X, y_0, \epsilon)$ as the restriction of the real analytic vector field $Re(X)$ to $[y = y_0] \cap [x < \epsilon]$ for $y_0 \in B(0,\delta)$. If ϵ or y_0 are implicit we just write $\xi(X, y_0)$ or $\xi(X)$ for shortness.

Let $P = (x_0, y_0) \in SingX$ such that $[y = y_0] \not\subset SingX$. We denote by $\nu_X(P)$ the order of the vector field $X_{|y=y_0}$ at $x = x_0$. Our conditions imply that $\nu_X(P) \geq 2$ for all $P \in SingX$. As a consequence

COROLLARY 2.2.1. *Let* $y_0 \in B(0,\delta)$. *If* $y_0 \neq 0$ *then the vector field* $\xi(X, y_0, \epsilon)$ *is a flower type vector field. Moreover, if* $m = 0$ *then* $\xi(X, 0, \epsilon)$ *is also a flower type vector field.*

We can describe the nature of $(\alpha,\omega)^{-1}(P,P)$ for $P \in Sing\xi(X, y_0, \epsilon)$.

LEMMA 2.2.1. *Let* $P \in Sing\xi(X, y_0, \epsilon)$ *for a flower type vector field* $\xi(X, y_0, \epsilon)$. *Then* $(\alpha,\omega)^{-1}(P,P) \setminus \{P\}$ *has exactly* $2(\nu_X(P) - 1)$ *connected components.*

PROOF. We denote $P = (x_0, y_0)$. Consider the partition $\{V_l\}_{l\in\Theta}$ associated to $\xi = \xi(X, y_0, \epsilon)$ at P (proposition 2.2.1). We choose a point x_l in V_l for all $l \in \Theta$; we consider $\epsilon' \leq \min(\min_{l\in\Theta} |x_l - x_0|, \epsilon - x_0)$. For $l \in \Theta_+$ let γ_l be the unique connected component of $\Gamma_{\xi,+}^{|x|<\epsilon}[x_l] \cap [|x - x_0| < \epsilon']$ contained in $\omega_{\xi,|x-x_0|<\epsilon'}^{-1}(P)$. By replacing ω with α we define γ_l for $l \in \Theta_-$. The set $[|x - x_0| < \epsilon'] \setminus \cup_{l\in\Theta}\gamma_l$ has $2(\nu_X(P) - 1)$ connected components; we denote by A_l ($l \in \Theta$) the one adhering to the closed arc in $(r,\lambda) \in \{0\} \times \mathbb{S}^1$ going from l to $NE(l)$ in counter clock-wise sense. Let $\{W_l\}_{l\in\Theta}$ be a partition associated to ξ at P and whose sets are contained in $|x - x_0| < \epsilon'$. By construction $A_l \cap W_k \neq \emptyset$ if and only if $k \in \{l, NE(l)\}$. Therefore $A_l \cap (W_l \cup W_{NE(l)})$ is a neighborhood of 0 in A_l. As a consequence we obtain that $A_l \cap W_l \cap W_{NE(l)} \neq \emptyset$ because otherwise $(A_l \cap W_l) \cup (A_l \cap W_{NE(l)})$ induces a partition in non-empty disjoint open sets of every sufficiently small connected neighborhood of 0 in A_l. We choose $Q_l \in A_l \cap W_l \cap W_{NE(l)}$.

Let $l \in \Theta$; we define C_l the component of $(\alpha,\omega)_{[|x|<\epsilon]}^{-1}(P,P) \setminus \{P\}$ containing Q_l. By conditions (4) and (6) (resp. (5) and (7)) in proposition 2.2.1 the mapping l_+ (resp. l_-) is locally constant in C_l. Therefore, we have

$$l_{\xi,+}^{|x|<\epsilon}[C_l] \cup l_{\xi,-}^{|x|<\epsilon}[C_l] = \{l, NE(l)\}.$$

Moreover, the component C_l adheres to the directions in the closed arc going from l to $NE(l)$ in counter clock-wise sense by construction of Q_l. We deduce that $C_l \neq C_{l'}$ if $l, l' \in \Theta$ and $l \neq l'$.

Suppose there is a connected component C of $(\alpha, \omega)_{[|x|<\epsilon]}^{-1}(P, P) \setminus \{P\}$ such that $C \neq C_l$ for all $l \in \Theta$. On the one hand C adheres to at least two directions $l' \in \Theta_-$ and $l'' \in \Theta_+$ at P. On the other hand C is contained in a connected component of $[|x| < \epsilon] \setminus [\cup_{l \in \Theta} \overline{C_l}]$ and then C adheres at most to a direction at P. We obtain a contradiction. $\square$

CHAPTER 3

A Clockwork Orange

Let $X = f\partial/\partial x$ be a (NSD) vector field. We want to split U_ϵ in several pieces where the dynamics of $Re(X)$ is simple. We define the number $N = SingX \cap U_\epsilon \cap [y = y_0]$ for a generic $y_0 \in B(0, \delta)$. For $N = 0$ the dynamics is simple. Since there are no singular points then

$$(U_\epsilon \cap [y = s]) = (\alpha_{\xi(X(\lambda),s,\epsilon)}, \omega_{\xi(X(\lambda),s,\epsilon)})^{-1}_{|x|<\epsilon}(\infty, \infty)$$

for all $s \in B(0, \delta) \setminus \{0\}$. As a consequence $U_\epsilon \cap [y = s]$ is the only component of $[|x| < \epsilon] \setminus \mathcal{H}^{|x|<\epsilon}_{\xi(X(\lambda),s)}$ and its boundary contains no critical trajectories. Therefore, we obtain $\sharp T^{|x|<\epsilon}_{\xi(X(\lambda),s),-} \equiv 2$. The dynamics is represented in picture 1. The tricky

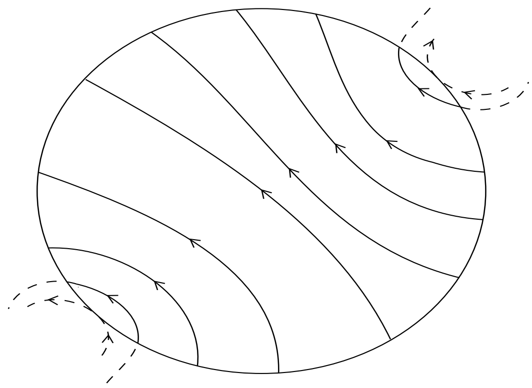

Figure 1. Dynamics of $\xi(X(\lambda), s, \epsilon)$ for $N = 0$

dynamics is attached to the case $N > 0$.

3.1. The tangent set

Since our approach is based in study $Re(X)$ in a fixed domain $U_{\epsilon,\delta}$ then it is natural to study the set where $Re(X)$ and ∂U_ϵ are tangent.

We define the sets $T^\epsilon_X(s) = T^{|x|<\epsilon}_{\xi(X,s,\epsilon)}$ and $T^\epsilon_X = \cup_{s\in B(0,\delta)} T^\epsilon_X(s)$. The set $T^\epsilon_X(0)$ is not defined if $[y = 0] \subset SingX$. Let $f_1^{n_1} \dots f_p^{n_p} y^m$ be the decomposition in irreducible factors of f. We denote $\nu(f_1^{n_1} \dots f_p^{n_p}(x, 0))$ by $\tilde{\nu}(X)$. We define the vector field $X(\lambda) = \lambda(f/y^m)\partial/\partial x$ for $\lambda \in \mathbb{S}^1$.

Proposition 3.1.1. *Let $X = f\partial/\partial x$ be a (NSD) vector field. There exists $\epsilon_0 > 0$ such that if $\epsilon < \epsilon_0$ then $\sharp T^\epsilon_{X(\lambda)}(s) = 2|\tilde{\nu}(X) - 1|$ for all s in a neighborhood of 0 and all $\lambda \in \mathbb{S}^1$.*

PROOF. The points in $T^{\epsilon}_{X(\lambda)}$ are those in U_ϵ where $x\partial/\partial x$ and $X(\lambda)$ are orthogonal, i.e.

$$T^{\epsilon}_{X(\lambda)} \equiv \left\{ \begin{array}{c} Re\left(\frac{\lambda f(x,y)}{xy^m}\right) = 0 \\ |x| = \epsilon. \end{array} \right.$$

We denote $\tilde{\nu}(X)$ by ν. We have $(f/y^m)(x,0) = a_\nu x^\nu + a_{\nu+1}x^{\nu+1} + \ldots$ where $a_\nu \neq 0$. We define

$$\begin{array}{cccc} \Lambda^{\epsilon}_X : & \partial U_\epsilon \times \mathbb{S}^1 & \to & \mathbb{S}^1 \\ & (P,\lambda) & \mapsto & \lambda\frac{(f/xy^m)(P)}{|(f/xy^m)(P)|}. \end{array}$$

We define $\arg^{\epsilon}_X = \ln(\Lambda^{\epsilon}_X)/i$; it is well defined up to a multiple of 2π. If $\epsilon > 0$ is small enough then $\Lambda_X((x,0),\lambda_0)$ is a locally injective $|\nu-1|$ to 1 function for all $\lambda_0 \in \mathbb{S}^1$. We identify ∂U_ϵ with $\mathbb{S}^1 \times (\mathbb{C},0)$. The derivative of $\arg^{\epsilon}_X((x,0),\lambda)$ with respect to $\arg(x)$ is then well-defined and it tends uniformly to $\nu - 1$ when $\epsilon \to 0$. The derivative of $\arg^{\epsilon}_X((x,\eta),\lambda)$ with respect to $\arg(x)$ tends uniformly to the derivative of $\arg^{\epsilon}_X((x,0),\lambda)$ when $\eta \to 0$. Therefore, there exists $\epsilon_0 > 0$ such that for all $\epsilon < \epsilon_0$, all $\lambda_0 \in \mathbb{S}^1$ and $|y_0| < \delta_0(\epsilon)$ the mapping $\Lambda^{\epsilon}_X((x,y_0),\lambda_0)$ is $|\nu-1|$ to 1.

Since

$$T^{\epsilon}_{X(\lambda)}(y_0) = \{(x,y_0) :\ \Lambda^{\epsilon}_X((x,y_0),\lambda) \in \{-i,i\}\}$$

then $\sharp T^{\epsilon}_{X(\lambda)}(y_0) = 2|\nu-1|$ for all $\epsilon < \epsilon_0$, $\lambda \in \mathbb{S}^1$ and $|y_0| < \delta_0(\epsilon)$. $\square$

COROLLARY 3.1.1. *Let $X = f\partial/\partial x$ be a (NSD) vector field. If $p \geq 1$ all the points in $T^{\epsilon}_{X(\lambda)}$ are convex, otherwise they are all concave.*

PROOF. A tangent point is convex (resp. concave) if the function $\arg^{\epsilon}_X$ is locally increasing (resp. decreasing) with respect to $\arg(x)$. By choosing ϵ and δ small enough we make the derivative of $\arg^{\epsilon}_X$ with respect to $\arg(x)$ sufficiently close to $\tilde{\nu}(X) - 1$. Hence, the tangent points are convex if $p \geq 1$ since the (NSD) conditions imply $\tilde{\nu}(X) \geq 2$. If $p = 0$ then $\tilde{\nu}(X) = 0$ and all the tangent points are concave. $\square$

REMARK 3.1.1. *If $m = 0$ then $X = X(1)$, otherwise the trajectories of $Re(X)$ and $Re(X(y^m/|y|^m))$ coincide. Therefore, the statements in proposition 3.1.1 and corollary 3.1.1 are valid outside of $y^m = 0$ when we replace $X(\lambda)$ with X.*

Let $\pi : (\mathbb{R}^+ \cup \{0\}) \times \mathbb{S}^1 \to \mathbb{R}^2$ be the real blow-up of the origin. We can lift $\pi(r,\lambda) = r\lambda$ to the universal covering of $(\mathbb{R}^+ \cup \{0\}) \times \mathbb{S}^1$ to obtain a mapping $\tilde{\pi} : \mathbb{R}_{\geq 0} \times \mathbb{R} \to \mathbb{R}^2$ such that $\tilde{\pi}(r,\theta) = re^{i\theta}$.

PROPOSITION 3.1.2. *The set T^{ϵ}_X is the union of $2|\tilde{\nu}(X)-1|$ real analytic sets $T^{\epsilon,1}_X(r,\theta)$, …, $T^{\epsilon,2|\tilde{\nu}(X)-1|}_X(r,\theta)$ defined in $[0,r_0] \times \mathbb{R}$ for some $r_0 > 0$.*

PROOF. Consider a local chart $x = \epsilon e^{i\zeta}$ of the manifold $|x| = \epsilon$. The mappings $\Lambda^{\epsilon}_X(r,\zeta,\theta,\lambda)$ and $\arg^{\epsilon}_X(r,\zeta,\theta,\lambda)$ are real analytic. We choose

$$\lambda = y^m/|y|^m = e^{im\theta}.$$

As a consequence we can consider Λ^{ϵ}_X and $\arg^{\epsilon}_X$ as real analytic functions of (r,ζ,θ). Moreover, the choice of λ implies that

$$T^{\epsilon}_X = (\Lambda^{\epsilon}_X)^{-1}\{-i,i\}.$$

We can make $\partial \arg_X^\epsilon/\partial\zeta$ sufficiently close to $\tilde{\nu}(X)-1$ if $r << 1$. As a consequence we can suppose $[\partial \arg_X^\epsilon/\partial\zeta](r,\zeta,\theta) \neq 0$ for all (r,ζ,θ) in $(\mathbb{R}_{\geq 0}, 0) \times \mathbb{R} \times \mathbb{R}$. The thesis of the lemma is now a consequence of the implicit function theorem. □

REMARK 3.1.2. *If $[y=0] \not\subset SingX$ then $T_X^{\epsilon,j}(y)$ is a real analytic function for $1 \leq j \leq 2|\tilde{\nu}(X)-1|$. The proof is almost the same than the proof of proposition 3.1.2; the difference being that since $\lambda \equiv 1$ is a function of y then $\arg_X^\epsilon$ can be consider as a function of (ζ, y).*

3.2. Exterior dynamics

3.2.1. Existence of the integral of the time form. This paper is based on a basic fact: the dynamics of the real part of a (NSD) vector field can be described both qualitatively and quantitatively. The qualitative study can be enriched with quantitative estimates provided by the analysis of the integrals of the time form of the (NSD) vector field.

Let Y be a holomorphic vector field Y defined over a 1 dimensional analytic variety. We can associate to Y a unique meromorphic 1-form ω_Y such that $\omega_Y(Y) = 1$; this is the time form. At any $P \in SingY$ the 1-form ω_Y has attached a residue $Res_Y(P)$. An integral ψ_Y of the time form ω_Y is a multi-valued function defined outside $SingY$ and such that

$$Y(\psi_Y) = 1 \ \Leftrightarrow \ \psi_Y = \int \omega_Y(z)dz.$$

As a consequence we have

$$\psi_Y \circ \exp(tY) = \psi_Y + t \text{ for all } t \in \mathbb{C}$$

where the last equality is defined.

We can associate to $X = f\partial/\partial x$ a 1-form ω_X in the relative cohomology of the vector field $\partial/\partial x$. The expression of ω_X in coordinates (x,y) is equal to $(1/f)dx$. We denote by ψ_X an integral of ω_X for every fiber $y = y_0$ in a neighborhood of $y_0 = 0$. For any $P \in SingX$ we denote by $Res_X(P)$ the residue of the form $(\omega_X)_{|y=y(P)}$ at P.

REMARK 3.2.1. *For any component $\beta \neq (y=0)$ of $f=0$ the function Res_X is holomorphic in $\beta \setminus \{(0,0)\}$. On the other hand, in general the function $(Res_X)_{|\beta}$ is not continuous at $(0,0)$. Let $X = x^2(x-y)^2\partial/\partial x$. For $(0,y) \in [x=0] \setminus \{(0,0)\}$ we have $Res_X(0,y) = 2/y^3$ whereas $Res_X(0,0) = 0$.*

We denote by Res_X^β the restriction of Res_X to $\beta\setminus\{(0,0)\}$. Consider $f_1^{n_1} \ldots f_p^{n_p} y^m$ the decomposition of f in irreducible factors. The number

$$N_j = \sharp([f_j = 0] \cap [y = y_0])$$

does not depend on y_0 for y_0 in a small pointed neighborhood of 0. We define the ramification

$$R = (x, y^{N_1 \ldots N_p}).$$

Then $f \circ R = 0$ has $N = \sum_{j=1}^p N_j$ irreducible components κ_1, …, κ_N different than $y = 0$. These curves are smooth and transversal to $\partial/\partial x$, hence they can be parameterized by y. We denote $Res_{R^*X}^{\kappa_j}$ by $Res_X^{\kappa_j}$ for simplicity. We have

PROPOSITION 3.2.1. *For all $1 \leq j \leq N$ there exist P_j and $Q_j \neq 0$ in $\mathbb{C}\{y\}$ such that $Res_X^{\kappa_j} = P_j(y)/Q_j(y)$ on κ_j.*

PROOF. Let us fix $j \in \{1, \ldots, N\}$. Since κ_j is parameterized by y we can suppose $\kappa_j \equiv [x = 0]$ up to a change of coordinates. We have $f \circ R = a_\nu(y)x^\nu + a_{\nu+1}(y)x^{\nu+1} + \ldots$ where $\nu \geq 1$ and $a_\nu \not\equiv 0$. Let q be the order of $a_\nu(y)$. Consider the transformation

$$H : \left\{ \begin{array}{rcl} x & = & zy^q \\ y & = & y. \end{array} \right.$$

We have $f \circ R \circ H = y^{q(\nu+1)}([a_\nu(y)/y^q]z^\nu + O(z^{\nu+1}))$. Since $a_\nu(y)/y^q$ is a unit then $Y = (H^* R^* X)/y^{q\nu}$ satisfies $\nu_Y(P) = \nu$ for every P in $z = 0$. As a consequence $Res_Y^{z=0}$ is holomorphic. The transformation H is biholomorphic outside $y = 0$, therefore it preserves the residues. Hence, we obtain $Res_X^{\kappa_j}(0, y) = Res_Y^{z=0}(0, y)/y^{q\nu}$. $\square$

Let $g_j = 0$ be an irreducible equation of κ_j. Let $g_1^{l_1} \ldots g_N^{l_N} y^{mN_1 \ldots N_p}$ be the irreducible decomposition of $f \circ R$. We are looking for a holomorphic ψ_{R^*X} of the form

$$\psi_{R^*X} = \alpha(x, y) + \sum_{j=1}^{N} \frac{P_j(y)}{Q_j(y)} \ln g_j(x, y).$$

This equation is equivalent to

$$\frac{\partial \alpha(x, y)}{\partial x} = \frac{1}{f \circ R} - \sum_{j=1}^{N} \frac{P_j(y)}{Q_j(y)} \frac{\partial g_j}{\partial x} \frac{1}{g_j(x, y)}. \tag{3.1}$$

A solution α is an integral of the relatively closed meromorphic form obtained by multiplying the right hand side of equation 3.1 by dx. The equation 3.1 is free of residues.

LEMMA 3.2.1. *There exists a solution α of equation 3.1 of the form*

$$\frac{\beta}{g_1^{l_1-1} \ldots g_N^{l_N-1} y^{m_0}}$$

where $\beta \in \mathbb{C}\{x, y\}$ and $m_0 \leq \max(mN_1 \ldots N_p, \max_{1 \leq j \leq N} \nu(Q_j))$.

PROOF. Let us consider a simply connected domain $U \times D \subset U_{\epsilon,\delta}$ where $0 \notin U$ and $0 \in D$. We also request $(U \times D) \cap (f \circ R = 0)$ to be either the empty set if $m = 0$ or $U \times \{0\}$ if $m > 0$. The equation

$$\frac{\partial \rho}{\partial x} = \frac{1}{f \circ R}$$

admits a solution $a(x, y)/y^{mN_1 \ldots N_p}$ for a holomorphic function a defined over $U \times D$. We can extend ρ as a multi-valuated function to

$$V = ([|x| < \epsilon] \times D) \setminus \cup_{j=1}^{N} \kappa_j.$$

The function $\alpha_0 = \rho - \sum_{j=1}^{N} (P_j/Q_j) \ln g_j$ is single valued; it is meromorphic in V and holomorphic in $V \setminus [y = 0]$. Moreover α_0 is a solution of equation 3.1. Let m_0 be the order of pole of α_0 at the curve $y = 0$.

Consider a point P in $\kappa_j \setminus \{(0, 0)\}$. The curve $f \circ R = 0$ can be transformed into the curve $x = 0$ up to a change of coordinates $(H_P(x, y), y)$ defined over a neighborhood of P. It is straightforward to find at P a local solution α_P of equation

3.1 such that $\alpha_P g_j^{l_j-1}$ is holomorphic. Since $\partial(\alpha_0-\alpha_P)/\partial x=0$ then $(\alpha_0-\alpha_P)(y)$ is holomorphic in a neighborhood of $y=y(P)$. As a consequence

$$\beta=\alpha_0 g_1^{l_1-1}\dots g_N^{l_N-1}y^{m_0}$$

is holomorphic in $([|x|<\epsilon]\times D)\setminus\{(0,0)\}$; this function is holomorphic in a neighborhood of the origin by Hartogs' theorem. □

REMARK 3.2.2. *The expression*

$$\psi_{R^*X}=\frac{\beta}{g_1^{l_1-1}\dots g_N^{l_N-1}y^{m_0}}+\sum_{j=1}^{N}\frac{P_j(y)}{Q_j(y)}\ln g_j$$

shows the holomorphic dependance of ψ_{R^*X} on the parameter y outside of $y=0$. The holomorphic character of ψ_{R^*X} in the neighborhood of $y=0$ is provided by the proof of last proposition.

We denote by $\mu(B)$ the *order of pole* of a meromorphic function B defined in a neighborhood of 0. Let A be a multi-valued function defined in a pointed neighborhood of 0. Suppose there exists $k\in\mathbb{N}$ such that $A(y^k)$ is meromorphic in the neighborhood of 0. We define the *order of pole* $\mu(A)$ of A as $\mu(A(y^k))/k$. The definition does not depend on k. In our case we have

$$Res_X^{f_j=0}=\frac{P(y^{1/N_j})}{Q(y^{1/N_j})}$$

for some $P,Q\in\mathbb{C}\{y\}$. Let M_j be the generic number of pre-images of $Res_X^{f_j=0}=cte$; this number coincides with $|\nu(P/Q)|$ if $|\nu(P/Q)|\geq 1$. Therefore if $\mu(Res_X^{f_j=0})\neq 0$ then $\mu(Res_X^{f_j=0})=M_j/N_j$.

3.2.2. Dynamics at the limit line. In order to describe the dynamics we study the behavior of the critical trajectories at $y=0$.

PROPOSITION 3.2.2. *Let $\lambda\in\mathbb{S}^1$. There are no critical tangent cords for $\xi(X(\lambda),0,\epsilon)$.*

PROOF. If $SingX$ is contained in $y=0$ then all the tangent points are concave (corollary 3.1.1) and we are done. Otherwise, we consider the connected components $C_1(\lambda)$, …, $C_l(\lambda)$ of

$$C(\lambda)\stackrel{def}{=}(\alpha_{\xi(X(\lambda),0)},\omega_{\xi(X(\lambda),0)})^{-1}_{|x|<\epsilon}(0,0)\setminus\{0\}.$$

We have $l=2(\nu_{X(\lambda)}(0)-1)=2(\tilde{\nu}(X)-1)$ by lemma 2.2.1. The number of tangent points of $\xi(X(\lambda),0,\epsilon)$ is also $2(\tilde{\nu}(X)-1)$ by proposition 3.1.1. We have $\overline{C_j(\lambda)}\cap\overline{C_k(\lambda)}\cap\partial U_\epsilon=\emptyset$ for $1\leq j<k\leq l$ and $\overline{C_j(\lambda)}\cap\partial U_\epsilon\neq\emptyset$ for all $1\leq j\leq l$. Since the number of tangent points and l coincide then $\sharp(\overline{C_j(\lambda)}\cap\partial U_\epsilon)=1$ for all $1\leq j\leq l$. Hence, the trajectories of $\xi(X(\lambda),0)$ in $|x|\leq\epsilon$ passing through a tangent point do not contain other points in ∂U_ϵ. Therefore, there are no critical tangent cords. □

REMARK 3.2.3. *If $SingX\not\subset(y=0)$ we proved that*

$$(\alpha_{\xi(X(\lambda),0)},\omega_{\xi(X(\lambda),0)})_{[|x|<\epsilon]\cup\{Q\}}(Q)=(0,0)$$

for all $Q\in T^{|x|<\epsilon}_{\xi(X(\lambda),0)}$.

Suppose that $SingX \neq [y=0]$. The remark 3.2.3 implies that the dynamics of $Re(X(\lambda))$ in $U_\epsilon \cap [y=0]$ is as described in picture 2.

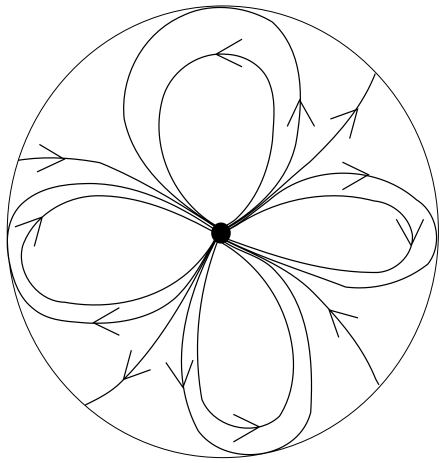

FIGURE 2. Dynamics in $y=0$

3.2.3. Dynamics far away from the singular points. Far away from the singular points, we can not distinguish them; basically they can be replaced by a single singular point. We exploit this fact to show that the dynamics in the exterior part of a domain $U_{\epsilon,\delta}$ depends nicely on the parameter.

We suppose $N \geq 1$, otherwise the dynamics is trivial. Let $f = y^m f_1^{n_1} \ldots f_p^{n_p}$ be the the decomposition of f in irreducible factors. Throughout the rest of this chapter and up to a ramification we suppose that $f_j = 0$ is transversal to $\partial/\partial x$ for all $1 \leq j \leq p$ and then $N = p$. This hypothesis is not restrictive since the results we deal with in this chapter are invariant by a ramification $(x,y) \to (x, y^k)$. We split $\overline{U_\epsilon}$ in two sets

$$U_\epsilon^{\eta,-} = U_\epsilon \cap [|x| \leq \eta |y|] \text{ and } U_\epsilon^{\eta,+} = \overline{U_\epsilon} \cap [|x| \geq \eta |y|].$$

We claim that roughly speaking the dynamics at $U_\epsilon^{\eta,+}$ is trivial whereas $U_\epsilon^{\eta,-}$ can be subdivided to obtain a simple description of the dynamics

The remaining part of this section is devoted to prove that $Re(X(\lambda^m))(x, r\lambda)$ is dynamically similar to $Re(X(\lambda^m))(x,0)$ in $U_\epsilon^{\eta,+}$. We consider $\eta > 0$ big enough to guarantee that $U_\epsilon^{\eta,+} \setminus [y=0]$ does not contain singular points.

LEMMA 3.2.2. *Suppose $N > 0$. There exists $\eta_0 > 0$ such that for all $\eta > \eta_0$ the set $T_{X(\lambda_0)}^{|x|<\eta|y_0|}(y_0)$ is composed by $2(\tilde{\nu}(X) - 1)$ convex points for all y_0 in a pointed neighborhood of* 0 *and all* $\lambda_0 \in \mathbb{S}^1$.

PROOF. Since $f_j = 0$ is parameterized by y then $f_j/(x - g_j(y))$ is a unit for some $g_j \in \mathbb{C}\{y\}$. Therefore, there exists a unit $u \in \mathbb{C}\{y\}$ such that X is of the form

$$X = u(x,y) y^m (x - g_1(y))^{n_1} \ldots (x - g_N(y))^{n_N} \partial/\partial x.$$

Up to consider the transformation

$$\begin{cases} x &= yw \\ y &= y \end{cases}$$

the vector field $(wy, y)^* X$ is equal to $y^{m+n_1+\ldots+n_N-1} Y$ where

$$Y = u(yw, y)(w - g_1(y)/y)^{n_1} \ldots (w - g_N(y)/y)^{n_N} \partial/\partial w.$$

Thus $(r\lambda, w)^* X(\lambda_0)_{|y=r\lambda} = r^{n_1+\ldots n_N-1} Y(\lambda_0 \lambda^{n_1+\ldots n_N-1})_{|y=r\lambda}$. We have

$$\lim_{w \to \infty} \frac{(w - g_1(y)/y)^{n_1} \ldots (w - g_N(y)/y)^{n_N}}{w^{n_1+\ldots n_N}} = 1.$$

The limit is uniform in $y \in B(0, \delta)$. The derivative of

$$\arg[u(yw, y)(w - g_1(y)/y)^{n_1} \ldots (w - g_N(y)/y)^{n_N}]_{|(|w|=\eta)\cap(y=y_0)}$$

with respect to $\arg(w)$ tends to $\tilde{\nu}(X) - 1$ if $\eta \to \infty$ and $y_0 \to 0$. Since we have $(|x| = \eta|y|) \equiv (|w| = \eta)$ then the result is a consequence of applying proposition 3.1.1 and corollary 3.1.1 to $Y(\lambda_0 \lambda^{n_1+\ldots+n_N-1})_{|y=r\lambda}$. $\square$

From now on we suppose that $\eta > 0$ is big enough. Next we provide a qualitative description of the dynamics of $Re(X(\lambda_0))$ in $U_\epsilon^{\eta,+}$.

Lemma 3.2.3. *Suppose $N \geq 1$. Consider $\lambda_0 \in \mathbb{S}^1$ and a point (x_0, y_0) in $T_{X(\lambda_0)}^{|x|<\epsilon}$. Then, for $s \in \{+, -\}$ the closure of $\Gamma_{\xi(X(\lambda_0)),s}^{\eta|y_0|\leq|x|\leq\epsilon}[x_0, y_0]$ contains a unique point (x_0', y_0) in $(|x| = \eta|y_0|) \cup (|x| = \epsilon)$ different than (x_0, y_0). Moreover $(x_0', y_0) \in (|x| = \eta|y_0|)$.*

Proof. If $y_0 = 0$ the lemma is true (see remark 3.2.3). Suppose $y_0 \neq 0$; we denote $A = [\eta|y_0| \leq |x| \leq \epsilon] \subset [y = y_0]$ and $\xi = \xi(X(\lambda_0), y_0, \epsilon)$. Consider the set $\mathcal{H} = \cup_{P \in T_{X(\lambda_0)}^{\epsilon}(y_0)} \Gamma_\xi^A[P]$. Intuitively , the set $\mathcal{H}$ is the union of the critical trajectories of $Re(X)$ in A. With respect to A the points of $T_\xi^{|x|<\epsilon}$ are convex (corollary 3.1.1) whereas the points of $T_\xi^{|x|<\eta|y_0|}$ are concave (lemma 3.2.2). Moreover, we have

$$\sharp T_{X(\lambda_0)}^{\epsilon}(y_0) = \sharp T_{X(\lambda_0)}^{\eta|y_0|}(y_0) = 2(\tilde{\nu}(X) - 1)$$

by proposition 3.1.1 and lemma 3.2.2.

We proceed like in proposition 3.2.2. Since $A \cap Sing X = \emptyset$ then

$$A \subset (\alpha_\xi, \omega_\xi)_A^{-1}(\infty, \infty).$$

For $P \in T_\xi^{|x|<\eta|y_0|}$ we denote by C_P the unique connected component of $A \setminus \mathcal{H}$ such that $P \in \overline{C_P}$. We can define $end_A^+(S) = \Gamma_{\xi,+}^A[S] \cap \partial A$ for $S \in C_P$; the definition of end_A^- is analogous. The sets $end_A^+(C_P)$ and $end_A^-(C_P)$ are connected and contained in $\partial A \setminus (T_\xi^{|x|<\epsilon} \cup T_\xi^{|x|<\eta|y_0|})$. There are two connected components of $\partial A \setminus T_\xi^{|x|<\eta|y_0|}$ whose closure contains P. Since $P \in \partial C_P$ the set $end_A^+(C_P)$ is contained in one of those components whereas $end_A^-(C_P)$ is contained in the other one. As a consequence of this discussion $\overline{C_P} \cap \overline{C_Q} = \emptyset$ for $Q \in T_\xi^{|x|<\eta|y_0|} \setminus \{P\}$.

The set $\partial C_P \setminus (end_A^+(C_P) \cup end_A^-(C_P))$ has two connected components. One of them is $\{P\}$ and since $\overline{C_P} \cap (T_\xi^{|x|<\eta|y_0|} \setminus \{P\}) = \emptyset$ we deduce that the other component is contained in $\mathcal{H}$. As a consequence we obtain that $\overline{C_P} \cap T_\xi^{|x|<\epsilon} \neq \emptyset$. Moreover, the latter set is a singleton since $\sharp T_\xi^{|x|<\epsilon} = \sharp T_\xi^{|x|<\eta|y_0|}$. We deduce that for $(x_0, y_0) \in T_\xi^{|x|<\epsilon} \cap \overline{C_P}$ and $s \in \{+, -\}$ the set

$$\Gamma_{\xi(X(\lambda_0),s)}^A[x_0, y_0] \cap (\partial A \setminus \{(x_0, y_0)\})$$

is a singleton contained in $\overline{end^s_A(C_P)}$ and then in $[|x| = \eta|y_0|]$. □

Since $\sharp T^{|x|<\epsilon}_{X(\lambda_0)}(y_0) = \sharp T^{|x|<\eta|y_0|}_{X(\lambda_0)}(y_0) = 2(\tilde{\nu}(X)-1)$ then the dynamics of $Re(X(\lambda_0))_{|y=y_0}$ in $\eta|y_0| \leq |x| \leq \epsilon$ is as represented in figure 3. The dynamics in the exterior zone of

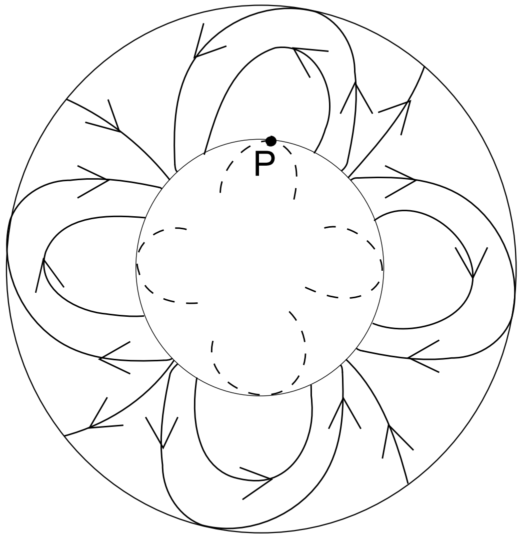

FIGURE 3. Dynamics of $Re(X(\lambda_0))$ in $U_\epsilon^{\eta,+}$

$\xi(X(\lambda_0), y, \epsilon)$ is qualitatively equal to the dynamics of $\xi(X(\lambda_0), 0, \epsilon)$. We are also interested in a quantitative comparison.

Let $X_0 = (f/y^m)(x,0)\partial/\partial x$. The series $(f/y^m)(x,0)$ is of the form $a_{\tilde{\nu}(X)}x^{\tilde{\nu}(X)}+$ $h.o.t$ where $a_{\tilde{\nu}(X)} \neq 0$. We define $X_{00} = a_{\tilde{\nu}(X)}x^{\tilde{\nu}(X)}\partial/\partial x$. For $(y_0, \lambda_0) \in B(0,\delta)\times\mathbb{S}^1$ we consider the set

$$[\eta|y_0| < |x| < \epsilon] \setminus \left(\cup_{P\in T^{|x|<\epsilon}_{X(\lambda_0)}(y_0)} \Gamma^{\eta|y_0|\leq|x|\leq\epsilon}_{\xi(X(\lambda_0),y_0)}[P] \right)$$

An *exterior region* at (y_0, λ_0) is the closure $R^{\epsilon,\eta}_{X(\lambda_0)}(y_0)$ of a component of the previous set. The exteriors regions depend continuously on (y_0, λ_0); a priori we can have $R^{\epsilon,\eta}_{X(\lambda_0)}(y_0) \neq R^{\epsilon,\eta}_{X(e^{2\pi i}\lambda_0)}(y_0)$ but anyway the monodromy is finite since we have $R^{\epsilon,\eta}_{X(\lambda_0)}(y_0) = R^{\epsilon,\eta}_{X(e^{2\pi i(\tilde{\nu}(X)-1)}\lambda_0)}(y_0)$. We denote $(\lambda^{\tilde{\nu}(X)-1})^*\mathbb{S}^1$ by $\mathbb{S}^1_{\tilde{\nu}(X)}$. Fix a region $R^{\epsilon,\eta}_{X(\lambda)}(y)$. If the set $T^{|x|<\epsilon}_{X(\lambda)}(y) \cap R^{\epsilon,\eta}_{X(\lambda)}(y)$ is a singleton for all $(y,\lambda) \in B(0,\delta)\times\mathbb{S}^1_{\tilde{\nu}(X)}$ we denote its element by $T^{\epsilon,1}_{X(\lambda)}(y)$; we say that $R^{\epsilon,\eta}_{X(\lambda)}(y)$ is an "a" exterior region. Otherwise $R^{\epsilon,\eta}_{X(\lambda)}(y)$ is a "b" exterior region. It satisfies $T^{|x|<\epsilon}_{X(\lambda)}(y) \cap R^{\epsilon,\eta}_{X(\lambda)}(y) = \{T^{\epsilon,1}_{X(\lambda)}(y), T^{\epsilon,2}_{X(\lambda)}(y)\}$.

We have that

$$T^{|x|<\epsilon}_{X_{00}(\lambda)}(y) = \epsilon \sqrt[\tilde{\nu}(X)-1]{\frac{i|a_{\tilde{\nu}(X)}|}{\lambda a_{\tilde{\nu}(X)}}}\{e^{\frac{i\pi 0}{\tilde{\nu}(X)-1}} = 1, \ldots, e^{\frac{i\pi[2(\tilde{\nu}(X)-1)-1]}{\tilde{\nu}(X)-1}}\},$$

in particular $T^{|x|<\epsilon}_{X_{00}(\lambda)}(y)$ depends on λ but it does not depend on y.

Lemma 3.2.4. *Suppose $N \geq 1$. Let $0 < \zeta \leq \pi/[2(\tilde{\nu}(X)-1)]$. For $\epsilon << 1$ and $\delta(\epsilon) << 1$ we have that there is exactly one point of $T_{X(\lambda)}^{|x|<\epsilon}(y)$ in $e^{i(-\zeta,\zeta)}z$ for all $z \in T_{X_{00}(\lambda)}^{|x|<\epsilon}$ and $\lambda \in \mathbb{S}^1$.*

Proof. Consider the function $\arg_X^\epsilon : \partial U_\epsilon \times \mathbb{S}^1 \to \mathbb{R}$ defined in the proof of proposition 3.1.1. Since $(f/y^m)(x,0) = a_{\tilde{\nu}(X)}x^{\tilde{\nu}(X)}(1+h.o.t)$ then for all $(x,0) \in \partial U_\epsilon$ and $\lambda \in \mathbb{S}^1$ we have

$$|\arg_X^\epsilon((x,0),\lambda) - \arg_{X_{00}}^\epsilon((x,0),\lambda)| \leq h(\epsilon),$$

where $h : (\mathbb{R}^+,0) \to \mathbb{R}^+$ satisfies $\lim_{\epsilon\to 0} h(\epsilon) = 0$. We also have that the derivative of $\arg_X^\epsilon((x,0),\lambda)$ with respect to $\arg(x)$ tends to $\tilde{\nu}(X)-1 > 0$ when $\epsilon \to 0$; the limit is uniform in $\lambda \in \mathbb{S}^1$. We choose ϵ_0 such that for $\epsilon < \epsilon_0$ we have $\partial(\arg_X^\epsilon((x,0),\lambda))/\partial(\arg(x)) > (\tilde{\nu}(X)-1)/2$ and $h(\epsilon) < \zeta(\tilde{\nu}(X)-1)/4$. These properties imply that there is exactly one point of $T_{X(\lambda)}^{|x|<\epsilon}(0)$ in $e^{i(-\zeta/2,\zeta/2)}z$ for all $z \in T_{X_{00}(\lambda)}^{|x|<\epsilon}$ and $\lambda \in \mathbb{S}^1$. We can extend the result to $y \in B(0,\delta)$ by continuity. □

Let $0 < \zeta \leq \pi/[2(\tilde{\nu}(X)-1)]$. Consider a region $R = R_{X(\lambda)}^{\epsilon,\eta}(y)$. For $(y_0,\lambda_0) \in B(0,\delta) \times \mathbb{S}^1_{\tilde{\nu}(X)}$ we have $T_{X(\lambda_0)}^{\epsilon,1}(y_0) \in e^{i(-\zeta,\zeta)}T_{X_{00}(\lambda_0)}^{\epsilon,1}$ for a unique $T_{X_{00}(\lambda_0)}^{\epsilon,1}$ in $T_{X_{00}(\lambda_0)}^{|x|<\epsilon}$. If $R_{X(\lambda)}^{\epsilon,\eta}(y)$ is of "a" type we define

$$D_R^{\epsilon,\eta}(\lambda_0) = D_R^{\epsilon,\eta}(y_0,\lambda_0) = (\eta|y| \leq |x| \leq \epsilon) \setminus (T_{X_{00}(\lambda_0)}^{\epsilon,1}\mathbb{R}^-).$$

If the type is "b" then $T_{X(\lambda_0)}^{\epsilon,2}(y_0) \in e^{i\pi/(\tilde{\nu}(X)-1)}e^{i(-\zeta,\zeta)}T_{X_{00}(\lambda_0)}^{\epsilon,1}$. We define

$$D_R^{\epsilon,\eta}(\lambda_0) = D_R^{\epsilon,\eta}(y_0,\lambda_0) = (\eta|y| \leq |x| \leq \epsilon) \setminus (T_{X_{00}(\lambda_0)}^{\epsilon,1}e^{i\pi/[2(\tilde{\nu}(X)-1)]}\mathbb{R}^-).$$

The shape of $D_R^{\epsilon,\eta}(\lambda)$ is as presented in picture 4.

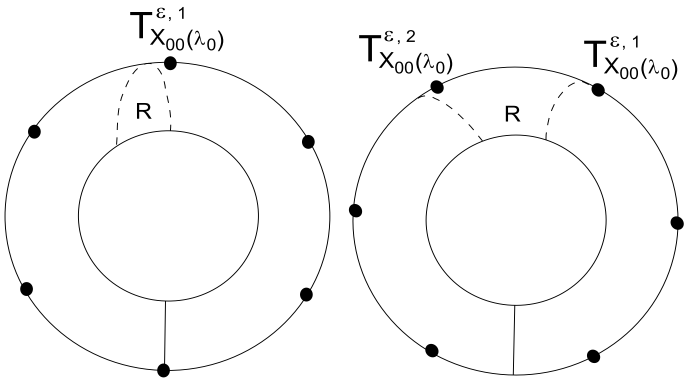

Figure 4. $D_R^{\epsilon,\eta}(\lambda)$

3.2.4. Behavior of the integral of the time form. We denote by ψ_{00} a meromorphic integral of the time form of X_{00}; that is possible because $Res_{X_{00}}(0,y) \equiv 0$. We denote by ψ_0^R and ψ^R integrals of the time forms of X_0 and $X(1)$ respectively defined in the set

$$D_R^{\epsilon,\eta} \equiv [(x,y,\lambda) \in D_R^{\epsilon,\eta}(\lambda) \times \{\lambda\}] \cap [\lambda \in \mathbb{S}^1_{\tilde{\nu}(X)}].$$

Fix $\epsilon_0 << 1$; we choose ψ^R and ψ_0^R such that

$$\psi^R(T_{X_{00}(\lambda)}^{\epsilon_0,1}, y) = \psi_0^R(T_{X_{00}(\lambda)}^{\epsilon_0,1}, y) = \psi_{00}(T_{X_{00}(\lambda)}^{\epsilon_0,1}, y)$$

for all $(y,\lambda) \in B(0,\delta) \times \mathbb{S}^1_{\tilde{\nu}(X)}$. Our approach is proving that the dynamics of $Re(X(\lambda))$ and $Re(X_{00}(\lambda))$ are similar by comparing ψ^R and ψ_{00} in $U_\epsilon^{\eta,+}$.

LEMMA 3.2.5. *Suppose $N \geq 1$. Let $\zeta > 0$. Consider any exterior region $R(y,\lambda) = R_{X(\lambda)}^{\epsilon,\eta}(y)$. Then $|\psi_0^R/\psi_{00} - 1| < \zeta$ in $D_R^{\epsilon,\eta}$ for $\epsilon << 1$. Moreover, we have*

$$\left| \frac{\psi^R}{\psi_{00}} - 1 \right| < \zeta$$

in $D_R^{\epsilon,\eta}$ for $\epsilon << 1$, $\eta >> 0$ and $\delta << 1$.

PROOF. We denote $\nu = \tilde{\nu}(X)$. We choose a determination for $\ln x$ in the simply connected set $D_R^{\epsilon,\eta}(1)$ and then we extend $\ln x$ analytically to $D_R^{\epsilon,\eta}$. Since $\mathbb{S}^1_{\tilde{\nu}(X)} \equiv \mathbb{S}^1$ is compact there exists a constant $J > 0$ such that $|Img(\ln x)| \leq J$ if (x,y,λ) belongs to $D_R^{\epsilon,\eta}$. The function $\psi_{00}(x,y,\lambda)$ is equal to $-1/(a_\nu(\nu-1)x^{\nu-1})$ and

$$\psi_0^R(x,y,\lambda) = \psi_{00} + b\ln x + \frac{1}{x^{\nu-2}}H(x) + C(\lambda)$$

where $b = Res_{X_0}(0)$; the continuous functions $C(\lambda)$ and $H(x)$ are defined in $\mathbb{S}^1_{\tilde{\nu}(X)}$ and a neighborhood of $\overline{B(0,\epsilon)}$ respectively. The function C is a continuous function defined in a compact set and then bounded. We obtain

$$\frac{\psi_0^R}{\psi_{00}} - 1 = -a_\nu(\nu-1)[bx^{\nu-1}\ln x + xH(x) + C(\lambda)x^{\nu-1}]$$

Since $|Img(\ln x)| \leq J$ then the right hand side is a $o(1)$.

Let us focus on ψ^R/ψ_0^R. We define $K(x,y,\lambda) : D_R^{\epsilon_0,\eta} \to \mathbb{C}$ such that $K(x,y,\lambda) = \psi^R(x,y,\lambda) - \psi_0^R(x,y,\lambda)$. We have $K(T_{X_{00}(\lambda)}^{\epsilon_0,1}, y, \lambda) \equiv 0$ by choice. Consider the decomposition $u(x,y)y^m(x-g_1(y))^{n_1}\ldots(x-g_N(y))^{n_N}$ of f in irreducible factors. Since

$$\frac{\partial \psi^R}{\partial x} = \frac{y^m}{f(x,y)} \quad \text{and} \quad \frac{\partial \psi_0^R}{\partial x} = \frac{1}{u(x,0)x^{n_1+\ldots+n_N}}$$

then K satisfies

$$\frac{f(x,y)}{y^m}\frac{\partial K}{\partial x} = 1 - \frac{u(x,y)(x-g_1(y))^{n_1}\ldots(x-g_N(y))^{n_N}}{u(x,0)x^{n_1+\ldots+n_N}}.$$

For $\eta >> 0$ and $\delta << 1$ we have $|(f/y^m)\partial K/\partial x| \leq A_1|y/x|$ in $D_R^{\epsilon_0,\eta}$ for some $A_1 > 0$. That leads us to

$$\left|\frac{\partial K}{\partial x}(x,y,\lambda)\right| \leq A_2 \frac{|y/x|}{|x|^{n_1+\ldots n_N}}$$

in $D_R^{\epsilon_0,\eta}$ for some $A_2 > 0$. We denote $x_0(\lambda) = T_{X_{00}(\lambda)}^{\epsilon_0,1}$. For any point $(x,y,\lambda) \in D_R^{\epsilon_0,\eta}$ we can express x as $x = (r/\epsilon_0)e^{i\theta}x_0(\lambda)$ for $r \in [\eta|y|, \epsilon_0]$ and $|\theta| < 2\pi$. Let $x_1 = x_0(\lambda)e^{i\theta}$; we obtain

$$|K(x_1,y,\lambda) - K(x_0(\lambda),y,\lambda)| \leq \left|\int_{x_0(\lambda)}^{x_1} \frac{\partial K}{\partial x}dx\right| \leq \frac{2\pi A_2|y|}{\epsilon_0^{n_1+\ldots+n_N}}.$$

Consider the path $\gamma : [0,1] \to D_R(y,\lambda)$ defined by

$$\gamma(t) = (x_1[(1-t) + tr/\epsilon_0], y, \lambda).$$

We obtain

$$|K(x,y,\lambda) - K(x_1,y,\lambda)| \leq \left| \int_\gamma \frac{\partial K}{\partial x} dx \right| \leq \left| \int_0^1 \frac{\partial K}{\partial x}(\gamma(t))\gamma'(t) dt \right|$$

The previous expression implies

$$|K(x,y,\lambda) - K(x_1,y,\lambda)| \leq \frac{A_2(\epsilon_0 - r)}{\eta} \int_0^1 \frac{1}{[(1-t)\epsilon_0 + tr]^{n_1+\ldots+n_N}} dt.$$

By integration we obtain

$$|K(x,y,\lambda) - K(x_1,y,\lambda)| \leq \frac{A_3}{\eta} \left(\frac{1}{r^{n_1+\ldots+n_N-1}} - \frac{1}{{\epsilon_0}^{n_1+\ldots+n_N-1}} \right)$$

where $A_3 = A_2/(n_1 + \ldots + n_N - 1)$. As a consequence

$$|K(x,y,\lambda)| \leq A_4 \left(|y| + \frac{1}{\eta} \right) \frac{1}{|x|^{n_1+\ldots+n_N-1}}$$

in $D_R^{\epsilon,\eta}$ for $A_4 = \max(2\pi A_2, A_3)$. By the first part of the lemma we have $A_5 \leq |\psi_0^R||x|^{n_1+\ldots+n_N-1}$ for some $A_5 > 0$ and $\epsilon << 1$. Therefore

$$\left| \frac{\psi^R}{\psi_0^R} - 1 \right| = \left| \frac{K}{\psi_0^R} \right| \leq \frac{A_4}{A_5} \left(|y| + \frac{1}{\eta} \right) < \frac{A_4}{A_5} \left(\delta + \frac{1}{\eta} \right).$$

□

For $N = 1$ the behavior of $Re(X(\lambda))$ in U_ϵ is analogous to the one we obtain in the exterior regions.

Lemma 3.2.6. *Suppose $f = y^m x^n$ for some $n > 0$. Let $\zeta > 0$. Consider any exterior region $R(y,\lambda) = R^{\epsilon,0}_{X(\lambda)}(y)$. Then $|\psi_0^R/\psi_{00} - 1| < \zeta$ in $D_R^{\epsilon,0}$ for $\epsilon << 1$. Moreover, we have*

$$\left| \frac{\psi^R}{\psi_{00}} - 1 \right| < \zeta$$

in $D_R^{\epsilon,0}$ for $\epsilon << 1$ and $\delta << 1$.

Proof. The first part of the proof is analogous to the first part of the proof of lemma 3.2.5. For the second part of the proof we proceed as in lemma 3.2.5 but with improved inequalities. It is straightforward to check out that the function $K(x,y,\lambda)$ satisfies

$$\left| \frac{\partial K}{\partial x}(x,y,\lambda) \right| \leq A_1 \frac{|y|}{|x|^{n_1+\ldots n_N}}$$

in $D_R^{\epsilon,0} \cap [y \in B(0,\delta)]$ for some $A_1 > 0$ and $\delta << 1$. As a consequence there exists $A > 0$ such that

$$\left| \frac{\psi^R}{\psi_0^R} - 1 \right| = \left| \frac{K}{\psi_0^R} \right| \leq A|y|$$

in $D_R^{\epsilon,0}$ for $\epsilon << 1$ and $\delta << 1$. □

3.2.5. Variation. The exterior region $R^{\epsilon,\eta}_{X(\lambda_0)}(y_0) \subset [y = y_0]$ is simply connected. Therefore the function $\ln x$ is uni-valuated in $R^{\epsilon,\eta}_{X(\lambda_0)}(y_0)$ and it is unique up to an additive constant. We define

$$Var(R^{\epsilon,\eta}_{X(\lambda_0)}(y_0)) = \max_{x_0,x_1 \in R^{\epsilon,\eta}_{X(\lambda_0)}(y_0)} |Img(\ln x_1) - Img(\ln x_0)|.$$

The function $Var(R^{\epsilon,\eta}_{X(\lambda)}(y)) : B(0,\delta) \times \mathbb{S}^1_{\tilde{\nu}(X)} \to \mathbb{R}^+$ is well-defined and continuous. By controlling the variation we assure that the trajectories in the exterior zone do not spiral around the singular points of X. In next lemma we find an analogue of $Var(R^{\epsilon,\eta}_{X_{00}(\lambda)}(y)) \leq \pi/(\tilde{\nu}(X) - 1)$ valid for $Var(R^{\epsilon,\eta}_{X(\lambda)}(y))$.

PROPOSITION 3.2.3. *Suppose $N \geq 1$. Let $\zeta > 0$. Consider an exterior region $R^{\epsilon,\eta}_{X(\lambda)}(y)$. For $\epsilon << 1$, $\delta << 1$ and $\eta >> 1$ we have*

$$Var(R^{\epsilon,\eta}_{X(\lambda)}(y)) \leq \frac{\pi}{\tilde{\nu}(X) - 1} + \zeta$$

for all $(y,\lambda) \in B(0,\delta) \times \mathbb{S}^1_{\tilde{\nu}(X)}$. In particular $R^{\epsilon,\eta}_{X(\lambda)}(y) \subset D_R(\lambda)$ for all $(y,\lambda) \in B(0,\delta) \times \mathbb{S}^1_{\tilde{\nu}(X)}$.

PROOF. Let $\zeta < \pi$. Let $(y_0,\lambda_0) \in B(0,\delta) \times \mathbb{S}^1_{\tilde{\nu}(X)}$. We denote $\nu = \tilde{\nu}(X)$ and $T^{\epsilon,1}_{X(\lambda_0)}(y_0)$ by $z_1(y_0,\lambda_0)$. We also define

$$\gamma(y_0,\lambda_0) = \Gamma^{\eta|y_0| \leq |x| \leq \epsilon}_{\xi(X(\lambda_0),y_0)}[z_1(y_0,\lambda_0), y_0].$$

Suppose $R_{X(\lambda)}$ is an "a" exterior region. The function $Img(\ln x)$ is harmonic in $R^{\epsilon,\eta}_{X(\lambda_0)}(y_0)$; therefore the minimum and the maximum are attained in $\partial[R^{\epsilon,\eta}_{X(\lambda_0)}(y_0)]$. The set of extrema of $Img(\ln x)$ restricted to the arc $R^{\epsilon,\eta}_{X(\lambda_0)}(y_0) \cap [|x| = \eta|y_0|]$ is $\partial[R^{\epsilon,\eta}_{X(\lambda_0)}(y_0) \cap [|x| = \eta|y_0|]]$. As a consequence we have

$$Var(R^{\epsilon,\eta}_{X(\lambda_0)}(y_0)) = \max_{x_0,x_1 \in \gamma(y_0,\lambda_0)} |Img(\ln x_1) - Img(\ln x_0)|.$$

We denote $h(\lambda_0) = T^{\epsilon,1}_{X_{00}(\lambda_0)}$; this point satisfies

$$\frac{\lambda_0 a_\nu x^\nu}{x}(h(\lambda_0)) \in i\mathbb{R} \implies \lambda_0 a_\nu (h(\lambda_0))^{\nu-1} \in i\mathbb{R}.$$

We obtain

$$\frac{\psi_{00}}{\lambda_0}(h(\lambda_0)) = \frac{-1}{\nu - 1} \frac{1}{\lambda_0 a_\nu (h(\lambda_0))^{\nu-1}} \in i\mathbb{R}.$$

Therefore $Img[\ln(\psi_{00}/\lambda_0)(h(\lambda_0))] \in \{-\pi/2, \pi/2\}$; we can suppose it is $\pi/2$ because otherwise we would replace X with $-X$. If (ϵ,δ,η) is close enough to $(0,0,\infty)$ lemmas 3.2.4 and 3.2.5 imply that

$$Img \circ \ln\left[\frac{\psi^R}{\lambda_0}(z_1(y_0,\lambda_0), y_0)\right] \in [-\zeta/4 + \pi/2, \zeta/4 + \pi/2].$$

Let $t_0 \in \mathbb{R}^+$ such that $\gamma(y_0,\lambda_0)[0,t_0) \subset D^{\epsilon,\eta}_R(\lambda_0)$. Let $t_1 \in [0,t_0)$; we have

$$Img \circ \ln\left[\frac{\psi^R}{\lambda_0}(z_1(y_0,\lambda_0), y_0) + t_1\right] \in (0, \zeta/4 + \pi/2].$$

The equation

$$\frac{\psi_{00}}{\lambda_0}(\gamma(y_0,\lambda_0)(t_1)) = \left[\frac{\psi_R}{\lambda_0}(z_1(y_0,\lambda_0), y_0) + t_1\right] \frac{\psi_{00}}{\psi^R}(\gamma(y_0,\lambda_0)(t_1)) \tag{3.2}$$

and lemma 3.2.5 imply that

$$Img \circ \ln \left[\frac{\psi_{00}}{\lambda_0} (\gamma(y_0, \lambda_0)(t_1)) \right] \in (-\zeta/2, \zeta/2 + \pi/2]$$

if (ϵ, δ, η) is close enough to $(0, 0, \infty)$. We deduce that

$$Img \circ \ln x \circ \gamma(y_0, \lambda_0)(t_1) - Img \circ \ln(h(\lambda_0)) \in \left[\frac{-\zeta}{2(\nu-1)}, \frac{\zeta+\pi}{2(\nu-1)} \right).$$

Since $\zeta < \pi$ and $\nu \geq 2$ we deduce that $\gamma(y_0, \lambda_0)(t_0) \in D_R(\lambda_0)$. We just proved that $\Gamma_{\xi(X(\lambda_0)),+}^{\eta|y_0| \leq |x| \leq \epsilon}[z_1(y_0, \lambda_0), y_0]$ is contained in $D_R(\lambda_0)$. In an analogous way we obtain $\Gamma_{\xi(X(\lambda_0)),-}^{\eta|y_0| \leq |x| \leq \epsilon}[z_1(y_0, \lambda_0), y_0] \subset D_R(\lambda_0)$; moreover if $\gamma(y_0, \lambda_0)[-t_0, 0]$ is contained in $[\eta|y_0| \leq |x| \leq \epsilon]$ for some $t_0 \in \mathbb{R}^+$ then

$$Img \circ \ln x(\gamma(y_0, \lambda_0)(-t_0)) - Img \circ \ln(h(\lambda_0)) \in \left(\frac{-(\zeta+\pi)}{2(\nu-1)}, \frac{\zeta}{2(\nu-1)} \right].$$

Therefore, the variation function satisfies

$$Var(R_{X(\lambda_0)}^{\epsilon,\eta}(y_0)) < \frac{\pi}{\tilde{\nu}(X)-1} + \frac{\zeta}{\tilde{\nu}(X)-1}.$$

Suppose $\sharp[R_{X(\lambda)}^{\epsilon,\eta}(y) \cap T_{X(\lambda)}^{\epsilon}(y)] \equiv 2$. We proceed in a similar way, we stress the main steps of the proof. We consider the arc

$$arc(y_0, \lambda_0) = R_{X(\lambda_0)}^{\epsilon,\eta}(y_0) \cap \partial U_\epsilon.$$

Suppose $Re(X(\lambda_0))$ points towards U_ϵ in the interior of $arc(y_0, \lambda_0)$; otherwise we replace X with $-X$. The arc $arc(y_0, \lambda_0)$ satisfies

$$arc(y_0, \lambda_0) \subset T_{X_{00}(\lambda_0)}^{\epsilon,1} e^{i\left[\frac{-\zeta}{4(\nu-1)}, \frac{\zeta}{4(\nu-1)} + \frac{\pi}{\nu-1}\right]}$$

for (ϵ, δ, η) in the neighborhood of $(0, 0, \infty)$ by lemma 3.2.4. As a consequence

$$Img \circ \ln x \circ \frac{\psi_{00}}{\lambda_0}(arc(y_0, \lambda_0)) \subset [-\zeta/4 - \pi/2, \zeta/4 + \pi/2].$$

Again we use equation 3.2 and lemma 3.2.5 to prove that

$$Img \circ \ln x \circ \frac{\psi_{00}}{\lambda_0}(R_{X(\lambda_0)}^{\epsilon,\eta}(y_0)) \subset [-\zeta/2 - \pi/2, \zeta/2 + \pi/2]$$

for (ϵ, δ, η) near $(0, 0, \infty)$. The last equation implies

$$Var(R_{X(\lambda_0)}^{\epsilon,\eta}(y_0)) \leq \frac{\pi}{\tilde{\nu}(X)-1} + \frac{\zeta}{\tilde{\nu}(X)-1}.$$

That implies $R_{X(\lambda_0)}^{\epsilon,\eta}(y_0) \subset D_R(\lambda_0)$. □

We can adapt the proof of proposition 3.2.3 to obtain

LEMMA 3.2.7. *Suppose $N = 1$ and $f = y^m x^n$. Consider an exterior region $R_{X(\lambda)}^{\epsilon,0}(y)$. For $\epsilon << 1$ and $\delta << 1$ we have*

$$Var(R_{X(\lambda)}^{\epsilon,0}(y)) \leq \frac{\pi}{\tilde{\nu}(X)-1} + \zeta$$

for all $(y, \lambda) \in B(0, \delta) \times \mathbb{S}^1_{\tilde{\nu}(X)}$.

Next proposition implies that the trajectories in the exterior zone do not spiral around the singular points of X.

PROPOSITION 3.2.4. *Let $N \geq 1$ and $\zeta > 0$. Let $f_j = 0$ be an irreducible component of $f/y^m = 0$. Consider an exterior region $R_{X(\lambda)}^{\epsilon,\eta}(y)$. For $\epsilon << 1$, $\delta << 1$ and $\eta >> 1$ we have*

$$|Img \circ lnf_j(x_1, y_0) - Img \circ lnf_j(x_0, y_0)| \leq \frac{\pi}{\tilde{\nu}(X) - 1} + \zeta$$

for $(x_0, y_0), (x_1, y_0) \in R_{X(\lambda_0)}^{\epsilon,\eta}(y_0)$ and for all $(y_0, \lambda_0) \in B(0, \delta) \times \mathbb{S}^1_{\tilde{\nu}(X)}$.

PROOF. We have $f_j = u_j(x, y)(x - g_j(y))$ for some unit $u_j \in \mathbb{C}\{x, y\}$. We can suppose that $u_j(0, 0) = 1$ by replacing f_j with $f_j/u_j(0, 0)$. Since $|g_j(y)| < D|y|$ for $y \in B(0, \delta)$ and $\delta << 1$ we have that

$$\ln f_j - \ln x = \ln u_j(x, y) + \ln\left(1 - \frac{g_j(y)}{x}\right)$$

tends to 0 if $(\epsilon, \delta, \eta) \to (0, 0, \infty)$. The result of the lemma is then a consequence of proposition 3.2.3. $\square$

3.3. The magnifying glass

We want to understand the behavior of $Re(X(\lambda))$ in U_ϵ. We consider the sets $U_\epsilon^{\eta,+}$ and $U_\epsilon^{\eta,-}$ for suitable $\epsilon > 0$ and $\eta > 0$. We pointed out in lemma 3.2.5 and proposition 3.2.3 that the dynamics of $Re(X(\lambda))$ and $Re(X_{00}(\lambda))$ in $U_\epsilon^{\eta,+}$ are analogous. As a consequence, we can focus in the dynamical behavior in the magnifying glass $U_\epsilon^{\eta,-}$.

Let $(x, y) = (wy, y)$; we consider $Y = [(yw, y)^* X]/y^{m+n_1+\ldots+n_N-1}$. More precisely, if $X = u(x, y)y^m(x - g_1(y)^{n_1} \ldots (x - g_N(y)^{n_N} \partial/\partial x$ then

$$Y = u(wy, y)\left(w - \frac{g_1(y)}{y}\right)^{n_1} \ldots \left(w - \frac{g_N(y)}{y}\right)^{n_N} \frac{\partial}{\partial w}.$$

Moreover, we have

$$(yw, y)^* X(\lambda) = |y|^{n_1+\ldots+n_N-1} Y(e^{i(n_1+\ldots+n_N-1)\arg(y)}\lambda).$$

The set $U_\epsilon^{\eta,-} \setminus [y = 0]$ is equal to $[|w| \leq \eta] \setminus [y = 0]$. As a consequence to describe the behavior of $Re(X(\lambda))$ in $U_\epsilon^{\eta,-}$ for all $\lambda \in \mathbb{S}^1$ it is enough to describe the behavior of $Re(Y(\lambda))$ in $[|w| \leq \eta]$ for all $\lambda \in \mathbb{S}^1$. The curve $w = g_j(y)/y$ intersects $y = 0$ at the point $(w, y) = ((\partial g_j/\partial y)(0), 0)$ for $1 \leq j \leq N$. We consider the set

$$F = \left\{\frac{\partial g_1}{\partial y}(0), \ldots, \frac{\partial g_N}{\partial y}(0)\right\}.$$

We choose $\eta > 0$ such that $F \subset [|w| < \eta]$.

Let $c \in F$; there are two cases depending whether or not

$$N_c \stackrel{def}{=} \sharp\{j \in \{1, \ldots, N\} \ : \ (c, 0) \in [w - g_j(y)/y = 0]\}$$

is equal to 1. If $N_c > 1$ we consider $V_{c,k(c)} = [|w - c| \leq k(c)]$ for some $k(c) > 0$ small enough. Otherwise $c = (\partial g_{j_0}/\partial y)(0)$ for a unique $1 \leq j_0 \leq N$ and we define $V_{c,k(c)} = [|w - g_{j_0}(y)/y| \leq k(c)]$ for some $k(c) > 0$ small enough. The dynamics of $Re(Y(\lambda))$ in $V_{c,k(c)}$ is simple for $N_c = 1$ and $k(c) << 1$ because of lemmas 3.2.6 and 3.2.7.

Between the exterior zone and the sets $V_{c,k(c)}$ $(c \in F)$ there is a set VC such that VC is the closure of $[|w| \leq \eta] \setminus \cup_{c \in F} V_{c,k(c)}$ deprived of $y = 0$. Since $[|w| \leq \eta] \setminus \cup_{c \in F} V_{c,k(c)}$ is compact in (w, y) coordinates we say that VC is a *compact-like* basic set. The set VC does not contain singular points of X; hence the dynamics of $Re(Y(\lambda))$ is simple in VC. Then, we are down to the point of describing the dynamics of $Re(Y(\lambda))$ in $V_{c,k(c)}$ for $N_c > 1$; this task is pretty much the original one just replacing (X, U_ϵ) with $(Y, V_{c,k(c)})$. Fortunately, the latter goal is easier because we can separate all the components of $f/y^m = 0$ by repeating this process a finite number of times. Indeed, we are just desingularizing the curve $f_1 \dots f_N = 0$. At the end of the process we have only exterior sets, compact-like sets and domains of the form $[|w| < k]$ such that $[|w| < k] \cap SingX = [w = 0]$ in some coordinates (w, y) . These latter sets behave like exterior sets and then the domain U_ϵ is partitioned in exterior and compact-like sets. All the sets in the partition are called basic sets; they are dynamically simple.

Example: Let $f = x^2 (x-y)^2 (x-y^2)^2$. The first exterior zone is of the form $U_\epsilon^{\eta,+}$ for some $\eta > 1$. We have $F = \{0, 1\}$. The curves $w = 0$ and $w = y$ pass through $(w, y) = (0, 0)$ whereas $w = 1$ pass through $(w, y) = (1, 0)$. Thus, for $k(0) > 0$ and $k(1) > 0$ small enough we have

$$V_{0,k(0)} = [|w| \leq k(0)] \text{ and } V_{1,k(1)} = [|w-1| \leq k(1)]$$

We have $VC = [|w| \leq \eta] \setminus ([|w| < k(0)] \cup [|w-1| < k(1)])$. Since V_{0,k_0} contains two irreducible components of $SingX$ then we consider the exterior zone $(U_\epsilon^{\eta',+})' = [|y|\eta' \leq |w| \leq k(0)]$ for some $\eta' >> 0$. For $k'(0) > 0$ and $k'(1) > 0$ small enough we define the sets

$$V'_{0,k'(0)} = [|w'| \leq k'(0)] \text{ and } V'_{1,k'(1)} = [|w'-1| \leq k'(1)].$$

in the system of coordinates (w', y) given by $(w, y) = (w'y, y)$. We also define $VC' = [|w'| \leq \eta'] \setminus ([|w'| < k'(0)] \cup [|w'-1| < k'(1)])$. The basic sets are

$$U_\epsilon^{\eta,+} \ , \ VC \ , \ V_{1,k(1)} \ , \ (U_\epsilon^{\eta',+})' \ , \ VC' \ , \ V'_{0,k'(0)} \text{ and } V'_{1,k'(1)}.$$

The picture 5 corresponds to this example.

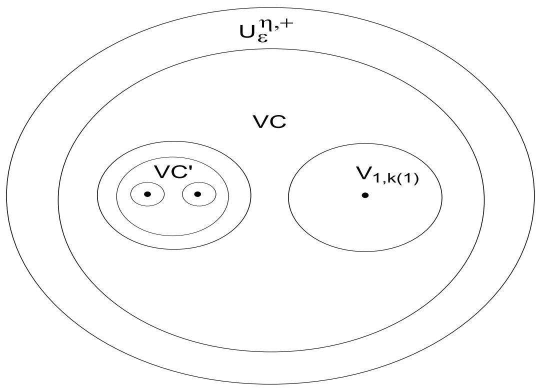

FIGURE 5. Partition of $U_\epsilon \cap [y = s]$ in basic sets

3.3.1. Dynamical finiteness of the partition.

Any trajectory

$$\Gamma_{\xi(X(\lambda))}^{|x|\leq\epsilon}[x, y][t_0, t] \subset \overline{U_\epsilon}$$

is divided in several sub-trajectories entirely contained in the basic sets. More precisely there exists a sequence $t_0 < t_1 < \ldots < t_k = t$ such that

- $\Gamma_{\xi(X(\lambda))}^{|x|\leq\epsilon}[x,y][t_j,t_{j+1}] \subset B_j$ for a basic B_j and all $0 \leq j \leq k-1$.
- $B_j \neq B_{j+1}$ for all $0 \leq j \leq k-2$.

We denote $split(\Gamma_{\xi(X(\lambda))}^{|x|\leq\epsilon}[x,y][t_0,t]) = k$. The definition implies that $Re(X(\lambda))$ is transversal to ∂B_j at $\Gamma_{\xi(X(\lambda))}^{|x|\leq\epsilon}[x,y](t_j)$ for $1 \leq j \leq k-1$.

LEMMA 3.3.1. *Suppose $N \geq 1$. There exists $K > 0$ such that*

$$split(\Gamma_{\xi(X(\lambda))}^{|x|\leq\epsilon}[x,y][t_0,t]) \leq K$$

for all possible trajectories of $\xi(X(\lambda))$ in $\overline{U_\epsilon} \cap [y \in B(0,\delta)]$.

PROOF. We have $split(\Gamma_{\xi(X(\lambda))}^{|x|\leq\epsilon}[x,0][t_0,t]) = 1$ since there is only one basic set at $y=0$. Consider a connected component C of the boundary of a basic set B. The set $C \cap [y=y_0]$ for $y_0 \neq 0$ encloses some singular points, namely $(g_{j_1}(y_0),y_0)$, $\ldots$, $(g_{j_r}(y_0),y_0)$. The indexes j_1, $\ldots$, j_r do not depend on y_0. Now consider $Tg(C) = 2(n_{j_1} + \ldots + n_{j_r} - 1)$. By construction the set $C \cap [y=y_0]$ is tangent to $Re(X(\lambda))$ in $Tg(C)$ points for all $y_0 \in B(0,\delta) \setminus \{0\}$ and all $\lambda \in \mathbb{S}^1$. We define $Tg = \sum_{C \in J} Tg(C)$ where J is the sets of boundaries of basic sets except ∂U_ϵ. For all $(y_0,\lambda) \in (B(0,\delta)\setminus\{0\}) \times \mathbb{S}^1$ the set $\cup_{C\in J} C$ is a union of Tg points and Tg open arcs which are transversal to $Re(X(\lambda))$. Therefore

$$split(\Gamma_{\xi(X(\lambda))}^{|x|\leq\epsilon}[x,y_0][t_0,t]) \leq Tg + 1$$

by the Rolle property. □

3.3.2. The variation is uniformly bounded. We define $X_g^V(y_0,\lambda)$ the set of couples of the form $((x_0,y_0)(x_1,y_0))$ such that $(x_0,y_0) \in V \setminus [g=0]$ and $(x_1,y_0) \in \Gamma_{\xi(X(\lambda)),+}^{[|x|\leq\epsilon]\cap V}[x_0,y_0]$. We define

$$Var_g((x_0,y),(x_1,y)) = |Img \circ \ln \circ g(x_1,y) - Img \circ \ln \circ g(x_0,y)|$$

and

$$Var_g^V(X)(y_0,\lambda) = \sup_{((x_0,y),(x_1,y)) \in X_g^V(y_0,\lambda)} Var_g((x_0,y),(x_1,y)).$$

Finally we define $Var_g^V(X) = \sup_{(y_0,\lambda)\in B(0,\delta)\times\mathbb{S}^1} Var_g^V(X)(y_0,\lambda)$. We denote $Var_g^V(X)$ by $Var_g^{\epsilon,\delta}(X)$ if $V = \overline{U_\epsilon} \cap [y \in B(0,\delta)]$. The decomposition of the dynamics in basic sets provides the basis to bound the variation $Var_{f_j}^{\epsilon,\delta}(X)$.

PROPOSITION 3.3.1. *Suppose $N \geq 1$. Let $1 \leq j \leq N$; then we have $Var_{f_j}^{\epsilon,\delta}(X) < \infty$ for $\epsilon << 1$ and $\delta(\epsilon) << 1$.*

PROOF. By proposition 3.2.4 we have that $Var_{f_j}^{\epsilon,\delta}(X)(0,\lambda)$ is bounded by any constant greater than $\pi/(\tilde{\nu}(X)-1)$ if we make $\epsilon > 0$ small enough. It is enough to bound $Var_{f_j}^{\epsilon,\delta}(X)(y,\lambda)$ in $(B(0,\delta)\setminus\{0\}) \times \mathbb{S}^1$.

We have $f_j = u_j(x,y)(x - g_j(y))$ for some unit $u_j \in \mathbb{C}\{x,y\}$. Since $\ln u_j(x,y)$ is a holomorphic function in $U_{\epsilon,\delta}$ for $\epsilon << 1$ and $\delta << 1$ then we can suppose that $f_j = x - g_j(y)$.

Consider $(x_1,y_0) = \Gamma_{\xi(X(\lambda))}^{|x|\leq\epsilon}[x_0,y_0](t)$ for some $t > 0$ and $y_0 \neq 0$. There exists $0 = t_0 < \ldots < t_k = t$ such that $\Gamma_{\xi(X(\lambda))}^{|x|\leq\epsilon}[x,y_0][t_j,t_{j+1}]$ is contained in a basic set for $0 \leq j \leq k-1$. Moreover, we can suppose $k < K$ for a constant $K > 0$ only depending on X by lemma 3.3.1. As a consequence it is enough to prove $Var_{f_j}(X)((x_0,y_0),(x_1,y_0)) < D_B$ for a constant $D_B > 0$ depending only on X if $\Gamma^{|x|\leq\epsilon}_{\xi(X(\lambda))}[x_0,y_0][0,t]$ is contained in a basic set B.

If B is the first exterior set then we can choose D_B to be any positive number greater than $\pi/(\tilde{\nu}(X)-1)$ by proposition 3.2.4. Otherwise, we define the vector field $Y = (wy,y)^* X / y^{m+n_1+\ldots+n_p-1}$ and $f_j' = w - g_j'(y) = w - g_j(y)/y$. Since

$$\ln(x - g_j(y))(wy,y) = \ln y + \ln(w - g_j(y)/y)$$

then $Var_{f_j}((x_0,y_0),(x_1,y_0)) = Var_{f_j'}((w_0,y_0)(w_1,y_0))$ where we denote $w_l = x_l/y_0$ for $l \in \{0,1\}$. Moreover, we have

$$(w_1,y_0) \in \Gamma^{|x|\leq\epsilon}_{\xi(Y(e^{i(n_1+\ldots n_N-1)\arg(y_0)}\lambda))}[w_0,y_0][0,t|y_0|^{n_1+\ldots+n_N-1}] \subset B.$$

As a consequence it is enough to bound $Var^B_{f_j'}(Y)(y,\lambda)$ for all (y,λ) in $B(0,\delta)\times\mathbb{S}^1$. If B is the first compact-like set VC we remark that $SingY \cap VC = \emptyset$ and that $VC = [|w| \leq \eta] \setminus \cup_{c\in F} \overset{o}{V}_{c,k(c)}$ is compact. Therefore $Var^B_{f_j'}(Y)(y,\lambda)$ is un upper semi-continuous function and then bounded in the compact set $\overline{B}(0,\delta/2)\times\mathbb{S}^1$.

If $c \in F \setminus \{g_j'(0)\}$ then f_j' is a unit in the simply connected set $V_{c,k(c)}$ and then $\ln f_j'$ is holomorphic. We can choose

$$D_B = \max_{P\in V_{c,k(c)}\cap[y\in B(0,\delta)]} Img \circ \ln f_j'(P) - \min_{P\in V_{c,k(c)}\cap[y\in B(0,\delta)]} Img \circ \ln f_j'(P)$$

for all $B \subset V_{c,k(c)}$.

If $c = g_j'(0)$ and $B \subset V_{c,k(c)}$ then we just iterate the process. In this way we find a bound D_B for all basic set B. □

We just proved that spiraling wildly around the singular points is excluded for $Re(X)$ if X is a (NSD) vector field. Because of the absence of irregular behavior the topological type of X can be characterized in terms of the residue functions.

3.3.3. The compact-like sets. The only basic sets which can support non topologically trivial dynamics (with respect to y) are the compact-like sets. These sets are the places where the interesting phenomena regarding the evolution of the dynamics are located.

Let $X = u(x,y)y^m(x-g_1(y))^{n_1}\ldots(x-g_N(y))^{n_N}\partial/\partial x$. We denote $c_j = (\partial g_j/\partial y)(0)$. Let $X^{00} = u(0,0)(x-c_1y)^{n_1}\ldots(x-c_Ny)^{n_N}\partial/\partial x$. Since

$$T^{|x|<\epsilon}_{X_{00}(e^{i(\arg(y_0)+\theta)m})} = e^{-\frac{m}{\tilde{\nu}(X)-1}i\theta} T^{|x|<\epsilon}_{X_{00}(e^{i\arg(y_0)m})}$$

then the points in $T^{|x|<\epsilon}_{X_{00}(e^{i\arg(y)m})}$ move at speed $-m/(\tilde{\nu}(X)-1)$ with respect to $\arg(y)$. We have $X = |y|^m X(e^{i\arg(y)m})$; hence the situation for $T^{|x|<\epsilon}_X(y)$ is very similar because the derivative of $\arg^\epsilon_X$ with respect to $\arg x$ at $((x,y),\lambda)$ tends to $\tilde{\nu}(X)-1$ if $(\epsilon,y)\to 0$. As a consequence the points in $T^{|x|<\epsilon}_X(y)$ move at a speed tending to $-m/(\tilde{\nu}(X)-1)$ with respect to $\arg(y)$ if $(\epsilon,y)\to 0$.

We consider

$$Y = u(wy,y)(w-g_1(y)/y)^{n_1}\ldots(w-g_N(y)/y)^{n_N}\partial/\partial w$$

and

$$Y_0 = Y_{00} = u(0,0)(w-c_1)^{n_1}\ldots(w-c_N)^{n_N}\frac{\partial}{\partial w}.$$

The vector field $Y(e^{i(m+n_1+\ldots+n_p-1)\arg(y)})$ is equal to $(wy,y)^*X$ up to a positive multiplicative function. Since the limit of the dynamics of $Y(e^{i(m+\tilde{\nu}(X)-1)\arg(y)})$ when $y \to 0$ is $Y_{00}(e^{i(m+\tilde{\nu}(X)-1)\arg(y)})$ we will focus in the latter vector field. We remark that $(wy,y)^*X^{00}(e^{im\arg(y)})$ is equal to $Y_{00}(e^{i(m+\tilde{\nu}(X)-1)\arg(y)})$ up to a multiplicative positive function. Therefore, studying the behavior of Y_{00} and X^{00} in the first compact-like zone VC are equivalent goals.

For $y_1 = y_0 e^{(i\pi k)/(m+\tilde{\nu}(X)-1)}$ we have that

$$Y_{00}(e^{i(m+\tilde{\nu}(X)-1)\arg(y_1)}) = (-1)^k Y_{00}(e^{i(m+\tilde{\nu}(X)-1)\arg(y_0)}).$$

We have that $Re(Y_{00}(e^{i(m+\tilde{\nu}(X)-1)\arg(y_1)}))$ and $Re(Y_{00}(e^{i(m+\tilde{\nu}(X)-1)\arg(y_0)}))$ are topologically equivalent by the mapping

$$H_k : (w,y) \mapsto (w, e^{i\frac{\pi k}{m+\tilde{\nu}(X)-1}} y).$$

This mapping is equal to

$$H_k : (x,y) \mapsto (xe^{i\frac{\pi k}{m+\tilde{\nu}(X)-1}}, e^{i\frac{\pi k}{m+\tilde{\nu}(X)-1}} y).$$

expressed in (x,y) coordinates. Suppose $k=1$. We have

$$H_1(T^{\epsilon,j}_{X^{00}(e^{im\arg(y_0)})}(y_0)) = e^{i\frac{\pi}{m+\tilde{\nu}(X)-1}} T^{\epsilon,j}_{X^{00}(e^{im\arg(y_0)})}(y_0).$$

We also have

$$T^{\epsilon,j}_{X^{00}(e^{im\arg(y_1)})}(y_1) \sim e^{i\frac{-m\pi}{(\tilde{\nu}(X)-1)(m+\tilde{\nu}(X)-1)}} T^{\epsilon,j}_{X^{00}(e^{im\arg(y_0)})}(y_0)$$

since the speed of the tangent points of $T^{|x|<\eta|y|}_{X^{00}(e^{im\arg(y)})}(y)$ move at speed close to $-m/(\tilde{\nu}(X)-1)$ with respect to $\arg(y)$ for $\eta >> 0$. Then

$$T^{\epsilon,j+1}_{X^{00}(e^{im\arg(y_1)})}(y_1) \sim e^{i\left(\frac{-m\pi}{(\tilde{\nu}(X)-1)(m+\tilde{\nu}(X)-1)}+\frac{\pi}{\tilde{\nu}(X)-1}\right)} T^{\epsilon,j}_{X^{00}(e^{im\arg(y_0)})}(y_0)$$

implies

$$T^{\epsilon,j+1}_{X^{00}(e^{im\arg(y_1)})}(y_1) = H_1(T^{\epsilon,j}_{X^{00}(e^{im\arg(y_0)})}(y_0)).$$

By iteration we obtain

$$T^{\epsilon,j+k}_{X^{00}(e^{im\arg(y_1)})}(y_1) = H_k(T^{\epsilon,j}_{X^{00}(e^{im\arg(y_0)})}(y_0)).$$

for $k \geq 0$. The application H_k changes the roles of the tangent points, forcing dynamics to rotate. The dynamics is not topologically trivial in VC with respect to the parameter y except if all the tangent points in $T^{|x|<\eta|y|}_{X^{00}(e^{im\arg(y)})}(y)$ play the same role, i.e. $c_a = c_b$ for all $a, b \in \{1, \ldots, N\}$. Since the irreducible components of $f=0$ are separated by the desingularization process then for $N > 1$ there are compact-like basic sets supporting non topologically trivial dynamics.

CHAPTER 4

The T-sets

4.1. Unstable set and bi-tangent cords

We define $UN_X^\epsilon \subset B(0,\delta)$ such that $y_0 \in B(0,\delta) \setminus UN_X^\epsilon$ if there exists a continuous family $\sigma_y : [|x| \leq \epsilon] \to [|x| \leq \epsilon]$ of oriented homeomorphisms for y in a neighborhood W of y_0 such that

- $\sigma_{y_0} \equiv Id$
- $\xi(X, y_0, \epsilon)$ and $\xi(X, s, \epsilon)$ are topologically equivalent by σ_s.

Consider the projections π_S and π_T obtained by restraining to $SingX$ and T_X^ϵ respectively the mapping $(x, y) \mapsto y$. The mappings π_S and π_T are ramified coverings in their domains of definition. Their ramification places satisfy $(ram(\pi_S) \cup ram(\pi_T)) \cap (B(0,\delta) \setminus \{0\}) = \emptyset$ by the choice of the domain $U_{\epsilon,\delta}$ and proposition 3.1.2. As a consequence we obtain

$$\begin{array}{rcl} SingX \cap [y = re^{i\theta}] \cap U_\epsilon & = & \{S_X^1(r,\theta), \ldots, S_X^N(r,\theta)\} \\ T_X^\epsilon \cap [y = re^{i\theta}] & = & \{T_X^{\epsilon,1}(r,\theta), \ldots, T_X^{\epsilon,2(\tilde{\nu}(X)-1)}(r,\theta)\} \end{array}$$

for $0 \leq r << 1$ and $\theta \in \mathbb{R}$. The sections S_X^j and $T_X^{\epsilon,k}$ are real analytic. The list $L_X^\epsilon(s)$ associated to $\xi(X, s, \epsilon)$ is composed of sets of the types

$$\{S^a(s), T^{\epsilon,b}(s)\} \ , \ \{T^{\epsilon,a,a+1}(s), T^{\epsilon,b}(s)\} \text{ and } \{T^{\epsilon,a}(s), T^{\epsilon,b}(s)\}.$$

When we vary the parameter s the first two types persist locally. On the other hand, the sets of type $\{T^{\epsilon,a}, T^{\epsilon,b}\}$ are unstable. We call *bi-tangent cords* the critical trajectories containing two tangent points. We will describe the set of parameters containing a bi-tangent cord; this set is the natural candidate to be $UN_X^\epsilon \setminus \{0\}$.

4.1.1. Partitions of the singular points and the basic formula. We suppose $SingX \not\subset [y = 0]$; otherwise there are no singular points to deal with. Let $0 < \epsilon' < \epsilon$; there exists a small $c(\epsilon') > 0$ such that $SingX$ and $[\epsilon'/2 \leq |x| \leq \epsilon] \times [0 < |y| \leq c(\epsilon')]$ are disjoint. Let (x_0, y_0, λ) be an element of the set $[\epsilon' \leq |x| \leq \epsilon] \times \overline{B(0, c(\epsilon'))} \times \mathbb{S}^1$; we define $ML(x_0, y_0, \lambda)$ the maximum non negative number such that

$$\Gamma_{\xi(X(\lambda)),+}^{\epsilon' \leq |x| \leq \epsilon}[x_0, y_0][0, ML(x_0, y_0, \lambda)] \subset [\epsilon' \leq |x| \leq \epsilon].$$

It is straightforward to check out that ML is upper semi-continuous and then it attains its maximum in $[\epsilon' \leq |x| \leq \epsilon] \times \overline{B(0, c(\epsilon'))} \times \mathbb{S}^1$. We denote this maximum by $MX(\epsilon')$.

Consider a trajectory $\gamma : [0, t] \to \overline{U_\epsilon} \cap [y = y_0]$ of $Re(X(\lambda_0))$ for some $(y_0, \lambda_0) \in \overline{B(0, c(\epsilon'))} \times \mathbb{S}^1$. Suppose also that $\gamma(0), \gamma(t) \in [|x| \geq \epsilon']$ and that $t > MX(\epsilon')$. We claim that γ splits the singular points. We notice that γ intersects $|x| = \epsilon'$ since otherwise we would have $t \leq MX(\epsilon')$. Moreover $\sharp(\gamma \cap [|x| = \epsilon']) \geq 2$, this is a consequence of the convexity of the tangent points. Suppose $\sharp(\gamma \cap [|x| = \epsilon']) = 2$, let $\gamma(a)$ and $\gamma(b)$ $(0 \leq a < b \leq t)$ be the points in $\gamma \cap [|x| = \epsilon']$. We denote $\beta_{\epsilon'} = \gamma[a,b]$. Let κ_γ be a path in $\partial U_{\epsilon'} \cap [y = y_0]$ going from $\gamma(a)$ to $\gamma(b)$ in counter clock wise sense. The path $\beta_{\epsilon'}\kappa_\gamma^{-1}$ encloses a connected component $C_-(y_0)$ of $(U_{\epsilon'} \cap [y = y_0]) \setminus \beta_{\epsilon'}$, the latter set has another connected component that we denote by $C_+(y_0)$. We define

$$E_-(y_0) = C_-(y_0) \cap SingX \quad \text{and} \quad E_+(y_0) = C_+(y_0) \cap SingX.$$

Since $SingX \cap [y = y_0] \subset U_{\epsilon'}$ then $(E_-(y_0), E_+(y_0))$ induces a partition of the singular points. We can extend continuously E_- and E_+ to the set $[0 \leq r \leq c(\epsilon')] \cap [\theta \in \mathbb{R}]$; more precisely if $E_-(r_0 e^{i\theta_0})$ is equal to $\{S_X^{j_1}(r_0,\theta_0), \ldots, S_X^{j_d}(r_0,\theta_0)\}$ then $E_-(r,\theta) = \{S_X^{j_1}(r,\theta), \ldots, S_X^{j_d}(r,\theta)\}$.

We can play basically the same game if $\sharp(\gamma \cap [|x| = \epsilon']) > 2$; let $a = t_1 \leq \ldots \leq t_{2k} = b$ the sequence of times in which $\gamma[0,t]$ intersects $\partial U_{\epsilon'}$. For $1 \leq j < k$ we choose $t_{2j} = t_{2j+1}$ if $\gamma(t_{2j}) \in T_{X(\lambda_0)}^{\epsilon'}(r_0,\theta_0)$. We choose $t_{2j-1} < t_{2j}$ for all $1 \leq j \leq k$. Consider a couple (t_{2j}, t_{2j+1}) for $1 \leq j < k$. We have $\gamma[t_{2j}, t_{2j+1}] \subset |x| \geq \epsilon'$; we define $\gamma(t_l')$ as

$$\{\gamma(t_l')\} = \Gamma_{\xi(X(\lambda_0),y_0),(-1)^{l+1}}^{\epsilon'/2 \leq |x| \leq \epsilon}[\gamma(t_l)] \cap \partial U_{\epsilon'/2}$$

for $l \in \{2j, 2j+1\}$. The trajectory $\gamma[t_{2j}', t_{2j+1}']$ is homotopic in the set $\overline{U_\epsilon} \setminus SingX$ to a path γ^j contained in $\partial U_{\epsilon'/2}$ and whose initial and ending points are $\gamma(t_{2j}')$ and $\gamma(t_{2j+1}')$ respectively (see picture 1). The path $\beta_{\epsilon'} = \gamma[t_1, t_2']\gamma^1\gamma[t_3', t_4'] \ldots \gamma^{k-1}\gamma[t_{2k-1}', t_{2k}]$

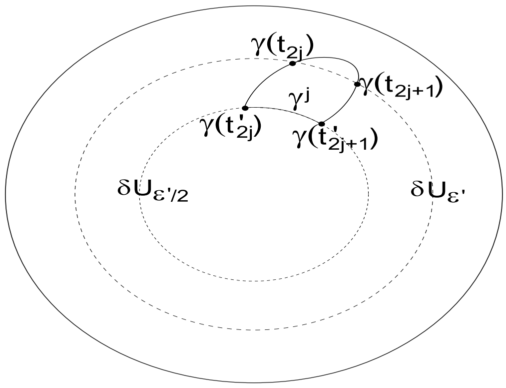

FIGURE 1. Changing $\gamma[t_{2j}, t_{2j+1}]$ by γ^j

is contained in $\overline{U_{\epsilon'}}$; moreover $\beta_\epsilon' \setminus \{\gamma(a), \gamma(b)\} \subset U_{\epsilon'}$. We stress that γ does not cut twice any exterior region in $\epsilon'/2 \leq |x| \leq \epsilon$ because of the Rolle property. As a consequence the path $\beta_{\epsilon'}$ is simple. We can nowdefine C_-, C_+, E_- and E_+ in an analogous way than for the case $\sharp(\gamma \cap [|x| = \epsilon']) = 2$.

If we consider $\epsilon' < \epsilon'' < \epsilon$ then the partitions of the singular points induced by $\beta_{\epsilon'}$ is the same one than the partition induced by $\beta_{\epsilon''}$. Of course, the same result holds for $\epsilon'' < \epsilon'$ if $|y_0| < c(\epsilon'')$.

We introduce the formula that is going to allow us to make a qualitative description of the dynamics of $Re(X)$. We remind the reader that m is the only non-negative integer such that $y^m | f$ but $y^{m+1} \not| f$. Let $\psi_0(., y_0)$ be an integral of the time form of $X(1)$ defined in a neighborhood of $\gamma(0)$ in $y = y_0$; we extend ψ_0

analytically along the path $\gamma[0,a]\kappa_\gamma\gamma[b,t]$ to obtain an integral $\psi_1(.,y_0)$ of the time form of $X(1)$ defined in the neighborhood of $\gamma(t)$ in $y=y_0$. Let $\psi_1'(.,y_0)$ be the integral of the time form of $X(1)$ defined in the neighborhood of $\gamma(t)$ and obtained by analytic continuation along $\gamma[0,t]$. By the properties of the integral of the time form we have

$$t = \frac{\psi_1'}{\lambda_0}(\gamma(t)) - \frac{\psi_0}{\lambda_0}(\gamma(0)).$$

The theorem of residues implies that

$$t = \frac{\psi_1}{\lambda}(\gamma(t)) - 2\pi i \sum_{P\in E_-(y_0)} Res_{X(\lambda)}(P) - \frac{\psi_0}{\lambda}(\gamma(0)).$$

We will use the right hand side of the previous formula to calculate the time that $Re(X(\lambda))$ spends to join two points in the same trajectory.

There is a reason to replace ψ_1' with ψ_1. Suppose we have a sequence of trajectories $\gamma_n[0,t_n]\subset \overline{U_\epsilon}\cap[y=y_n]$ such that $y_n\neq 0$ for all $n\in\mathbb{N}$ and $\lim_{n\to\infty} y_n=0$. We also ask γ_n to fulfill that $I=\lim_{n\to\infty}\gamma_n(0)$ and $L=\lim_{n\to\infty}\gamma_n(t_n)$ exist and that they are both different than $(0,0)$. Consider $0<\epsilon'<\epsilon$ such that $\overline{U_{\epsilon'}}$ contains neither I nor L. The limit of the paths $\gamma_n[0,t_n]$ does not necessarily exist, moreover if it exists it can be non-simple. Despite of this, the limit of $\gamma[0,a_n]\kappa_{\gamma_n}\gamma[b_n,t_n]$ has always a limit; the limit is a path σ. We can now define ψ_0 to be an integral of the time form of $X(1)$ defined in the neighborhood of I in $\mathbb{C}^2$ whereas we define ψ_1 to be the analytic continuation along σ. The formula

$$\text{Time} = \frac{\psi_1}{\lambda}(\text{final pt.}) - 2\pi i \sum_{P\in E_-(r,\theta)} Res_{X(\lambda)}(P) - \frac{\psi_0}{\lambda}(\text{initial pt.})$$

involves holomorphic functions ψ_0 and ψ_1 whereas ψ_1' can not be chosen holomorphic in the neighborhood of L. In this way we relate the complexity of the dynamics with the residue functions.

4.1.2. Cords. We consider sections of the form $S:(\mathbb{R}_{\geq 0},0)\times\mathbb{R}$ such that

- $S(r,\theta)\in\overline{U_\epsilon}\cap[y=re^{i\theta}]$ for all $(r,\theta)\in(\mathbb{R}_{\geq 0},0)\times\mathbb{R}$.
- $S(0,\theta)\neq(0,0)$ for all $\theta\in\mathbb{R}$.
- $S(r,\theta+2\pi k)=S(r,\theta)\ \forall(r,\theta)\in(\mathbb{R}_{\geq 0},0)\times\mathbb{R}$ and some $k\in\mathbb{N}$.
- $S(r,\theta)$ is real analytic in $(\mathbb{R}_{\geq 0},0)\times\mathbb{R}$.

We call them *nice* sections for X in U_ϵ.

Example: A trivial example is $S'(r,\theta)=(x_0,re^{i\theta})$ for some $x_0\neq 0$. The standard example is $S(r,\theta)=T_X^{\epsilon,j}(r,\theta)$; in this case $\theta\to\theta+2\pi$ induces a permutation in $T_X^\epsilon(re^{i\theta})$. We obtain $S(r,\theta)=S(r,\theta+2\pi k)$ for some $k\in\mathbb{N}$; moreover, we can choose $k=|\tilde{\nu}(X)-1|$.

For two nice sections $S_0(r,\theta)$ and $S_1(r,\theta)$ we say that they have no finite connection on $H\subset\mathbb{R}$ if

- $\omega_{\xi(X(e^{i\theta m})),([|x|<\epsilon]\cup\{S_0(0,\theta)\})}(S_0(0,\theta))=(0,0)$ for all $\theta\in H$.
- $\alpha_{\xi(X(e^{i\theta m})),([|x|<\epsilon]\cup\{S_1(0,\theta)\})}(S_1(0,\theta))=(0,0)$ for all $\theta\in H$.
- $S_1(0,\theta)\notin\Gamma^{|x|\leq\epsilon}_{\xi(X(e^{i\theta m})),+}[S_0(0,\theta)]$ for all $\theta\in H$.

We will always suppose that H is closed and invariant by $\theta\to\theta+2\pi k$ for some $k\in\mathbb{Z}\setminus\{0\}$. We say that S_0 and S_1 have no finite connection if they have no finite connection on $\mathbb{R}$. As a consequence we obtain

LEMMA 4.1.1. *Let S_0 and S_1 be two nice sections for X in U_ϵ with no finite connection on H. Then, for all $C > 0$ there exists $K(C) > 0$ such that*

$$S_1(r,\theta) \notin \Gamma^{|x|\leq\epsilon}_{\xi(X(e^{i\theta m})),+}[S_0(r,\theta)][0,C]$$

for all $(r,\theta) \in B(0,K(C)) \times [\cup_{\theta' \in H} B(\theta', K(C))]$.

REMARK 4.1.1. *By last lemma the trajectories of $Re(X)$ from $S_0(r,\theta)$ to $S_1(r,\theta)$ induce a partition of $SingX$ for (r,θ) close to $\{0\} \times H$.*

Consider two nice sections S_0 and S_1 with no finite connection on H for X in U_ϵ. We can define a holomorphic integral ψ_0 of the time form of $X(1)$ in an open set containing $S_0(r,\theta)$ for $r << 1$ and $\theta \in \mathbb{R}$; we just define ψ_0 in a neighborhood of $S_0(0,0)$ and then we make analytic continuation. We choose $\epsilon' > 0$ such that $S_j(0,\theta) \notin \overline{U_{\epsilon'}}$ for $j \in \{0,1\}$; by lemma 4.1.1 we can use the process in subsection 4.1.1 to define a holomorphic ψ_1 for parameters in the neighborhood of $\{0\} \times H$. We consider a continuous partition $(E_-(r,\theta), E_+(r,\theta))$ of the singular points. We define

$$I_{S_0,S_1,E}(r,\theta) = \frac{\psi_1}{e^{i\theta m}}(S_1(r,\theta)) - 2\pi i r^m \sum_{P \in E_-(s)} Res_X(P) - \frac{\psi_0}{e^{i\theta m}}(S_0(r,\theta)).$$

The function $I_{S_0,S_1,E}(r,\theta)$ is real analytic outside $r = 0$ where it is maybe not defined because $\sum_{P \in E_-(s)} Res_X(P)$ is the ramification of a meromorphic function. We denote by $T_{S_0,S_1,E}$ the set of parameters (r_0,θ_0) such that there exists a trajectory $\gamma[0,t]$ in $\overline{U_\epsilon} \cap [y = r_0 e^{i\theta_0}]$ of $Re(X)$ satisfying $\gamma(0) = S_0(r_0,\theta_0)$, $\gamma(1) = S_1(r_0,\theta_0)$ and inducing the partition $(E_-(r_0,\theta_0), E_+(r_0,\theta_0))$. We have $(r_0,\theta_0) \notin \{0\} \times H$ by the no finite connection hypothesis. We obtain $t = I_{S_0,S_1,E}(r_0,\theta_1)/r_0^m$ since we have $X = |y|^m X(e^{i\theta m})$. The next lemma is an immediate consequence of the previous discussion.

LEMMA 4.1.2. *Let S_0 and S_1 be nice sections for X in U_ϵ with no finite connection on H and let $E = (E_-, E_+)$ be a continuous partition of $SingX$. Then the germ of $T_{S_0,S_1,E}$ at $\{0\} \times H$ is contained in $I^{-1}_{S_0,S_1,E}(\mathbb{R}^+)$.*

We define $T_{S_0,S_1} = \cup_{E \in J} T_{S_0,S_1,E}$ where J is the set of continuous partitions (E_-, E_+) of $SingX$.

PROPOSITION 4.1.1. *If $\mu(\sum_{P \in E_-(s)} Res_X(P)) \leq m$ then the germ of the set $T_{S_0,S_1,E}$ at $\{0\} \times H$ is empty.*

PROOF. We choose $0 < \epsilon' < \epsilon$ such that $\epsilon' < \min_{(j,\theta) \in \{0,1\} \times \mathbb{R}} S_j(0,\theta)$. Since the length of the trajectories of $Re(X(e^{i\theta m}))$ is bounded by $MX(\epsilon')$ on $\epsilon' \leq |x| \leq \epsilon$ and $[|x| = \epsilon'] \cap [|y| \leq c(\epsilon')]$ is compact then $[\psi_1(S_1(r,\theta)) - \psi_0(S_0(r,\theta))]/e^{i\theta m}$ is bounded for (r,θ) belonging to $T_{S_0,S_1,E} \cap V$ for some neighborhood V of $\{0\} \times H$. Therefore, the hypothesis implies that there exists $C > 0$ such that $I_{S_0,S_1,E}(r,\theta) < C$ if $(r,\theta) \in T_{S_0,S_1,E} \cap V'$ for some neighborhood V' of $\{0\} \times H$. We deduce that $T_{S_0,S_1,E} = \emptyset$ by lemma 4.1.1. □

We can focus on the partitions satisfying $\mu(\sum_{P \in E_-(s)} Res_X(P)) > m$.

LEMMA 4.1.3. *Suppose $\mu(\sum_{P \in E_-(s)} Res_X(P)) > m$. Then $I^{-1}_{S_0,S_1,E}(\mathbb{R}^+)$ is a finite union of branches of analytic sets.*

PROOF. We denote $I_{S_0,S_1,E}$ by I. The hypothesis of the lemma is invariant under ramification as well as the real analytic sets. Hence, up to ramify by $R = (x, y^{N_1 \dots N_p})$ we can suppose that $\sum_{P \in E_-(y)} Res_X(P)$ is a meromorphic function. We have

$$-2\pi i \sum_{P \in E_-(y)} Res_X(P) = \frac{C}{y^d} + \sum_{j \in \mathbb{Z}_{>-d}} C_j y^j = \frac{C}{y^d} + O\left(\frac{1}{y^{d-1}}\right).$$

for some $d \in \mathbb{Z}_{>m}$ and $C \in \mathbb{C}^*$. Hence, we obtain

$$I(r,\theta) r^{d-m} = Ce^{-i\theta d} + O(r).$$

Moreover, the function $I(r,\theta)r^{d-m}$ is real analytic. Since $I \in \mathbb{R}^+$ coincides with $Ir^{d-m} \in \mathbb{R}^+$ the set $I^{-1}(\mathbb{R}^+)$ adheres to the set

$$DL = \left\{ j \in \mathbb{Z} : \left(0, \frac{\arg C}{d} + \frac{2\pi j}{d}\right) \right\}.$$

Since S_0 and S_1 are nice we have $S_j(r, \theta + 2\pi k) = S_j(r,\theta)$ for some $k \in \mathbb{N}$ and all $j \in \{0,1\}$. As a consequence $I^{-1}(\mathbb{R}^+)$ is invariant by $(r,\theta) \to (r, \theta + 2\pi k)$. We deduce that $I^{-1}(\mathbb{R}^+)$ is also the union of the irreducible components of $I^{-1}(\mathbb{R}^+)$ adhering the finite set

$$DL' = \{0 \le j < kd : (0, \arg(C)/d + (2\pi j)/d\}.$$

Then it is enough to prove that in the neighborhood of a point $(0,\theta_0)$ in DL the set $I^{-1}(\mathbb{R}^+)$ is a branch of a real analytic set. Since

$$I(r,\theta) r^{d-m} = |C| e^{-i(\theta-\theta_0)d} + O(r)$$

then $Re(I(r,\theta)r^{d-m}) \in \mathbb{R}^+$ in the neighborhood of $(0,\theta_0)$. Moreover, we obtain

$$Img(I(r,\theta)r^{d-m}) = -|C|(\theta - \theta_0)d + O(r + (\theta-\theta_0)^2)$$

and then $Img(I(r,\theta)r^{d-m}) = 0$ is a smooth real analytic curve parameterized by r. It is still smooth in the y plane since it is transversal to the divisor $r = 0$. Moreover, its branch $[Img(I(r,\theta)r^{d-m}) = 0] \cap [r > 0]$ coincides with the germ of $I^{-1}(\mathbb{R}^+)$ at $(0,\theta_0)$. ☐

REMARK 4.1.2. *If all the components of $SingX$ different than $y = 0$ are parameterized by the coordinate y then $I^{-1}_{S_0,S_1,E}(\mathbb{R}^+)$ is a finite union of branches of smooth real analytic sets.*

4.1.3. Definition, analyticity and finiteness of the T-sets. Let S_0, S_1 be two nice sections (for X in U_ϵ) with no finite connection on H. We consider the set of curves $I^B_{S_0,S_1} = \{\beta_j\}_{j \in J}$ such that $\beta_j \in I^B_{S_0,S_1}$ if there exists a triple (L_0, L_1, E) such that

- $L_j \in \{S_j\} \cup_{l=1}^{2|\tilde{\nu}(X)-1|} \{T_X^{\epsilon,l}\}$ for $j \in \{0,1\}$.
- $\mu(\sum_{P \in E_-(s)} Res_X(P)) > m$.
- β_j is an irreducible component of $I^{-1}_{L_0,L_1,E}(\mathbb{R}^+)$.

The nice sections L_0 and L_1 do not have finite connections on H; this is a consequence of the definition of no finite connection for S_0, S_1 and the remark 3.2.3. If we restrict (L_0, L_1) to be (S_0, S_1) in the previous definition we obtain the set I_{S_0,S_1}; it clearly satisfies $I_{S_0,S_1} \subset I^B_{S_0,S_1}$. Since the tangent sections and the continuous partitions of $SingX$ are both finite sets then J is a finite set.

LEMMA 4.1.4. *Consider two nice sections S_0, S_1 for X in U_ϵ with no finite connection on H and a continuous partition $E = (E_-, E_+)$ of $SingX$. Let β be a semi-analytic curve such that $\overline{(re^{i\theta})^{-1}(\beta)} \cap [r = 0]$ is contained in $\{0\} \times H$. Then $\beta \cap T_{S_0,S_1,E} \neq \emptyset$ implies $\beta \subset T_{S_0,S_1,E}$.*

PROOF. We can suppose that $\beta \cap \beta_j \neq \emptyset$ implies $\beta \subset \beta_j$ by considering $U_{\epsilon,\delta}$ for a smaller $\delta > 0$. We can suppose $\mu(\sum_{P \in E_-(s)} Res_X(P)) > m$ by proposition 4.1.1. Since $\beta \cap I^{-1}_{S_0,S_1,E}(\mathbb{R}^+) \neq \emptyset$ then $\beta \subset I^{-1}_{S_0,S_1,E}(\mathbb{R}^+)$. Let $(r_0, \theta_0) \in \beta \cap T_{S_0,S_1,E}$. Consider the piece of trajectory $\gamma[t_0, t_1]$ of $Re(X)$ in $\overline{U_\epsilon} \cap [y = r_0 e^{i\theta_0}]$ such that $\gamma(t_j) = S_j(r_0, \theta_0)$ for $j \in \{0, 1\}$ and $\gamma[t_0, t_1]$ induces the partition $(E_-(r_0, \theta_0), E_+(r_0, \theta_0))$ of the equilibrium points. We consider the finite set $\gamma(t_0, t_1) \cap \partial U_\epsilon$ whose elements are

$$\gamma(d_1) = T_X^{\epsilon,a_1}(r_0, \theta) \ , \ \gamma(d_2) = T_X^{\epsilon,a_2}(r_0, \theta_0) \ , \ \dots \ , \ \gamma(d_h) = T_X^{\epsilon,a_h}(r_0, \theta_0)$$

for some $h \geq 0$. We suppose $t_0 = d_0 < d_1 < \dots < d_h < d_{h+1} = t_1$. The no finite connection hypothesis implies that $\gamma[a_k, a_{k+1}]$ induces a partition E_k of the equilibrium points for $0 \leq k \leq h$. We denote $A_0 = S_0$, $A_{h+1} = S_1$ and $A_k = T_X^{\epsilon,a_k}$ for $1 \leq k \leq h$.

For all $k \in \{0, \dots, h\}$ we define the set $H_k \subset T_{A_k,A_{k+1},E_k}$ composed by the lines $y_1 \in \beta$ such that there exists a trajectory $\gamma_k[c, d]$ of $Re(X)$ in $\overline{U_\epsilon} \cap [y = y_1]$ satisfying

$$\gamma_k(c) = A_k(y_1) \ , \ \gamma_k(d) = A_{k+1}(y_1) \text{ and } \gamma_k(c, d) \cap \partial U_\epsilon = \emptyset.$$

We have $r_0 e^{i\theta_0} \in H_k \subset T_{A_k,A_{k+1},E_k} \subset I^{-1}_{A_k,A_{k+1},E_k}(\mathbb{R}^+)$ for all $0 \leq k \leq h$. By proposition 4.1.1 we have that $\mu(\sum_{P \in E_{k,-}(s)} Res_X(P)) > m$; therefore $\beta \subset I^{-1}_{A_k,A_{k+1},E_k}(\mathbb{R}^+)$ for all $0 \leq k \leq h$. We deduce that every set H_k is open in β by continuity of the flow. As a consequence $T_{S_0,S_1,E}$ is open in β.

It is enough to prove that H_k is closed in β for all $0 \leq k \leq h$ because then $T_{S_0,S_1,E} \subset \beta$ by connectedness. Suppose there exists y_1 in $\beta \cap [\overline{H_k} \setminus H_k]$, then $\overline{U_\epsilon} \cap [y = y_1]$ contains a trajectory $\gamma'_k[c, d]$ of $Re(X)$ satisfying

$$\gamma'_k(c) = A_k(y_1) \ , \ \gamma'_k(d) = A_{k+1}(y_1) \text{ and } \gamma'_k(c, d) \cap \partial U_\epsilon \neq \emptyset.$$

We choose a point $\gamma'_k(e) = T_X^{\epsilon,a_{k+1/2}}(y_1)$ in $\gamma'_k(c, d) \cap \partial U_\epsilon$. We denote $T_X^{\epsilon,a_{k+1/2}}$ by $A_{k+1/2}$. We denote by F and G the partitions of the equilibrium points induced by $\gamma'_k[c, e]$ and $\gamma'_k[e, d]$ respectively. By the first part of the proof the sets $T_{A_k,A_{k+1/2},F}$ and $T_{A_{k+1/2},A_{k+1},G}$ are open in β_j. Hence $y_1 \notin \overline{H_k}$, that is a contradiction. □

COROLLARY 4.1.1. *Let S_0, S_1 be nice sections with no finite connection on H for $Re(X)$ in U_ϵ. Then, the germ of T_{S_0,S_1} at $\{0\} \times H$ is a finite union of semi-analytic sets.*

PROOF. The set I_{S_0,S_1} is a finite union of branches of real analytic sets by lemma 4.1.3. We are done, since by lemma 4.1.4 the germ of T_{S_0,S_1} at $\{0\} \times H$ is the union of some branches of I_{S_0,S_1}. □

COROLLARY 4.1.2. *Let S_0, S_1 be nice sections with no finite connection for $Re(X)$ in U_ϵ. Then T_{S_0,S_1} is a finite union of semi-analytic sets.*

By definition a *T-set* is a connected component of the set of parameters $\cup_{j \neq k} T_{T_X^{\epsilon,j}, T_X^{\epsilon,k}}$ containing a bi-tangent cord. The results in this section imply

PROPOSITION 4.1.2. *Let $X = f\partial/\partial x$ be a germ of vector field defined in $U_{\epsilon,\delta}$ and satisfying the (NSD) conditions. Every T-set is a branch of real analytic curve. Moreover, there are finitely many T-sets.*

COROLLARY 4.1.3. *$UN_X^\epsilon \setminus \{0\}$ is the union of the T-sets*

4.2. Dynamical instability

So far we did not prove the existence of a (NSD) vector field X having at least one T-set; this is the aim of this section.

4.2.1. Definition and properties of zones. We call zones the connected components of $B(0,\delta) \setminus UN_X^\epsilon$. We can enumerate the T-sets $\beta_1, \ldots, \beta_l, \beta_{l+1} = \beta_1$ by using a counter clock wise order. If $UN_X^\epsilon \setminus \{0\} = \emptyset$ then there is only one zone $Z_{X,1}^\epsilon$. Otherwise there are exactly l zones; we denote by $Z_{X,j}^\epsilon$ $(1 \leq j \leq l)$ the zone whose boundary contains the set $\beta_j \cup \beta_{j+1}$. We will use the notation Z_j^ϵ if the vector field X is implicitly known.

A zone Z_j^ϵ adheres to either a point or to a closed arc of directions. In the former case it is a *narrow* zone, otherwise it is a *wide* zone.

LEMMA 4.2.1. *Suppose $\tilde{\nu}(X) > 0$. Let Z_X^ϵ be a wide zone. Then for all $y_0 \in Z_X^\epsilon$ we have*

$$(\alpha_{\xi(X,y_0)}, \omega_{\xi(X,y_0)})_{|x|<\epsilon}^{-1}(\infty,\infty) = \emptyset.$$

PROOF. We define

$$D(y_1) = (\alpha_{\xi(X,y_1)}, \omega_{\xi(X,y_1)})_{|x|<\epsilon}^{-1}(\infty,\infty)$$

for $y_1 \in B(0,\delta)$. Suppose that there exists $y_0 \in Z^\epsilon$ such that $D(y_0) \neq \emptyset$. In such a case $D(y_1) \neq \emptyset$ for all $y_1 \in Z^\epsilon$ because $Z^\epsilon \cap UN_X^\epsilon = \emptyset$. We replace X with $-X$ if necessary to obtain a point $T_X^{\epsilon,a}(y)$ such that

$$\gamma^y \stackrel{def}{=} \overline{\Gamma_{\xi(X),+}^{(|x|<\epsilon)\cup\{T_X^{\epsilon,a}(y)\}}[T_X^{\epsilon,a}(y)]}$$

is a critical tangent cord (see subsection 2.1.4) for all $y \in Z^\epsilon$. The continuous curve $y \to \gamma^y$ induces a continuous partition $E(y)$ of the equilibrium points. Let $Q(y)$ be the only point in $(\gamma^y \cap \partial U_\epsilon) \setminus \{T_X^{\epsilon,a}(y)\}$ for all $y \in Z^\epsilon$. We define

$$I(y) = \frac{\psi_1}{e^{i\theta m}}(Q(y)) - 2\pi i r^m \sum_{P \in E_-(y)} Res_X(P) - \frac{\psi_0}{e^{i\theta m}}(T_X^{\epsilon,a}(y))$$

for $y = re^{i\theta} \in Z^\epsilon$. Let $d = \mu(\sum_{P\in E_-(s)} Res_X(P))$. We proceed as in the proof of proposition 4.1.1 to show that $[\psi_1(Q(y)) - \psi_0(T_X^{\epsilon,a}(y))]/e^{i\theta m}$ is bounded in Z^ϵ. Moreover, we can also obtain that if $d \leq m$ then there exists $D > 0$ such that $I < D$ in Z^ϵ. Since $\partial U_\epsilon \cap \gamma^y(0,D] = \emptyset$ by continuity of the flow and remark 3.2.3 then $d > m$. We obtain $I(y)|y|^{d-m} = Ce^{-i\theta d} + O(r^{1/q})$ for some $C \in \mathbb{C}^*$ and $q \in \mathbb{N}$ like in the proof of lemma 4.1.3. The set $I^{-1}(\mathbb{R}^+)$ adheres to the d directions in $C\lambda^{-d} \in \mathbb{R}^+$. Hence Z^ϵ adheres to a finite set of directions; since Z^ϵ is connected then it is a narrow zone. □

We explain now that the graph $\mathcal{G}_{\xi(X,y_0)}^{|x|<\epsilon}$ is connected for most of the parameters.

COROLLARY 4.2.1. *Let Z_X^ϵ be a wide zone. Then for all $y_0 \in Z_X^\epsilon$ the graph $\mathcal{G}_{\xi(X,y_0)}^{|x|<\epsilon}$ is connected.*

PROOF. If $\tilde{\nu}(X) = 0$ then the graph has no vertexes and it is clearly connected. Otherwise, lemma 4.2.1 and proposition 2.1.3 imply that the graph is connected. $\square$

4.2.2. The graph does not have permanent edges. Since $\mathcal{G}_{\xi(X,y)}^{|x|<\epsilon}$ is connected for most of the parameters the absence of permanent edges will imply the existence of T-sets for $\tilde{\nu}(X) > 0$.

PROPOSITION 4.2.1. *There is not an edge* $S_X^j(y) \to S_X^k(y)$ *in* $\mathcal{G}_{\xi(X,y)}^{|x|<\epsilon}$ *for all* $y \in B(0,\delta) \setminus \{0\}$.

We clarify the statement. We consider a point $r_0 e^{i\theta_0} \in B(0,\delta) \setminus \{0\}$ and an edge $S_X^j(r_0,\theta_0) \to S_X^k(r_0,\theta_0)$. The equilibrium points $S_X^j(r,\theta)$ and $S_X^k(r,\theta)$ are obtained by analytical prolongation. Hence, the proposition only makes real sense in $[0 < r < \delta] \cap [\theta \in \mathbb{R}]$.

PROOF. Up to ramify by R we can suppose that all the components of $SingX$ different than $y = 0$ are parameterized by y. Suppose there is a permanent edge $(\Delta_1(y), y) \to (\Delta_2(y), y)$ for all $y \in B(0,\delta) \setminus \{0\}$. The vector field X can be expressed in the form

$$X = (x - \Delta_1(y))^{l_1} (x - \Delta_2(y))^{l_2} h(x,y) \frac{\partial}{\partial x}$$

where $l_j \geq 2$ and $\gcd(h(x,y), x - \Delta_j(y)) = 1$ for all $j \in \{1,2\}$. We denote by X_j the germ of X at $x = \Delta_j(y)$ for $j \in \{1,2\}$. We have that

$$\Theta_1^-(y) \stackrel{def}{=} \Theta^-(\xi(X_1,y)) = \zeta_1(y)\{1, e^{(2\pi i)/(l_1-1)}, \ldots, e^{(2\pi i)(l_1-2)/(l_1-1)}\}$$

where

$$\zeta_1(y) = \sqrt[l_1-1]{\frac{|\Delta_1(y) - \Delta_2(y)|^{l_2} |h(\Delta_1(y),y)|}{(\Delta_1(y) - \Delta_2(y))^{l_2} h(\Delta_1(y),y)}}.$$

The directions in $\Theta_1^-(y_0)$ turn

$$-C_1 = -\frac{\nu((\Delta_2(y) - \Delta_1(y))^{l_2} h(\Delta_1(y),y))}{l_1 - 1}$$

times (in counter clock wise sense) when y travels along $\theta \mapsto y_0 e^{2\pi i\theta}$ ($\theta \in [0,1]$). By convention to turn a negative amount of radians in counter clock wise sense is the same thing than turning in clock wise sense. In an analogous way the directions in $\Theta^+(\xi(X_2,y_0))$ turn

$$-C_2 \stackrel{def}{=} -\frac{\nu((\Delta_2(y) - \Delta_1(y))^{l_1} h(\Delta_2(y),y))}{l_2 - 1}$$

times around $x = \Delta_2(y_0)$ when y goes along the path $\theta \mapsto y_0 e^{2\pi i\theta}$ ($\theta \in [0,1]$). We define

$$D(y_0) = (\alpha_{\xi(X)}, \omega_{\xi(X)})_{|x|<\epsilon}^{-1}((\Delta_1(y_0), y_0), (\Delta_2(y_0), y_0)).$$

We denote by $D'(y_0)$ the set of trajectories of $\xi(X, y_0, \epsilon)$ contained in $D(y_0)$. The set $D(y_0)$ is connected for all $y_0 \in B(0,\delta) \setminus \{0\}$ by lemma 2.1.7. Thus $D(y_0)$ adheres to unique directions $\lambda_1(y_0) \in \Theta^-(\xi(X_1,y_0))$ and $\lambda_2(y_0) \in \Theta^+(\xi(X_2,y_0))$ by proposition 2.2.1.

Consider the real blow-up ρ of the curves $x = \Delta_1(y)$ and $x = \Delta_2(y)$. If $\gamma \in D'(y_0)$ we define $\tilde{\gamma} = \overline{\rho^{-1}(\gamma)}$. The starting point of $\tilde{\gamma}$ is $\lambda_1(y_0)$ whereas the ending point of $\tilde{\gamma}$ is $\lambda_2(y_0)$. Let γ_1, γ_2 in $D'(y_0)$; by lemma 2.1.9 there exists an homotopy $\tilde{\gamma}_{1+c}$ ($c \in [0,1]$) where $\gamma_{1+c} \in D'(y_0)$ for all $c \in [0,1]$. We denote by $\tilde{\gamma}(y)$ the unique homotopy class induced by the liftings of the elements of $D'(y)$.

Fix $y_0 \in B(0,\delta) \setminus \{0\}$ and consider the path $\theta \mapsto y_0 e^{2\pi i \theta}$ ($\theta \in [0,1]$). Since the starting points of $\tilde{\gamma}(y_0)$ and $\tilde{\gamma}(y_0 e^{2\pi i})$ are equal then $C_1 \in \mathbb{N}$. In an analogous way we obtain that $C_2 \in \mathbb{N}$. For $j \in \{1,2\}$ we choose a loop σ_j in $\rho^{-1}(\Delta_j(y_0), y_0)$ turning once in counter clock wise sense; we also ask σ_j for having $\lambda_j(y_0)$ as initial and ending point. We define $D_j = C_j + \nu(\Delta_1(y) - \Delta_2(y))$ for $j \in \{1,2\}$. We travel along the path $\theta \to \tilde{\gamma}(y_0 e^{2\pi i \theta})$ ($\theta \in [0,1]$) to obtain

$$\tilde{\gamma}(y_0 e^{2\pi i}) = \sigma_1^{D_1} \tilde{\gamma}(y_0) \sigma_2^{D_2}.$$

We also know that $\tilde{\gamma}(y_0) = \tilde{\gamma}(y_0 e^{2\pi i})$. This is a contradiction, since the topological type of $\rho^{-1}(y = y_0)$ is a figure eight and $D_1 \neq 0 \neq D_2$. $\square$

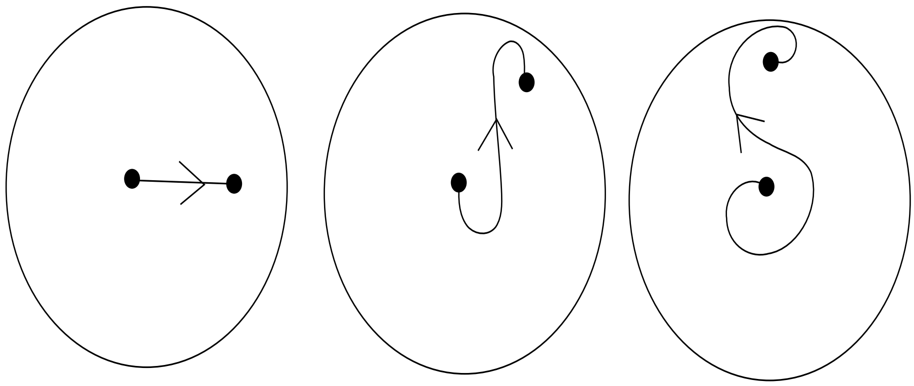

FIGURE 2. $X = x^2 (x-y)^2 \partial/\partial x$. Parameters $\theta = 0, 1/8, 1/4$

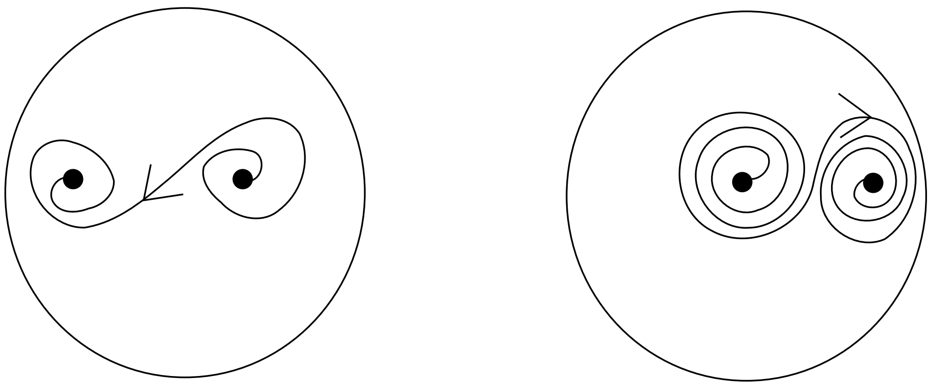

FIGURE 3. Parameters $\theta = 1/2$ and $\theta = 1$

Example: We consider $X = x^2 (x-y)^2 \partial/\partial x$. For all $y_0 \in \mathbb{R}^+$ the real line is invariant by $\xi(X, y_0, \epsilon)$. Moreover $(0, y_0) \to (y_0, y_0)$ belongs to $\mathcal{G}_{\xi(X,y_0)}^{|x|<\epsilon}$. The pictures 2 and 3 illustrate the evolution of $\tilde{\gamma}(e^{2\pi\theta i} y_0)$ supposed $(0,y) \to (y,y)$ is a permanent edge of the graph. We have $\tilde{\gamma}(e^{2\pi i}y_0) = \sigma_1^3\tilde{\gamma}(y_0)\sigma_2^3$ and as a consequence the paths $\tilde{\gamma}(e^{2\pi i}y_0)$ and $\tilde{\gamma}(y_0)$ are not homotopic.

We defined N as $\sharp(SingX \cap U_\epsilon \cap [y = y_0])$ for $y_0 \in B(0,\delta) \setminus \{0\}$; the number N does not depend on y_0.

COROLLARY 4.2.2. *Let X be a (NSD) vector field defined in $U_{\epsilon,\delta}$. If $N > 1$ then there is at least a T-set, i.e. $UN_X^\epsilon \cap (B(0,\delta) \setminus \{0\}) \neq \emptyset$.*

PROOF. If there are no T-sets then the only zone is $UN_X^\epsilon \setminus \{0\}$. Since it is wide the graph is connected. Therefore, there is at least a permanent edge in the graphs $\mathcal{G}_{\xi(x,y)}^{|x|<\epsilon}$. That contradicts proposition 4.2.1. $\square$

Next lemma focuses on the evolution of the dynamics in the neighborhood of a point in the limit fiber $y = 0$.

LEMMA 4.2.2. *Suppose that $N > 1$ and $(y = 0) \not\subset SingX$. Let $(x_0, 0)$ be a point contained in $U_\epsilon \setminus \{(0,0)\}$ such that $\omega_{\xi(X),|x|<\epsilon}(x_0, 0) = (0,0)$. Then the set*

$$\{y \in B(0,\delta) : \omega_{\xi(X),|x|<\epsilon}(x_0, y) = \infty\}$$

adheres to 0.

PROOF. Suppose the result is false. Then $(\Delta(y), y) = \omega_{\xi(X),|x|<\epsilon}(x_0, y)$ belongs to $SingX$ for y in some neighborhood $B(0,\eta)$ of 0. The mapping Δ is continuous by remark 2.2.1; hence Δ is an analytic function. The vector field X can be expressed in the form

$$X = (x - \Delta(y))^l h(x,y)\frac{\partial}{\partial x}.$$

We consider the real blow-up ρ of the curve $x = \Delta(y)$. We define

$$\tilde{\gamma}(y) = \overline{\rho^{-1}\left(\Gamma_{\xi(X),+}^{|x|<\epsilon}[x_0, y]\right)}.$$

for all $y \in B(0,\eta)$. The curve $\tilde{\gamma}(y)$ intersects $\rho^{-1}(\Delta(y), y)$ at a point $\lambda(y)$. Fix $y_0 \in B(0,\eta) \setminus \{0\}$. Let $\sigma[0,1]$ be the loop obtained by turning once in counter clock wise sense in $\rho^{-1}(\Delta(y_0), y_0)$ and such that $\sigma(0) = \sigma(1) = \lambda(y_0)$. We define $C = -\nu(h(\Delta(y), y))/(l-1)$. We can proceed as in the proof of proposition 4.2.1 to obtain that $C \in \mathbb{Z}$; we have

$$\tilde{\gamma}(y_0)\sigma^C \sim \tilde{\gamma}(y_0 e^{2\pi i}) = \tilde{\gamma}(y_0).$$

On the one hand $N > 1$ implies $C < 0$, on the other hand $\rho^{-1}(\Delta(y_0), y_0)$ has the homotopical type of $\mathbb{S}^1$, thus $\tilde{\gamma}(y_0)\sigma^C \sim \tilde{\gamma}(y_0)$ implies $C = 0$. That is a contradiction. $\square$

4.3. Disassembling the graph

Let $\mathcal{G}$ be an oriented graph. We denote by $Sing(\mathcal{G})$ and $\Gamma(\mathcal{G})$ the sets of vertexes and edges of $\mathcal{G}$ respectively. By definition $\mathcal{G} \subset \mathcal{G}'$ if $Sing(\mathcal{G}) \subset Sing(\mathcal{G}')$ and $\Gamma(\mathcal{G}) \subset \Gamma(\mathcal{G}')$. We define a graph $\mathcal{G}\&\mathcal{G}'$ such that $Sing(\mathcal{G}\&\mathcal{G}') = Sing(\mathcal{G}) \cap Sing(\mathcal{G}')$ and $\Gamma(\mathcal{G}\&\mathcal{G}') = \Gamma(\mathcal{G}) \cap \Gamma(\mathcal{G}')$.

Let $\mathcal{G}$ be an oriented graph such that $Sing(\mathcal{G}) \subset Sing\xi(X, y_0, \epsilon)$. We can associate a graph $\mathcal{G}(s)$ to any s contained in the universal covering of $B(0,\delta) \setminus \{0\}$. By definition the vertex $S_X^j(s)$ is in $Sing(\mathcal{G}(s))$ if $S_X^j(y_0)$ is in $Sing(\mathcal{G}(y_0))$. In an analogous way $S_X^j(s) \to S_X^k(s)$ is in $\Gamma(\mathcal{G}(s))$ if $S_X^j(y_0) \to S_X^k(y_0)$ is in $\Gamma(\mathcal{G})$.

We define $\mathcal{G}_{y_0} = \mathcal{G}_{\xi(X,y_0)}^{|x|<\epsilon}$. Next result is a consequence of remark 2.2.1.

LEMMA 4.3.1. *Let $y_0 \in B(0,\delta) \setminus \{0\}$. Let $\mathcal{G}$ be an oriented graph whose set of vertexes is $Sing\xi(X, y_0, \epsilon)$. Then $\mathcal{G} \subset \mathcal{G}_{y_0}$ implies*

$$\mathcal{G}(s) \subset \mathcal{G}_s$$

for all s in some neighborhood of y_0.

REMARK 4.3.1. *By considering $\mathcal{G} = \mathcal{G}_{y_0}$ in the previous lemma we obtain that the mapping $y \mapsto \mathcal{G}_y$ is lower semicontinuous.*

LEMMA 4.3.2. *Let $\lambda : [0,1] \to B(0,\delta) \setminus \{0\}$ be a path such that $\lambda[0,1]$ is completely contained in either $B(0,\delta) \setminus UN_X^\epsilon$ or in UN_X^ϵ. Then $\mathcal{G}_{\lambda(0)}(\lambda(1)) = \mathcal{G}_{\lambda(1)}$.*

PROOF. We define the set $UN_\lambda \subset [0,1]$ such that $t_0 \notin UN_\lambda$ if there is a continuous family of oriented homeomorphisms $\sigma_t : [|x| \leq \epsilon] \to [|x| \leq \epsilon]$ for t in a neighborhood W of t_0 in $[0,1]$ satisfying that

- $\sigma_{t_0} \equiv Id$
- $\xi(X, \lambda(t_0), \epsilon)$ and $\xi(X, \lambda(t), \epsilon)$ are topol. equivalent by σ_t.

We have that $t_0 \in UN_\lambda$ if there exists $\{T^{\epsilon,a}(\lambda(t_0)), T^{\epsilon,b}(\lambda(t_0))\}$ in $L_X^\epsilon(\lambda(t_0))$ but $\{T^{\epsilon,a}(\lambda(t)), T^{\epsilon,b}(\lambda(t))\}$ does not belong to $L_X^\epsilon(\lambda(t))$ for all t in a neighborhood of t_0 in $[0,1]$. By hypothesis $UN_\lambda = \emptyset$, thus the list $L_X^\epsilon(\lambda(t))$ is constant for $t \in [0,1]$. Since the list determines the graph (proposition 2.1.4) then $\mathcal{N}\mathcal{G}_{\lambda(0)}(\lambda(t)) = \mathcal{N}\mathcal{G}_{\lambda(t)}$ for all $t \in [0,1]$. Hence $\mathcal{G}_{\lambda(0)}(\lambda(0)) = \mathcal{G}_{\lambda(0)}$ implies $\mathcal{G}_{\lambda(0)}(\lambda(t)) = \mathcal{G}_{\lambda(t)}$ for all $t \in [0,1]$ since the orientation of an edge remains constant in connected sets. □

We enumerate the T-sets $\beta_1, \ldots, \beta_l$ and the zones $Z_{X,1}^\epsilon, \ldots, Z_{X,l}^\epsilon$ as in section 4.2. Let $y_0 \in Z_1^\epsilon$, we define the graph

$$\mathcal{G}^1(s) = \mathcal{G}_{y_0}(s)$$

for all $s \in \overline{Z_1^\epsilon} \setminus \{0\}$. This definition does not depend on y_0 by lemma 4.3.2. If $l = 0$ we define

$$\mathcal{G}^1(s) = \mathcal{G}^2(s) = \mathcal{G}^3(s) = \ldots$$

for all $s \in B(0,\delta) \setminus \{0\}$. For $l \geq 1$ we provide an inductive definition. Suppose we already defined $\mathcal{G}^j(s)$ for $s \in \overline{Z_j^\epsilon} \setminus \{0\}$. Let $y_1 \in \beta_{j+1}$. We define $\mathcal{G}^{j+1}(y_1) = \mathcal{G}_{y_1} \& \mathcal{G}^j(y_1)$. For $y \in \overline{Z_{j+1}^\epsilon} \setminus \{0\}$ the graph $\mathcal{G}^{j+1}(y)$ is obtained by continuous prolongation of $\mathcal{G}^{j+1}(y_1)$. The definition does not depend on y_1 by lemma 4.3.2.

LEMMA 4.3.3. *For all $j > 1$ and $y \in \beta_j$ we have $\mathcal{G}^j(y) \subset \mathcal{G}_y$. For all $j \geq 1$ and $y \in Z_j^\epsilon$ we have $\mathcal{G}^j(y) \subset \mathcal{G}_y$.*

PROOF. The first statement is a direct consequence of the construction. The second statement is trivial for $j = 1$. Suppose $j > 1$ and let $y_1 \in \beta_j$; we have $\mathcal{G}^j(y_1) \subset \mathcal{G}_{y_1}$. Since $y_1 \in \overline{Z_j^\epsilon}$ there exists $y_2 \in Z_j^\epsilon$ such that $\mathcal{G}^j(y_2) \subset \mathcal{G}_{y_2}$ by lemma 4.3.1. Thus we obtain

$$\mathcal{G}^j(y) = [\mathcal{G}^j(y_2)](y) \subset [\mathcal{G}_{y_2}](y) = \mathcal{G}_y$$

for all $y \in Z_j^\epsilon$ by lemma 4.3.2. □

Consider the sequence of graphs $\{\mathcal{G}^{jl+1}(y_0)\}_{j \geq 0}$. We have

PROPOSITION 4.3.1. *There exists $M \in \mathbb{N} \cup \{0\}$ such that $\mathcal{G}^{Ml+1}(y_0)$ does not have any edge.*

PROOF. We denote by M_j the number of edges of the graph $\mathcal{G}^{jl+1}(y_0)$; by construction we have $M_j \geq M_{j+1}$ for all $j \geq 0$. Suppose the lemma is false, then there exists $k \geq 0$ such that $M_j = D > 0$ for all $j \geq k$. Since

$$(S_X^a(y_0) \to S_X^b(y_0)) \in \mathcal{G}^{kl+1}(y_0) \Rightarrow (S_X^a(s) \to S_X^b(s)) \in \mathcal{G}_{\xi(X,s)}^{|x|<\epsilon}$$

for all s in the universal covering of $B(0,\delta) \setminus \{0\}$ our assumption contradicts proposition 4.2.1. $\square$

The next couple of lemmas is devoted to study what kind of splitting induces $\mathcal{G}_y$ in $\mathcal{G}^j(y)$ when $y \in \beta_{j+1}$.

LEMMA 4.3.4. *Suppose $UN_X^\epsilon \setminus \{0\} \neq \emptyset$. Let $j \geq 1$ and $y_1 \in \beta_{j+1}$. Let C be a connected component of $\mathcal{G}^j(y_1)$. Then $\xi(X, y_1, \epsilon)$ separates the connected components of $\mathcal{G}^{j+1}(y_1)$ whose sets of vertexes are contained in $Sing(C)$.*

PROOF. Let $C_1 \subset C$ and $C_2 \subset C$ be two non-empty connected components of $\mathcal{G}^{j+1}(y_1)$. We have $C_k \subset \mathcal{G}_{y_1}$ for all $k \in \{1,2\}$ since $\mathcal{G}^{j+1}(y_1) \subset \mathcal{G}_{y_1}$. Suppose $\xi(X, y_1, \epsilon)$ does not separate C_1 and C_2, then there exists a connected subgraph D of $\mathcal{G}_{y_1}$ such that $C_k \subset D$ for $k \in \{1,2\}$. We ask D for having as few vertexes as possible. The graph D is unique because of the absence of cycles in $\mathcal{N}\mathcal{G}_{y_1}$ (lemma 2.1.11). We have $D(y) \subset \mathcal{G}_y$ for all y in a neighborhood of y_1 by lemma 4.3.1. Since $\mathcal{N}\mathcal{G}_y$ has no cycles then $D(y) \subset C(y)$ for all y in Z_j^ϵ sufficiently close to y_1. We deduce that $D \subset C$. Since $D \subset C \subset \mathcal{G}^j(y_1)$ and $D \subset \mathcal{G}_{y_1}$ we obtain $D \subset \mathcal{G}^{j+1}(y_1)$. The connectedness of D implies $C_1 = C_2 = D$. $\square$

LEMMA 4.3.5. *Suppose $UN_X^\epsilon \setminus \{0\} \neq \emptyset$. Let $j \geq 1$ and $y_1 \in \beta_{j+1}$. Let γ be a critical tangent cord of $\xi(X, y_1, \epsilon)$. Then for every connected component C of $\mathcal{G}^j(y_1)$ except at most one, the set $Sing(C)$ is contained in a connected component of $(|x| < \epsilon) \setminus \gamma$.*

PROOF. Let $E = S_b^X(y_1) \to S_c^X(y_1)$ be an edge of $\mathcal{G}^j(y_1)$. We have $E(y) \subset \mathcal{G}_y$ for all $y \in Z_j^\epsilon$ by lemma 4.3.3. We define the set

$$D(y_0) = (\alpha_{\xi(X)}, \omega_{\xi(X)})_{|x|<\epsilon}^{-1}(S_b^X(y_0), S_c^X(y_0))$$

for all $y_0 \in Z_j^\epsilon$. The set $\partial D(y_0) \cap \partial U_\epsilon$ contains a convex tangent point $T_X^{\epsilon,a}(y_0)$ for all $y_0 \in Z_j^\epsilon$. We have

$$(\alpha_{\xi(X)}, \omega_{\xi(X)})_{|x|\leq\epsilon}(T_X^{\epsilon,a}(y_1)) = (S_b^X(y_1), S_c^X(y_1))$$

by continuity of the flow.

Let C be a connected component of $\mathcal{G}^j(y_1)$ such that $Sing(C)$ is not contained in a connected component of $(|x| < \epsilon) \setminus \gamma$. We choose E to be an edge $S_d^X(y_1) \to S_e^X(y_1)$ joining two points of $Sing(C)$ located in different connected components of $(|x| < \epsilon) \setminus \gamma$. By our previous discussion we have $(\alpha, \omega)(\Gamma_{\xi(X)}^{|x|\leq\epsilon}[Q]) = (S_d^X(y_1), S_e^X(y_1))$ for some $Q \in \partial U_\epsilon \cap (y = y_1)$. Since $(|x| \leq \epsilon) \setminus \overline{\gamma}$ has two connected components then $\Gamma_{\xi(X)}^{|x|\leq\epsilon}[Q] \cap \gamma \neq \emptyset$. We deduce that $\gamma \subset \Gamma_{\xi(X)}^{|x|\leq\epsilon}[Q]$ because γ is a piece of trajectory. We obtain

$$(\alpha_{\xi(X)}, \omega_{\xi(X)})_{|x|\leq\epsilon}(\gamma) \in Sing(C) \times Sing(C).$$

The last relation implies the uniqueness of C among the connected components of $\mathcal{G}^j(y_1)$ divided by γ. $\square$

CHAPTER 5

The L-limits

The previous chapter provides the first glimpse of a more general phenomenon: the limit of trajectories γ_n passing through the points $(x_n, y_n) \to (\zeta, 0)$ is not necessarily the trajectory passing through $(\zeta, 0)$. We will prove that for $N > 1$ the limit of the dynamics of $Re(X)_{|y=s}$ when $s \to 0$ is the complex flow of $X_{|y=0}$. This chapter is devoted to make rigorous the previous statement as well as to prove it.

5.1. Setup and non-oscillation properties

Throughout this section we define some concepts we will use to define the L-limits and to prove their main properties. We denote $y = a + ib$.

Let β_1, β_2 be semi-analytic curves; indeed they are branches of real analytic curves. The curve β_j adheres to a unique direction $\lambda = \lambda(\beta_j)$. Next, we define the *order of contact* $I(\beta_1, \beta_2)$. If $\lambda(\beta_1) \neq \lambda(\beta_2)$ then we define $I(\beta_1, \beta_2) = 1$. Otherwise, up to linear change of coordinates we have $\lambda(\beta_1) = \lambda(\beta_2) = 1$. There exists a Puiseux expantion $b = P_j(a)$ for $j \in \{1, 2\}$. We define $I(\beta_1, \beta_2) = \nu(P_1(a) - P_2(a))$, this is a positive rational number. Since $\lambda_j = 1$ then $\nu(P_j) > 1$ for $j \in \{1, 2\}$; as a consequence $I(\beta_1, \beta_2) > 1$ if β_1 and β_2 adhere to the same direction.

We will deal with meromorphic functions $A(y)$ up to a ramification $y \mapsto y^k$. Such a function does not oscillate when restricted to a semi-analytic curve.

Lemma 5.1.1. *Let β be a connected real semi-analytic curve in a neighborhood of $y = 0$ in $\mathbb{C}$. Consider a meromorphic complex analytic function $A(y^k)$ in a neighborhood of $y = 0$. For all $d \in \mathbb{N} \cup \{0\}$ we have*

$$\lim_{y \in \beta,\ y \to 0} |y|^d |A(y)| \neq \infty \implies \lim_{y \in \beta,\ y \to 0} |y|^d A(y) \in \mathbb{C},$$

$$\lim_{y \in \beta,\ y \to 0} |Img(|y|^d A(y))| \neq \infty \implies \lim_{y \in \beta,\ y \to 0} Img(|y|^d A(y)) \in \mathbb{R}.$$

Proof. If $A \equiv 0$ the result is obvious. Otherwise $A = \alpha y^c + o(y^c)$ for some $c \in \mathbb{Q}$ and $\alpha \in \mathbb{C} \setminus \{0\}$. If $c + d < 0$ then $\lim_{y \in \beta,\ y \to 0} |y|^d |A(y)| = \infty$ whereas if $c + d > 0$ then $\lim_{y \in \beta,\ y \to 0} |y|^d A(y) = 0$. If $c + d = 0$ we obtain

$$\lim_{y \in \beta,\ y \to 0} |y|^d A(y) = \alpha \lambda(\beta)^{-d}.$$

Let us prove the second property. There exists a sequence $y_k \in \beta$ such that $\lim_{k \to \infty} y_k = 0$ and $\lim_{k \to \infty} Img(|y_k|^d A(y_k))$ exists; we denote this limit by c. We define $e = \max(\mu(A(y)), d)$. Let η be any positive real number. The curves

$$I = [Img(|y|^e A(y)) = (c - \eta)|y|^{e-d}],\ D = [Img(|y|^e A(y)) = (c + \eta)|y|^{e-d}]$$

are real-analytic in coordinates $(r^{1/k}, \lambda)$. The curve β does not cut neither I nor D in a neighborhood of $(r, \lambda) = (0, \lambda(\beta))$; otherwise we obtain two semi-analytic curves intersecting each other infinitely many times. That implies $|Img(|y|^d A(y)) - c| < \eta$ for all $y \in \beta$ close to 0. Hence, we obtain $\lim_{y \in \beta,\ y \to 0} Img(|y|^d A(y)) = c$. $\square$

We focus now on evolution properties. Consider a meromorphic function $A(y^k)$ such that $\mu(A) > d$. Suppose that $\lim_{y \in \beta,\ y \to 0} Img(|y|^d A(y))$ exists. For $C \in \mathbb{R}$ we define the set of *contact curves* Υ_A^C as the set of semi-analytic curves such that $\beta' \in \Upsilon_A^C$ if $\lambda(\beta') = \lambda(\beta)$ and

$$\lim_{y \in \beta,\ y \to 0} Img(|y|^d A(y)) - \lim_{y \in \beta',\ y \to 0} Img(|y|^d A(y)) = C.$$

A *compact wedge* W of width $M \geq 0$ is by definition a connected, simply connected set W containing β such that $W = \cup_{C \in [-M,M]} \sigma_A^C$ where $\sigma_A^C \in \Upsilon_A^C$ for all $C \in [-M, M]$.

We prove next the existence of contact curves and compact wedges. We suppose $\lambda(\beta) = 1$ up to a linear change of coordinates. As a consequence the Puiseux expansion $b = P(a)$ of β satisfies $\nu(P) > 1$. For $\Delta \in \mathbb{R}$ we consider the curves $\beta(\Delta) : \mathbb{R}^+ \to \mathbb{C}$ such that

$$\beta(\Delta, a) = a + i[P(a) + \Delta a^{\mu(A)-d+1}].$$

Proposition 5.1.1. *Let $d < \mu(A)$. Suppose $\lim_{\beta \ni y \to 0} Img(|y|^d A(y)) \in \mathbb{R}$. Then, there exists $K \in \mathbb{R} \setminus \{0\}$ such that $\beta(CK)$ belongs to Υ_A^C for all $C \in \mathbb{R}$. Let $M > 0$. The set $\cup_{L \in [-M,M]} \beta(LK)$ is a compact wedge of width M. Moreover, the function*

$$(\Delta, a) \mapsto [|y|^d A(y)] \circ \beta(0, a) - [|y|^d A(y)] \circ \beta(\Delta, a)$$

is continuous in $[-M|K|, M|K|] \times [0 \leq a < \delta']$ for $\delta' > 0$ small enough.

Proof. We have

$$A = \frac{h_{-\mu(A)}}{y^{\mu(A)}} + \sum_{j \in J} \frac{h_{-j}}{y^j} + H(y) + \dots \quad (h_{\mu(A)} \neq 0)$$

where $J \subset [d, \mu(A)) \cap \mathbb{Q}$ is a finite set and H is a sum of monomials of degree bigger than $-d$. Let $F_j = h_{-j}/y^j$ for $d \leq j \leq \mu(A)$; we have

$$|y|^d F_j(y) = \left(\frac{|y|}{y}\right)^d \frac{h_{-j}}{y^{j-d}}.$$

By simple calculations we obtain

$$\frac{h_{-j}}{y^{j-d}} \circ \beta(0, a) - \frac{h_{-j}}{y^{j-d}} \circ \beta(\Delta, a) = i h_{-j} \Delta (j - d) a^{\mu(A)-j} + o(a^{\mu(A)-j})$$

where $lim_{\Delta \in E,\ a \to 0} o(a^{\mu(A)-j})/a^{\mu(A)-j} = 0$ for any compact set $E \subset \mathbb{R}$.

We have $\lim_{y \to 0} |y|^d H(y) = 0$, thus $(|y|^d H(y)) \circ \beta(\Delta, a)$ is continuous in $E \times (\mathbb{R}_{\geq 0}, 0)$ for any compact set $E \subset \mathbb{R}$. The analysis of A implies that $(|y|^d A(y)) \circ \beta(0, a) - (|y|^d A(y)) \circ \beta(\Delta, a)$ is of the form

$$i\Delta[h_{-\mu(A)}(\mu(A) - d)] + o(1)$$

for Δ in a compact set E. It is a continuous function in $E \times (\mathbb{R}_{\geq 0}, 0)$ for all compact set $E \subset \mathbb{R}$. We define $K = 1/(h_{-\mu(A)}[\mu(A) - d])$; we have that $\beta(CK)$ belongs to Υ_A^C for all $C \in \mathbb{R}$. $\square$

REMARK 5.1.1. *Suppose that besides* $\lim_{y\in\beta,\ y\to 0} Img(|y|^d A(y)) \in \mathbb{R}$ *we have* $\lim_{y\in\beta,\ y\to 0} Re(|y|^d A(y)) = +\infty$ *As a consequence we have*

$$h_{-\mu(A)} = lim_{y\to 0}|y|^{\mu(A)}A(y) \in \mathbb{R}^+ \cup \{0\}.$$

Since $h_{-\mu(A)} \neq 0$ *we obtain* $h_{-\mu(A)} \in \mathbb{R}^+$ *and then* $K \in \mathbb{R}^+$.

5.2. Definition of the L-limit

Let $X = f\partial/\partial x$ be a (NSD) vector field defined in a neighborhood of $\overline{U_\epsilon}$. Consider a semi-analytic curve β and a point $0 < x_0 \leq \epsilon$ such that $\omega_{\xi(X),|x|\leq\epsilon}(x_0, 0) = (0, 0)$. We are interested on describing the limit of $\Gamma_{\xi(X),+}[x_0, y]$ when $y \in \beta$ and $y \to 0$. Consider the decomposition $y^m f_1^{n_1} \dots f_p^{n_p}$ of f in irreducible factors. Let $0 < |x_1| \leq \epsilon$ be a point satisfying that there exists a sequence $\{(x_1^j, y_j)\}_{j\in\mathbb{N}}$ contained in $\mathbb{C}\times\beta$ and such that

- $(x_1, 0) = \lim_{j\to\infty}(x_1^j, y_j)$.
- $(x_1, 0) \notin \Gamma^{|x|\leq\epsilon}_{\xi(X(\lambda(\beta)^m)),+}[x_0, 0]$.
- For all $\eta > 0$ there exists $j(\eta) \in \mathbb{N}$ such that for all $j \geq j(\eta)$ we have $(x_1^j, y_j) \in \Gamma^{|x|<\epsilon+\eta}_{\xi(X,y_j,\epsilon+\eta),+}[x_0, y_j]$.

The set of points satisfying the previous conditions will be denoted by $L^{+,\epsilon}_{\beta,x_0}$; it is the positive L-limit associated to x_0, ϵ and β. We can define $L^{-,\epsilon}_{\beta,x_0}$ by replacing in the definition the positive trajectories with the negative ones. Next lemma is obvious.

LEMMA 5.2.1. *A L-limit* $L^{+,\epsilon}_{\beta,x_0}$ *is contained in* $\overline{U_\epsilon} \cap [y = 0]$. *Moreover* $L^{+,\epsilon}_{\beta,x_0}$ *is invariant by* $\xi(X(\lambda(\beta)^m), 0, \epsilon)$, *more precisely*

$$Q \in L^{+,\epsilon}_{\beta,x_0} \implies \Gamma^{|x|\leq\epsilon}_{\xi(X(\lambda(\beta)^m))}[Q] \subset L^{+,\epsilon}_{\beta,x_0}.$$

5.2.1. True sections and virtual sections.

5.2.1.1. *Existence of virtual sections.* A L-limit is so far a definition. Throughout this section we justify the term. In order to achieve this goal we define the virtual sections. We denote by A_{E_-} the function $-2\pi i \sum_{P\in E_-(y)} Res_X(P)$.

PROPOSITION 5.2.1. *Let* β *be a semi-analytic curve. Consider* $x_1 \in L^{+,\epsilon}_{\beta,x_0}$. *There exists a compact wedge* $\beta \subset W$ *(width*$(W) > 0$*), a continuous section* $\sigma : W \to \mathbb{C}^2$, *a sequence* $\{y_k\}_{k\in\mathbb{N}} \subset \beta$, $y_k \to 0$ *and a continuous partition* $E = (E_-, E_+)$ *of the equilibrium points such that*

1. W *is associated to* $|y|^m A_{E_-}(y)$ *and* β.
2. $\mu(A_{E_-}) > m$.
3. $\sigma(s) \in (y = s)$ *for all* $s \in W$ *and* $\lim_{y\in\beta,\ y\to 0}\sigma(y) = (x_1, 0)$.
4. $T(s) \stackrel{def}{=} \psi_1(\sigma(s))/s^m + A_{E_-}(s) - \psi_0(x_0, s)/s^m \in \mathbb{R}^+$ *for* $s \in W$.
5. $\Gamma^{|x|<\epsilon+\eta}_{\xi(X),+}[x_0, y_k](T(y_k)) = \sigma(y_k)$ *for all* $\eta > 0$ *and* $k > k(\eta)$.
6. $\Gamma^{|x|<\epsilon+\eta}_{\xi(X),+}[x_0, y_k][0, T(y_k)]$ *induces the partition* $E(y_k)$ *for all* $\eta > 0$ *and* $k > k(\eta)$.

As usual ψ_0 is an integral of the time form of $X(1)$ defined in a neighborhood of $(x_0, 0)$ whereas ψ_1 is obtained from ψ_0 by applying the method in subsection 4.1.1. By definition a section σ satisfying the conditions in proposition 5.2.1 is called a *virtual section*. Roughly speaking for a virtual section σ the points (x_0, y) and $\sigma(y)$ are candidates to be connected by a trajectory spending time $T(y)$ to go from (x_0, y) to $\sigma(y)$. If that connection really exists, i.e. if conditions (5) and (6) are satisfied for all $y \in W$ close to 0 then σ is a *true section*.

PROOF. Let $\lambda_0 = \lambda(\beta)^m$. Let (x_1^j, y_j) be the sequence provided in the definition of the L-limit. Consider a transversal Tr to $Re(X(\lambda))$ passing through $(x, y, \lambda) = (x_1, 0, \lambda(\beta)^m)$. We can suppose that Tr contains the point $(x_1^j, y_j, (y_j/|y_j|)^m)$ for all $j >> 0$ by replacing (x_1^j, y_j) with a point in the same trajectory of $Re(X)$. We have that $(x_1^j, y_j) \in \Gamma_{\xi(X),+}^{|x|<\epsilon+\eta}[x_0, y_j]$ for all $\eta > 0$ and $j \geq j(\eta)$. For $j > 0$ big enough the piece of trajectory of $\xi(X, y_j, \epsilon + \eta)$ from (x_0, y_j) to (x_1^j, y_j) induces a partition of the singular points. By taking a sub-sequence we can suppose that the partition is always the same, we denote it by E. We have

$$I_{x_0,j,E} \stackrel{def}{=} |y_j|^m \left(\frac{\psi_1}{y^m}(x_1^j, y_j) + A_{E_-}(y_j) - \frac{\psi_0}{y^m}(x_0, y_j) \right) \in \mathbb{R}^+$$

for all $j \geq 0$. As a consequence $\mu(A_{E_-}) > m$ because otherwise

$$\lim_{j \to \infty} I_{x_0,j,E}(y_j) = (\psi_1(x_1, 0) + h_{-m} - \psi_0(x_0, 0))\lambda_0^{-1} \in \mathbb{R}^+ \cup \{0\}$$

implies that $(x_1, 0) \in \Gamma_{\xi(X(\lambda_0))}^{|x|\leq\epsilon}[x_0, 0](\alpha)$ for some $\alpha \geq 0$. That contradicts $x_1 \in L_{\beta,x_0}^{+,\epsilon}$. We have

$$\lim_{j \to \infty} Img(|y_j|^m A_{E_-}(y_j)) = -Img(\psi_1(x_1, 0)\lambda_0^{-1} - \psi_0(x_0, 0)\lambda_0^{-1}).$$

Hence $\lim_{y \in \beta,\ y \to 0} Img(|y|^m A_{E_-}(y)) \in \mathbb{R}$ by lemma 5.1.1. By proposition 5.1.1 and the implicit function theorem we obtain $\sigma : W \cup \{0\} \to Tr$ such that

$$\psi_1(\sigma(s))/s^m + A_{E_-}(s) - \psi_0(x_0, s)/s^m \in \mathbb{R}^+$$

for all $y \in W$. By the uniqueness obtained from the implicit function theorem we have $\sigma(y_k) = (x_1^k, y_k)$ for all $k >> 0$. Therefore σ is a virtual section. ☐

Propositions 5.1.1 and 5.2.1 imply immediately the next remarks.

REMARK 5.2.1. *If $width(M) > 0$ then the section $\sigma : W \cup \{0\} \to \mathbb{C}^2$ is not continuous at 0. In fact for $\beta' \in \Upsilon_{A_{E_-}}^C$ we have*

$$\lim_{y \in \beta',\ y \to 0} Img(\psi_1(\sigma(y))\lambda(\beta)^{-m}) - \lim_{y \in \beta,\ y \to 0} Img(\psi_1(\sigma(y))\lambda(\beta)^{-m}) = C.$$

REMARK 5.2.2. *We have $\lim_{y \in W,\ y \to 0} |y|^m T(y) = +\infty$.*

REMARK 5.2.3. *Let $M > 0$. Suppose $\xi(iX(\lambda(\beta)^m), 0, \epsilon)[x_1, 0][-M, M]$ is contained in U_ϵ. Then W can be chosen in proposition 5.2.1 to have width at least M.*

5.2.1.2. *Existence of true sections of zero width.* There is no difference between virtual and true sections when the width of the wedge is 0.

PROPOSITION 5.2.2. *A virtual section $\sigma : \beta \cup \{0\} \to \mathbb{C}^2$ associated to a semi-analytic β and points $0 < |x_0| \leq \epsilon$, $x_1 \in L_{\beta,x_0}^{+,\epsilon}$ is a true section.*

Proof. Let $\lambda_0 = \lambda(\beta)^m$. Fix $\eta > 0$. We define

$$F = \{y \in \beta : \sigma(y) = \Gamma_{\xi(X)}^{|x|<\epsilon+\eta}[x_0, y](T(y))\}.$$

We have $y_j \in F$ for all $j > 0$ big enough. The set F is open. If the germs of F and β at 0 coincide then we are done. Otherwise, consider the connected component F_k of F containing y_k. There exists a sequence $s_k \to 0$ such that $s_k \in \partial F_k$. The points s_k satisfy that

$$\sigma(s_k) = \Gamma_{\xi(X)}^{|x|\leq\epsilon+\eta}[x_0, s_k](T(s_k))$$

for all $k >> 0$ but $\Gamma_{\xi(X)}^{|x|\leq\epsilon+\eta}[x_0, s_k][0, T(s_k))$ contains a tangent point $T_X^{\epsilon+\eta,a}(s_k)$. A priori a depends on k but we can suppose that a is constant by taking a subsequence. Consider the set

$$G = \{y \in \beta : T_X^{\epsilon+\eta,a}(y) \in \Gamma_{\xi(X),+}^{|x|\leq\epsilon+\eta}[x_0, y]\}.$$

We have $s_k \in G$ for all $k >> 0$. The lemma 4.1.4 applied to $S_0 = (x_0, y)$, $S_1 = T_X^{\epsilon+\eta,a}(y)$ and $H = \{\lambda(\beta)\}$ implies $G = \beta$. By Rolle's property there exists a unique function $T' : \beta \to \mathbb{R}^+$ such that

$$T_X^{\epsilon+\eta,a}(y) = \Gamma_{\xi(X),+}^{|x|\leq\epsilon+\eta}[x_0, y](T'(y))$$

and $T'(s_k) < T(s_k)$ for all $k >> 0$. The function T' is of the form

$$T'(y) = \frac{\psi_1'}{y^m}(T_X^{\epsilon+\eta,a}(y)) + A_{E'_-}(y) - \frac{\psi_0}{y^m}(x_0, y)$$

for ψ_1' defined in the neighborhood of $\lim_{y\in\beta,\ y\to 0} T_X^{\epsilon+\eta,a}(y) \in T_{X(\lambda_0)}^{\epsilon+\eta}(0)$. By lemma 5.1.1 we have that

$$c \stackrel{def}{=} \lim_{y\in\beta,\ y\to 0} |y|^m (T(y) - T'(y)) \in \mathbb{R}_{\geq 0} \cup \{+\infty\}.$$

If $c = 0$ then $(x_1, 0) = \lim_{y\in\beta,\ y\to 0} T_X^{\epsilon+\eta,a}(y) \in (|x| = \epsilon + \eta)$; that is not possible. Therefore $c > 0$; as a consequence $T'(y_k) < T(y_k)$ for all $k >> 0$. We deduce that the trajectory

$$\Gamma_{\xi(X)}^{|x|<\epsilon+\eta}[x_0, y_k][0, T(y_k)] \subset U_{\epsilon+\eta}$$

contains a point in $\partial U_{\epsilon+\eta}$ for all $k >> 0$. That is a contradiction. □

The existence of true sections defined over β justifies the term limit for the L-limit. The set $\{0\} \cup \Gamma_{\xi(X(\lambda(\beta)^m)),+}^{|x|\leq\epsilon}[x_0, 0] \cup L_{\beta,x_0}^{+,\epsilon}$ is the limit of the trajectories passing through (x_0, y) when $y \in \beta$ tends to 0.

5.3. Structure of the L-limit

5.3.1. Dynamics supporting non-empty L-limits. Roughly speaking the L-limit phenomenon appears when the limit of trajectories passing through some points is not the trajectory passing through the limit point. Therefore, the existence of L-limits is associated with complicated dynamics. We claim that the complexity of the dynamics depends on whether $N \leq 1$ or $N > 1$. We remind the reader that N is the generic number of points in $U_\epsilon \cap [y = c] \cap SingX$.

Proposition 5.3.1. *Suppose $N \leq 1$. For any choice of the data we have $L_{\beta,x_0}^{+,\epsilon} = \emptyset$.*

Proof. Consider a partition E of the singular points. We claim that $y^m A_{E_-}(y) = y^m(-2\pi i)\sum_{P\in E_-(y)} Res_X(P)$ is a holomorphic function. If $E_-(y)=\emptyset$ then $A_{E_-}\equiv 0$, otherwise $X(1)=u(x,y)(x-g(y))^\nu \partial/\partial x$ and $X=y^m X_1$ for $u\in\mathbb{C}\{x,y\}\setminus(x,y)$. The order of $X(1)$ along $x=g(y)$ is constant and equal to ν; thus $-2\pi i Res_{X(1)}(g(y),y) = y^m A_{E_-}(y)$ is a holomorphic function. By remark 5.2.2 we obtain $L^{+,\epsilon}_{\beta,x_0}=\emptyset$. □

Proposition 5.3.2. *Suppose $N>1$. There exists a semi-analytic β and $0<|x_0|\leq\epsilon$ such that $L^{+,\epsilon}_{\beta,x_0}\neq\emptyset$.*

Proof. We know that $UN^\epsilon_X\setminus\{0\}\neq\emptyset$ by corollary 4.2.2. We choose β to be a T-set. Let $\lambda_0=\lambda(\beta)^m$. There exist $T^{\epsilon,a}_X$, $T^{\epsilon,b}_X$ and $T:\beta\to\mathbb{R}^+$ such that $\Gamma^{|x|\leq\epsilon}_{\xi(X),+}[T^{\epsilon,a}_X(s)](T(s)) = T^{\epsilon,b}_X(s)$ for all $s\in\beta$. The limit $(c_j,0)=\lim_{y\in\beta,\ y\to 0} T^{\epsilon,j}_X(y)$ exists and it is contained in $T^\epsilon_{X(\lambda_0)}(0)$ for all $j\in\{a,b\}$. We have $(c_b,0)\in L^{+,\epsilon}_{\beta,c_a}$ by proposition 3.2.2. □

Lemma 5.3.1. *Suppose $N>1$ and $m=0$. Let $x_0\in(0<|x|\leq\epsilon)$. Then there exists a semi-analytic β such that $L^{+,\epsilon}_{\beta,x_0}\cup L^{-,\epsilon}_{\beta,x_0}\neq\emptyset$.*

Proof. We have that either

$$\alpha_{\xi(X),|x|\leq\epsilon}(x_0,0)=(0,0) \ \text{ or } \ \omega_{\xi(X),|x|\leq\epsilon}(x_0,0)=(0,0).$$

Suppose without lack of generality that we are in the latter case. We can suppose that $\omega_{\xi(X),|x|<\epsilon}(x_0,0)=(0,0)$ by replacing $(x_0,0)$ with $\Gamma^{|x|\leq\epsilon}_{\xi(X)}[x_0,0](t)$ for some $t>>0$ if necessary. We define

$$F=\{y\in B(0,\delta)\setminus\{0\}:\Gamma^{|x|\leq\epsilon}_{\xi(X),+}[x_0,y]\cap T^\epsilon_X(y)\neq\emptyset\}.$$

The set F is a finite union of semi-analytic curves by corollary 4.1.2. Let β be a semi-analytic curve contained either in F or in a zone Z of $B(0,\delta)\setminus(F\cup\{0\})$ such that $\omega_{\xi(X),|x|<\epsilon}(x_0,y)=\infty$ for all $y\in Z$. Such a curve exists by lemma 4.2.2. Since $\Gamma^{|x|\leq\epsilon}_{\xi(X),+}[x_0,y]\cap\partial U_\epsilon\neq\emptyset$ for all $y\in\beta$ and $\Gamma^{|x|\leq\epsilon}_{\xi(X),+}[x_0,0]\subset U_\epsilon$ then $L^{+,\epsilon}_{\beta,x_0}\cap(|x|=\epsilon)\neq\emptyset$. □

5.3.2. Nature of the L-limit. A L-limit satisfies the Rolle property. Let β be a semi-analytic germ of curve.

Lemma 5.3.2. *Let $(x_1,0)\neq(x_2,0)$ in $L^{+,\epsilon}_{\beta,x_0}\cup\Gamma^{|x|\leq\epsilon}_{\xi(X(\lambda(\beta)^m))}[x_0,0]$. Then there is no a connected transversal $I\subset U_\epsilon\cap(y=0)$ to $\xi(X(\lambda(\beta)^m),0,\epsilon)$ containing both $(x_1,0)$ and $(x_2,0)$.*

Proof. Let $\lambda_0=\lambda(\beta)^m$ and $\eta>0$ small enough. Suppose the result is false. The set $I\times V\times W$ is a transversal to $Re(X(\lambda))$ for some neighborhood V of 0 and some neighborhood W of λ_0. For $y\in\beta$ sufficiently close to 0 the trajectory $\Gamma^{|x|\leq\epsilon+\eta}_{\xi(X)}[x_0,y]$ cuts $I\times\{y\}$ at points $(x_1(y),y)$ and $(x_2(y),y)$ such that $\lim_{y\in\beta,\ y\to0}(x_j(y),y)=(x_j,0)$ for all $j\in\{1,2\}$. As a consequence the Rolle property is violated for $y\in\beta$ sufficiently close to 0. □

Next we describe the structure of a particular L-limit.

Proposition 5.3.3. *The L-limit $L^{+,\epsilon}_{\beta,x_0}$ is a finite collection $\rho_1<\ldots<\rho_l$ of trajectories of $\xi(X(\lambda(\beta)^m),0,\epsilon)$ in $(|x|\leq\epsilon)\times\{0\}$. The number of connected* *components of* $L^{+,\epsilon}_{\beta,x_0}$ *is at most* $\tilde{\nu}(X)-1$. *The order is provided by the time of the flow.*

Let $(x_l,0)$ be a point of ρ_l. By propositions 5.2.1 and 5.2.2 there exists a true section $\sigma_l : \beta \cup \{0\} \to \mathbb{C}^2$ such that $\sigma_l(y) = \Gamma^{|x|<\epsilon+\eta}_{\xi(X),+}[x_0,y](T_l(y))$ for all $y \in \beta$ and $\sigma_l(0) = (x_l,0)$. The function $T_l : \beta \to \mathbb{R}^+$ is continuous and $\lim_{y\in\beta,\ y\to 0} |y|^m T_l(y) = \infty$. We say that $\rho_l < \rho_{l+1}$ if $\lim_{y\in\beta,\ y\to 0} |y|^m (T_{l+1}(y) - T_l(y)) = \infty$. This order does not depend on the choice of the sections σ_l and σ_{l+1}. Indeed, for a different choice σ_l' the function $T_l' : \beta \to \mathbb{R}^+$ satisfies that $|y|^m |T_{l'}(y) - T_l(y)|$ is bounded.

Proof. Let $\lambda_0 = \lambda(\beta)^m$. We claim that the order is a total one. Let ρ_1, ρ_2 be two connected components of $L^{+,\epsilon}_{\beta,x_0}$. Let $E^j(s)$ be the partition of $SingX$ induced by $\Gamma^{|x|<\epsilon+\eta}_{\xi(X),+}[x_0,s][0,T_j(s)]$ for any $\eta > 0$. If

$$\lim_{y\in\beta,\ y\to 0} |y|^m |T_2(y) - T_1(y)| = \infty$$

then either $\rho_1 < \rho_2$ or $\rho_2 < \rho_1$. Otherwise the limit

$$lim_{y\in\beta,\ y\to 0}\left[2\pi i|y|^m\left(\sum_{P\in E^1_-(y)} Res_X(P) - \sum_{P\in E^2_-(y)} Res_X(P)\right)\right]$$

exists by lemma 5.1.1. Hence $c = \lim_{y\in\beta,\ y\to 0} |y|^m (T_2(y) - T_1(y))$ exists. We deduce that $(x_2,0) = \Gamma^{|x|\le\epsilon}_{\xi(X(\lambda_0)),+}[x_1,0](c)$ and then $\rho_1 = \rho_2$.

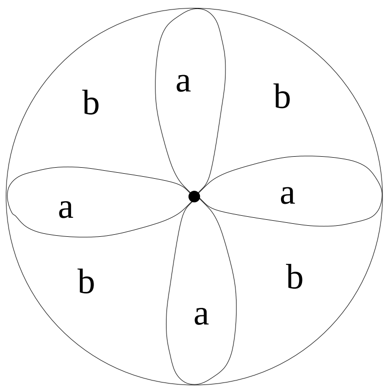

Figure 1.

Consider $\mathcal{H}^{|x|<\epsilon}_{\xi(X(\lambda_0),0)}$ (see picture 1) and the sequence

$$\Gamma^{|x|\le\epsilon}_{\xi(X(\lambda_0))}[x_0,0] = \rho_0 < \rho_1 < \ldots < \rho_l$$

of connected components of $L^{+,\epsilon}_{\beta,x_0}$. For every $1 \le j \le l$ we have $\alpha_{\xi(X(\lambda_0)),|x|\le\epsilon}(\rho_j) = \{0\}$; otherwise there is no component lesser than ρ_j. Moreover, for all $0 \le j < l$ we have $\omega_{\xi(X(\lambda_0)),|x|\le\epsilon}(\rho_j) = \{0\}$.

We call "aba" set a union of three contiguous regions labeled "a", "b" and "a" respectively. There are $2(\tilde{\nu}(X)-1)$ "aba" sets. It is straightforward to prove that the trajectories in "aba" sets can be connected by connected transversals. Hence an "aba" set can not contain more than one component of $L^{+,\epsilon}_{\beta,x_0} \cup \rho_0$. For all $1 \leq j < l-1$ the component ρ_j is contained in an "a" set and then in two "aba" sets. The components ρ_0 and ρ_l are contained in at least one "aba" set. Hence, we obtain $2 + 2(l-1) \leq 2(\tilde{\nu}(X) - 1) \implies l \leq \tilde{\nu}(X) - 1$. □

The first component of a L-limit is the only one which is invariant by reduction of the domain of definition.

LEMMA 5.3.3. *Consider $0 < |x_0| \leq \epsilon$. Suppose $L^{+,\epsilon}_{\beta,x_0} \neq \emptyset$ and let ρ be a component of $L^{+,\epsilon}_{\beta,x_0}$. Suppose that for all $0 < \epsilon' < \epsilon$ there exist points $(x_0', 0) \in \Gamma^{|x| \leq \epsilon}_{\xi(X(\lambda(\beta)^m)),+}[x_0, 0] \cap U_{\epsilon'}$ and $(x_1, 0) \in \rho \cap U_{\epsilon'}$ such that $x_1 \in L^{+,\epsilon'}_{\beta,x_0'}$. Then ρ is the first component of $L^{+,\epsilon}_{\beta,x_0}$.*

PROOF. Let ρ_1 be the first component of $L^{+,\epsilon}_{\beta,x_0}$. Let $\epsilon' > 0$ be a constant such that $\overline{U_{\epsilon'}}$ does not contain ρ_1. As a consequence $\overline{U_{\epsilon'}} \cap \rho_1$ has more than a connected component. Therefore $L^{+,\epsilon'}_{\beta,x_0'} \subset \rho_1$ for all $(x_0', 0)$ in $\Gamma^{|x| \leq \epsilon}_{\xi(X(\lambda(\beta)^m)),+}[x_0, 0] \cap U_{\epsilon'}$. That implies $\rho \cap \rho_1 \neq \emptyset$ and then $\rho = \rho_1$. □

5.4. Evolution of the L-limit

5.4.1. Virtual evolution. Up to a linear change of coordinates we suppose $\lambda(\beta) = 1$. Let $\rho_1 < \ldots < \rho_l$ be the connected components of the L-limit $L^{+,\epsilon}_{\beta,x_0}$ and consider $\rho_0 = \Gamma^{|x| \leq \epsilon}_{\xi(X(1))}[x_0, 0]$.

Let $\sigma_0(y) = (x_0, y)$. For $1 \leq j \leq l$ the couple (ρ_0, ρ_j) has associated a true section $\sigma_j : \beta \cup \{0\} \to \mathbb{C}^2$, a partition E_j of the singular points and a time function $T_j : \beta \to \mathbb{R}^+$ such that $\sigma_j(0) \in \rho_j$ and

$$T_j(y) = \frac{\psi_1^j}{y^m}(\sigma_j(y)) + A_{E_j,-}(y) - \frac{\psi_0^j}{y^m}(x_0, y).$$

Since $\sigma_k(0) \in L^{+,\epsilon}_{\beta,\sigma_j(0)}$ for $j < k$ then (ρ_j, ρ_k) has associated a partition $E_{j,k}$ of $Sing(X)$ and a time function $T_{j,k} : \beta \to \mathbb{R}^+$ such that

$$T_{j,k}(y) = \frac{\psi_1^{j,k}}{y^m}(\sigma_k(y)) + A_{E_{j,k},-}(y) - \frac{\psi_0^{j,k}}{y^m}(\sigma_j(y)).$$

We denote E_j by $E_{0,j}$ and T_j by $T_{0,j}$.

Fix $L \in \mathbb{Q}_{>m}$. Let $co(E_{j,k}, L)$ be the coefficient of y^{-L} in $A_{E_{j,k},-}$.

LEMMA 5.4.1.

$co(E_{j,k}, L) + co(E_{k,r}, L) = co(E_{j,r}, L)$ for all $0 \leq j < k < r \leq l$.
$\mu(A_{E_{j,k},-}) \leq \mu(A_{E_{j',k'},-})$ if $j' \leq j$ and $k \leq k'$.
$co(E_{j,k}, L) \geq 0$ if $\mu(A_{E_{j,k},-}) \leq L$.
$co(E_{j,k}, L) > 0$ if $\mu(A_{E_{j,k},-}) = L$ for all $0 \leq j < k \leq l$.

PROOF. The first relation is a consequence of $T_{j,k} + T_{k,r} = T_{j,r}$.

Suppose $j' \leq j$, $k \leq k'$ and $(j,k) \neq (j',k')$, thus $|y|^m (T_{j',k'} - T_{j,k})$ tends to ∞ when $y \in \beta$ tends to 0. As a consequence

$$|y|^m |A_{E_{j,k},-}(y)| < |y|^m |A_{E_{j',k'},-}(y)|$$

in β since $y \mapsto |y|^m T_{a,b}(y) - |y|^m |A_{E_{a,b,-}}(y)|$ is a bounded function of β for $0 \leq a < b \leq l$. Therefore, we obtain $\mu(A_{E_{j,k,-}}) \leq \mu(A_{E_{j',k',-}})$.

If $\mu(A_{E_{j,k,-}}) < L$ then $co(E_{j,k}, L) = 0$. If $\mu(A_{E_{j,k,-}}) = L$ then we obtain $co(E_{j,k}, L) > 0$ by remarks 5.1.1 and 5.2.2. □

By lemma 5.4.1 we have $\mu(A_{E_{1,-}}) \leq \ldots \leq \mu(A_{E_{l,-}})$. We consider the components $\rho_1 < \ldots < \rho_q$ such that

$$\mu(A_{E_{1,-}}) \leq \ldots \leq \mu(A_{E_{q,-}}) \leq L.$$

Let $a \mapsto (a, P(a))$ the Puiseux parametrization of β. We define

$$\beta(\Delta, a) = a + i[P(a) + \Delta a^{L-m+1}]$$

and let us consider $W(M) = \cup_{\Delta \in [-M,M]} \beta(\Delta)$. Our aim is describing the evolution of the L-limit $L^{+,\epsilon}_{\beta(\Delta),x_0}$. We denote by $width_{j,k}(W(M))$ the width of $W(M)$ as a compact wedge relative to (ρ_j, ρ_k), or in other words relative to the function $|y|^m A_{E_{j,k,-}}$.

Lemma 5.4.2. *The curve $\beta(\Delta)$ belongs to $\Upsilon^{co(E_{j,k},L)(L-m)\Delta}_{A_{E_{j,k,-}}}$ if we have $0 \leq j < k \leq q$. Then, $width_{j,k}(W(M)) = co(E_{j,k}, L)(L-m)M$.*

If $co(E_{j,k}, L) = 0$ the statement of the lemma means that $W(M)$ does not contain a compact wedge relative to (ρ_j, ρ_k) of positive width. We will not prove explicitly the lemma because we are just rephrasing some of the results in proposition 5.1.1.

For $1 \leq j \leq q$ the curves ρ_j belongs to $\alpha^{-1}_{\xi(X(1)),|x|\leq\epsilon}(0,0)$ whereas ρ_k belongs to $\omega^{-1}_{\xi(X(1)),|x|\leq\epsilon}(0,0)$ for $0 \leq k \leq q-1$. If $\rho_j \subset \alpha^{-1}_{\xi(X(1)),|x|\leq\epsilon}(0,0)$ then ρ_j is contained in a repulsive petal $V_{l_j^-}$ for some $l_j^- \in \Theta^-(X(1)_{|y=0})$. There is an integral $\psi^-_{j,0}$ of the time form of $X(1)_{|y=0}$ defined in $V_{l_j^-}$. We define the curve

$$\rho^{\Delta}_{j,-} = \left(\psi^-_{j,0}\right)^{-1}(\psi^-_{j,0}(\rho_j) + \mathbb{R} + i\Delta(L-m)co(E_{0,j}, L)).$$

If $\rho_j \subset \omega^{-1}_{\xi(X(1)),|x|\leq\epsilon}(0,0)$ we can use the same construction with the attractive petal $V_{l_j^+}$ containing ρ and the integral $\psi^+_{j,0}$ of the time form of $X(1)$ defined in $V_{l_j^+}$. We define

$$\rho^{\Delta}_{j,+} = \left(\psi^+_{j,0}\right)^{-1}(\psi^+_{j,0}(\rho_j) + \mathbb{R} + i\Delta(L-m)co(E_{0,j}, L)).$$

Let $h^L_\Delta = \inf(\{j \in \{1, \ldots, q-1\} : \rho^\Delta_{j,-} \neq \rho^\Delta_{j,+}\} \cup \{q\})$.

5.4.2. Evolution with respect to the base curve. This subsection is devoted to prove the next result:

Proposition 5.4.1. *Let $L \in \mathbb{Q}_{>m}$. Consider $\rho_1 < \ldots < \rho_q$ the components of $L^{+,\epsilon}_{\beta,x_0}$ such that $\mu(A_{E_{j,-}}) \leq L$ for all $1 \leq j \leq q$. Then, for all $\Delta \in \mathbb{R}$ the first h^L_Δ components of $L^{+,\epsilon}_{\beta(\Delta),x_0}$ are*

$$\rho^\Delta_{1,-} < \ldots < \rho^\Delta_{h^L_\Delta,-}.$$

We will prove the result step by step. Let us define $\alpha_{j,k} = (L-m)co(E_{j,k}, L)$. Fix $M > 0$. We choose points $(x_{j,+}, 0)$ in ρ_j and $(x_{k,-}, 0)$ in ρ_k for $0 \leq j \leq q-1$ and $1 \leq k \leq q$ such that

(1) $S_j^+ \stackrel{def}{=} \Gamma^{|x|<\epsilon}_{\xi(iX(1))}[x_{j,+},0][-M\alpha_{0,j}-1, M\alpha_{0,j}+1]$ is well defined.
(2) $S_k^- \stackrel{def}{=} \Gamma^{|x|<\epsilon}_{\xi(iX(1))}[x_{k,-},0][-M\alpha_{0,k}-1, M\alpha_{0,k}+1]$ is well defined.
(3) $\omega_{\xi(X(1)),|x|<\epsilon}(P) = \{(0,0)\}$ for all $P \in S_j^+$.
(4) $\alpha_{\xi(X(1)),|x|<\epsilon}(P) = \{(0,0)\}$ for all $P \in S_k^-$.

To obtain $(x_{j,+},0)$ we can try at first with any $(x_{j,+},0) \in \rho_j$. If it does not hold the previous conditions then we replace $(x_{j,+},0)$ with $\Gamma^{|x|<\epsilon}_{\xi(X(1))}[x_{j,+},0](t)$ for some $t >> 0$. A similar method provides $(x_{j+1,-},0)$. Let $y = re^{\lambda}$. The previous properties imply that

- $Tr_j^+(y) \stackrel{def}{=} \Gamma^{|x|<\epsilon}_{\xi(iX(\lambda^m))}[x_{j,+},y][-M\alpha_{0,j}-1, M\alpha_{0,j}+1]$ is well defined for (r,λ) in a neighborhood of $(0,1)$.
- $Tr_k^-(y) \stackrel{def}{=} \Gamma^{|x|<\epsilon}_{\xi(iX(\lambda^m))}[x_{k,-},y][-M\alpha_{0,k}-1, M\alpha_{0,k}+1]$ is well defined for (r,λ) in a neighborhood of $(0,1)$.

Let $\psi_{j,+}$ be an integral of the time form of $X(1)$ defined in a neighborhood of Tr_j^+ whereas $\psi_{j+1,-}$ is an integral of the time form of $X(1)$ defined in a neighborhood of Tr_{j+1}^- obtained by prolongating $\psi_{j,+}$ (see subsection 4.1.1). For any point $z_{j,+}$ in $Tr_j^+(0)$; we define

$$\Delta(z_{j,+}) = \psi_{j,+}(z_{j,+},0) - \psi_{j,+}(x_{j,+},0) = \psi_{j,0}^+(z_{j,+},0) - \psi_{j,0}^+(x_{j,+},0).$$

Let $z_{j,+}$ be a point in $Tr_j^+(0)$ such that $|\Delta(z_{j,+})| \leq M\alpha_{0,j}$. We define

$$\rho_{j,j+1}^{\Delta}(z_{j,+}) = \left(\psi_{j+1,0}^-\right)^{(-1)}(\psi_{j+1,0}^-(x_{j+1,-},0) + \Delta(z_{j,+}) + i\Delta\alpha_{j,j+1} + \mathbb{R}).$$

Lemma 5.4.3. *Let $z_{j,+}$ be a point in $Tr_j^+(0)$ such that $|\Delta(z_{j,+})| \leq M\alpha_{0,j}$. If $\Delta \in [-M,M]$ then $\rho_{j,j+1}^{\Delta}(z_{j,+})$ is the first component of $L^{+,\epsilon}_{\beta(\Delta),z_{j,+}}$.*

Proof. There is a virtual section $\sigma_{j+1,-} : W(M) \to Tr_{j+1}^-$ such that

$$T_{j+1,-}(y) = \frac{\psi_{j+1,-}}{y^m}(\sigma_{j+1,-}(y)) + A_{E_{j,j+1,-}}(y) - \frac{\psi_{j,+}}{y^m}(x_{j,+},y)$$

and $\lim_{y\in\beta,\ y\to 0}\sigma_{j+1,-}(y) = (x_{j+1,-},0)$. We know that $\sigma_{j+1,-|\beta}$ is a true section by proposition 5.2.2. Let $\eta \geq 0$; we consider the trajectory

$$\gamma(y) = \Gamma^{|x|<\epsilon+\eta}_{\xi(X)}[x_{j,+},y][0,T_{j+1,-}(y)]$$

for all $y = r\lambda \in \beta$. We have

$$\lim_{y\in\beta,\ y\to 0}\gamma(y) = \Gamma^{|x|\leq\epsilon}_{\xi(X(1))),+}[x_{j,+},0] \cup \Gamma^{|x|\leq\epsilon}_{\xi(X(1))),-}[x_{j+1,-},0].$$

Since $\lim_{y\in\beta,\ y\to 0}\gamma(y)$ does not contain points in ∂U_ϵ by the choice of $x_{j,+}$ and $x_{j+1,-}$ then $\gamma(y)$ is contained in $U_{\epsilon'}$ for some $\epsilon' < \epsilon$ and $y = r\lambda \in \beta$ in a neighborhood of 0. Moreover

$$\Gamma^{|x|<\epsilon}_{\xi(iX(\lambda^m))}[\gamma(y)](C) \subset U_{\epsilon''} \tag{5.1}$$

for all $C \in [-M\alpha_{0,j}-1, M\alpha_{0,j}+1]$, some $\epsilon'' < \epsilon$ and $y \in \beta$ in a neighborhood of 0; it is a consequence of the conditions on $x_{j,+}$ and $x_{j+1,-}$. Hence $\rho_{j,j+1}^0(z_{j,+})$ is the first component of $L^{+,\epsilon}_{\beta,z_{j,+}}$. There exists a virtual section $\sigma'_{j+1,-} : W(M) \to Tr^{-}_{j+1}$ such that

$$T'_{j+1,-}(y) = \frac{\psi_{j+1,-}}{y^m}(\sigma'_{j+1,-}(y)) + A_{E_{j,j+1,-}}(y) - \frac{\psi_{j,+}}{y^m}(z_{j,+}, y)$$

and $\lim_{y\in\beta,\ y\to 0} \sigma'_{j+1,-}(y) \in \rho^0_{j,j+1}(z_{j,+})$. By proposition 5.1.1 we have that

$$\lim_{y\in\beta(\Delta),\ y\to 0} \sigma'_{j+1,-}(y) \in \rho^{\Delta}_{j,j+1}(z_{j,+})$$

for all $\Delta \in [-M, M]$. We prove next that $\sigma'_{j+1,-}$ is a true section. Let us define the set

$$G = \{r\lambda \in W(M) : \Gamma^{|x|\leq\epsilon}_{\xi(X)}[z_{j,+}, r\lambda][0, T'_{j+1,-}(r\lambda)] \subset U_\epsilon\}.$$

Then β is contained in the open set G because of the relation 5.1 applied to $C = Img(\psi_{j,+}(z_{j,+}, r\lambda) - \psi_{j,+}(x_{j,+}, r\lambda))$. Let F be the connected component of G containing β; we denote by ∂F the boundary of F in $W(M)$. If $y \in \partial F$ then $\Gamma^{|x|\leq\epsilon}_{\xi(X)}[z_{j,+}, y][0, T'_{j+1,-}(y)]$ is contained in $\overline{U_\epsilon}$ but its intersection with ∂U_ϵ is not empty. We deduce that ∂F is contained in the set

$$H = \{y \in B(0, \delta) \setminus \{0\} : \Gamma^{|x|\leq\epsilon}_{\xi(X),+}[z_{j,+}, y] \cap T^{\epsilon}_X \neq \emptyset\}.$$

By corollary 4.1.1 the restriction of H to a neighborhood of $(r, \lambda) = (0, 1)$ is a finite union of semi-analytic curves.

Let ξ be a connected component of H such that $\xi \cap \partial F$ contains infinitely many points in every neighborhood of 0. Thus there exists $T^{\epsilon,a}_X$ and $T' : \xi \to \mathbb{R}^+$ such that $T^{\epsilon,a}_X(y) = \Gamma^{|x|\leq\epsilon}_{\xi(X)}[z_{j,+}, y](T'(y))$ for all $y \in \xi$ and $T'(y_k) < T'_{j+1,-}(y_k)$ for a subsequence $\{y_k\} \subset \xi \cap \partial F$ such that $\lim_{k\to\infty} y_k = 0$. We denote $(c, 0) = \lim_{y\in\xi, y\to 0} T^{\epsilon,a}_X(y)$ and $(d, 0) = \lim_{y\in\xi,\ y\to 0} \sigma'_{j+1,-}(y)$. We have $(d, 0) \in S^{-}_{j+1}$ since

$$|\psi_{j+1,-}(d, 0) - \psi_{j+1,-}(x_{j+1,-}, 0)| \leq \alpha_{0,j} M + \alpha_{j,j+1} M = \alpha_{0,j+1} M.$$

By the condition on S^{-}_{j+1} we deduce that $d \in L^{+,\epsilon}_{\xi,c}$. As a consequence we obtain $|y|^m (T'_{j+1,-}(y) - T'(y)) \to +\infty$ when $y \in \xi$ and $y \to 0$. The point $(z_{j,+}, 0)$ is in S^{+}_j; that implies $\lim_{y\in\xi,\ y\to 0} |y|^m T'(y) = +\infty$. Since $T'(y) < T'_{j+1,-}(y)$ for $y \in \xi$ we deduce that $\xi \subset \partial F$. As a consequence the set ∂F is a union of at most 2 semi-analytic curves.

Let $\xi \subset \partial F$. Consider a transversal Tr_c passing through the point $(x, y, \lambda) = (c, 0, 1)$. There exists a virtual section $\sigma_c : \overline{F} \to Tr_c$ such that $\lim_{y\in\xi,\ y\to 0} \sigma_c(y) = (c, 0)$. The section σ_c has associated a function $T_c : \overline{F} \to \mathbb{R}^+$ and a partition E_c of the singular points such that

$$T_c(y) = \frac{\psi_c}{y^m}(\sigma_c(y)) + A_{E_{c,-}}(y) - \frac{\psi_{j,+}}{y^m}(z_{j,+}, y).$$

Moreover, we have $T_c(y) = T'(y)$ for all $y \in \xi$. By lemma 5.4.1 we have $\mu(E_c) \leq \mu(E_{j,j+1}) \leq L$. By proposition 5.1.1

$$(|y|^m A_{E,-}) \circ \beta(0, a) - (|y|^m A_{E,-}) \circ \beta(\Delta, a)$$

is bounded in $[-M, M] \times \mathbb{R}_{\geq 0}$ for $E \in \{E_c, E_{j,j+1}\}$. Then

$$\lim_{y\in\overline{F},\ y\to 0} |y|^m (T'_{j+1,-}(y) - T_c(y)) = \lim_{y\in\xi,\ y\to 0} |y|^m (T'_{j+1,-}(y) - T_c(y)) = \infty$$

and

$$\lim_{y \in \overline{F},\ y \to 0} |y|^m T_c(y) = \lim_{y \in \xi,\ y \to 0} |y|^m T_c(y) = \infty.$$

We define $(e, 0) = \lim_{y \in \beta,\ y \to 0} \sigma_c(y)$. Since $\lim_{y \in \beta, y \to 0} |y|^m T_c(y) = \infty$ then $e \in L^{+,\epsilon}_{\beta, z_{j,+}}$. Moreover $\lim_{y \in \beta, y \to 0} |y|^m (T'_{j+1,-}(y) - T_c(y)) = \infty$ implies $\rho^0_{j,j+1}(z_{j,+}) \subset L^{+,\epsilon}_{\beta,e}$. We proved that $\rho^0_{j,j+1}(z_{j,+})$ is not the first component of $L^{+,\epsilon}_{\beta,z_{j,+}}$. Since we also proved the opposite statement then $\partial F = \emptyset$ and $G = F = W(M)$. Therefore $\sigma'_{j+1,-}$ is a true section.

We claim that $\rho^{\Delta}_{j,j+1}(z_{j,+})$ is the first component of $L^{+,\epsilon}_{\beta(\Delta), z_{j,+}}$. Let $0 < \epsilon_1 < \epsilon$. For $t >> 0$ we replace S^+_j and S^-_{j+1} with

$$\Gamma^{|x|<\epsilon}_{\xi(X(\lambda^m))}[S^+_j](t) \quad \text{and} \quad \Gamma^{|x|<\epsilon}_{\xi(X(\lambda^m))}[S^-_{j+1}](-t)$$

respectively. The choice of $t > 0$ is intended to satisfy the four conditions on S^+_j and S^-_{j+1} for ϵ_1 instead of ϵ. We define

$$(z'_{j,+}, 0) = \Gamma^{|x|<\epsilon}_{\xi(X(\lambda^m))}[z_{j,+}, 0](t).$$

We already proved that $\emptyset \neq L^{+,\epsilon_1}_{\beta(\Delta), z'_{j,+}} \subset \rho^{\Delta}_{j,j+1}(z_{j,+})$. We are done by lemma 5.3.3. $\square$

We can now prove the main result in this subsection.

PROOF OF PROPOSITION 5.4.1. Let $M = |\Delta|$; for $0 \leq j \leq h^L_{\Delta} - 1$ we choose $x_{j,+}, x_{j+1,-}$ satisfying the four conditions on $(x_{j,+}, 0)$ in ρ_j and $(x_{j+1,-}, 0)$ in ρ_{j+1}. We also consider the transversals Tr^+_j, Tr^-_{j+1} and the integrals $\psi_{j,+}$, $\psi_{j+1,-}$ of the time form of $X(1)$ for all $0 \leq j \leq h^L_{\Delta} - 1$.

By lemma 5.4.3 the first component of $L^{+,\epsilon}_{\beta(\Delta), x_0}$ is $\rho^{\Delta}_{1,-}$. Moreover, the only point $(z_{1,-}, 0)$ in $\rho^{\Delta}_{1,-} \cap Tr^-_1(0)$ satisfies

$$\psi^-_{1,0}(z_{1,-}, 0) - \psi^-_{1,0}(x_{1,-}, 0) = \psi_{1,-}(z_{1,-}, 0) - \psi_{1,-}(x_{1,-}, 0) = i\Delta\alpha_{0,1}.$$

Suppose now that for $1 \leq j < h_{\Delta}$ we have that the first j components of $L^{+,\epsilon}_{\beta(\Delta), x_0}$ are $\rho^{\Delta}_{1,-} < \ldots < \rho^{\Delta}_{j,-}$. We also suppose that there is a unique point $(z_{j,-}, 0)$ in $\rho^{\Delta}_{j,-} \cap Tr^-_j(0)$ such that

$$\psi^-_{j,0}(z_{j,-}, 0) - \psi^-_{j,0}(x_{j,-}, 0) = \psi_{j,-}(z_{j,-}, 0) - \psi_{j,-}(x_{j,-}, 0) = i\Delta\alpha_{0,j}.$$

Since $\rho^{\Delta}_{j,-} = \rho^{\Delta}_{j,+}$ there exists a unique point $(z_{j,+}, 0) \in \rho^{\Delta}_{j,-} \cap Tr^+_j(0)$. This point satisfies that $\Delta(z_{j,+}) = i\Delta\alpha_{0,j}$ and then $|\Delta(z_{j,+})| = M\alpha_{0,j}$. By lemma 5.4.3 the next component of $L^{+,\epsilon}_{\beta(\Delta), x_0}$ is $\rho^{\Delta}_{j,j+1}(z_{j,+})$. Since $\alpha_{0,j+1} = \alpha_{0,j} + \alpha_{j,j+1}$ then $\rho^{\Delta}_{j+1,-} = \rho^{\Delta}_{j,j+1}(z_{j,+})$. The proposition is proved by induction. $\square$

REMARK 5.4.1. *Suppose $N > 1$ and $m = 0$. Let $0 < |x_1| < \epsilon$. Then either we have $\alpha_{\xi(X), |x|<\epsilon}(x_1, 0) = (0, 0)$ or $\omega_{\xi(X), |x|<\epsilon}(x_1, 0)$. Suppose without lack of generality that we are in the former case. There exists a semi-analytic β such that $L^{-,\epsilon}_{\beta, x_1} \neq \emptyset$ by the proof of lemma 5.3.1. Let $(x_0, 0)$ be a point in the first component of $L^{-,\epsilon}_{\beta, x_1}$, thus x_1 belongs to the first component ρ_1 of $L^{+,\epsilon}_{\beta, x_0}$. Let $L = \mu(A_{E_{1,-}})$. For any neighborhood $V \subset \mathbb{R}$ we have that $\cup_{\Delta \in V} \rho^{\Delta}_{1,-}$ is a neighborhood in $\mathbb{C}$ of x_1. As a consequence the real flow of X generates the complex flow of $X_{|y=0}$ at every point of $U_\epsilon \cap [y = 0]$.*

CHAPTER 6

Topological Conjugation of (NSD) Vector Fields

We described so far the behavior of a (NSD) vector field X. From now on we will use this information to compare two different (NSD) vector fields and to characterize whether or not they are topologically conjugated.

Our aim is comparing two (NSD) vector fields in a set

$$\mathcal{H}_f = \{uf\partial/\partial x \ / \ u \text{ is a unit}\}.$$

In order to assure that the elements of $\mathcal{H}_f$ are (NSD) vector fields we ask f for fulfilling the (NSD) conditions. We are going to describe whether the real flows of $X_1 = u_1 f\partial/\partial x$ and $X_2 = u_2 f\partial/\partial x$ are topologically conjugated by a homeomorphism σ such that

- $\sigma_{|[(SingX)\setminus(y=0)]} \equiv Id$.
- $y \circ \sigma \equiv y$.

A mapping σ satisfying the two previous conditions will be called special. We impose the special conditions because we are interested in comparing the dynamics of $Re(X_1)$ and $Re(X_2)$; whether they are topologically conjugated for a certain fiber in a neighborhood of a singular point and if the evolution of the dynamics with respect to the parameter is compatible.

We say that $X_1 \overset{sp}{\sim} X_2$ if there exists a special germ of homeomorphism σ such that σ conjugates $Re(X_1)$ and $Re(X_2)$.

6.1. Orientation

Consider $f \in \mathbb{C}\{x, y\}$ satisfying the (NSD) conditions. Let X_1, X_2 in $\mathcal{H}_f$. Suppose $X_1 \overset{sp}{\sim} X_2$ by a homeomorphism σ defined in a neighborhood of $\overline{U_{\epsilon,\delta}}$. For every $s \in B(0, \delta)$ there exists a mapping

$$\sigma(s)_* : \pi_1((U_\epsilon \cap [y = s]) \setminus (f = 0)) \to \pi_1((\sigma(U_\epsilon) \cap [y = s]) \setminus (f = 0))$$

induced by $\sigma_{|y=s}$. Since $\sigma_{|f=0} \equiv Id$ then the fundamental groups

$$\pi_1((U_{\epsilon,\delta} \cap [y = s]) \setminus (f = 0)) \quad \text{and} \quad \pi_1((\sigma(U_{\epsilon,\delta}) \cap [y = s]) \setminus (f = 0))$$

are canonically identified. We claim that the mapping $\sigma_{|y=s}$ preserves the orientation for $s \in B(0, \delta)$.

Proposition 6.1.1. *Suppose $N > 1$. The mapping $\sigma(s)_*$ is the identity for all $s \in B(0, \delta)$.*

Proof. The result is invariant under a ramification $(x, y) \mapsto (x, y^k)$, so we can suppose that the irreducible components of the set $f = 0$ are $x = g_1(y)$, ..., $x = g_N(y)$ and maybe $y = 0$. We consider a loop $\xi[0, 1] : \theta \mapsto re^{2\pi i\theta}$ for $0 < r < \delta$. We define $\kappa(x, y) = y$. Let

$$\sigma_j : \pi_1(\kappa^{-1}(\xi) \setminus (x = g_j(y))) \to \pi_1(\kappa^{-1}(\xi) \setminus (x = g_j(y)))$$

be the mapping induced by $\sigma_{|\kappa^{-1}(\xi)\setminus(x=g_j(y))}$. It is enough to prove that $\sigma_j \equiv Id$ for all $1 \leq j \leq p$. The space $\kappa^{-1}(\xi) \setminus [x = g_j(y)]$ is homotopic to a torus whose fundamental group is isomorphic to $\mathbb{Z}\times\mathbb{Z}$. We choose a loop $\alpha_{1,0} \subset \kappa^{-1}(\xi)\setminus(f = 0)$ such that $\alpha_{1,0} \sim 0$ in $\pi_1(U_{\epsilon,\delta}\setminus(f = 0))$ and $\kappa(\alpha_{1,0})$ turns once around 0. Let $\alpha_{0,1}$ be a loop in $\kappa^{-1}(r)\setminus[x = g_j(y)]$ turning once around $(g_j(r), r)$. The choice of generators $\alpha_{1,0}$ and $\alpha_{0,1}$ induces an isomorphism from $\mathbb{Z} \times \mathbb{Z}$ to $\pi_1(\kappa^{-1}(\xi) \setminus [x = g_j(y)])$. The isomorphism σ_j is of the form

$$\begin{array}{rccc} \sigma_j : & \mathbb{Z}\times\mathbb{Z} & \to & \mathbb{Z}\times\mathbb{Z} \\ & (a,b) & \mapsto & (a, c_j a + d_j b) \end{array}$$

because σ preserves the fibration $y = cte$. Moreover, we have $c_j \in \mathbb{Z}$ and $d_j \in \{-1, 1\}$. Fix $k \in \{1, \ldots, p\} \setminus \{j\}$, we denote $\nu(g_j(y) - g_k(y))$ by ν. Let us consider $\xi_1[0, 1] : \theta \mapsto (g_k(re^{2\pi i\theta}), re^{2\pi i\theta})$. The loop ξ_1 is contained in $\kappa^{-1}(\xi) \setminus [x = g_j(y)]$ and since $\sigma_{|f=0} \equiv Id$ then $\sigma_j(\xi_1) = \xi_1$. Therefore, we have $\sigma_j(1, \nu) = (1, \nu)$.

Consider a continuous function $l_1^j[0, 1] : \theta \to \Theta(X_1^j(re^{2\pi i\theta}))$ where $X_1^j(s)$ is the germ of $X_{1|y=s}$ at $(g_j(s), s)$. The function l_1^j is determined by $l_1^j(0)$. The direction $l_1^j(\theta)$ turns $t \in \mathbb{Q}_{<0}$ times around $\{0\} \times \mathbb{S}^1$ (see proof of proposition 4.2.1). The number t does not depend on $l_1^j(0)$; moreover, it does not depend on X_1 but on f (see proof of proposition 4.2.1 for a explicit calculation). Since σ preserves basins of attraction and repulsion then σ induces a mapping from $\Theta(X_1^j(s))$ to $\Theta(X_2^j(s))$. We define $l_2^j(\theta) = \sigma(l_1^j(\theta))$. The function l_2^j is determined by $l_2^j(0)$; it turns t times around $\{0\} \times \mathbb{S}^1$ since t depends on f. Let $u \in \mathbb{N}$ such that $-tu \in \mathbb{N}$. Then $l_2^j = \sigma(l_1^j)$ implies $\sigma_j(u, tu) = (u, tu)$. We have

$$\left\{ \begin{array}{rcl} \sigma_j(u,tu) & = & (u,tu) \\ \sigma_j(1,\nu) & = & (1,\nu) \end{array} \right.$$

It is straightforward to prove that the previous system can only be satisfied if $c_j = 0$ and $d_j = 1$. As a consequence $\sigma_j \equiv Id$. □

REMARK 6.1.1. *Suppose $N > 1$. Let X_1, $X_2 \in \mathcal{H}_f$ be vector fields such that $X_1 \overset{sp}{\sim} X_2$ by a special homeomorphism σ. A priori, if a trajectory γ of $\xi(X, y, \epsilon)$ induces a partition (E_-, E_+) the trajectory $\sigma(\gamma)$ induces either (E_-, E_+) or (E_+, E_-) depending on whether the orientation is preserved or reversed. We are in the former case by proposition 6.1.1. Therefore γ and $\sigma(\gamma)$ induce the same partition of the singular points.*

6.2. Comparing residues

Let $X_1 \overset{sp}{\sim} X_2$ be conjugated by σ. We can suppose that X_1, X_2 and σ are defined in the neighborhood of both $\overline{U_{\epsilon,\sigma}}$ and $\sigma(\overline{U_{\epsilon,\delta}})$. This section is devoted to prove that the existence of σ forces the residue functions of X_1 and X_2 to be related.

LEMMA 6.2.1. *Let X_1, $X_2 \in \mathcal{H}_f$ such that $X_1 \overset{sp}{\sim} X_2$ by a special germ of homeomorphism σ. Consider a non-empty L-limit $L^{+,\epsilon}_{\beta,x_0}$ associated to X_1 and a component ρ of $L^{+,\epsilon}_{\beta,x_0}$. Let E be the partition induced by (x_0, ρ); then*

$$\mu\left(\sum_{P\in E_-(y)} [Res_{X_1}(P) - Res_{X_2}(P)]\right) \leq m.$$

PROOF. Throughout this proof the L-limits and sections will be referred to X_1. Let $x_1 \in \rho$. There exists a true section $S : \beta \cup \{0\} \to \mathbb{C}^2$ such that $S(0) = x_1$. We have $T : \beta \to \mathbb{R}^+$ such that

$$T(y) = \frac{\psi_{1,X_1}}{y^m}(S(y)) + A_{E_-,X_1}(y) - \frac{\psi_{0,X_1}}{y^m}(x_0, y).$$

Let $\eta > 0$ such that X_1 and σ are defined in $U_{\epsilon+\eta,\delta}$. We define

$$\gamma(y) = \Gamma^{|x|<\epsilon+\eta}_{\xi(X_1),+}[x_0, y][0, T(y)]$$

for $y \in \beta$. By remark 6.1.1 the partition induced by $\gamma(y)$ and $\sigma(\gamma(y))$ is the same. Therefore, we have

$$T(y) = \frac{\psi_{1,X_2}}{y^m}(\sigma(S(y))) + A_{E_-,X_2}(y) - \frac{\psi_{0,X_2}}{y^m}(\sigma(x_0, y))$$

for all $y \in \beta$. This relation implies

$$\lim_{y\in\beta,\ y\to 0} y^m (A_{E_-,X_1}(y) - A_{E_-,X_2}(y)) \in \mathbb{C}.$$

Finally, we obtain $\mu(A_{E_-,X_1}(y) - A_{E_-,X_2}(y)) \leq m$. $\square$

REMARK 6.2.1. *The result in the previous lemma is also true if we replace E_- with E_+ because for any (NSD) vector field X the function*

$$s^m \sum_{P\in SingX\cap(y=s)} Res_X(P) = s^m \sum_{P\in E_-(s)} Res_X(P) + s^m \sum_{P\in E_+(s)} Res_X(P)$$

is holomorphic in a neighborhood of $s = 0$. This statement is a consequence of the formula

$$s^m \sum_{P\in SingX\cap(y=s)} Res_X(P) = \sum_{P\in SingX(1)\cap(y=s)} Res_{X(1)}(P) = \int_{\epsilon\mathbb{S}^1\times\{s\}} \psi$$

where ψ is a multi-valuated integral of the time form of $X(1)$ defined in the neighborhood of ∂U_ϵ. The function $\int_{\epsilon\mathbb{S}^1\times\{s\}} \psi$ is holomorphic because $SingX(1) \cap \partial U_\epsilon = \emptyset$.

Let $X_1, X_2 \in \mathcal{H}_f$ such that $X_1 \overset{sp}{\sim} X_2$. Fix $s \in B(0,\delta) \setminus \{0\}$. The graph $\mathcal{G}^{|x|<\epsilon}_{\xi(X_1,s)}$ has several connected components $G_1, \ldots, G_l$ whose singular points we denote by $E_1, \ldots, E_l$ respectively.

LEMMA 6.2.2.

$$\mu(A_{E_j,X_1}(y) - A_{E_j,X_2}(y)) \leq m \quad \text{for all } 1 \leq j \leq l.$$

PROOF. The T-sets, zones, L-limits and sections in this proof are referred to X_1. If $TC^{|x|<\epsilon}_{\xi(X_1,s)} = \emptyset$ then the result is true by the remark 6.2.1. Otherwise we consider the set $\Xi(G_j) \subset TC^{|x|<\epsilon}_{\xi(X_1,s),\sim}$ (see subsection 2.1.4 for definitions). We choose $a \in \Xi(G_j)$; the lemma 2.1.12 implies

$$A_{E_j,X_1}(y) - A_{E_j,X_2}(y) =$$
$$= (A_{E_a^{G_j,1},X_1}(y) - A_{E_a^{G_j,1},X_2}(y)) - \sum_{b\in\Xi(G)\setminus\{a\}} (A_{E_b^{G_j,2},X_1}(y) - A_{E_b^{G_j,2},X_2}(y)).$$

It is enough to prove that the right hand side is a summation of functions whose order is less or equal than m. Either s belongs to a T-set β or it belongs to a zone $Z^\epsilon_{X_1}$; in the latter case we choose a semi-analytic curve β contained in Z. A critical tangent cord $c(y) \in \Xi(G_j(y))$ contains a point $T_{X_1}^{\epsilon,d}(y)$ and another point $P(y) \in \partial U_\epsilon$. Since $\mathbb{S}^1$ is compact there exists $(x_1, 0)$ and a sequence $y_k \in \beta$, $y_k \to 0$ such that $(x_1, 0) = \lim_{k\to\infty} P(y_k)$. We denote $(x_0, 0) = \lim_{y\in\beta, y\to 0} T_{X_1}^{\epsilon,d}(y)$. We have that $x_1 \in L_{\beta,x_0}^{+,\epsilon} \cup L_{\beta,x_0}^{-,\epsilon}$. Moreover, the partition induced by (x_0, x_1) is the same partition induced by $c(s)$. Therefore, by lemma 6.2.1 we deduce that $\mu(A_{E_c^{G_j,k},X_1}(y) - A_{E_c^{G_j,k},X_2}(y)) \le m$ for $k \in \{1, 2\}$. □

We consider the T-sets $\beta_1, \ldots, \beta_l$ and the zones $Z_{X_1,1}^\epsilon, \ldots, Z_{X_1,l}^\epsilon$ associated to the vector field X_1. We consider the sequence of graphs $\mathcal{G}^1, \mathcal{G}^2, \ldots$ associated to X_1 (see section 4.3). Let $G_1^j, \ldots, G_{l_j}^j$ be the connected components of $\mathcal{G}^j$. We define $E_k^j = Sing(G_k^j)$. We have

LEMMA 6.2.3. *For all $j \ge 1$ and all $1 \le k \le l_j$ we have*

$$\mu(A_{E_k^j,X_1}(y) - A_{E_k^j,X_2}(y)) \le m.$$

PROOF. In this proof T-sets, zones and graphs are referred to X_1. If there are no T-sets the result is obvious. The result for $j = 1$ is implied by lemma 6.2.2. Suppose it is true for $j = j_0$. Consider $s \in \beta_{j_0+1}$. Let C be a connected component of $\mathcal{G}^{j_0}$. By varying C it is enough to prove the result for the connected components of $\mathcal{G}^{j_0+1}$ contained in C. By lemma 4.3.4 the critical tangent cords in $TC_{\xi(X_1,s,\epsilon),\sim}^{|x|<\epsilon}$ separate the connected components of $\mathcal{G}^{j_0+1}$ contained in C. Let $\gamma \subset [y = s]$ be a critical tangent cord dividing C; it induces a partition $(E_{\gamma,-}, E_{\gamma,+})$ in $Sing(C)$ and a partition $(E'_{\gamma,-}, E'_{\gamma,+})$ in $f = 0$. It is enough to prove that

$$\mu(A_{E_{\gamma,-},X_1}(y) - A_{E_{\gamma,-},X_2}(y)) \le m, \quad \mu(A_{E_{\gamma,+},X_1}(y) - A_{E_{\gamma,+},X_2}(y)) \le m$$

because then we can proceed as we did in lemma 6.2.2. Since γ does not split any component $\mathcal{G}^{j_0}$ other than C (lemma 4.3.5) then we have

$$A_{E_{\gamma,-},X_1}(y) - A_{E_{\gamma,-},X_2}(y) =$$

$$(A_{E'_{\gamma,-},X_1}(y) - A_{E'_{\gamma,-},X_2}(y)) - \sum_{d\in J}(A_{Sing(C_d),X_1}(y) - A_{Sing(C_d),X_2}(y))$$

for a certain subset $\{C_d\}_{d\in J}$ of components of $\mathcal{G}^{j_0}$ other than C. We obtain $\mu(A_{E_{\gamma,-},X_1}(y) - A_{E_{\gamma,-},X_2}(y)) \le m$ by lemmas 6.2.1, 6.2.2 and hypothesis of induction. The proof for the + case is analogous. □

PROPOSITION 6.2.1. *Let $X_1, X_2 \in \mathcal{H}_f$ such that there exists a special germ of homeomorphism conjugating $Re(X_1)$ and $Re(X_2)$. Consider a continuous multi-valuated section $S : B(0,\delta) \setminus \{0\} \to (f = 0)$ such that $S(s) \in [y = s]$ for all $s \in B(0,\delta) \setminus \{0\}$. Then*

$$\mu(Res_{X_1}(S(y)) - Res_{X_2}(S(y))) \le m.$$

PROOF. By proposition 4.3.1 we can apply lemma 6.2.3 to the graph with no edges. Since the connected components are singletons we are done. □

6.2.1. Rigidity of the special conjugation at $y = 0$. Let $X_1 \stackrel{sp}{\sim} X_2$ be conjugated by σ. We argued in remark 5.4.1 that the real flow generates the complex flow at $y = 0$. We make rigorous that statement in order to prove that $\sigma_{|U_\epsilon \cap [y=0]}$ is complex analytic if $N > 1$ or $m > 0$.

LEMMA 6.2.4. *Let $X_1, X_2 \in \mathcal{H}_f$ such that $X_1 \overset{sp}{\sim} X_2$ by a special germ of homeomorphism σ. If $[y=0] \subset Sing X_1$ then $\sigma_{|U_\epsilon \cap [y=0]}$ is holomorphic, moreover it conjugates $X_1(1)_{|y=0}$ and $X_2(1)_{|y=0}$.*

For $t \in \mathbb{C}$ we define $\exp(tX)(x_0, y_0)$ the point obtained by following the vector field X from (x_0, y_0) during time t. For t close to 0 we have $\exp(tX)(x_0, y_0) = \Gamma_{\xi(e^{i\arg(t)}X, y_0, \epsilon)}[x_0, y_0](|t|)$.

PROOF. We have $X_1 = y^m X_1(1)$ and $X_2 = y^m X_2(1)$. Let $\eta > 0$ such that X_1 and σ are defined in $U_{\epsilon+\eta,\delta}$ whereas X_2 is defined in $\sigma(U_{\epsilon+\eta,\delta})$. Let $(x_0, 0) \in U_\epsilon$; there exists $A > 0$ such that the complex flows $\exp(tX_1)(x_0, 0)$ and $\exp(tX_2)(\sigma(x_0, 0))$ are well defined for $|t| < 2A$. Our goal is proving

$$\sigma(\exp(tX_1)(x_0, 0)) = \exp(tX_2)(\sigma(x_0, 0))$$

for all $t \in B(0, A)$. This statement implies that $\sigma_{|U_\epsilon \cap [y=0]}$ is holomorphic except maybe at 0 and then Riemann's theorem implies that $\sigma_{|U_\epsilon \cap [y=0]}$ is holomorphic.

Let $t \in B(0, A) \setminus \{0\}$ and consider $\lambda_0 \in \mathbb{S}^1$ such that $t/|t| = \lambda_0^m$. We restrict our parameters to the line $y \in \lambda_0 \mathbb{R}^+$. In $y = r\lambda_0$ the vector fields $Re(X_1)$ and $Re(X_2)$ are topologically conjugated. We obtain

$$Re(\lambda_0^m X_1(1))_{|y=r\lambda_0} \sim Re(\lambda_0^m X_2(1))_{|y=r\lambda_0}$$

By making $r \to 0$ we have

$$\sigma(\exp(h\lambda_0^m X_1(1))(x_0, 0)) = \exp(h\lambda_0^m X_2(1))(\sigma(x_0, 0))$$

for all $0 \leq h < A$. Therefore

$$\sigma(\exp(tX_1(1))(x_0, 0)) = \exp(tX_2(1))(\sigma(x_0, 0)).$$

□

We remind the reader that N is the generic number of points in $[f = 0] \cap [y = s]$.

LEMMA 6.2.5. *Let $X_1, X_2 \in \mathcal{H}_f$ such that $X_1 \overset{sp}{\sim} X_2$ by a special germ of homeomorphism σ. Suppose $N > 1$. Then $\sigma_{|U_\epsilon \cap [y=0]}$ is holomorphic, moreover it conjugates $X_{1|y=0}$ and $X_{2|y=0}$.*

This lemma is a consequence of the evolution of the L-limits.

PROOF. In this proof the L-limits, virtual and true sections are referred to X_1. By lemma 6.2.4 we can suppose that $[y = 0] \not\subset Sing(X_1)$. Let $x_1 \in B(0, \epsilon) \setminus \{0\}$. We proceed as in remark 5.4.1. Consider $M > 0$ such that $\exp(B(0, 2M)X_1)(x_1, 0)$ is contained in $\cup_{\Delta \in \mathbb{R}} \rho_{1,-}^{\Delta}$. There exists a true section

$$\Sigma : W(M/[(L-m)co(E_1, L)]) \to \mathbb{C}^2$$

and a function $T : W \to \mathbb{R}^+$ such that

$$T(y) = \psi_{1,X_1}(\Sigma(y)) + A_{E_{1,-},X_1}(y) - \psi_{0,X_1}(x_0, y).$$

Moreover, we know that

$$Img\left(\psi_{1,X_1}\left(\lim_{y \in \beta(\Delta),\ y \to 0} \Sigma(y)\right) - \psi_{1,X_1}(x_1, 0)\right) = \Delta(L-m)co(E_1, L).$$

Let $(x_2, 0) = \exp(KX_1)(x_1, 0)$ for $K \in B(0, M)$; we define

$$\Delta_0 = Img(K)/[(L-m)co(E_1, L)].$$

and $r = \psi_{1,X_1}(x_2, 0) - \psi_{1,X_1}(\lim_{y \in \beta(\Delta_0),\ y \to 0} \Sigma(y))$. We have $r \in \mathbb{R}$. Now consider $\Sigma_{x_2}(y) = \exp(rX_1)(\Sigma(y))$. We use that σ is a conjugation between the real flows to obtain

$$\begin{aligned} \psi_{1,X_2}(\sigma(\Sigma_{x_2}(y))) + A_{E_{1,-},X_2}(y) - \psi_{0,X_2}(\sigma(x_0, y)) = \\ = \psi_{1,X_1}(\Sigma_{x_2}(y)) + A_{E_{1,-},X_1}(y) - \psi_{0,X_1}(x_0, y). \end{aligned} \tag{6.1}$$

Since by proposition 6.2.1 the function $A_{E_{1,-},X_1}(y) - A_{E_{1,-},X_2}(y)$ is holomorphic up to a finite ramification then there exists $C \in \mathbb{C}$ such that

$$C = \lim_{y \to 0} (A_{E_{1,-},X_1}(y) - A_{E_{1,-},X_2}(y)).$$

We define $D = C + \psi_{0,X_2}(\sigma(x_0, 0)) - \psi_{0,X_1}(x_0, 0)$. By taking $y \in \beta(\Delta_0)$ and making $y \to 0$ we obtain

$$\psi_{1,X_2}(\sigma(\exp(KX_1)(x_1, 0))) - \psi_{1,X_1}(\exp(KX_1)(x_1, 0)) = D$$

for all $K \in B(0, M)$. We substract from the previous one the expression we have for $K = 0$. Therefore, the expression

$$\psi_{1,X_2}(\sigma(\exp(KX_1)(x_1, 0))) - \psi_{1,X_2}(\sigma(x_1, 0)) = K$$

is satisfied for all $K \in B(0, M)$. The last equation is equivalent to

$$\sigma(\exp(KX_1)(x_1, 0)) = \exp(KX_2)(\sigma(x_1, 0))$$

for all $K \in B(0, M)$. As a consequence $\sigma_{|y=0}$ is holomorphic in the neighborhood of $(x_1, 0)$. By changing $(x_1, 0)$ we deduce that $\sigma_{|U_\epsilon \cap [y=0]}$ is holomorphic except maybe at 0. By Riemman's theorem the mapping $\sigma_{|U_\epsilon \cap [y=0]}$ is holomorphic. $\square$

Let $f_1^{n_1} \dots f_p^{n_p} y^m$ be the decomposition of f in irreducible factors. The previous lemmas imply

PROPOSITION 6.2.2. *Let $X_1, X_2 \in \mathcal{H}_f$ such that $X_1 \overset{sp}{\sim} X_2$ by a special germ of homeomorphism σ. Suppose $(N, m) \neq (1, 0)$. Then $\sigma_{|U_\epsilon \cap [y=0]}$ is holomorphic, moreover it conjugates $X_1(1)_{|U_\epsilon \cap [y=0]}$ and $X_2(1)_{|U_\epsilon \cap [y=0]}$.*

6.2.2. Comparing the residues revisited. We can improve the results we obtained early in this section. The rigidity of the special conjugation at $y = 0$ implies a stronger relation on the residues.

LEMMA 6.2.6. *Let $X_1,\ X_2 \in \mathcal{H}_f$ such that $X_1 \overset{sp}{\sim} X_2$ by a special germ of homeomorphism σ. Consider a L-limit $L^{+,\epsilon}_{\beta,x_0} \neq \emptyset$ associated to X_1. Consider a component ρ of $L^{+,\epsilon}_{\beta,x_0}$ and let E be the partition induced by (x_0, ρ). Then*

$$\lim_{y \to 0} y^m \left(\sum_{P \in E_-(y)} [Res_{X_1}(P) - Res_{X_2}(P)] \right) = 0.$$

PROOF. We use the same notations than in the proof of lemma 6.2.1. There exists $C \in \mathbb{C}$ such that

$$C = \lim_{y \in \beta,\ y \to 0} y^m (A_{E_-,X_1}(y) - A_{E_-,X_2}(y)).$$

We have

$$C = [\psi_{1,X_2}(\sigma(S(0))) - \psi_{1,X_1}(S(0))] - [\psi_{0,X_2}(\sigma(x_0, 0)) - \psi_{0,X_1}(x_0, 0)].$$

Since $\sigma_{|y=0}$ is holomorphic (prop. 6.2.2) then $\psi_{0,X_2} \circ \sigma_{|y=0} - \psi_{0,X_1|y=0} \equiv D$ for some $D \in \mathbb{C}$. The function ψ_{1,X_1} is the prolongation of ψ_{0,X_1} along a path $\gamma \subset \mathbb{C}^* \times \{0\}$

going from $(x_0, 0)$ to $S(0)$ in counter clock wise sense. The function ψ_{1,X_2} is the prolongation of ψ_{0,X_2} along $\sigma(\gamma)$. Hence, the prolongation of $\psi_{0,X_2} \circ \sigma_{|y=0} = \psi_{0,X_1|y=0} + D$ along γ is $\psi_{1,X_2} \circ \sigma_{|y=0}$ and then $\psi_{1,X_2} \circ \sigma_{|y=0} = \psi_{1,X_1|y=0} + D$. As a consequence the constant C is equal to $D - D = 0$. $\square$

PROPOSITION 6.2.3. *Suppose* $(N, m) \neq (1, 0)$. *Let* $X_1, X_2 \in \mathcal{H}_f$ *such that* $X_1 \overset{sp}{\sim} X_2$ *by a special germ of homeomorphism. Consider a continuous multi-valuated section* $S : B(0,\delta) \setminus \{0\} \to (f = 0)$ *such that we have* $S(s) \in [y = s]$ *for all* $s \in B(0,\delta) \setminus \{0\}$. *Then*

$$\lim_{y \to 0} y^m (Res_{X_1}(S(y)) - Res_{X_2}(S(y))) = 0.$$

PROOF. The lemma 6.2.1 is the key to prove proposition 6.2.1. Lemmas 6.2.2 and 6.2.3 are intended to prove that the partitions can be chosen to be singletons. In an analogous way the lemma 6.2.6 leads us to prove the proposition 6.2.3 for $N > 1$. If $N = 1$ and $m > 0$ then we have $Res_{X_1(1)}(0,0) = Res_{X_2(1)}(0,0)$ by proposition 6.2.2. That implies $[y^m(Res_{X_1}(S(y))](0) = [y^m(Res_{X_2}(S(y))](0)$ for the unique continuous section S of $f = 0$. $\square$

6.3. Topological invariants

Let $X \in \mathcal{H}_f$. The set of topological invariants $SP(X)$ of X for the $\overset{sp}{\sim}$ conjugation is by definition

- $SP(X) = \emptyset$ if $N = 0$ or $(N, m) = (1, 0)$.
- Otherwise we consider the parts of degree less or equal than 0 of every function $y^m(Res_X(S(y)))$ associated to some continuous section $S : B(0,\delta) \setminus \{0\} \to Sing X$.

We say that $X \overset{ana}{\sim} Y$ for $X, Y \in \mathcal{H}_f$ if X and Y are conjugated by a special analytic diffeomorphism. By definition we denote $X \overset{ana}{\sim} Y$ if $X, Y \in \mathcal{H}(\mathbb{C}, 0)$ are analytically conjugated.

LEMMA 6.3.1. *Let* $X, Y \in \mathcal{H}_f$. *Suppose* $(N, m) \neq (1, 0)$*; then*

$$SP(X) = SP(Y) \implies X(1)_{|y=0} \overset{ana}{\sim} Y(1)_{|y=0}.$$

Moreover, if $N = 1$ *and* $m > 0$ *we have*

$$SP(X) = SP(Y) \Leftrightarrow X(1)_{|y=0} \overset{ana}{\sim} Y(1)_{|y=0}.$$

PROOF. If $N = 0$ the result is obvious because $X(1)_{|y=0}$ and $Y(1)_{|y=0}$ are both regular. Otherwise, since for $Z = X$ or $Z = Y$ we have

$$Res_{Z(1)}(0,0) = \lim_{s \to 0} s^m \sum_{P \in [f=0] \cap [y=s]} Res_Z(P).$$

Then $SP(X) = SP(Y)$ implies $Res_{X(1)}(0,0) = Res_{Y(1)}(0,0)$. As a consequence $X(1)_{|y=0} \overset{ana}{\sim} Y(1)_{|y=0}$ since the only analytic invariants are the order and the residue and $\nu(X(1)_{|y=0}) = \nu(Y(1)_{|y=0}) = \nu((f/y^m)(x,0))$. For $N = 1$ and $m > 0$ the part of degree less or equal than 0 of $y^m Res_Z(S(y))$ associated to the unique continuous section $S(y)$ is equal to $Res_{Z(1)}(0,0)$ for $Z \in \mathcal{H}_f$. As a consequence $X(1)_{|y=0} \overset{ana}{\sim} Y(1)_{|y=0}$ implies $SP(X) = SP(Y)$. $\square$

THEOREM 6.1. *Let $f \in \mathbb{C}\{x,y\}$ satisfying the (NSD) conditions. Let $X, Y \in \mathcal{H}_f$. Then*

$$X \overset{sp}{\sim} Y \Leftrightarrow SP(X) = SP(Y).$$

PROOF OF THE IMPLICATION $\Rightarrow$. The invariants coincide by proposition 6.2.3. $\square$

Our next goal is proving the $\Leftarrow$ implication in theorem 6.1.

6.3.1. Proof of theorem 6.1 for the case $N = 0$, $m > 0$.

PROPOSITION 6.3.1. *Let $X_j = u_j(x,y)y^m\partial/\partial x \in \mathcal{H}_{y^m}$ for $j \in \{1,2\}$ and $m \geq 0$. Then $X \overset{ana}{\sim} Y$.*

PROOF. It is enough to prove the existence of an analytic diffeomorphism $(\xi(x,y), y)$ conjugating $X_1(1)$ and $X_2(1)$. The mapping

$$\exp_Z(t,y) = \exp_Z(tZ(1))(0,y)$$

is a germ of analytic diffeomorphism for all $Z \in \mathcal{H}_{y^m}$. Moreover $\exp_Z$ conjugates $\partial/\partial x$ and $Z(1)$. As a consequence $(\xi(x,y), y) = \exp_Y \circ \exp_X^{-1}$ conjugates $X(1)$ and $Y(1)$. $\square$

6.3.2. Case $N \geq 1$. Strips. Let $x = x_1 + ix_2$. Consider X_1, X_2 in $\mathcal{H}_f$ such that $SP(X_1) = SP(X_2)$. A good candidate to be a special conjugation is

$$(\psi_2^{-1}(x,y), y) \circ (\psi_1(x,y), y)$$

where ψ_j is an integral of the time form of X_j for $j \in \{1,2\}$. This conjugation is well-defined only if $Res_{X_1}(P) = Res_{X_2}(P)$ for all P in $[(f=0) \setminus (y=0)]$ and then it is analytic. We will modify the integral of the time form of X_1 in order to make this strategy works.

Consider the decomposition $X_1 = (1/2)(\Re X_1 - i\Im X_1)$ in real and imaginary parts. We have $\Re X_1(\psi_1) = 1$ and $\Im X_1(\psi_1) = i$ whereas an integral ψ_1' of the time form of $\Re X_1$ only satisfies $\Re X_1(\psi_1') = 1$. That provides a motivation to replace ψ_1 with ψ_1' such that

1. $y^m\psi_1'$ is multi-valuated and continuous in $V \setminus (f/y^m = 0)$ for some set V. Moreover ψ_1' is C^∞ in $V \setminus (yf = 0)$.
2. $\Re X_1(\psi_1') = 1$ and $\Im X_1(\psi_1')$ is uni-valuated and bounded.
3. $f(\psi_2 - \psi_1')$ is a complex uni-valuated continuous function defined in V. It satisfies $[f(\psi_2 - \psi_1')]_{|(f/y^m)=0} \equiv 0$.
4. If $N + m > 1$ then $f(\psi_2 - \psi_1')_{|V \cap [y=0]} = f(\psi_2 - \psi_1)_{|V \cap [y=0]}$.
5. If $N + m > 1$ then $\lim_{\eta \to 0} \sup_{P \in B(0,\eta)} |\Im X_1(\psi_1')(P) - i| = 0$.
6. $\partial(f[\psi_2 - \psi_1'])/\partial x_j$ is continuous in $V \setminus (f/y^m = 0)$ for $j \in \{1,2\}$.

We say that ψ_1' is a modification of ψ_1 with respect to X_2. The set V is typically of the form $U_\epsilon \cap [y \in W \cup \{0\}]$; the set $W \subset B(0,\delta)$ is always a simply connected open set such that $0 \in \overline{W}$. The modification will take effect in strips. Consider a continuous section $T_{X_1}^{\epsilon,a} : W \to T_{X_1}^{\epsilon}$ and a circular arc $arc(s) = T_{X_1}^{\epsilon,a}(s)T_{X_1}^{\epsilon,a+1}(s)$ such that

$$\omega_{\xi(X),(|x|<\epsilon)\cup\{x_0\}}(x_0,y_0) \in (f=0), \ \forall (x_0,y_0) \in arc(y_0) \text{ and } \forall y_0 \in W.$$

We have $\omega_{\xi(X),|x| \leq \epsilon}(arc(s)) = F(s)$ where $F(s)$ is a continuous section of $SingX_1$ defined over W. We say that $S = \cup_{s \in W} \Gamma_{\xi(X),+}^{|x| \leq \epsilon}[arc(s)]$ is a positive strip over W with vertex at F.

We define a C^∞ function H defined over $\mathbb{C}$ such that

- $H : \mathbb{C} \to [0,1]$ is an increasing function of $Img(z)$.
- $H(z) = 0$ if $Img(z) \leq 0$ whereas $H(z) = 1$ if $Img(z) \geq 1$.

We define $M_S(x,y)/(2\pi i)$ as

$$(Res_{X_2}(F(y)) - Res_{X_1}(F(y)))H\left(\frac{\psi_1(x,y) - \psi_1(T_X^{\epsilon,a}(y))}{Img(\psi_1(T_X^{\epsilon,a+1}(y)) - \psi_1(T_X^{\epsilon,a}(y)))}\right)$$

for $(x,y) \in S$. The function M_S can be extended to a C^∞ multi-valuated function defined in $(U_\epsilon \cap [y \in W]) \setminus (f = 0)$. We define $\Re X_1(M_S) \equiv 0$ and $\Im X_1(M_S) \equiv 0$ outside of S. Since

$$\Im X_1(0) \equiv \Re X_1(\Im X_1(M_S))$$

then we use the couple $\Re X_1$, $\Im X_1$ to obtain M_S by C^∞ prolongation.

In next lemma W is a neighborhood of 0 if $(N,m) = (1,0)$; otherwise we suppose $0 \notin W$. Anyway $[(f/y^m) = 0] \cap [y \in W]$ is composed by N continuous sections $(g_j(y), y) : W \to SingX_1$ for $1 \leq j \leq N$. Suppose there exists a positive strip S^j over W with vertex at $(g_j(y), y)$ for all $1 \leq j \leq N$. Then, we define

$$\psi_1' = \psi_1 + \sum_{j=1}^{N} M_{S^j}.$$

LEMMA 6.3.2. *Let $X_1, X_2 \in \mathcal{H}_f$ such that $SP(X_1) = SP(X_2)$. Then ψ_1' is a modification of ψ_1 in $U_\epsilon \cap [y \in W \cup \{0\}]$ with respect to X_2.*

PROOF. Up to ramify by $(x,y) \mapsto (x, y^k)$ we can suppose that $(f/y^m) = 0$ is the union of N curves $x = g_j(y)$ for $1 \leq j \leq N$. It is enough to prove the lemma in this setting because conditions (1) through (6) are invariant by $(x,y) \mapsto (x,y^k)$.

Let $V = U_\epsilon \cap [y \in W \cup \{0\}]$. The function ψ_1' is C^∞ in $V \setminus [yf = 0]$ by construction. The construction also implies that $\Re X_1(\psi_1') = 1$. We define ψ_1' such that $\psi_1'(\epsilon, y) = \psi_1(\epsilon, y)$ for all $y \in W$. There exists $K > 0$ such that $Var_{x-g_j(y)}^{\epsilon,\delta} < K$ for all $1 \leq j \leq N$ by proposition 3.3.1. Proposition 3.2.4 implies that

$$|Img \ln(x_1 - g_j(y)) - Img \ln(x_0 - g_j(y))| < 2\pi + K$$

for all (x_0, y), $(x_1, y) \in S^k$ and all $1 \leq j, k \leq N$.

We define $R_{1,2}^j(y) = Res_{X_2}(g_j(y), y) - Res_{X_1}(g_j(y), y)$. We have that $D = \psi_2 - \psi_1 - \sum_{j=1}^N R_{1,2}^j(y) \ln(x - g_j(y))$ is a solution of

$$\frac{\partial D}{\partial x} = \frac{1}{u_2 f} - \frac{1}{u_1 f} - \sum_{j=1}^{N} \frac{\partial (R_{1,2}^j(y) \ln[x - g_j(y)])}{\partial x}.$$

This equation is free of residues. Moreover, the right hand side is of the form h/f for some $h \in \mathbb{C}\{x,y\}$. By lemma 3.2.1 the function $\psi_2 - \psi_1$ can be expressed in the form

$$\frac{\beta(x,y)}{(x - g_1(y))^{n_1 - 1} \ldots (x - g_N(y))^{n_N - 1} y^m} + \sum_{j=1}^{N} R_{1,2}^j(y) \ln(x - g_j(y))$$

for some $\beta \in \mathbb{C}\{x, y\}$.

Let $(x, y_0) \in \overline{U_\epsilon} \setminus [yf = 0]$. We can obtain $(\psi_2 - \psi_1')(x, y_0)$ by continuous extension of a path $\gamma : [0, 1] \to \overline{U_\epsilon} \cap [y = y_0]$ such that $\gamma(0) = (\epsilon, y_0)$ and $\gamma(1) = (x, y_0)$. Consider the universal covering $\tilde{U}_\epsilon(y_0)$ of $(\overline{U_\epsilon} \cap [y = y_0]) \setminus [f = 0]$. We can choose γ such that the lifting $\tilde{\gamma}$ of γ cuts at most one connected component of $\tilde{S}^j(y_0)$ for all $1 \leq j \leq N$. As a consequence

$$|Img \ln(x - g_j(y))(\gamma(t)) - Img \ln(x - g_j(y))(\epsilon, y)| < 2\pi + (2\pi + K)$$

for all $1 \leq j \leq N$ and all $t \in [0, 1]$. We deduce that for $\delta << 1$ the choice of γ satisfies

$$Img \ln(x - g_j(y))(\gamma[0, 1]) \in [-(6\pi + K), 6\pi + K]$$

for all $(x, y_0) \in \overline{U_\epsilon} \setminus [yf = 0]$ and all $1 \leq j \leq N$.

By continuous extension we obtain

$$|y_0^m(\psi_1 - \psi_1')\gamma(1)| \leq 2\pi \sum_{1 \leq j \leq N} |y_0^m R_{1,2}^j(y_0)|$$

for all $y_0 \in W$. If $SP(X_1) = SP(X_2)$ the right hand side is bounded when $y_0 \to 0$. Hence $[y^m(\psi_1' - \psi_1)](\gamma(1))$ is bounded independently of (x, y_0). Moreover, if $(N, m) \neq (1, 0)$ the right hand side is a $O(y)$.

We have that

$$\left| f(x, y) \sum_{j=1}^{N} R_{1,2}^j(y) \ln(x - g_j(y)) \right| (\gamma(1))$$

is less or equal than

$$\left| f(x, y_0) \sum_{j=1}^{N} R_{1,2}^j(y_0) \ln|x - g_j(y_0)| \right| + (6\pi + K) \left| f(x, y_0) \sum_{j=1}^{N} R_{1,2}^j(y_0) \right|$$

As a consequence

$$[f(x, y)(\psi_2 - \psi_1')](x, y) = O((x - g_1(y)) \dots (x - g_N(y)))$$

in $\overline{U_\epsilon} \cap [y \in W]$. We have $y^m R_{1,2}^j \in (y)$ for $N + m > 1$ and all $1 \leq j \leq N$ since $SP(X) = SP(Y)$. As a consequence for $N + m > 1$ we have that

$$[f(x, y)(\psi_2 - \psi_1')](x, y) - \beta(x, y)(x - g_1(y)) \dots (x - g_N(y))$$

is a $O((x - g_1(y))^{n_1 - 1} \dots (x - g_N(y))^{n_N - 1} y)$ in $\overline{U_\epsilon} \cap [y \in W]$. We extend the function $f(\psi_2 - \psi_1')$ to $[f/y^m = 0]$ as 0 whereas for $N + m > 1$ we extend $f(\psi_2 - \psi_1')$ to $\overline{U_\epsilon} \cap [y = 0]$ as $\beta(x, 0)x^N$. This definition implies conditions (3) and (4). Since $y^m \psi_1' = y^m(\psi_1' - \psi_2) + y^m \psi_2$ the proof of condition (1) is now complete.

Since $\Im X_1(\psi_1) \equiv i$ then

$$\Im X_1(\psi_1') - i = \sum_{j=1}^{N} \Im X_1(M_{S^j}).$$

By making calculations in the system of coordinates provided by ψ_1 we obtain

$$|\Im X_1(\psi_1') - i| \leq D \sum_{j=1}^{N} \frac{|Res_{X_2}(g_j(y), y) - Res_{X_1}(g_j(y), y)|}{|Img(\psi_1(T_X^{\epsilon, a_j + 1}(y)) - \psi_1(T_X^{\epsilon, a_j}(y)))|}$$

where $D = 2\pi \sup_{z\in\mathbb{C}} |\partial H/\partial Img(z)|$. The function

$$Gap(a,\lambda) \stackrel{def}{=} \left| Img\left((|y|^m \psi_1)(T^{\epsilon,a+1}_{X(\lambda)}(0)) - (|y|^m \psi_1)(T^{\epsilon,a}_{X(\lambda)}(0)) \right) \right|$$

is defined over $J = \{1,\ldots,2(\tilde{\nu}(X_1)-1)\} \times \mathbb{S}^1$. It is strictly positive; hence $C = \inf_{(a,\lambda)\in J} Gap(a,\lambda)$ belongs to $\mathbb{R}^+$. We have

$$|\Im X_1(\psi_1')(x,y) - i| \leq \frac{2D}{C} \sum_{j=1}^{N} |y^m Res^j_{1,2}(y)|$$

for all $x \in B(0,\epsilon)$ and $y \in W$ close to 0. This equation is analogous to the one we obtained for $|y^m(\psi_1' - \psi_1)|$. We deduce that $\Im X_1(\psi_1')$ is bounded. Moreover $\Im X_1(\psi_1')$ extends continuously to $V \cap [y=0]$; for $(N,m) = (1,0)$ is obvious, otherwise we define $\Im X_1(\psi_1')(x,0) \equiv i$. As a consequence $\Im X_1(\psi_1')$ is continuous, uni-valuated and bounded in $V \setminus [f/y^m = 0]$. Condition (5) is a consequence of $y^m Res^j_{1,2} \in (y)$ for $N+m>1$ and all $1 \leq j \leq N$.

The only condition still to prove is (6). We suppose $(N,m) \neq (1,0)$, otherwise it is trivial. Condition (6) is equivalent to the function $\partial(y^m[\psi_1 - \psi_1'])/\partial x_j$ extending continuously to $(V \cap [y=0]) \setminus \{0\}$ as the zero function for $j \in \{1,2\}$. Since $\Re X_1(\psi_1 - \psi_1') \equiv 0$ and $|\Im X_1(\psi_1 - \psi_1')| \leq \eta|y|$ for some $\eta > 0$ we have

$$\begin{array}{rcl} Re(u_1 f)\partial(\psi_1 - \psi_1')/\partial x_1 + Img(u_1 f)\partial(\psi_1 - \psi_1')/\partial x_2 & = & 0 \\ -Img(u_1 f)\partial(\psi_1 - \psi_1')/\partial x_1 + Re(u_1 f)\partial(\psi_1 - \psi_1')/\partial x_2 & = & \eta_1 \end{array}$$

where $|\eta_1(x,y)| \leq \eta|y|$. By solving the system we deduce that

$$\left| \frac{\partial(y^m[\psi_1 - \psi_1'])}{\partial x_j} \right| \leq \frac{\eta|y|}{|u_1||f/y^m|}$$

for all $j \in \{1,2\}$. The inequalities imply condition (6). □

REMARK 6.3.1. *The constant C depends on ϵ and $\lim_{\epsilon\to 0} C(\epsilon) = \infty$. As a consequence we can choose $\Im X_1(\psi_1')$ as close to i as desired just by taking (ϵ,δ) close to $(0,0)$.*

6.3.3. Existence of strips. Case $N=1$. In this case the set $f/y^m = 0$ is equal to a curve $x = f_1(y)$.

LEMMA 6.3.3. *Let $N=1$, $m=0$ and $X \in \mathcal{H}_f$. There exists a strip over $B(0,\delta)$ with vertex at $x = f_1(y)$.*

PROOF. We claim there exists an arc $arc(0) = T^{\epsilon,a}_X(0)T^{\epsilon,a+1}_X(0)$ such that

$$\omega_{\xi(X),(|x|<\epsilon)\cup\{x_0\}}(x_0,0) = (0,0),\ \forall (x_0,0) \in arc(0).$$

We choose $1 \leq a \leq 2(\tilde{\nu}(X)-1)$ such that in the interior of $arc(0)$ the vector field $Re(X)$ points towards the interior of $|x| \leq \epsilon$. By Rolle property the trajectory $\Gamma^{(|x|<\epsilon)\cup\{x_0\}}_{\xi(X),+}[x_0,0]$ for $(x_0,0) \in arc(0)$ is contained in the bounded region enclosed by the curve

$$\Gamma^{|x|\leq\epsilon}_{\xi(X),+}[T^{\epsilon,a}_X(0)] \cup \Gamma^{|x|\leq\epsilon}_{\xi(X),+}[T^{\epsilon,a+1}_X(0)] \cup arc(0) \cup \{(0,0)\}.$$

It is a Jordan curve by remark 3.2.3. Then

$$\omega_{\xi(X),(|x|<\epsilon)\cup\{x_0\}}(x_0,y) \in (f=0),\ \forall (x_0,y) \in arc(y) \text{ and } \forall y \in B(0,\delta)$$

since the basins of attraction and repulsion of $x = f_1(y)$ are open by remark 2.2.1. □

The same proof implies the existence of strips for $N = 1$ and $m > 0$.

LEMMA 6.3.4. *Suppose* $N = 1$ *and* $m > 0$. *Consider* $X \in \mathcal{H}_f$. *There exists a strip over* $B(0,\delta) \setminus (\lambda_0 \mathbb{R}^+ \cup \{0\})$ *with vertex at* $x = f_1(y)$ *for all* $\lambda_0 \in \mathbb{S}^1$.

Next we prove the existence of modifications for $N = 1$.

LEMMA 6.3.5. *Fix* $\eta > 0$. *Let* $N = 1$. *Consider* $X_1, X_2 \in \mathcal{H}_f$ *such that* $SP(X_1) = SP(X_2)$. *There exists a modification* ψ_1' *of* ψ_1 *in* $U_{\epsilon,\delta}$ *with respect to* X_2. *If* $m = 0$ *we can choose* ψ_1' *to be* C^∞ *in* $U_{\epsilon,\delta} \setminus [f = 0]$; *moreover, for* $\epsilon > 0$ *and* $\delta(\epsilon) > 0$ *small we have* $|\Im X_1(\psi_1') - i| < \eta$.

PROOF. If $m = 0$ the lemma 6.3.3 guarantees the existence of strips. Then we use lemma 6.3.2 to build a modification in $U_{\epsilon,\delta}$ by taking $W = B(0,\delta)$. The function ψ_1' is C^∞ in $U_{\epsilon,\delta} \setminus [f = 0]$ by construction. Moreover $|\Im X_1(\psi_1') - i|$ can be made as small as desired by remark 6.3.1.

If $m > 0$ we define $W_+ = B(0,\delta) \setminus \mathbb{R}_{\leq 0}$ and $W_- = B(0,\delta) \setminus \mathbb{R}_{\geq 0}$. By lemmas 6.3.4 and 6.3.2 there exists a modification $\psi_{1,+}$ of ψ_1 with respect to X_2 in $U_\epsilon \cap [y \in W_+ \cup \{0\}]$. By replacing $+$ with $-$ in the previous argument we obtain $\psi_{1,-}$. Consider a partition of the unit ξ_+, ξ_- of $B(0,\delta) \setminus \{0\}$ with respect to the covering $W_+ \cup W_-$. It is straightforward to check that $\psi_1'(x,y) = \xi_+(y)\psi_{1,+}(x,y) + \xi_-(y)\psi_{1,-}(x,y)$ is a modification of ψ_1 with respect to X_2 in $U_{\epsilon,\delta}$. □

REMARK 6.3.2. *The properties* $\Re X_1(\psi_1') = 1$ *and* $|\Im X_1(\psi_1') - i| < 1$ *imply that* ψ_1' *is locally injective. That is a necessary condition in order to make*

$$(\psi_2(x,y),y)^{-1} \circ (\psi_1'(x,y),y)$$

well-defined.

6.3.4. Existence of strips. Case $N > 1$**.** Let $X \in \mathcal{H}_f$. We have $UN_X^\epsilon \setminus \{0\} \neq \emptyset$ by corollary 4.2.2. We denote by β_1, ..., β_l the T-sets and by $Z_{X,1}^\epsilon$, ..., $Z_{X,l}^\epsilon$ the zones as we did in section 4.2. If $l = 1$ we choose a semi-analytic fake T-set β_2 such that $\beta_2 \neq \beta_1$. Then we can suppose that $l \geq 2$. As a consequence the set

$$Z_X^{\epsilon,j} \stackrel{def}{=} Z_{X,j}^\epsilon \cup \beta_{j+1} \cup Z_{X,j+1}^\epsilon$$

is contained in $B(0,\delta) \setminus \{0\}$ and it is simply connected; then there are N sections $x = g_j(y)$ in $Z_X^{\epsilon,j}$ of $SingX$ for $1 \leq j \leq N$.

LEMMA 6.3.6. *Let* $1 \leq k \leq l$. *For all* $1 \leq j \leq N$ *there is a strip* S_k^j *over* $Z_X^{\epsilon,k}$ *with vertex at* $x = g_j(y)$.

PROOF. Fix $s \in \beta_{k+1}$. There exists a connected component D of

$$(\alpha,\omega)^{-1}_{\xi(X),|x|<\epsilon}((g_j(s),s),(g_j(s),s)) \subset [|x| < \epsilon]$$

by lemma 2.2.1. The set ∂D is of the form $\gamma_0 \cup \{(g_j(s),s)\}$ where γ_0 is a trajectory of $\xi(X)$ in $|x| \leq \epsilon$. There exist times t_0, $t_1 \in \mathbb{R}$ such that $\gamma_0(t_q) \in T_X^\epsilon(s)$ for $q \in \{0,1\}$ and $[\gamma_0(-\infty,t_0) \cup \gamma_0(t_1,\infty)] \cap T_X^\epsilon(s) = \emptyset$. The sub-trajectory $\gamma_0(t_1,\infty)$ is the boundary of two connected components of $(|x| < \epsilon) \setminus \mathcal{H}_{\xi(X,s)}^{|x|<\epsilon}$, namely D and a component D_0 contained in $(\alpha,\omega)^{-1}_{\xi(X),|x|<\epsilon}(\infty,(g_j(s),s))$. Let $arc_0(s) = T_X^{\epsilon,a_0}(s)T_X^{\epsilon,a_0+1}(s)$ be the unique arc containing $end^-_{\xi(X,s,\epsilon)}(D_0)$. We have that

$\gamma_0(t_1)$ is either $T_X^{\epsilon,a_0}(s)$ or $T_X^{\epsilon,a_0+1}(s)$. We suppose without lack of generality that we are in the former case. Then either

$$\omega_{\xi(X),(|x|<\epsilon)\cup\{x_0\}}(x_0,s) \in (f=0) \text{ for all } (x_0,s) \in arc_0(s)$$

or there exists $Q_{1/2} \in arc_0(s) \setminus \{T_X^{\epsilon,a_0}(s)\}$ such that $\gamma_1 = \Gamma_{\xi(X,s,\epsilon),+}^{|x|\leq\epsilon}[Q_{1/2}]$ contains a point $T_X^{\epsilon,a_1}(s)$ different than $Q_{1/2}$. Let t_1^1 be the unique real number such that $\gamma_1(t_1^1) = T_X^{\epsilon,a_1}(s)$ for some $1 \leq a_1 \leq 2(\tilde{\nu}(X)-1)$ and $\gamma_1(t_1^1,\infty) \cap T_X^{\epsilon}(s) = \emptyset$. The sub-trajectory $\gamma_1(t_1^1,\infty)$ is in the boundary of two connected components of $(|x|<\epsilon) \setminus \mathcal{H}_{\xi(X,s)}^{|x|<\epsilon}$, namely D_0 and a component $D_1 \subset (\alpha,\omega)^{-1}(\infty,(g_j(s),s))$.

We can iterate the process; we claim that at some point we obtain some $arc(s) = T_X^{\epsilon,a}(s)T_X^{\epsilon,a+1}(s)$ such that

$$\omega_{\xi(X),(|x|<\epsilon)\cup\{x_0\}}(x_0,s) \in (f=0) \text{ for all } (x_0,s) \in arc_0(s).$$

Otherwise we build an infinite sequence D_0, D_1, $\ldots$ of components of $(|x|<\epsilon) \setminus \mathcal{H}_{\xi(X,s)}^{|x|<\epsilon}$ contained in $(\alpha,\omega)_{\xi(X),|x|<\epsilon}^{-1}(\infty,(g_j(s),s))$ (see picture 1). This sequence

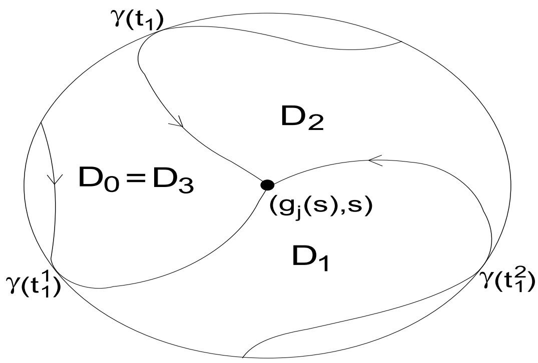

FIGURE 1.

is periodic, in particular $\cup_{q\in\mathbb{N}}\overline{D_q}$ is a neighborhood of $(g_j(s),s)$. But that is a contradiction since $[\cup_{q\in\mathbb{N}}\overline{D_q}] \cap D = \emptyset$.

We consider $arc(y) = T_X^{\epsilon,a}(y)T_X^{\epsilon,a+1}(y)$ for $y \in Z_X^{\epsilon,k}$. We claim that

$$E = \{z \in Z_X^{\epsilon,k} : \omega_{\xi(X),(|x|<\epsilon)\cup\{x_0\}}(x_0,z) \in (f=0)\ \forall (x_0,z) \in arc(z)\}$$

is equal to $Z_X^{\epsilon,k}$. We already proved that $s \in E$. If $E \neq Z_X^{\epsilon,k}$ then the set of parameters containing a bitangent cord joining a point in $\{T_X^{\epsilon,a}(s), T_X^{\epsilon,a+1}(s)\}$ with another tangent point is a non-empty union of T-sets intersecting $Z_X^{\epsilon,k}$ and disjoint from $Z_X^{\epsilon,k} \setminus \beta_{k+1}$; therefore it contains β_{k+1}. Since $s \in \beta_{k+1} \cap E$ we obtain a contradiction. As a consequence the set $S_k^j = \cup_{y\in Z_X^{\epsilon,k}} \Gamma_{\xi(X),+}^{|x|\leq\epsilon}[arc(y)]$ is a strip over $Z_X^{\epsilon,k}$ with vertex at $x = g_j(y)$. $\square$

LEMMA 6.3.7. *Let $N>1$. Let $X_1, X_2 \in \mathcal{H}_f$ be vector fields such that $SP(X_1) = SP(X_2)$. There exists a modification ψ_1' of ψ_1 in $U_{\epsilon,\delta}$ with respect to X_2.*

Proof. By lemma 6.3.6 we can define

$$\psi_{1,k} = \psi_1 + \sum_{j=1}^{N} M_{S_k^j}$$

for $1 \leq k \leq l$. The function $\psi_{1,k}$ is a modification of ψ_1 with respect to X_2 defined in $U_{\epsilon,\delta} \cap [y \in Z_X^{\epsilon,k} \cup \{0\}]$ by lemma 6.3.2. Now we can define a modification ψ_1' of ψ_1 with respect to X_2 in $U_{\epsilon,\delta}$. We just have to consider a partition of the unit associated to the covering $\cup_{1 \leq k \leq l} Z_X^{\epsilon,k}$ of $B(0,\delta) \setminus \{(0,0)\}$ and then to proceed like in lemma 6.3.5. $\square$

6.3.5. End of the proof of theorem 6.1.

Proof. Let $X_1 = X$ and $X_2 = Y$. We have $X_j = u_j f \partial/\partial x$ for all $j \in \{0,1\}$. If $N = 0$ the result is true by proposition 6.3.1. We define

$$X_{1+\xi} = u_{1+\xi} f \frac{\partial}{\partial x} = \frac{u_1 u_2}{u_2(1-\xi) + u_1 \xi} f \frac{\partial}{\partial x}.$$

Let $\xi_0 = u_2(0,0)/(u_2(0,0) - u_1(0,0))$; we have $\xi_0 \in \mathbb{C} \cup \{\infty\}$. The vector field $X_{1+\xi}$ belongs to $\mathcal{H}_f$ if $\xi \in \mathbb{C} \setminus \{\xi_0\}$. The integral of the time form of $X_{1+\xi}$ is $(1-\xi)\psi_1 + \xi\psi_2$. As a consequence any couple of vector fields $X_{1+\xi}$ and $X_{1+\xi'}$ satisfy that $SP(X_{1+\xi}) = SP(X_{1+\xi'})$. Suppose we can prove $X_1 \overset{sp}{\sim} X_2$ under the hypothesis $\xi_0 \notin [0,1]$. Then we are done because if $\xi_0 \in [0,1]$ we consider the families

$$X_{1+\xi}^1 = \frac{u_1 u_{1+i}}{u_{1+i}(1-\xi) + u_1 \xi} f \frac{\partial}{\partial x} \quad \text{and} \quad X_{1+\xi}^2 = \frac{u_{1+i} u_2}{u_2(1-\xi) + u_{1+i} \xi} f \frac{\partial}{\partial x}.$$

Since $u_{1+i}(0,0)/(u_{1+i}(0,0) - u_1(0,0))$ and $u_2(0,0)/(u_2(0,0) - u_{1+i}(0,0))$ do not belong to $[0,1]$ we obtain $X_1 \overset{sp}{\sim} X_{1+i} \overset{sp}{\sim} X_2$.

We choose $U_{\epsilon,\delta}$ such that $C_0 < |u_{1+\xi}(x,y)| < C_1$ for (x,y,ξ) in $U_{\epsilon,\delta} \times [0,1]$ and some positive constants C_0 and C_1. Let $x = x_1 + ix_2$. Let ψ_1' be the modification of ψ_1 with respect to X_2 provided by lemmas 6.3.5 and 6.3.7. We can choose $U_{\epsilon,\delta}$ and ψ_1' to satisfy $|\Im X_1(\psi_1' - \psi_1)| < \eta$ for some $0 < \eta < 1$ we will precise later on. We want to find a vector field $Z = \partial/\partial\xi + a\partial/\partial x_1 + b\partial/\partial x_2$ such that

$$\left(\frac{\partial}{\partial \xi} + a(x,y,\xi)\frac{\partial}{\partial x_1} + b(x,y,\xi)\frac{\partial}{\partial x_2}\right)((1-\xi)\psi_1' + \xi\psi_2) = 0.$$

We want a and b to be continuous functions satisfying

- a and b are real continuous functions defined in $U_{\epsilon,\delta} \times [0,1]$.
- $a_{|(f/y^m=0)\times[0,1]} = b_{|(f/y^m=0)\times[0,1]} \equiv 0$.

Supposed Z exists then the mapping

$$\sigma(x,y) = \exp\left(\frac{\partial}{\partial \xi} + a(x,y,\xi)\frac{\partial}{\partial x_1} + b(x,y,\xi)\frac{\partial}{\partial x_2}\right)(x,y,0)$$

is a special germ of homeomorphism such that $\psi_1' = \psi_2 \circ \sigma$. Therefore we obtain that $X_1 \overset{sp}{\sim} X_2$ by σ.

Let us find Z. The equation for Z is equivalent to

$$\left(a\frac{\partial}{\partial x_1} + b\frac{\partial}{\partial x_2}\right)((1-\xi)\psi_1' y^m + \xi\psi_2 y^m) = \psi_1' y^m - \psi_2 y^m.$$

Let $f' = f/y^m$. We define $\psi'_{1,m} = \psi'_1 y^m$ and $\psi_{j,m} = \psi_j y^m$ for j in $\{1,2\}$. We define $H = H_1 + iH_2 = (1-\xi)\psi'_{1,m} + \xi\psi_{2,m}$. We remark that $\psi'_{1,m} - \psi_{2,m}$ and $\partial((1-\xi)\psi'_{1,m} + \xi\psi_{2,m})/\partial x_j$ ($j \in \{1,2\}$) are uni-valuated and continuous in $(U_{\epsilon,\delta} \setminus [f'=0]) \times [0,1]$. We obtain a system

$$\begin{array}{rcl} a\partial H_1/\partial x_1 + b\partial H_1/\partial x_2 & = & Re(\psi'_{1,m} - \psi_{2,m}) \\ a\partial H_2/\partial x_1 + b\partial H_2/\partial x_2 & = & Img(\psi'_{1,m} - \psi_{2,m}) \end{array}$$

whose solutions

$$a = \frac{\begin{vmatrix} Re(\psi'_{1,m} - \psi_{2,m}) & \partial H_1/\partial x_2 \\ Im(\psi'_{1,m} - \psi_{2,m}) & \partial H_2/\partial x_2 \end{vmatrix}}{\begin{vmatrix} \partial H_1/\partial x_1 & \partial H_1/\partial x_2 \\ \partial H_2/\partial x_1 & \partial H_2/\partial x_2 \end{vmatrix}} \quad b = \frac{\begin{vmatrix} \partial H_1/\partial x_1 & Re(\psi'_{1,m} - \psi_{2,m}) \\ \partial H_2/\partial x_1 & Im(\psi'_{1,m} - \psi_{2,m}) \end{vmatrix}}{\begin{vmatrix} \partial H_1/\partial x_1 & \partial H_1/\partial x_2 \\ \partial H_2/\partial x_1 & \partial H_2/\partial x_2 \end{vmatrix}}$$

satisfy that the numerators and denominator in the previous expressions are continuous in $(U_{\epsilon,\delta} \setminus [f'=0]) \times [0,1]$. We denote $\psi'_{1,m} - \psi_{1,m}$ by $\rho = \rho_1 + i\rho_2$ and $(1-\xi)\psi_{1,m} + \xi\psi_{2,m}$ by $h = h_1 + ih_2$. The denominator of the previous expressions can be developed as

$$\sum_{j=1}^{4} D_j \stackrel{def}{=} \begin{vmatrix} \partial h_1/\partial x_1 & \partial h_1/\partial x_2 \\ \partial h_2/\partial x_1 & \partial h_2/\partial x_2 \end{vmatrix} + (1-\xi)\begin{vmatrix} \partial h_1/\partial x_1 & \partial \rho_1/\partial x_2 \\ \partial h_2/\partial x_1 & \partial \rho_2/\partial x_2 \end{vmatrix} +$$

$$+(1-\xi)\begin{vmatrix} \partial \rho_1/\partial x_1 & \partial h_1/\partial x_2 \\ \partial \rho_2/\partial x_1 & \partial h_2/\partial x_2 \end{vmatrix} + (1-\xi)^2\begin{vmatrix} \partial \rho_1/\partial x_1 & \partial \rho_1/\partial x_2 \\ \partial \rho_2/\partial x_1 & \partial \rho_2/\partial x_2 \end{vmatrix}.$$

Since h is holomorphic we can use the Cauchy-Riemann's equation to obtain

$$\begin{vmatrix} \partial h_1/\partial x_1 & \partial h_1/\partial x_2 \\ \partial h_2/\partial x_1 & \partial h_2/\partial x_2 \end{vmatrix} = \left(\frac{\partial h_1}{\partial x_1}\right)^2 + \left(\frac{\partial h_2}{\partial x_1}\right)^2 = \left|\frac{\partial h}{\partial x}\right|^2$$

We have $\partial h/\partial x = y^m/(u_{1+\xi} f)$, therefore $D_1 = |D_1| \geq 1/(|f'|^2 C_1^2)$ in $U_{\epsilon,\delta} \times [0,1]$. We have

$$\left|\frac{\partial h_j}{\partial x_k}\right| \leq \left|\frac{\partial h}{\partial x}\right| \leq \frac{1}{|f'|C_0}$$

for all $j \in \{1,2\}$ and $k \in \{1,2\}$. We want to estimate $|\partial\rho_j/\partial x_k|$, the relations $\Re X_1(\rho_j) = 0$ and $|\Im X_1(\rho_j)| \leq |y|^m \eta$ provide the system

$$\begin{array}{rcl} Re(u_1 f)\partial\rho_j/\partial x_1 + Img(u_1 f)\partial\rho_j/\partial x_2 & = & 0 \\ -Img(u_1 f)\partial\rho_j/\partial x_1 + Re(u_1 f)\partial\rho_j/\partial x_2 & = & \eta_1 \end{array}$$

where $|\eta_1(x,y)| \leq |y|^m \eta$ for $(x,y) \in U_{\epsilon,\delta}$. By using Kramer's rule we deduce that $|\partial\rho_j/\partial x_k| \leq \eta/(|f'|C_0)$. Therefore, we can choose $\eta > 0$ to have

$$\left\|\begin{matrix} \partial H_1/\partial x_1 & \partial H_1/\partial x_2 \\ \partial H_2/\partial x_1 & \partial H_2/\partial x_2 \end{matrix}\right\| \geq \frac{1}{|f'|^2 C_1^2} - \frac{4\eta}{|f'|^2 C_0^2} - \frac{2\eta^2}{|f'|^2 C_0^2} \geq \frac{1}{2|f'|^2 C_1^2}.$$

As a consequence a and b are continuous in $(U_{\epsilon,\delta} \setminus [f'=0]) \times [0,1]$. It is enough to prove that

$$(f')^2 \begin{vmatrix} Re(\psi'_{1,m} - \psi_{2,m}) & \partial H_1/\partial x_k \\ Im(\psi'_{1,m} - \psi_{2,m}) & \partial H_2/\partial x_k \end{vmatrix}$$

is a continuous function in $U_{\epsilon,\delta}$ for $k \in \{1,2\}$ whose restriction to $f' = 0$ is identically 0. We have

$$\left|\frac{\partial H_j}{\partial x_k}\right| \leq \frac{1}{|f'|C_0} + \frac{\eta}{|f'|C_0} = \frac{1+\eta}{|f'|C_0}$$

for $(x, y, \xi) \in U_{\epsilon,\delta} \times [0, 1]$ and $j, k \in \{1, 2\}$. Condition (3) on ψ_1' concludes the proof since $f'(\psi_{1,m}' - \psi_{2,m}) = f(\psi_1' - \psi_2)$. □

COROLLARY 6.3.1. *Let $f \in \mathbb{C}\{x, y\}$ satisfying the (NSD) conditions. Let $X, Y \in \mathcal{H}_f$. If $SP(X) = SP(Y)$ then $Re(X)$ and $Re(Y)$ are conjugated by a germ of special homeomorphism σ such that*

- *σ is analytic in a neighborhood of $(0, 0)$ if $N = 0$.*
- *σ and $\sigma^{(-1)}$ are C^∞ outside $f = 0$ if $(N, m) = (1, 0)$.*
- *σ and $\sigma^{(-1)}$ are C^∞ outside $yf = 0$ if $N \geq 1$ and $N + m > 1$.*

PROOF. The result for $N = 0$ is a consequence of proposition 6.3.1. The proof of theorem 6.1 has a modification ψ_1' as an input and a special continuous conjugation σ as an output. For $(N, m) = (1, 0)$ the modification ψ_1' is C^∞ in $U_{\epsilon,\delta} \setminus [f = 0]$; therefore σ is C^∞ in a neighborhood of $(0, 0)$ minus $f = 0$. For $N + m > 1$ the modification ψ_1' is C^∞ in the complementary of $yf = 0$, this property is shared by σ. □

CHAPTER 7

Families of Diffeomorphisms without Small Divisors

We already classified the topological behavior of the (NSD) vector fields. By definition $\varphi \in \text{Diff}\,(\mathbb{C}^2, 0)$ is a (NSD) diffeomorphism if it can be expressed in the form $\varphi(x, y) = (x + f(x, y), y)$ for a (NSD) function f. We will show that a (NSD) diffeomorphism has a flow-like behavior.

7.1. Normal form and residues

By definition $\varphi \in \text{Diff}\,(\mathbb{C}^n, 0)$ is unipotent if for all $k \in \mathbb{N}$ the linear isomorphism

$$\begin{array}{cccc} \varphi_k : & m/m^{k+1} & \to & m/m^{k+1} \\ & g + m^{k+1} & \mapsto & g \circ \varphi + m^{k+1} \end{array}$$

is unipotent where m is the maximal ideal of $\mathbb{C}[[x_1, \ldots, x_n]]$. We denote by $\text{Diff}_u(\mathbb{C}^n, 0)$ the subgroup of $\text{Diff}\,(\mathbb{C}^n, 0)$ of unipotent diffeomorphisms. It is easy to check out that φ is unipotent if and only if φ_1 is unipotent. Since a (NSD) diffeomorphism φ satisfies $j^1\varphi = (x + \rho y, y)$ for some $\rho \in \mathbb{C}$ then the (NSD) diffeomorphisms are unipotent.

We consider the set of formal vector fields $\hat{\mathcal{H}}(\mathbb{C}^n, 0)$ whose elements are of the form $\sum_{j=1}^{n} \hat{a}_j(x_1, \ldots, x_n)\partial/\partial x_j$ where $\hat{a}_j \in \mathbb{C}[[x_1, \ldots, x_n]]$ and $\hat{a}_j(0) = 0$ for all $1 \leq j \leq n$. We denote by $\hat{\mathcal{H}}_n(\mathbb{C}^n, 0)$ the set of nilpotent formal vector fields. The set of formal diffeomorphisms $\widehat{\text{Diff}}\,(\mathbb{C}^n, 0)$ is composed of elements $\hat{\varphi} = (\hat{\varphi}_1, \ldots, \hat{\varphi}_n)$ where $\hat{\varphi}_j \in \mathbb{C}[[x_1, \ldots, x_n]]$, $\hat{\varphi}_j(0) = 0$ for $1 \leq j \leq n$ and $j^1\hat{\varphi}$ is a linear isomorphism.

By definition

$$\exp(t\hat{X}) = \left(\sum_{j=0}^{\infty} t^j \frac{\hat{X}^j(x_1)}{j!}, \ldots, \sum_{j=0}^{\infty} t^j \frac{\hat{X}^j(x_n)}{j!} \right)$$

is the exponential of $\hat{X} \in \hat{\mathcal{H}}_n(\mathbb{C}^n, 0)$. We have $\hat{X}^0(x_k) = x_k$ whereas $\hat{X}^{j+1}(x_k) = \hat{X}(\hat{X}^j(x_k))$ for all $j \geq 0$. The components $x_k \circ \exp(\hat{X})$ $(1 \leq k \leq n)$ converges in the Krull topology for $\hat{X} \in \hat{\mathcal{H}}_n(\mathbb{C}^n, 0)$. Moreover, we obtain the next well known result:

Proposition 7.1.1. *The exponential mapping* $\exp(1\cdot)$ *establishes a bijection from* $\hat{\mathcal{H}}_n(\mathbb{C}^n, 0)$ *onto* $\widehat{\text{Diff}}_u(\mathbb{C}^n, 0)$. *Moreover, for all* $\hat{X} \in \hat{\mathcal{H}}_n(\mathbb{C}^n, 0)$ *and* $1 \leq k \leq n$ *we have* $x_k \circ \exp(t\hat{X}) \in \mathbb{C}[t][[x_1, \ldots, x_n]]$.

We denote by $\log \varphi$ the unique nilpotent formal vector field such that $\varphi = \exp(\log \varphi)$.

Proposition 7.1.2. *Let* $\varphi = (x + f(x, y), y)$ *be a (NSD) diffeomorphism. Then* $\log \varphi$ *is of the form* $\hat{u} f \partial/\partial x$ *for some formal unit* $\hat{u} \in \mathbb{C}[[x, y]]$.

PROOF. Since $y \circ \varphi = y$ we obtain $y \circ \exp(t \log \varphi) = y$ for all $t \in \mathbb{Z}$. The series $y \circ \exp(t \log \varphi) - y$ belongs to $\mathbb{C}[t][[x,y]]$ and it vanishes at $\mathbb{Z}$; therefore $y \circ \exp(t \log \varphi) \equiv y$. We have

$$\log \varphi(y) = \lim_{t \to 0} \frac{y \circ \exp(t \log \varphi) - y}{t} = 0;$$

that implies $\log \varphi = \hat{g}\partial/\partial x$ for some $\hat{g} \in \mathbb{C}[[x,y]]$. We can develop $\exp(\log \varphi)$ to obtain that φ is of the form $(x + \hat{v}\hat{g}, y)$ where $\hat{v}(0) = 1$. As a consequence $\log \varphi = \hat{v}^{-1} f \partial/\partial x$. □

We provide next a convergent normal form for the logarithm of a (NSD) diffeomorphism.

PROPOSITION 7.1.3. *Let $\varphi = \exp(\hat{u} f \partial/\partial x)$ be a (NSD) diffeomorphism. Then there exists $u_k \in \mathbb{C}\{x,y\}$ such that $\hat{u} - u_k \in (f^k)$ for all $k \in \mathbb{N}$.*

PROOF. Let $f = y^m f_1^{n_1} \dots f_p^{n_p}$ be the decomposition of f in irreducible components. It is enough to prove that there exists $u_k^g \in \mathbb{C}\{x,y\}$ such that $\hat{u} - u_k^g \in (g^k)$ for $g \in \{f_1, \dots, f_p, y\}$ and $k \in \mathbb{N}$. Fix g; the result is obviously true for $k = 0$. Suppose it is true for $k = a$; we have

$$\varphi = \exp\left((u_a^g + g^a \hat{h}) f \frac{\partial}{\partial x}\right)$$

where $\hat{h} \in \mathbb{C}[[x,y]]$ by hypothesis. Since $g^2 | \log \varphi(g)$ we obtain

$$x \circ \varphi - x \circ \exp(u_a^g f \partial/\partial x) - g^a f \hat{h} \in (g^{a+1} f).$$

As a consequence the series $(x \circ \varphi - x \circ \exp(u_a^g f \partial/\partial x))/(g^a f)$ belongs to $\mathbb{C}\{x,y\}$; we denote it by v. We have $\hat{h} - v \in (g)$; thus we obtain $\hat{u} - u_{a+1}^g \in (g^{a+1})$ for $u_{a+1}^g = u_a^g + g^a v$. □

Let $\exp(\hat{u} f \partial/\partial x)$ be a (NSD) diffeomorphism. Then $X = u f \partial/\partial x$ is a *convergent normal form* of $\exp(\hat{u} f \partial/\partial x)$ if $\hat{u} - u \in (f^2)$. Proposition 7.1.3 implies

PROPOSITION 7.1.4. *Every (NSD) diffeomorphism has a convergent normal form.*

The normal form is not unique. It can be proved that φ is formally conjugated to every convergent normal form; the proof is beyond the scope of this work.

Let φ be a (NSD) diffeomorphism; we define $Res_\varphi(P) = Res_X(P)$ for $P \in Fix\varphi$ where X is a convergent normal form of φ. The residues are well defined since

LEMMA 7.1.1. *Let $X_j = u_j f \partial/\partial x \in \mathcal{H}_f$ for $j \in \{1,2\}$. If $u_1 - u_2 \in (f)$ then $Res_{X_1(1)}(P) = Res_{X_2(1)}(P)$ for all $P \in [f = 0]$.*

PROOF. Let $f = y^m f_1^{n_1} \dots f_p^{n_p}$ be the decomposition of f in irreducible components. Fix $(x_0, y_0) \in [f = 0]$; let $\nu = \nu_{x_0}(f_1^{n_1} \dots f_p^{n_p}(x, y_0))$. The residue $Res_{X_j(1)}(x_0, y_0)$ is a function of the jet of order $2\nu - 1$ of $X_j(1)_{|y=y_0}$ at the point $x = x_0$. Since

$$(u_1 f_1^{n_1} \dots f_p^{n_p})(x, y_0) - (u_2 f_1^{n_1} \dots f_p^{n_p})(x, y_0) \in ((x - x_0)^{2\nu})$$

we have $Res_{X_1(1)}(P) = Res_{X_2(1)}(P)$ for all $P \in [f = 0]$. □

7.2. Comparing a diffeomorphism and its normal form

Throughout this section let φ be a (NSD) diffeomorphism; consider a convergent normal form $X(\varphi)$ whose exponential $\exp(X(\varphi))$ we denote by α_φ. Let $\psi_{X(\varphi)}$ be an integral of the time form of X. We claim that φ and α have very similar dynamics. Indeed, we want to prove

THEOREM 7.1. *Let φ be a (NSD) diffeomorphism. There exist open neighborhoods $V \subset W$ of $(0,0)$ and a constant $C > 0$ such that*

$$\{\alpha^{(0)}(x,y) = (x,y), \dots, \alpha^{(j)}(x,y)\} \subset V$$

for $j \in \mathbb{Z}$ implies $\{\varphi^{(0)}(x,y), \dots, \varphi^{(j)}(x,y)\} \subset W$ and

$$|\psi_{X(\varphi)} \circ \varphi^{(j)}(x,y) - [\psi_{X(\varphi)}(x,y) + j]| \leq C.$$

Moreover, we can make C arbitrarily small by shrinking V.

This theorem is very powerful. We are claiming that the orbits by φ are very close to the orbits by α_φ, regardless of the number of iterations. Apparently the function $\psi_{X(\varphi)} \circ \varphi^{(j)} - (\psi_{X(\varphi)} + j)$ is not well defined since $\psi_{X(\varphi)}$ is multi-valuated but it is. Let $\Delta = \psi_{X(\varphi)} \circ \varphi - (\psi_{X(\varphi)} + 1)$; we define

$$\Delta_j = \psi_{X(\varphi)} \circ \varphi^{(j)} - (\psi_{X(\varphi)} + j).$$

Then $\Delta_j = \sum_{k=0}^{j-1} \Delta \circ \varphi^{(k)}$ if $j > 0$ and $\Delta_j = \sum_{k=1}^{|j|} \Delta \circ \varphi^{(-k)}$ if $j < 0$.

LEMMA 7.2.1. *The function Δ does not depend on the choice of $\psi_{X(\varphi)}$. Moreover Δ is a holomorphic function in $U_{\epsilon,\delta}$ which belongs to (f^2).*

PROOF. The function $y^m \psi_{X(\varphi)}$ is unique up to an additive holomorphic function depending only on the variable y. As a consequence Δ is a holomorphic function in defined $U \setminus [f = 0]$ for some neighborhood U of $(0,0)$. Since

$$x \circ \varphi - x \circ \alpha \in (f^3) \;\text{ and }\; \Delta = \psi_{X(\varphi)} \circ \varphi - \psi_{X(\varphi)} \circ \alpha$$

then $\Delta = O(y^{2m})$ in the neighborhood of the points in $[y = 0] \setminus \{(0,0)\}$. As a consequence Δ is holomorphic outside $f/y^m = 0$. Consider a point P in the set $[f = 0] \setminus [y = 0]$. Up to a change of coordinates in the neighborhood of P we can suppose that $f = x^n$ and $\varphi_j = (x + v_1 x^n, y)$ for $j \in \{1,2\}$. Moreover $u_1 - u_2 \in (f^2)$ implies $v_1 - v_2 \in (x^{2n})$. We obtain

$$\Delta \in O(x^{(3n-1)-(n-1)}) = O(f^2)$$

in the neighborhood of P since $\psi_{X(\varphi)} = O(1/x^{n-1})$. We deduce that Δ/f^2 is a bounded function in the neighborhood of $[f = 0] \setminus \{(0,0)\}$; hence Δ/f^2 is holomorphic in a pointed neighborhood of $(0,0)$. Since compact singularities can be removed then Δ/f^2 is holomorphic in the neighborhood of $(0,0)$. ☐

The previous lemma implies immediately the following corollary:

COROLLARY 7.2.1. *If $\{\varphi^{(0)}(P), \dots, \varphi^{(j)}(P)\} \subset U_{\epsilon,\delta}$ then*

$$\psi_{X(\varphi)} \circ \varphi^{(j)}(P) - (\psi_{X(\varphi)}(P) + j)$$

is well defined.

7.2.1. Comparing φ and α_φ in an exterior basic set. In order to prove theorem 7.1 we will use the division in basic sets that we introduced in chapter 3. Throughout subsections 7.2.1 and 7.2.2, and up to ramify we will suppose that the components of $f/y^m = 0$ are parameterized by y.

Let $X = X(\varphi)$. We study next the behavior of Δ_j in the exterior sets. We will use the concepts and notations defined in section 3.2. Suppose $N \geq 1$ and let $\lambda(y) = y^m/|y|^m$. Every trajectory $\exp([0,j]X)(Q)$ contained in $U_\epsilon^{\eta,+}$ is also contained in some exterior region $R_{X(\lambda)}^{\epsilon,\eta}(y)$. We have $R_{X(\lambda)}^{\epsilon,\eta}(y) \subset D_R^{\epsilon,\eta}(\lambda)$ by proposition 3.2.3. There exists an uni-valuated determination $\psi^R = \psi_{X(\varphi)}^R$ of $\psi_{X(\varphi)(1)}$ in $D_R^{\epsilon,\eta}(\lambda)$. We define

$$\psi_{X(\varphi)}^R(T_{X_{00}(\lambda)}^{\epsilon_0,1}, y) = \psi_0^R(T_{X_{00}(\lambda)}^{\epsilon_0,1}, y) = \psi_{00}(T_{X_{00}(\lambda)}^{\epsilon_0,1}, y)$$

for some $0 < \epsilon_0 << 1$ like in subsection 3.2.4.

LEMMA 7.2.2. *Suppose $N \geq 1$; let $\Delta = O(y^{a-mb}f^b)$. Fix $R_{X(\lambda)}^{\epsilon,\eta}(y)$. Then $\Delta = O(y^a/(\psi_{X(\varphi)}^R)^b)$ in $D_R^{\epsilon,\eta}(\lambda)$ for all $\epsilon << 1$, $\delta << 1$ and $\eta >> 0$.*

PROOF. Let $\nu = \tilde{\nu}(X(\varphi))$; the hypothesis $N \geq 1$ implies $\nu \geq 2$. Since $D_R^{\epsilon,\eta} \subset U_\epsilon^{\eta,+}$ then $\Delta = O(y^a x^{b\nu})$. Moreover $\psi^R \sim \psi_{00} \sim 1/x^{\nu-1}$ by lemma 3.2.5; hence $\Delta = O(y^a/(\psi^R)^{be})$ for $e = \nu/(\nu-1)$. □

Let $f = y^m f' = y^m (x - g_1(y))^{n_1} \dots (x - g_N(y))^{n_N}$ be the decomposition of f in irreducible factors. In the first exterior basic set $X(\varphi)/y^m$ never vanishes and $\Delta = O(y^{2m} {f'}^2)$ by lemma 7.2.1. For each point c in

$$F_1 = \{\partial g_1/\partial y(0), \dots, \partial g_N/\partial y(0)\}$$

there exists an exterior basic set E_c enclosing $(w,y) = (c,0)$ where $x = wy$. Let F_1^c be the set of indexes such that $j \in F_1^c$ if $\partial g_j/\partial y(0) = c$. Let $\nu_0 = \tilde{\nu}(X) = n_1 + \dots + n_N$. We have that $X(\varphi)/y^{m+\nu_0-1}$ is never singular in E_c whereas

$$\Delta = O(y^{2m+2\nu_0} \prod_{j \in F_1^c} (w - g_j(y)/y)^{2n_j}).$$

If $\sharp F_1^c \neq 1$ we have to continue the process; let $\nu_c = \sum_{j \in F_1^c} n_j$. For any next exterior basic set $E_{cc'}$ we have that $X(\varphi)/y^{m+\nu_0+\nu_c-2}$ is never singular and $\Delta = O(y^{2m+2\nu_0+2\nu_c} f_{cc'}^2)$ where $f_{cc'}$ is the strict transform of the curves in $f' = 0$ enclosed by $E_{cc'}$. It is easy to obtain expressions for $X(\varphi)$ and Δ in every basic set by induction. Fix an exterior basic set E; let $\nu_y^E(X)$ and $\nu_y^E(\Delta)$ be the non negative integers such that $X(\varphi)/y^{\nu_y^E(X)}$ and $\Delta/y^{\nu_y^E(\Delta)}$ are holomorphic and never vanishing in E. The previous discussion implies:

LEMMA 7.2.3. *Suppose $N \geq 1$. In any exterior basic set E we have $\Delta = O(y^{\nu_y^E(\Delta)} f_E^2)$ where $f_E = 0$ is the strict transform of the curves in $f' = 0$ enclosed by E. Moreover $\nu_y^E(\Delta) - \nu_y(X(\varphi)) \geq 0$; the inequality is strict if E is not the first exterior set or $m > 0$.*

We can now bound Δ_j in any exterior basic set. For simplicity we formulate the proposition for the first one.

PROPOSITION 7.2.1. *Suppose $N \geq 1$. Let $\nu = \nu_y(\Delta) - \nu_y(X)$. Fix $M > 0$ and $\eta >> 0$. Suppose $\Delta = O(f^2)$. For any $\xi > 0$ there exists $U_{\epsilon,\delta}$ such that the conditions*

- $|\psi_{X(\varphi)}(w,y) - \psi_{X(\varphi)}(x,y)| \leq M$ *where* $(x,y) \in U_\epsilon$.
- $\exp([0,j]X(\varphi))(x,y) \subset U_{\epsilon,\delta} \cap U_\epsilon^{\eta,+}$ *for some* $j \in \mathbb{N} \cup \{0\}$

imply

$$|\psi_X \circ \varphi^{(j+1)}(w,y) - \psi_X \circ \alpha^{(j+1)}(x,y)| \leq |\psi_X(w,y) - \psi_X(x,y)| + \xi |y|^\nu.$$

The condition $|\psi_{X(\varphi)}(w,y) - \psi_{X(\varphi)}(x,y)| \leq D$ for some constant $D > 0$ means that $(w,y) \in \exp(\overline{B}(0,D)X(\varphi))(x,y)$. The statement in the proposition is not completely rigorous. Technically, it would be necessary to say that there exists $U_{\epsilon',\delta} \supset U_{\epsilon,\delta}$ where $X(\varphi)$, $\psi_{X(\varphi)}$, α_φ and φ are defined and such that $\alpha^{(j)}(x,y) \in U_{\epsilon,\delta} \cap U_\epsilon^{\eta,+}$ implies $\varphi^{(j)}(w,y) \in U_{\epsilon',\delta}$. We think that this formulation is more natural. There is an analogous statement for $j < 0$, we omit the details.

Proof. Let $\xi < M$. We define $\gamma = \exp([0,j]X(\varphi))(x,y) \subset U_\epsilon^{\eta,+}$. Then γ is contained in some $R_{X(\lambda)}^{\epsilon,\eta} \subset D_R^{\epsilon,\eta}$. The integral $\psi_{X(\varphi)}^R$ of the time form of $X(\varphi)(1)$ is defined in $D_R^{2\epsilon,\eta/2}$ for $\epsilon << 1$ and $\eta >> 0$. We denote $\psi_{X(\varphi)}^R$ by ψ for simplicity. We remark that $\psi_{X(\varphi)} = \psi / y^{\nu_y(X)}$. For every $C > 0$ we can choose $\epsilon_C > 0$ such that $|\psi| > C$ in $U_\epsilon^{\eta,+} \cap D_R^{\epsilon,\eta}$ for $0 < \epsilon \leq \epsilon_C$.

We have $|\Delta| \leq K|y|^{\nu_y(\Delta)}/|\psi|^2$ in $D_R^{2\epsilon,\eta/2}$ for some $K > 0$ by lemma 7.2.2. Suppose $Q_1 \in U_{\epsilon(C)}^{\eta,+}$ and $|\psi(P_1) - \psi(Q_1)| < 2M|y|^{\nu_y(X)}$; we obtain

$$\frac{|\psi(Q_1)|}{|\psi(P_1)|} \leq 1 + \frac{2M|y|^{\nu_y(X)}}{|\psi(P_1)|} \leq 1 + \frac{2M|y|^{\nu_y(X)}}{C - 2M|y|^{2\nu_y(X)}}.$$

If $C \geq C_1$ for some $C_1 > 0$ then $P_1 \in D_R^{2\epsilon,\eta/2}$ and

$$|\Delta(P_1)| \leq K \frac{|y|^{\nu_y(\Delta)}}{|\psi(P_1)|^2} < 2K \frac{|y|^{\nu_y(\Delta)}}{|\psi(Q_1)|^2}.$$

Now consider $C_2 \geq C_1$ such that $C \geq C_2$ implies

$$2K \left(\frac{6\delta^{\nu_y(X)}}{C^2} + \left(\frac{4\sqrt{2}}{C} + \frac{2}{C^2} \right) \right) < \xi$$

We choose $\epsilon = \epsilon(C)$ for some $C \geq C_2$. We will prove the proposition by induction. The result is true for $j = 0$ since $2K\delta^{\nu_y(X)}/C^2 < \xi$. Suppose the result is true for 0, 1, $\ldots$, $j-1$; thus

$$|\psi_X \circ \varphi^{(k)}(w,y) - \psi_X \circ \alpha^{(k)}(x,y)| \leq |\psi_X(w,y) - \psi_X(x,y)| + \xi|y|^\nu < 2M$$

for all $0 \leq k \leq j$ and $\delta < 1$. As a consequence we obtain

$$|\Delta \circ \varphi^{(k)}(w,y)| \leq 2K \frac{|y|^{\nu_y(\Delta)}}{|\psi(x,y) + k y^{\nu_y(X)}|^2}$$

for all $0 \leq k \leq j$. We have $|\psi(x,y) + k y^{\nu_y(X)}| \geq C$ for $0 \leq k \leq j$ by the choice of ϵ. We define $\tau = \psi(x,y)|y|^{\nu_y(X)}/y^{\nu_y(X)}$; we have

$$\left| \sum_{k=0}^{j} \Delta \circ \varphi^{(k)}(w,y) \right| \leq 2K|y|^{\nu_y(\Delta)} \sum_{k=0}^{j} \frac{1}{|\tau + k|y|^{\nu_y(X)}|^2}.$$

We divide $\mathbb{C} \cap [|z| > C]$ in three sets, namely $E_1 = [Re(z) \geq |Img(z)|]$, $E_2 = [|Re(z)| \leq |Img(z)|]$ and $E_3 = -E_1$. Let S_l be an upper bound of $\sum_{k=0}^{j'} 1/|\tau' + k|y|^{\nu_y(X)}|^2$ supposed $\tau' + k|y|^{\nu_y(X)} \in E_l$ for $0 \leq k \leq j'$. Let $S = S_1 + S_2 + S_3$; we obtain

$$|\psi_X \circ \varphi^{(j+1)}(w, y) - \psi_X \circ \alpha^{(j+1)}(x, y)| \leq |\psi_X(w, y) - \psi_X(x, y)| + 2K|y|^{\nu_y(\Delta)} S.$$

We can calculate explicit values for S_1, S_2 and S_3. If $\tau' \in E_1$ then $Re(\tau') \geq C/\sqrt{2} > 0$; that implies

$$|\tau' + k|y|^{\nu_y(X)}|^2 \geq (Re(\tau') + k|y|^{\nu_y(X)})^2 \geq (C/\sqrt{2} + k|y|^{\nu_y(X)})^2.$$

As a consequence we have

$$S_3 = S_1 \leq \sum_{k=0}^{\infty} \frac{1}{(C/\sqrt{2} + k|y|^{\nu_y(X)})^2}.$$

The right hand side is smaller or equal than

$$\int_0^{\infty} \frac{dr}{(C/\sqrt{2} + r|y|^{\nu_y(X)})^2} + \frac{1}{(C/\sqrt{2})^2} \leq \frac{2}{C^2} + \frac{\sqrt{2}}{C} \frac{1}{|y|^{\nu_y(X)}}$$

If $\tau' \in E_2$ then $\tau' + 2|Img(\tau')| + 1 \in E_1 \setminus E_2$; moreover $Img(\tau') \geq C/\sqrt{2}$. We obtain

$$\sum_{k=0}^{j'} \frac{1}{|\tau' + k|y|^{\nu_y(X)}|^2} \leq \frac{(2|Img(\tau')| + 1)/|y|^{\nu_y(X)} + 1}{|Img(\tau')|^2}$$

and then

$$S_2 \leq \left(\frac{2\sqrt{2}}{C} + \frac{2}{C^2} \right) \frac{1}{|y|^{\nu_y(X)}} + \frac{2}{C^2}.$$

The inequality

$$2K|y|^{\nu_y(X)} S \leq 2K \left(\frac{6\delta^{\nu_y(X)}}{C^2} + \frac{4\sqrt{2}}{C} + \frac{2}{C^2} \right) < \xi$$

implies

$$|\psi_X \circ \varphi^{(j+1)}(w, y) - \psi_X \circ \alpha^{(j+1)}(x, y)| \leq |\psi_X(w, y) - \psi_X(x, y)| + |y|^{\nu} \xi.$$

□

7.2.2. Comparison in a compact-like basic set. We can proceed like in the exterior sets. For the first compact-like set VC_1 we have

$$\nu^1 = \nu_y^{VC_1}(\Delta) - \nu_y^{VC_1}(X(\varphi)) \geq m + \tilde{\nu}(X) + 1 > 0.$$

For any other compact-like basic set VC_l we obtain

$$\nu^l = \nu_y^{VC_l}(\Delta) - \nu_y^{VC_l}(X(\varphi)) > \nu_y^{VC_1}(\Delta) - \nu_y^{VC_1}(X(\varphi)) > 0.$$

PROPOSITION 7.2.2. *Fix $M > 0$ and a compact-like basic set VC_l. There exists a constant $K_l > 0$ such that*

- $|\psi_{X(\varphi)}(P) - \psi_{X(\varphi)}(Q)| \leq M$ *where* $\{P, Q\} \subset U_{\epsilon,\delta} \cap [y = s]$.
- $\exp([0, j]X(\varphi))(Q) \subset U_{\epsilon,\delta} \cap VC_l$ *for some* $j \in \mathbb{N} \cup \{0\}$.

imply $|\psi_X \circ \varphi^{(j+1)}(P) - \psi_X \circ \alpha^{(j+1)}(Q)| \leq |\psi_X(P) - \psi_X(Q)| + K_l|s|^{\nu^l}$.

PROOF. Let $\tau = \nu_y^{VC_l}(\Delta)$, $\zeta = \nu_y^{VC_l}(X(\varphi))$. We define

$$VC_l' = \exp(\overline{B}(0,2M)X(\varphi))(VC_l).$$

There exists $D > 0$ such that $|\Delta| \leq D|y|^\tau$ in VC_l'. Since VC_l is compact and $X(\varphi)/y^\zeta$ does not have singular points then $j \leq D'/|y|^\zeta$ for some $D' > 0$. Suppose that we have $\varphi^{(k)}(P) \in VC_l'$ for all $0 \leq k \leq j'$ and some $0 \leq j' \leq j$. We deduce that $|\psi_X \circ \varphi^{(j'+1)}(P) - \psi_X \circ \alpha^{(j'+1)}(Q)|$ is smaller or equal than

$$|\psi_X(P) - \psi_X(Q)| + D|s|^\tau \left(\frac{D'}{|s|^\zeta} + 1 \right).$$

We choose $\delta > 0$ such that $DD'\delta^{\tau-\zeta} + D\delta^\tau \leq M$. That implies $\varphi^{(j'+1)}(P) \in VC_l'$; we obtain $\varphi^{(k)}(P) \in VC_l'$ for all $0 \leq k \leq j+1$ by induction. We define $K_l = DD' + D\delta^\zeta$; it clearly satisfies the thesis of the proposition. □

7.2.3. Proof of theorem 7.1. Suppose $N = 0$. We can consider $U_{\epsilon,\delta}$ as a compact-like set since there are no fixed points outside $y = 0$. Since $\nu_y(\Delta) - \nu_y(X) \geq m$ for $N = 0$ then proposition 7.2.2 implies theorem 7.1 for some neighborhood $U_{\epsilon,\delta}$ of $(0,0)$.

Suppose $N \geq 1$ from now on. The hypotheses and theses in theorem 7.1 are invariant under ramification. As a consequence we can suppose that the components of $Fix\varphi$ different than $y = 0$ are parameterized by y. We can apply the results in subsections 7.2.1 and 7.2.2.

We suppose $j > 0$ without lack of generality. Fix $M > 0$. For $\{R, Q\} \subset U_{\epsilon,\delta} \cap [y = s]$ there exists $K > 0$ such that

$$\left|\psi_X \circ \varphi^{(j'+1)}(R) - \psi_X \circ \alpha^{(j'+1)}(Q)\right| \leq |\psi_X(R) - \psi_X(Q)| + K|s|^{m+1} \tag{7.1}$$

if $|\psi_X(R) - \psi_X(Q)| \leq M$ and $\exp([0,j']X(\varphi))(Q) \subset B$ for a basic set B different than the first exterior one E_1 and some $j' \geq 0$. This claim is a consequence of

$$\nu_y^B(\Delta) - \nu_y^B(X) > \nu_y^{E_1}(\Delta) - \nu_y^{E_1}(X) \geq m$$

for every basic set $B \neq E_1$ and propositions 7.2.1 and 7.2.2. We can choose the same $K > 0$ for every basic set because there are only finitely many such sets. Any trajectory of $\xi(X, s, \epsilon)$ splits in at most D sub-trajectories contained in the basic sets; the number $D > 0$ is provided by lemma 3.3.1. Let $C \in (0, M]$; the correction term $|\sum_{k=0}^{j'} \Delta \circ \varphi^{(k)}(P)|$ for E_1 can be made smaller than $C|y|^m/(2D)$ by shrinking $U_{\epsilon,\delta}$, making η bigger and using proposition 7.2.1.

Let $0 = j_0 < j_1 < \ldots < j_d = j - 1$ be the only sequence satisfying that

- $\exp([j_b, j_{b+1}]X)(P) \subset B_{b+1}$ for all $0 \leq b \leq d-1$
- B_b is a basic set for $1 \leq b \leq d$ and $B_b \neq B_{b+1}$ for all $1 \leq b \leq d-1$

We point out that $d \leq D$. Since j_k can be non-integer if $0 < k < d$ then we have to tweak a little bit the sequence. We define $k_0 = -1$, $k_1 = [j_1]$ where [] stands for integer part. Suppose we have defined

$$0 = k_0 + 1 \leq k_1 < k_1 + 1 \leq k_2 < k_2 + 1 \leq \ldots \leq k_l$$

such that $\exp([k_j + 1, k_{j+1}]X)(P)$ is contained in a basic set for all $0 \leq j \leq l-1$. If $k_l \neq j - 1$ we define $k_{l+1} = \inf\{[j_b] : j_b \geq k_l + 1\}$. The sequence $-1 = k_0 < k_1 <$

$\ldots < k_{d'} = j-1$ satisfies $d' \leq d$. Now we apply the equation 7.1 or its analogue for E_1 to the 3-uples

$$(R, Q, j') = (\varphi^{(k_b+1)}(P), \alpha^{(k_b+1)}(P), k_{b+1} - (k_b+1))$$

for $0 \leq b \leq d'-1$. By plugging each inequality in the following one we obtain

$$\left| \psi_{X(\varphi)} \circ \varphi^{(j)}(P) - \psi_{X(\varphi)} \circ \alpha^{(j)}(P) \right| \leq C|y|^m \left(1/2 + O(y)\right) \leq C|y|^m$$

for $\delta > 0$ small enough.

7.2.4. Some consequences of theorem 7.1. Basically the dynamics of a (NSD) diffeomorphism and its normal form are the same. For instance, we have

LEMMA 7.2.4. *Let φ be a (NSD) diffeomorphism and let $X(\varphi)$ be one of its normal forms. There exist $U_{\epsilon,\delta}$ and $\epsilon' > \epsilon$ such that*

$$\omega_{\xi(X(\varphi),y,\epsilon),|x|\leq\epsilon}(x,y) \in [f=0] \implies \{\varphi^{(j)}(x,y)\}_{j\in\mathbb{N}\cup\{0\}} \subset U_{\epsilon'}.$$

Moreover, we have $\lim_{j\to\infty} \varphi^{(j)}(x,y) = \omega_{\xi(X(\varphi),y,\epsilon),|x|\leq\epsilon}(x,y)$

In other words the basins of repulsion and attraction for a (NSD) diffeomorphism and its normal form can be considered to be the same.

PROOF. Fix $C > 0$. We can choose the domains V and W provided by theorem 7.1 in the form $V = U_{\epsilon,\delta}$ and $W = U_{\epsilon',\delta}$ for some $0 < \epsilon < \epsilon'$. We also want $\exp(tX(\varphi))(P)$ to be well-defined in $t \in B(0,2C)$ and such that $\overline{U_\epsilon}$ contains $\exp(B(0,2C)X(\varphi))(P) \subset U_{\epsilon'}$. That is possible by choosing a smaller $\epsilon > 0$. Since

$$|\psi_{X(\varphi)} \circ \varphi^{(j)}(P) - \psi_{X(\varphi)} \circ \exp(jX(\varphi))(P)| \leq C$$

then $\{P, \varphi(P), \varphi^{(2)}(P), \ldots\} \subset U_{\epsilon'}$. Moreover

$$\lim_{j\to\infty} \varphi^{(j)}(P) \in \exp(\overline{B}(0,C)X(\varphi))(\lim_{j\to\infty} \alpha_\varphi^{(j)}(P)) = \{\lim_{j\to\infty} \alpha_\varphi^{(j)}(P)\};$$

the last equality holds because $\lim_{j\to\infty} \alpha_\varphi^{(j)}(P)$ is a fixed point. □

We know that the analytic class of $X(1)_{|y=0}$ is a special invariant of a (NSD) vector field X if $(N,m) \neq (1,0)$ by lemma 6.3.1. That motivates us to look for the underlying complex structure associated to a (NSD) diffeomorphism φ at $y=0$. If $m > 0$ we define $\log \varphi_0(1) = X(\varphi)(1)_{|y=0}$; the definition does not depend on the choice of $X(\varphi)$. For $N > 0$ and $m > 0$ we can define $\varphi_0(1) = \exp(\log \varphi_0(1))$ since $\log \varphi_0(1)$ is singular at 0. For $N > 0$ and $m = 0$ we define $\varphi_0(1) = \varphi_{|y=0}$.

The L-limit phenomenon has a similar behavior for (NSD) diffeomorphisms and vector fields. Consider φ, ϵ, ϵ' and δ like in lemma 7.2.4. Let β be a semi-analytic curve and $x_0 \in [0 < |x| \leq \epsilon]$. Suppose for simplicity that the direction $\lambda(\beta)$ of β at 0 is 1. Let $x_1 \in U_\epsilon$ be a point in the first component ρ_1 of $L_{\beta,x_0}^{+,\epsilon}$. There exists a continuous partition (E_-, E_+) of $Fix\varphi$ and a true section $\chi : W(M) \to U_{\epsilon'}$ $(0 < M << 1)$ such that for $y \in W$ we have

$$T(y) = \frac{\psi_1}{y^m}(\chi(y)) + A_{E_-}(y) - \frac{\psi_0}{y^m}(x_0, y) \in \mathbb{R}^+$$

where $W = \cup_{r\in[-M,M]}\beta_r$ and $\beta_r \in \Upsilon_{A_{E_-}}^r$ for $r \in [-M,M]$. If we choose the section χ like in the proof of proposition 5.4.1 we obtain

$$\lim_{y\in\beta_r, y\to 0} \exp(sX(1))\chi(y) = \exp((s+ir)X(1))(x_1, 0)$$

for all $(s,r) \in [-M,M] \times [-M,M]$.

Fix $z = s + ir \in [-M,M] + i[-M,M]$. We define $T_z : \beta_r \to \mathbb{R}^+$ as $T_z(y) = T(y) + s/|y|^m$. Since $\lim_{y \to 0} T_z(y) = \infty$ we consider the sequence of points $\{y_n^z\}_{n \in \mathbb{N}}$ in β_r such that the germ of $T_z^{-1}(\mathbb{N})$ at 0 coincides with $\cup_{n \in \mathbb{N}} \{y_n^z\}$; we have $\lim_{n \to \infty} y_n^z = 0$. The question is what we can say about the sequence

$$\varphi^{(T_z(y_n))}(x_0, y_n^z).$$

The point $(x_0, 0)$ is in $\omega^{-1}_{\xi(X(1),0,\epsilon)}(0,0)$ whereas $(x_1,0)$ is in $\alpha^{-1}_{\xi(X(1),0,\epsilon)}(0,0)$. We defined in subsection 5.4.1 the integral $\psi_{0,0}^+$ of the time form of $X(1)_{|y=0}$ defined in the attractive petal $V_{l^+} \subset [y=0]$ containing $(x_0,0)$. In an analogous way we define $\psi_{1,0}^-$ in the repulsive petal $V_{l^-} \subset [y=0]$ containing $(x_1,0)$. By lemma 7.2.4 the domains V_{l^+} and V_{l^-} are still basins of attraction and repulsion respectively for $\varphi_0(1)$. By the one variable theory there exists an integral of the time form $\psi_{0,0}^{+,\varphi}$ of $\varphi_0(1)$ in V_{l^+}, in other words $\psi_{0,0}^{+,\varphi}$ satisfies

$$\psi_{0,0}^{+,\varphi} \circ \varphi_0(1) = \psi_{0,0}^{+,\varphi} + 1.$$

By definition $\psi_{0,0}^{+,\varphi} = \psi_{0,0}^{+} + \sum_{j=0}^{\infty} \Delta \circ \varphi^{(j)}$. There is also an integral $\psi_{1,0}^{-,\varphi}$ of the time form of φ in V_{l^-}; by definition $\psi_{1,0}^{-,\varphi} = \psi_{1,0}^{-} - \sum_{j=1}^{\infty} \Delta \circ \varphi^{(-j)}$.

PROPOSITION 7.2.3. *The limit* $\lim_{n \to \infty} \varphi^{(T_z(y_n))}(x_0, y_n^z)$ *exists for all complex number* $z = s + ir \in [-M,M] + i[-M,M]$. *Moreover*

$$\lim_{n \to \infty} |y_n^z|^m (T_z(y_n^z) - A_{E_-}(y_n^z)) = \psi_{1,0}^{-,\varphi}(\lim_{n \to \infty} \varphi^{(T_z(y_n^z))}(x_0, y_n^z)) - \psi_{0,0}^{+,\varphi}(x_0, 0)$$

and

$$\psi_{1,0}^{-,\varphi}(\lim_{n \to \infty} \varphi^{(T_z(y_n^z))}(x_0, y_n^z)) - \psi_{1,0}^{-,\varphi}(\lim_{n \to \infty} \varphi^{(T_0(y_n^0))}(x_0, y_n^0)) = z.$$

The first formula allows to estimate how much time φ spends to go from (x_0, y_n^z) to $\varphi^{(T_z(y_n))}(x_0, y_n^z)$. The second formula is the analogue of

$$\lim_{n \to \infty} \exp(T_z(y_n^z)X)(x_0, y_n^z) = \exp(zX(1))(x_1, 0)$$

for (NSD) diffeomorphisms. As a consequence the complex flow of $\varphi_0(1)$ is generated by φ for $N > 1$.

PROOF. Since

$$\left| \psi_{X(\varphi)}(\varphi^{(T_z(y_n^z))}(x_0, y_n^z)) - \psi_{X(\varphi)}(\alpha^{(T_z(y_n^z))}(x_0, y_n^z)) \right| \leq C$$

then the accumulation points of the sequence $\varphi^{(T_z(y_n^z))}(x_0, y_n^z)$ are contained in $\exp(\overline{B}(0,C)X(\varphi))(\exp(zX(1))(x_1,0))$. In particular

$$\lim_{n \to \infty} \varphi^{(T_z(y_n))}(x_0, y_n^z) = \exp(zX(1))(x_1, 0)$$

for $m > 0$; since for $m > 0$ we also have $\psi_{0,0}^{+,\varphi} = \psi_{0,0}^{+}$ and $\psi_{1,0}^{-,\varphi} = \psi_{1,0}^{-}$ then there is nothing to prove. We suppose $m = 0$ from now on. We can suppose that $\varphi^{(T_z(y_n^z))}(x_0, y_n^z)$ is convergent up to take a subsequence; we denote the limit by $(x_{1,z}, 0)$. Later on we will prove that $(x_{1,z}, 0)$ is the limit and not only an accumulation point. We have

$$T_z(y_n^z) = \psi_1(\alpha^{(T_z(y_n^z))}(x_0, y_n^z)) + A_{E_-}(y_n^z) - \psi_0(x_0, y_n^z).$$

We want to rewrite the previous expression in terms of $\varphi^{(T_z(y_n^z))}(x_0, y_n^z)$ instead of $\alpha^{(T_z(y_n^z))}(x_0, y_n^z)$. We obtain that $T_z(y_n^z)$ is equal to

$$\psi_1(\varphi^{(T_z(y_n^z))}(x_0, y_n^z)) + A_{E_-}(y_n^z) - \psi_0(x_0, y_n^z) - \sum_{j=0}^{T_z(y_n^z)-1} \Delta \circ \varphi^{(j)}(x_0, y_n^z).$$

We are interested in calculating the limit of the series in the previous expression when $n \to \infty$. Let an arbitrary $0 < \epsilon_1 < |x_1|$. We claim that for $n >> 0$ there exists $0 < a_1 < a_2 < T_z(y_n^z)$ such that

- $\exp([0, a_1]X)(x_0, y_n^z) \cup \exp([a_2, T_z(y_n^z)]X)(x_0, y_n^z) \subset [|x| \geq \epsilon_1]$.
- $\exp([a_1, a_2]X)(x_0, y_n^z) \subset \overline{U_{\epsilon_1}}$.

This is a consequence of $\exp(zX)(x_1, 0)$ belonging to the first component of $L_{\beta_r, x_0}^{\epsilon, +}$. By theorem 7.1 we have

$$\left| \sum_{j=[a_1]+1}^{[a_2]} \Delta \circ \varphi^{(j)}(x_0, y_n^z) \right| < D(\epsilon_1)$$

for a constant $D(\epsilon_1) > 0$ such that $\lim_{\epsilon_1 \to 0} D(\epsilon_1) = 0$. As a consequence

$$\sum_{j=0}^{T_z(y_n^z)-1} \Delta \circ \varphi^{(j)}(x_0, y_n^z) \to \sum_{j=0}^{\infty} \Delta \circ \varphi^{(j)}(x_0, 0) + \sum_{j=1}^{\infty} \Delta \circ \varphi^{(-j)}(x_{1,z}, 0)$$

when $n \to \infty$. We obtain

$$\lim_{n \to \infty} (T_z(y_n^z) - A_{E_-}(y_n^z)) = \psi_{1,0}^{-,\varphi}(x_{1,z}, 0) - \psi_{0,0}^{+,\varphi}(x_0, 0).$$

A different expression for the same limit provides

$$\psi_{1,0}^{-,\varphi}(x_{1,z}, 0) - \psi_{0,0}^{+,\varphi}(x_0, 0) = \psi_{1,0}^{-}(x_1, 0) - \psi_{0,0}^{+}(x_0, 0) + z.$$

Since every accumulation point of $\varphi^{(T_z(y_n^z))}(x_0, y_n^z)$ satisfies the previous expression then (x_1, z) is the only accumulation point, aka the limit. Substracting the expression for $z = 0$ we obtain

$$\psi_{1,0}^{-,\varphi}(x_{1,z}, 0) - \psi_{1,0}^{-,\varphi}(x_{1,0}, 0) = z$$

as we wanted to prove. □

Morally, the orbit $\varphi^{(j)}(x_0, y_n^z)$ $(0 \leq j \leq T_z(y_n^z))$ induces the same partition of the fixed points than $\exp([0, T_z(y_n^z)]X)(x_0, y_n^z)$. We explain how this is possible. Let $C > 0$; let V and W be the domains provided by theorem 7.1; we can suppose $V = U_{\epsilon,\delta}$ and $W = U_{\epsilon',\delta}$ without lack of generality. Moreover, we can suppose $t \mapsto \exp(tX)(P)$ is well defined in $t \in B(0, 3C)$ and its image is contained in $U_{\epsilon',\delta}$ for all $P \in U_{\epsilon,\delta}$. We stress that if $t \mapsto \exp(tX)(P)$ is well-defined in $B(0, 3C)$ and P does not belong to $[f = 0]$ then it is injective by the Rolle property.

First of all, we want to draw some sort of continuous path joining $\varphi^{(0)}(P)$ and $\varphi^{(1)}(P)$ for $P \in U_{\epsilon',\delta}$. We define

$$\kappa_0'(P, a) = (1 - a)\psi_{X(\varphi)}(P) + a\psi_{X(\varphi)}(\varphi(P))$$

for $a \in [0, 1]$. Since $|\kappa_0'(P, a) - \psi_{X(\varphi)}(\alpha^{(a)}(P))| \leq C$ then we define $\kappa_0(P, a) = \psi_{X(\varphi)}^{-1}(\kappa_0'(P, a))$. We define $\kappa_j(x_0, y_n^z) = \varphi^{(j)}(\kappa_0(x_0, y_n^z))$ for all $1 \leq j \leq T(y_n^z) - 1$. A possible choice for a path joining the points of the orbit is

$$\kappa = \kappa_0(x_0, y_n^z)\kappa_1(x_0, y_n^z) \ldots \kappa_{T_z(y_n^z)-1}(x_0, y_n^z)$$

Let $\kappa_{T_z(y_n^z)}(P,a) \subset [y = y(P)]$ be the path

$$a \to \psi_{X(\varphi)}^{-1}\left((1-a)\psi_{X(\varphi)}(\varphi^{(T_z(y_n^z))}(P)) + a\psi_{X(\varphi)}(\alpha^{(T_z(y_n^z))}(P))\right)$$

for all $a \in [0,1]$. We have

LEMMA 7.2.5. *The paths* $\kappa\kappa_{T_z(y_n^z)}(x_0, y_n^z)$ *and* $\exp([0, T_z(y_n^z)]X)(x_0, y_n^z)$ *are homotopic in* $[y = y_n^z] \setminus [f = 0]$.

PROOF. By construction we have

$$\kappa_0((x_0, y_n^z), a) \in \exp(\overline{B}(0, C)X)(\alpha^{(a)}(x_0, y_n^z))$$

for all $a \in [0,1]$. That implies

$$\kappa_l((x_0, y_n^z), a) \in \exp(\overline{B}(0, 2C)X)(\alpha^{(a+l)}(x_0, y_n^z))$$

for all $1 \leq l \leq T(y_n^z) - 1$ and all $a \in [0,1]$. Finally

$$\kappa_{T_z(y_n^z)}((x_0, y_n^z), a) \in \exp(\overline{B}(0, C)X)(\alpha^{(T_z(y_n^z))}(x_0, y_n^z))$$

for all $a \in [0,1]$. Since $\cup_{b \in [0, T_z(y_n^z)]} \overline{B}(0, 2C)$ is simply connected we are done. $\square$

The last lemma implies that κ and $\exp([0, T_z(y_n^z)]X)(x_0, y_n^z)$ induce the same partition in the fixed points set. Next, we are going to study the topological conjugation of diffeomorphisms. Since those conjugations do not conjugate normal forms we have to interpret partitions in terms of long orbits instead of long trajectories of the normal form.

CHAPTER 8

Topological Invariants of (NSD) Diffeomorphisms

We define the set

$$\mathcal{D}_f = \{(x + u(x,y)f(x,y), y) \ / \ u \text{ is a unit}\}$$

for any $f \in \mathbb{C}\{x,y\}$ satisfying the (NSD) conditions. The set $\mathcal{D}_f$ is the analogous of the set $\mathcal{H}_f$ for diffeomorphisms. We want to study when two elements of $\mathcal{D}_f$ are conjugated by a special homeomorphism.

Suppose that $\varphi_1, \varphi_2 \in \mathcal{D}_f$ are conjugated by the special homeomorphism σ. Fix convergent normal forms $X_1 = X(\varphi_1)$ and $X_2 = X(\varphi_2)$ respectively. Let $\alpha_j = \alpha_{\varphi_j}$ and $\psi_j = \psi_{X(\varphi_j)}$ for $j \in \{1,2\}$. Fix $C > 0$. For $j \in \{1,2\}$ there exist $0 < \tau_j < \tau_j'$ such that $\{P, \ldots, \alpha_j^{(k)}(P)\} \subset U_{\tau_j}$ for some $k \in \mathbb{Z}$ implies $\{P, \ldots, \varphi_j^{(k)}(P)\} \subset U_{\tau_j'}$ and

$$\left| \psi_j(\varphi_j^{(k)}(P)) - (\psi_j(P) + k) \right| \leq C.$$

The objects φ_j, α_j and ψ_j are defined in $U_{\tau_j'}$. By making $\tau_1 > 0$ smaller we can suppose that

- σ is defined in the neighborhood of $\overline{U_{\tau_1}}$.
- $t \mapsto \exp(tX_1)(P)$ is well-defined in $B(0,3C)$ for $P \in \overline{U_{\tau_1}}$.

By replacing (τ_1, σ, X_1) with $(\tau_2, \sigma^{(-1)}, X_2)$ in the previous conditions we obtain an analogous condition for τ_2. We choose $\epsilon < \kappa_1 < \tau_1$ and $\kappa_2 < \tau_2$ such that

- $\exp(B(0,6C)X_1)(U_\epsilon) \subset U_{\kappa_1} \subset U_{\tau_1}$.
- $\exp(B(0,6C+1)X_2)(\sigma(U_{\kappa_1})) \subset U_{\kappa_2} \subset U_{\tau_2}$.

8.1. Topological invariants

8.1.1. Orientation. We remind the reader that the mapping

$$\sigma(s)_* : \pi_1((U_{\tau_1} \cap [y=s]) \setminus (f=0)) \to \pi_1((\sigma(U_{\tau_1}) \cap [y=s]) \setminus (f=0))$$

is the one induced by $\sigma_{|y=s}$ for $s \in B(0,\delta)$.

Proposition 8.1.1. *Suppose $N > 1$. The mapping $\sigma(s)_*$ is the identity for all $s \in B(0,\delta)$.*

Proof. We can just copy the proof of proposition 6.1.1. In that proof we did not use that σ conjugates X_1 and X_2 but only that $\sigma_{|f=0} \equiv Id$ and that it satisfies

$$\sigma(\omega^{-1}_{\xi(X_1,y,\epsilon_1),|x|<\epsilon_1}(x,y)) \subset \omega^{-1}_{\xi(X_2,y,\epsilon_2),|x|<\epsilon_2}(x,y)$$

for some $0 < \epsilon_1 \leq \epsilon$, $0 < \epsilon_2$ and $\forall (x,y) \in (U_{\epsilon_1,\delta} \cap [f=0]) \setminus [y=0]$ (ditto for the α limit); the last result is a consequence of lemma 7.2.4. $\square$

8.1.2. Partition of the fixed points. Let $x_0 \in B(0,\epsilon) \setminus \{0\}$ and let β be a semi-analytic curve. Suppose $L_{\beta,x_0}^{+,\epsilon}(X_1) \neq \emptyset$; let $x_1 \in U_\epsilon$ be a point in the first component $\rho_{1,1}$ of $L_{\beta,x_0}^{+,\epsilon}(X_1)$. There exists a continuous partition (E_-, E_+) of $SingX_1$ and a true section $\chi : \beta \to U_{\kappa_1}$ such that for $y \in \beta$ we have

$$T(y) = \frac{\psi_{1,1}}{y^m}(\chi(y)) + A_{E_-,X_1}(y) - \frac{\psi_{1,0}}{y^m}(x_0, y) \in \mathbb{R}^+$$

and $\lim_{y\in\beta, y\to 0} \chi(y) = (x_1, 0)$. We remind the reader that $\psi_{1,1}$ and $\psi_{1,0}$ are integrals of the time form of $X_1(1)$. Consider the sequence $\{y_n\}$ of points in $T^{-1}(\mathbb{N})$. The orbit $\varphi_1^{(j)}(x_0, y_n)$ $(0 \leq j \leq T(y_n))$ is mapped onto $\varphi_2^{(j)}(\sigma(x_0, y_n))$ $(0 \leq j \leq T(y_n))$ since σ conjugates φ_1 and φ_2. Then

PROPOSITION 8.1.2. *We have that* $\lim_{y\in\beta,y\to 0} \exp(T(y)X_2)(\sigma(x_0, y))$ *exists. Let* $(x_1', 0)$ *be such a limit. Then* x_1' *belongs to the first component* $\rho_{2,1}$ *of* $L_{\beta,\sigma(x_0,0)}^{+,\kappa_2}(X_2)$. *Moreover, the partition of the fixed points induced by* $\rho_{2,1}$ *is* (E_-, E_+).

PROOF. We suppose $\lambda(\beta) = 1$ without lack of generality. We denote $\gamma_n = \exp([0, T(y_n)]X_2)(\sigma(x_0, y_n))$ and $a_n = \gamma_n(T(y_n))$. We have that $\varphi_1^{(j)}(x_0, y_n) \in \exp(\overline{B}(0, C)X_1)(\overline{U_\epsilon})$ for all $0 \leq j \leq T(y_n)$. Therefore $\varphi_2^{(j)}(x_0, y_n) \in \sigma(U_{\kappa_1})$ for all $0 \leq j \leq T(y_n)$ and then γ_n is contained in $\exp(B(0, 1 + C)X_1)(\sigma(U_{\kappa_1}))$. We define

$$b = \lim_{n\to\infty} \varphi_2^{(T(y_n))}(\sigma(x_0, y_n)) = \sigma\left(\lim_{n\to\infty} \varphi_1^{(T(y_n))}(x_0, y_n)\right);$$

the limit exists by proposition 7.2.3. The set of accumulation points of $\{a_n\}$ is contained in $\exp(\overline{B}(0, C)X_2)(b)$. Up to take a subsequence we can suppose that $\{a_n\}$ converges; we denote the limit by $(x_1', 0)$. Since $x_1 \in L_{\beta,x_0}^{+,\epsilon}(X_1)$ then $\lim_{n\to\infty} |y_n|^m T(y_n) = \infty$; as a consequence $x_1' \in L_{\beta,\sigma(x_0,0)}^{+,\kappa_2}(X_2)$. Let (E_-', E_+') the division induced by γ_n; we can suppose it is the same for all $n \in \mathbb{N}$ by refining the subsequence. We have that $\lim_{n\to\infty} |y_n|^m (A_{E_-,X_1}(y_n) - A_{E_-',X_2}(y_n))$ is equal to

$$(\psi_{2,1}(x_1', 0) - \psi_{1,1}(x_1, 0)) - (\psi_{2,0}(\sigma(x_0, 0)) - \psi_{1,0}(x_0, 0))$$

by comparing the formulas for $\exp([0, T(y_n)]X_1)(x_0, y_n)$ and γ_n. By lemma 5.1.1 the limit $\lim_{y\in\beta,y\to 0} |y^m|(A_{E_-,X_1}(y) - A_{E_-',X_2}(y))$ exists. That implies the existence of a true section $\zeta : \beta \cup \{0\} \to \mathbb{C}^2$ such that

$$|y|^m T(y) \equiv \psi_{2,1}(\zeta(y))\lambda^{-m} + |y|^m A_{E_-',X_2} - \psi_{2,0}(\sigma(x_0, y))\lambda^{-m}$$

where $\lambda = y/|y|$. Moreover we obtain $\zeta(0) = (x_1', 0)$ and $\zeta(y_n) = a_n$ for all $n >> 0$.

Suppose x_1' is not in the first component of $\rho_{2,1}$. Then there exists a function $T' : \beta \to \mathbb{R}^+$ such that

$$\lim_{y\in\beta} |y|^m T'(y) = \lim_{y\in\beta} |y|^m (T(y) - T'(y)) = \infty$$

and such that $\lim_{y\in\beta,y\to 0} \exp(T'(y)X_2)(\sigma(x_0, y))$ exists. Let $\{y_n'\}$ the sequence of points in $T'^{-1}(\mathbb{N})$. By analogous arguments to the already exposed we can prove that $\varphi_2^{(T'(y_n'))}(\sigma(x_0, y_n'))$ has an accumulation point different than $(0, 0)$. By applying $\sigma^{(-1)}$ we obtain that $\varphi_1^{(T'(y_n'))}(x_0, y_n')$ enjoys the same property and then $\exp(T'(y_n')X_1)(x_0, y_n')$. But such an accumulation point is in a component of $L_{\beta,x_0}^{+,\epsilon}(X_1)$ smaller than $\rho_{1,1}$ since $\lim_{y\in\beta} |y|^m (T(y) - T'(y)) = \infty$. That is impossible by hypothesis.

We still have to prove $(E_-, E_+) = (E'_-, E'_+)$. We consider the path $\kappa^1(n) = \kappa_0^1(x_0, y_n) \dots \kappa_{T(y_n)-1}^1(x_0, y_n)$ associated to the couple (φ_1, X_1) and defined in subsection 7.2.4. We also consider the path $\kappa^2(n) = \kappa_0^2(\sigma(x_0, y_n)) \dots \kappa_{T(y_n)-1}^2(\sigma(x_0, y_n))$ associated to (φ_2, X_2). Since (E_-, E_+) is induced by $\kappa^1(n)$ then it is also induced by $\sigma(\kappa^1(n))$ because σ preserves the orientation. Then it is enough to prove that $\sigma(\kappa^1(n))$ is homotopic to $\kappa^2(n)$ in $[y = y_n] \setminus [f = 0]$ because the latter path induces the partition (E'_-, E'_+). We remark that $\varphi_2(\sigma(\kappa_j^1)) = \sigma(\kappa_{j+1}^1)$ and $\varphi_2(\kappa_j^2) = \kappa_{j+1}^2$. It is enough to prove $\sigma(\kappa_{j_0}^1(x_0, y_n)) \sim \kappa_{j_0}^2(\sigma(x_0, y_n))$ for one $0 \leq j_0 \leq T(y_n) - 1$ since φ_2 preserves the fixed points and the orientation. We define

$$H_j(a) = \psi_2 \circ \sigma \circ \kappa_j^1(a) - \psi_2 \circ \kappa_j^2(a)$$

for $0 \leq j \leq T(y_n) - 1$ and $a \in [0, 1]$. The function H_0 is bounded since $[0, 1]$ is compact. Moreover, we have $|H_j(a) - H_0(a)| \leq 2C$ for $1 \leq j \leq T(y_n) - 1$. Therefore, we can suppose $|H_j(a)| < D$ for some $D > 0$ and all $0 \leq j \leq T(y_n) - 1$ and $a \in [0, 1]$. Since

$$\kappa_j^2(\sigma(x_0, y_n))(a) \in \exp(\overline{B}(0, 2C)X_2)(\alpha^{a+j}(\sigma(x_0, y_n)))$$

we deduce that $\sigma(\kappa_j^1(x_0, y_n)) \cup \kappa_j^2(\sigma(x_0, y_n))$ belongs to

$$\exp(B(0, 1 + 2C + D)X_2)(\alpha^j(\sigma(x_0, y_n))).$$

Let $\epsilon' > 0$ such that $\epsilon' \leq \min(|x| \circ \sigma(x_0, 0), |x'_1|)$ and $t \to \exp(tX_2)(P)$ is well defined in $t \in B(0, 1 + 2C + D)$ for all $P \in U_{\epsilon'}$. For all $n >> 0$ there exists $j_0(n)$ such that $\alpha^{j_0(n)}(\sigma(x_0, y_n)) \in U_{\epsilon'}$; otherwise we obtain $L^{+,\kappa_2}_{\beta,\sigma(x_0,0)}(X_2) = \emptyset$, that is a contradiction. Since $B(0, 1 + 2C + D)$ is simply connected then $\sigma(\kappa_{j_0(n)}^1(x_0, y_n)) \sim \kappa_{j_0(n)}^2(\sigma(x_0, y_n))$ for $n >> 0$; we are done. □

Last proposition and proposition 7.2.3 will be the key tools in order to prove that the topological invariants for the special conjugation of (NSD) diffeomorphisms are basically the same than for vector fields.

8.1.3. Rigidity of the special conjugation when $[y = 0] \subset [f = 0]$**.** In this subsection we prove that $\sigma_{|f=0}$ is analytic for $m > 0$ through the study of sectorial convergent logarithms.

A set $V_{a,b}(\upsilon_1, \upsilon_2) = [|x| < \upsilon_1] \cap \{y \in B(0, \upsilon_2) \setminus \{0\} : a < \arg y < b\}$ is called a sectorial domain; its aperture is $\theta = \theta(V) = b - a$.

Proposition 8.1.3 (Voronin (see [**I**$^+$**92**])). *Consider* $\varphi = \exp(\hat{u} y^m \partial/\partial x)$ *in* $\mathcal{D}_{y^m}$ *and* $X(\varphi) = u y^m \partial/\partial x$. *Let* $a < b$ *in* $\mathbb{R}$ *such that* $b - a < \pi/m$. *Then, there exist a sectorial domain* $S = V_{a,b}(\upsilon_1, \upsilon_2)$ *and a vector field* Y *defined in* S *such that*

- Y *is of the form* $y^m u'(x, y)\partial/\partial x$ *where* $u - u' = O(y^{2m})$.
- $\hat{u}$ *is the asymptotic development of* u' *in* S.
- $\varphi = \exp(Y)$.

The vector field Y is not unique. Anyway, any vector field fulfilling the previous properties will be called a *sectorial logarithm* of φ. Its existence implies:

Lemma 8.1.1. *Let* σ *be a special germ of homeomorphism conjugating* φ_1, $\varphi_2 \in \mathcal{D}_{y^m}$ *for* $m > 0$. *Then* $\sigma_{|y=0}$ *is a germ of analytic biholomorphism. Moreover* $\sigma_{|y=0}$ *conjugates* $\log \varphi_{1,0}(1)$ *and* $\log \varphi_{2,0}(1)$.

Proof. Let $\varphi_j = \exp(\hat{u}_j y^m \partial/\partial x)$ and $X_j = u_j y^m \partial/\partial x$ for $j \in \{1,2\}$. There exist $2m+1$ sectorial domains $V_{a_j,b_j}(\upsilon_1,\upsilon_2)$ $(1 \leq j \leq 2m+1)$ such that $b_j - a_j < \pi/m$ for $1 \leq j \leq 2m+1$ and

$$\cup_{1\leq j\leq 2m+1} V_{a_j,b_j}(\upsilon_1,\upsilon_2) = [|x| < \upsilon_1] \cap [0 < |y| < \upsilon_2].$$

Moreover we can suppose that φ_j has a sectorial logarithm Y_j^k in the domain $V_{a_k,b_k}(\upsilon)$ for $j \in \{1,2\}$ and $1 \leq k \leq 2m+1$. Let $\zeta > 0$ such that $\exp(B(0,\zeta)X_1(1))(0,0))$ is contained in U_{υ_1}. Consider $\zeta' \in B(0,\zeta)$; we define $\theta_0 = \arg(\zeta')/m$ and $r_n = (|\zeta'|/n)^{1/m}$. There exists k_0 such that $(0,\upsilon_2)e^{i\theta_0} \subset \pi_y(V_{a_{k_0},b_{k_0}}(\upsilon_1,\upsilon_2))$. Let $y_n = r_n e^{i\theta_0}$; we have

$$\sigma(\varphi_1^{(n)}(0,y_n)) = \varphi_2^{(n)}(\sigma(0,y_n)).$$

By developing $\varphi_1^{(n)}$ and $\varphi_2^{(n)}$ we obtain

$$\varphi_1^{(n)}(0,y_n) = \exp(nY_1^{k_0})(0,y_n) = \exp\left(\zeta' \frac{Y_1^{k_0}}{y^m}\right)(0,y_n)$$

and

$$\varphi_2^{(n)}(\sigma(0,y_n)) = \exp(nY_2^{k_0})(\sigma(0,y_n)) = \exp\left(\zeta' \frac{Y_2^{k_0}}{y^m}\right)(\sigma(0,y_n)).$$

We have

$$\sigma(\exp(\zeta' X_1(1))(0,0)) = \exp(\zeta' X_2(1))(\sigma(0,0))$$

by making $n \to \infty$. Since $X_1(1)_{|y=0}$ and $X_2(1)_{|y=0}$ are regular then $\sigma_{|y=0}$ is analytic in the neighborhood of $(0,0)$. $\square$

Proposition 8.1.4. *Let σ be a special germ of homeomorphism conjugating φ_1, $\varphi_2 \in \mathcal{D}_f$. Suppose $m > 0$. Then $\sigma_{|y=0}$ is a germ of analytic biholomorphism. Moreover $\sigma_{|y=0}$ conjugates* $\log \varphi_{1,0}(1)$ *and* $\log \varphi_{2,0}(1)$.

Proof. Let $(x_0,0) \in U_\epsilon \setminus \{(0,0)\}$ and $x_0' = x \circ \sigma(x_0,0)$. The mapping $\sigma(x,0)$ is analytic in a neighborhood of $(x_0,0)$ if and only if the mapping

$$\chi(x,y) = (x \circ \sigma(x+x_0,y) - x_0', y)$$

satisfies that $\chi(x,0)$ is a analytic in a neighborhood of $(0,0)$. Moreover χ conjugates

$$(x-x_0,y) \circ \varphi_1 \circ (x+x_0,y) \ \text{ and } \ (x-x_0',y) \circ \varphi_2 \circ (x+x_0',y);$$

both of these diffeomorphisms belong to $\mathcal{D}_{y^m}$. By lemma 8.1.1 the diffeomorphism $\chi(x,0)$ is analytic in a neigborhood of $(0,0)$. As a consequence $\sigma(x,0)$ is holomorphic in $[0 < |x| < \epsilon] \cap [y=0]$. Since σ is continuous then $\sigma(x,0)$ is holomorphic in $[|x| < \epsilon] \cap [y=0]$. $\square$

8.1.4. Definition of the Topological Invariants. Let $\varphi \in \mathcal{D}_f$. The set of topological invariants $SP(\varphi)$ of φ for the $\overset{sp}{\sim}$ conjugation is by definition empty if $N = 0$ or $(N,m) = (1,0)$. Otherwise $SP(\varphi)$ contains

- The parts of degree less or equal than 0 of every function $y^m(Res_\varphi(S(y)))$ associated to some continuous section
$$S : B(0,\delta) \setminus \{0\} \to Fix\varphi.$$
- The analytic class of $\varphi_0(1)$.

These invariants are analogous to the (NSD) vector fields ones. Even the analytic class of $X(1)_{|y=0}$ is a topological invariant for (NSD) vector fields (lemma 6.3.1).

The analytic class of $\varphi_0(1)$ can be replaced with the analytic class of $\varphi_{|y=0}$. If $m = 0$ it is clear since $\varphi_0(1) \equiv \varphi_{|y=0}$. Otherwise it is still true since $\varphi_{|y=0} \equiv Id$ and the analytic class of $\varphi_0(1)$ is determined by the invariants attached to the residue functions (lemma 6.3.1).

8.2. Theorem of topological conjugation

THEOREM 8.1. *Let $f \in \mathbb{C}\{x, y\}$ satisfying the (NSD) conditions. Let $\varphi_1, \varphi_2 \in \mathcal{D}_f$. Then*

$$\varphi_1 \overset{sp}{\sim} \varphi_2 \Leftrightarrow SP(\varphi_1) = SP(\varphi_2).$$

8.2.1. Theorem 8.1. Proof of the sufficient condition. We will prove first the sufficient condition. We will proceed in an analogous way than for proving the sufficient condition in theorem 6.1.

LEMMA 8.2.1. *Let φ_1, $\varphi_2 \in \mathcal{D}_f$ such that $\varphi_1 \overset{sp}{\sim} \varphi_2$ by a special germ of homeomorphism σ. Consider a non-empty L-limit $L^{+,\epsilon}_{\beta,x_0}(X(\varphi_1))$. Consider a component ρ of $L^{+,\epsilon}_{\beta,x_0}(X(\varphi_1))$ and let E be the partition induced by (x_0, ρ). Then*

$$\mu\left(\sum_{P \in E_-(y)} [Res_{X(\varphi_1)}(P) - Res_{X(\varphi_2)}(P)]\right) \leq m.$$

PROOF. The proof is analogous to the proof of lemma 6.2.1. Suppose $\lambda(\beta) = 1$ without lack of generality. Suppose ρ is the first component of $L^{+,\epsilon}_{\beta,x_0}(X_1)$. Let $x_1 \in \rho$. There exists a true section $\chi : \beta \cup \{0\} \to \mathbb{C}^2$ such that $\chi(0) = (x_1, 0)$ and

$$T(y) = \frac{\psi_{1,1}}{y^m}(\chi(y)) + A_{E_-,X_1}(y) - \frac{\psi_{1,0}}{y^m}(x_0, y)$$

for a function $T : \beta \to \mathbb{R}^+$. We consider the sequence of points $\{y_n\}$ contained in $T^{-1}(\mathbb{N})$. The limit $(z_1, 0) = \lim_{n\to\infty} \varphi_1^{(T(y_n))}(x_0, y_n)$ exists by proposition 7.2.3. Moreover, proposition 7.2.3 also implies

$$\lim_{n\to\infty} |y_n|^m (T(y_n) - A_{E_-,X_1}(y_n)) = \psi^{-,\varphi_1}_{1,0}(z_1, 0) - \psi^{+,\varphi_1}_{0,0}(x_0, 0).$$

By proposition 8.1.2 the limit $(x_1', 0) = \lim_{n\to\infty} \exp(T(y_n)X_2)(\sigma(x_0, y_n))$ exists and it is in the first component of $L^{+,\kappa_2}_{\beta,\sigma(x_0,0)}(X_2)$. Since

$$\lim_{n\to\infty} \varphi_2^{T(y_n)}(\sigma(x_0, y_n)) = \sigma(\lim_{n\to\infty} \varphi_1^{T(y_n)}(x_0, y_n)) = \sigma(z_1, 0)$$

we can proceed like we did previously to obtain

$$\lim_{n\to\infty} |y_n|^m (T(y_n) - A_{E_-,X_2}(y_n)) = \psi^{-,\varphi_2}_{1,0}(\sigma(z_1, 0)) - \psi^{+,\varphi_2}_{0,0}(\sigma(x_0, 0));$$

the partition of the fixed points coincide by proposition 8.1.2. Hence

$$\lim_{n\to\infty} |y_n|^m (A_{E_-,X_1}(y_n) - A_{E_-,X_2}(y_n)) \in \mathbb{C};$$

that clearly implies $\mu(A_{E_-,X_1} - A_{E_-,X_2}) \leq m$.

Let $\rho_1 < \ldots < \rho_k = \rho < \ldots$ be the decomposition of $L_{\beta,x_0}^{+,\epsilon}(X_1)$ in connected components. By the first part of the proof the partition of the fixed points (E_-^j, E_+^j) associated to (ρ_j, ρ_{j+1}) satisfies

$$\mu(A_{E_-^j,X_1} - A_{E_-^j,X_2}) \leq m \quad \text{and} \quad \mu(A_{E_+^j,X_1} - A_{E_+^j,X_2}) \leq m$$

for all $0 \leq j \leq k-1$. Let $(F_1, F_2, \ldots, F_l)$ be the partition whose elements are the sets of the form

$$E_{s_0}^0 \cap \ldots \cap E_{s_{k-1}}^{k-1}$$

where $(s_0, \ldots, s_{k-1}) \in \{+,-\}^k$. We can obtan $\mu(A_{F_j,X_1} - A_{F_j,X_2}) \leq m$ for $1 \leq j \leq l$ by proceeding like in lemma 6.2.2. Since

$$A_{E_-,X_1} - A_{E_-,X_2} = \sum_{j \in J}(A_{F_j,X_1} - A_{F_j,X_2})$$

for some subset $J \subset \{1, \ldots, l\}$ then the result is proved. □

Proposition 8.2.1. *Let $\varphi_1, \varphi_2 \in \mathcal{D}_f$ such that there exists a special germ of homeomorphism conjugating φ_1 and φ_2. Consider a continuous multi-valuated section $S : B(0,\delta) \setminus \{0\} \to (f=0)$ such that $S(s) \in [y=s]$ for all $s \in B(0,\delta) \setminus \{0\}$. Then*

$$\mu(Res_{\varphi_1}(S(y)) - Res_{\varphi_2}(S(y))) \leq m.$$

The proof of proposition 8.2.1 is obtained by copying the proofs of lemmas 6.2.2, 6.2.3 and proposition 6.2.1 with no change.

Proposition 8.2.2. *Suppose $N \neq 0$ and $(N,m) \neq (1,0)$. Let σ be a germ of special homeomorphism conjugating elements φ_1 and φ_2 in $\mathcal{D}_f$. Then $\sigma_{|y=0}$ is analytic, moreover it conjugates $\varphi_{1,0}(1)$ and $\varphi_{2,0}(1)$.*

Proof. If $m > 0$ then $\sigma_{|y=0}$ is analytic by proposition 8.1.4. Moreover $\sigma_{|y=0}$ conjugates $\exp(\log \varphi_{1,0}(1))$ and $\exp(\log \varphi_{2,0}(1))$.

If $m = 0$ then $N > 1$. Let $(x_1, 0) \in U_\epsilon \setminus \{(0,0)\}$. Suppose $\alpha_{\xi(X_1),|x|<\epsilon}(x_1,0) = (0,0)$ without lack of generality. Hence, there exists a L-limit $L_{\beta,x_1}^{-,\epsilon}(X_1) \neq \emptyset$. We can suppose $\lambda(\beta) = 1$. There exist (see proof of proposition 7.2.3) a point $(x_0, 0)$, a compact wedge $W = \cup_{r \in [-M,M]} \beta_r$ ($\beta_r \in \Upsilon_{A_{E_-}}^r$), a true section $\chi : W \to \mathbb{C}^2$ and a function $T : W \to \mathbb{R}^+$ such that

- $T(y) = \psi_{1,1}(\chi(y)) + A_{E_-,X_1}(y) - \psi_{1,0}(x_0, y)$.
- $\lim_{n\to\infty} \alpha_{\varphi_1}^{(-T(y_n))}(x_1, y_n)$ is in the first component of $L_{\beta,x_1}^{-,\epsilon}(X_1)$.
- $(x_1, 0) = \lim_{n\to\infty} \varphi_1^{(T(y_n))}(x_0, 0)$.
- $\lim_{y\in\beta_r, y\to 0} \chi(y) = \exp(irX_1)(\lim_{y\in\beta, y\to 0} \chi(y))$ for $r \in [-M, M]$.

For these conditions $\{y_n\}$ is the sequence of points in $T^{-1}(\mathbb{N}) \cap \beta$. We proceed like in the proof of proposition 7.2.3. Let $z = s + ir$ in the set $[-M,M] + i[-M,M]$; we define $T_z = T + s$, then we choose the sequence $\{y_n^z\}$ of points in $T_z^{-1}(\mathbb{N}) \cap \beta_r$. By proposition 7.2.3 the limit $(x_{1,z}, 0) = \lim_{n\to\infty} \varphi_1^{(T_z(y_n^z))}(x_0, y_n^z)$ exists, moreover we have

$$\lim_{n\to\infty}(T_z(y_n^z) - A_{E_-,X_1}(y_n^z)) = \psi_{1,0}^{-,\varphi_1}(x_{1,z}, 0) - \psi_{0,0}^{+,\varphi_1}(x_0, 0).$$

Since

$$\sigma(x_{1,z}, 0) = \lim_{n\to\infty} \sigma(\varphi_1^{(T_z(y_n^z))}(x_0, y_n^z)) = \lim_{n\to\infty} \varphi_2^{(T_z(y_n^z))}(\sigma(x_0, y_n^z))$$

proposition 8.1.2 allows to apply the same method to φ_2 to obtain

$$\lim_{n\to\infty}(T_z(y_n^z) - A_{E_-,X_2}(y_n^z)) = \psi_{1,0}^{-,\varphi_2}(\sigma(x_{1,z},0)) - \psi_{0,0}^{+,\varphi_2}(\sigma(x_0,0)).$$

The limit $D = \lim_{y\to 0}(A_{E_-,X_1}(y) - A_{E_-,X_2}(y))$ exists, it is a consequence of $\mu(A_{E_-,X_1}(y) - A_{E_-,X_2}(y)) = 0$. Therefore

$$\psi_{1,0}^{-,\varphi_2}(\sigma(x_{1,z},0)) - \psi_{1,0}^{-,\varphi_1}(x_{1,z},0) = D + (\psi_{0,0}^{+,\varphi_2}(\sigma(x_0,0)) - \psi_{0,0}^{+,\varphi_1}(x_0,0)).$$

Since $\psi_{1,0}^{-,\varphi_1}(x_{1,z},0) = \psi_{1,0}^{-,\varphi_1}(x_{1,0},0) + z$ the mapping $z \to x_{1,z}$ is a local biholomorphism. We deduce that $\sigma(x,0)$ is holomorphic in the neighborhood of $(x_1,0)$. That implies $\sigma(x,0)$ to be holomorphic in $(U_\epsilon \cap [y=0]) \setminus \{(0,0)\}$. Indeed, it is holomorphic in $U_\epsilon \cap [y=0]$, we can remove the singularity. □

Let V be a petal (either attracting or repelling) $V \subset [y=0]$ associated to $\varphi_{|y=0}$. We denote by $\psi_{V,0}^X$ the integral of the time form of $X(\varphi)(1)$ in V. We denote by $\psi_{V,0}^\varphi$ the integral of the time form of $\varphi_{|y=0}$ in V defined by

$$\psi_{V,0}^\varphi = \psi_{V,0}^X + \sum_{j=0}^{\infty} \Delta \circ \varphi^{(j)} \quad \text{or} \quad \psi_{V,0}^\varphi = \psi_{V,0}^X - \sum_{j=1}^{\infty} \Delta \circ \varphi^{(-j)}$$

depending on whether V is attracting or repelling.

Lemma 8.2.2. *Let φ_1, $\varphi_2 \in \mathcal{D}_f$ such that $\varphi_1 \overset{sp}{\sim} \varphi_2$ are conjugated by a special germ of homeomorphism σ. Suppose $N \neq 0$ and $(N,m) \neq (1,0)$. Let V be a petal for $\varphi_1(x,0)$ in $|x| < \epsilon$. Then, we have*

$$\psi_{\sigma(V),0}^{\varphi_2} \circ \sigma - \psi_{V,0}^{\varphi_1} \equiv L$$

for some constant $L \in \mathbb{C}$ which does not depend on V.

Proof. If $m > 0$ then $\psi_{V,0}^\varphi = \psi_{V,0}^X$ for every petal $V \subset [y=0]$; as a consequence the result is a trivial consequence of proposition 8.1.4. Suppose $m = 0$. Let $V \subset B(0,\epsilon) \setminus \{0\}$ be a petal for $\varphi_1(x,0)$. We can suppose V is repelling without lack of generality. Since there exists a non-empty $L_{\beta,x_1}^{-,\epsilon}(X_1)$ for some semi-analytic β then we can proceed like in proposition 8.2.2 to obtain

$$\psi_{1,0}^{-,\varphi_2}(\sigma(x_{1,z},0)) - \psi_{1,0}^{-,\varphi_1}(x_{1,z},0) \equiv cte$$

for $\psi_{1,0}^{-,\varphi_1} \equiv \psi_{V,0}^{\varphi_1}$, $\psi_{1,0}^{-,\varphi_2} \equiv \psi_{\sigma(V),0}^{\varphi_2}$ and z in a neighborhood of 0. We deduce that $\psi_{\sigma(V),0}^{\varphi_2} \circ \sigma(x,0) - \psi_{V,0}^{\varphi_1}(x,0)$ is locally constant; therefore

$$\psi_{\sigma(V),0}^{\varphi_2} \circ \sigma(x,0) - \psi_{V,0}^{\varphi_1}(x,0) \equiv L_V$$

in V for some constant $L_V \in \mathbb{C}$. Let V' be a petal next to V and let $x_n \in [|x| < \epsilon/n]$ such that $x_n \in V \cap V'$ and

$$(\alpha_{\xi(X_1,0,\epsilon/n)}, \omega_{\xi(X_1,0,\epsilon/n)})_{|x|<\epsilon/n}(x_n,0) = ((0,0),(0,0)).$$

By theorem 7.1 there exists $C(\epsilon/n) > 0$ such that $\lim_{n\to\infty} C(\epsilon/n) = 0$ and

$$|(\psi_{\sigma(V),0}^{\varphi_2} \circ \sigma(x_n,0) - \psi_{V,0}^{\varphi_1}(x_n,0)) - (\psi_{\sigma(V),0}^{X_2} \circ \sigma(x_n,0) - \psi_{V,0}^{X_1}(x_n,0))|$$

is lesser or equal than $2C(\epsilon/n)$. Since we can consider $\psi_{V,0}^{X_1} \equiv \psi_{V',0}^{X_1}$ and $\psi_{\sigma(V),0}^{X_2} \equiv \psi_{\sigma(V'),0}^{X_2}$ in $V \cap V'$ we deduce that $L_V \equiv L_{V'}$ by making $n \to \infty$. Therefore L_V does not depend on V. □

LEMMA 8.2.3. *Let φ_1, $\varphi_2 \in \mathcal{D}_f$ such that $\varphi_1 \overset{sp}{\sim} \varphi_2$ by a special germ of homeomorphism σ. Consider a component ρ of a non-empty L-limit $L^{+,\epsilon}_{\beta,x_0}(X(\varphi_1))$. Let E be the partition induced by (x_0, ρ). Then*

$$\lim_{y \to 0} y^m \left(\sum_{P \in E_-(y)} [Res_{X(\varphi_1)}(P) - Res_{X(\varphi_2)}(P)] \right) = 0.$$

PROOF. It is enough the proposition supposed ρ is the first component of $L^{+,\epsilon}_{\beta,x_0}(X_1)$; otherwise we proceed like in lemma 8.2.1 to extend the result to all the partitions induced by L-limits.

There exists $D \in \mathbb{C}$ such that

$$D = \lim_{y \in \beta, y \to 0} |y|^m (A_{E_-,X_1}(y) - A_{E_-,X_2}(y)))$$

by lemma 8.2.1. Let $x_1 \in \rho$; we have

$$\psi^{-,\varphi_2}_{1,0}(\sigma(x_{1,z}, 0)) - \psi^{-,\varphi_1}_{1,0}(x_{1,z}, 0) = D + (\psi^{+,\varphi_2}_{0,0}(\sigma(x_0, 0)) - \psi^{+,\varphi_1}_{0,0}(x_0, 0))$$

as we see in the proof of proposition 8.2.2. Let L be the constant provided by lemma 8.2.2; we obtain $L = D + L$ and then $D = 0$. $\square$

To end the proof we just need

PROPOSITION 8.2.3. *Suppose $N \neq 0$ and $(N, m) \neq (1, 0)$. Let $\varphi_1, \varphi_2 \in \mathcal{D}_f$ such that they are conjugated by a special germ of homeomorphism. Consider $S : B(0,\delta) \setminus \{0\} \to (f = 0)$ a continuous multi-valuated section such that $S(s) \in [y = s]$ for all $s \in B(0,\delta) \setminus \{0\}$. Then*

$$\lim_{y \to 0} y^m (Res_{\varphi_1}(S(y)) - Res_{\varphi_2}(S(y))) = 0.$$

We do not explicit the proof. It is completely analogous to the proof of proposition 6.2.3.

8.2.2. Proof of the necessary condition in theorem 8.1 when $N = 0$. Let $\varphi = \exp(\hat{u} y^m \partial/\partial x) \in \mathcal{D}_{y^m}$. Let $X(\varphi) = u y^m \partial/\partial x$ its convergent normal form. By theorem 6.1 it is enough to prove that φ is specially conjugated to $\exp(X(\varphi))$. Consider a sectorial domain $S = V_{a,b}(\upsilon_1, \upsilon_2)$ whose aperture $b - a$ is less than π/m. Let u'_S be the unit provided by proposition 8.1.3. An integral ψ_S of the time form of $u'_S \partial/\partial x$ in S is characterized by the equation

$$\frac{\partial(\psi_S - \psi)}{\partial x} = \frac{1}{u'_S y^m} - \frac{1}{u y^m} = \frac{1}{y^m}\left(\frac{u - u'_S}{u u'_S}\right).$$

Since the right hand side is a $O(y^m)$ there exists an integral ψ_S of the time form of $u' \partial/\partial x$ such that $\psi_S - \psi = O(y^m)$. Moreover, the equation $\varphi = \exp(u'_S y^m \partial/\partial x)$ implies $\psi_S \circ \varphi = \psi_S + 1$. Now consider $2m+1$ sectorial domains $S_j = V_{a_j,b_j}(\upsilon_1, \upsilon_2)$ $(1 \leq j \leq 2m+1)$ such that their union is $[|x| < \upsilon_1] \cap [0 < |y| < \upsilon_2]$. Let $\{\xi_j(y)\}_{j \in \{1,\dots,2m+1\}}$ be a partition of the unity associated to the covering $\cup \pi_y(S_j)$. We define

$$\psi_\varphi = \sum_{j=1}^{2m+1} \xi_j(y) \psi_{S_j}(x, y).$$

The following properties are straightforward:

- ψ_φ is a C^∞ function in $[|x| < \upsilon_1] \cap [0 < |y| < \upsilon_2]$.

- $\psi_\varphi \circ \varphi = \psi_\varphi + 1$.
- $\partial(\psi_\varphi - \psi)/\partial x_j = \sum_{j=1}^{2m+1} \xi_j(y) \partial(\psi_{S_j} - \psi)/\partial x_j = O(y^m)$.
- $\psi_\varphi - \psi = O(y^m)$.

The last two properties imply that ψ_φ, $\partial\psi_\varphi/\partial x_1$ and $\partial\psi_\varphi/\partial x_2$ admit a continuous extension to $U_{\upsilon_1,\upsilon_2}$. We look for a vector field

$$Z = a(x, y, \xi)\frac{\partial}{\partial x_1} + b(x, y, \xi)\frac{\partial}{\partial x_2} + \frac{\partial}{\partial \xi}$$

such that $Z((1-\xi)\psi + \xi\psi_\varphi) = 0$; we also require a and b to be continuous and to satisfy $a(0,0) = b(0,0) = 0$. If such a and b exist then $\exp(Z)(x, y, 1)$ conjugates $\exp(X(\varphi))$ and φ. We proceed like in the proof of theorem 6.1. For instance, we have

$$a = \frac{\begin{vmatrix} Re(y^m[\psi - \psi_\varphi]) & \partial H_1/\partial x_2 \\ Im(y^m[\psi - \psi_\varphi]) & \partial H_2/\partial x_2 \end{vmatrix}}{\begin{vmatrix} \partial H_1/\partial x_1 & \partial H_1/\partial x_2 \\ \partial H_2/\partial x_1 & \partial H_2/\partial x_2 \end{vmatrix}}$$

where $H = H_1 + iH_2 = (1-\xi)y^m\psi + \xi y^m \psi_\varphi$. The denominator is of the form $1/|u|^2 + O(y^{2m})$ whereas the numerator is a $O(y^{2m})$. As a consequence a is a $O(y^{2m})$ and then $a(x, 0) \equiv 0$. We can prove that b is continuous and it satisfies $b(x, 0) \equiv 0$ in an analogous way. The special mapping $\exp(Z)(x, y, 1)$ conjugates $\exp(X(\varphi))$ and φ; moreover $\exp(Z)(x, y, 1)$ is the identity by restriction to $y = 0$.

CHAPTER 9

Tangential Special Conjugations

9.1. The general plan

The remainder of the paper is devoted to prove the necessary condition in theorem 8.1 for $N > 0$. To conjugate φ_1 and φ_2 such that $SP(\varphi_1) = SP(\varphi_2)$ we consider a composition of special mappings

$$\sigma_2 \circ \sigma' \circ \sigma_1^{(-1)}$$

where σ' is a homeomorphism conjugating $Re(X(\varphi_1))$ and $Re(X(\varphi_2))$ and σ_j conjugates α_{φ_j} and φ_j for $j \in \{1, 2\}$. If the mapping σ_j is a germ of homeomorphism and $m = 0$ then $\varphi_{j,0}(1) \overset{ana}{\sim} \exp(X(\varphi_j)_{|y=0})$. That is not always possible since $\varphi_{j,0}(1)$ is not in general the exponential of a convergent vector field (or in other words $\varphi_{j,0}(1)$ is not always analytically trivial). This approach is hopeless if we do not enlarge the class of mappings we are considering.

We say that a mapping σ is *tangential special* (or tg-sp for shortness) if it satisfies that

- σ is a germ of homeomorphism defined in $(U_{\epsilon,\delta} \setminus [y = 0]) \cup \{(0, 0)\}$ for some $\epsilon, \delta > 0$.
- $y \circ \sigma = y$ and $\sigma_{|f/y^m=0} \equiv Id$.

Suppose $SP(\varphi_1) = SP(\varphi_2)$; we will prove the existence of a special analytic biholomorphism τ such that $\tau_{|y=0} \circ \varphi_{1,0}(1) = \varphi_{2,0}(1) \circ \tau_{|y=0}$. That will allow us to suppose that $\varphi_{1,0}(1)$ and $\varphi_{2,0}(1)$ coincide.

The diffeomorphisms α_{φ_j} and φ_j are conjugated by a tg-sp mapping σ_j. The mapping $\sigma = \sigma_2 \circ \sigma' \circ \sigma_1^{(-1)}$ is a tg-sp conjugation between φ_1 and φ_2. If $N = 1$ or $m > 0$ the conjugating mappings σ_j can be chosen to be defined in a neighborhood of $(0, 0)$. In other words, for $N = 1$ or $m > 0$ a (NSD) diffeomorphism is conjugated to its normal form by a special germ of homeomorphism. That implies theorem 8.1.

Suppose $N > 1$ and $m = 0$. Since $\varphi_{1,0}(1) = \varphi_{2,0}(1)$ then the mapping $\sigma'_{|y=0}$ can be chosen to be the identity map. We will provide a method to construct σ_j for $j \in \{1, 2\}$ such that σ can be extended to $y = 0$ as the identity map.

9.1.1. Preparation of φ_1 and φ_2. This subsection is of technical type; its purpose is showing that we can suppose $\varphi_{1,0}(1) = \varphi_{2,0}(1)$ when proving theorem 8.1. Moreover, in such a case $X(\varphi_1) = u_1 f\partial/\partial x$ and $X(\varphi_2) = u_2 f\partial/\partial x$ can be chosen such that $u_1 - u_2 \in (y)$.

PROPOSITION 9.1.1. *Let $\varphi_1, \varphi_2 \in \mathcal{D}_f$ such that $SP(\varphi_1) = SP(\varphi_2)$. Suppose $N > 0$ and $(N, m) \neq (1, 0)$. Then, there exists an analytic special germ of biholomorphism τ such that*

$$\tau_{|y=0}^{-1} \circ \varphi_{2,0}(1) \circ \tau_{|y=0} = \varphi_{1,0}(1).$$

PROOF. Suppose $N = 1$; that implies $f = y^m (x - g(y))^n$. There exists $h \in \mathrm{Diff}(\mathbb{C}, 0)$ conjugating $\varphi_{1,0}(1)$ and $\varphi_{2,0}(1)$. We can define

$$\tau = (h(x - g(y)) + g(y), y).$$

Suppose $N > 1$. Let $\varphi_j = \exp(\hat{u}_j f \partial/\partial x)$ and $X(\varphi_j) = u_j f \partial/\partial x$ for $j \in \{1, 2\}$. There exists $k \in \mathbb{N}$ such that

$$f(x, y^k) = y^{mk} (x - g_1(y))^{a_1} \ldots (x - g_N(y))^{a_N}.$$

We define

$$B = \sum_{j=1}^{N} \frac{y^{mk} (Res_{(x, y^{1/k}) \circ \varphi_1 \circ (x, y^k)} - Res_{(x, y^{1/k}) \circ \varphi_2 \circ (x, y^k)})(g_j(y), y)}{y^{mk} (x - g_j(y))}.$$

Since $SP(\varphi_1) = SP(\varphi_2)$ all the numerators in the previous expression belong to (y). Moreover $B(x, e^{(2\pi i)/k} y) \equiv B(x, y)$ and then the function $f(x, y) B(x, y^{1/k})$ is holomorphic in the neighborhood of the origin and it belongs to (y). We consider the unit $v_2 \in \mathbb{C}\{x, y\}$ satisfying

$$\frac{1}{v_2 f} = \frac{1}{u_1 f} - B(x, y^{1/k}).$$

By construction $Res_{X(\varphi_2)}(P) = Res_{v_2 f \partial/\partial x}(P)$ for $P \in [f/y^m = 0]$. Since no modification is required the special mapping ρ conjugating $v_2 f \partial/\partial x$ and $X(\varphi_2)$ and provided by theorem 6.1 is in fact analytic. Therefore, we can suppose that

$$\frac{1}{u_1(x, y)} - \frac{1}{u_2(x, y)} = f(x, y) B(x, y^{1/k})$$

up to replace φ_2 with $\rho^{(-1)} \circ \varphi_2 \circ \rho$. Thus $u_1(x, 0) \equiv u_2(x, 0)$ and then

$$(\hat{u}_1 - \hat{u}_2)(x, 0) = [(\hat{u}_1 - u_1) - (\hat{u}_2 - u_2)](x, 0) \in (f(x, 0)^2).$$

For $m > 0$ we are done, the identity conjugates $\varphi_{1,0}(1)$ and $\varphi_{2,0}(1)$. Otherwise $(f^2(x, 0)) = (x^{2\tilde{\nu}(X(\varphi_1))})$. As a consequence there exists h in $\mathrm{Diff}(\mathbb{C}, 0)$ such that $h \circ \varphi_{1,0}(1) = \varphi_{2,0}(1) \circ h$ and $h(x) - x \in (x^{2\tilde{\nu}(X(\varphi_1))+1})$. We define

$$H(x, \lambda_1, \ldots, \lambda_N) = \sum_{j=1}^{N} (h(\lambda_j) - \lambda_j) \prod_{k \in \{1, \ldots, N\} \setminus \{j\}} \frac{x - \lambda_k}{\lambda_j - \lambda_k}.$$

We can express it in the form

$$H = \frac{H'(x, \lambda_1, \ldots, \lambda_N)}{\prod_{1 \leq j < k \leq N} (\lambda_j - \lambda_k)^2}$$

where $H' \in \mathbb{C}[x]\{\lambda_1, \ldots, \lambda_N\}$ and the degree of H' as a polynomial in x is at most $N - 1$. It is clear that $(\lambda_j - \lambda_k) | H'$ for all $j \neq k$. Since $H(x, \lambda_1, \ldots \lambda_N) = H(x, \lambda_{b(1)}, \ldots, \lambda_{b(n)})$ for every $b \in S_n$ then the same property holds when we replace H with H'. As a consequence $(\lambda_j - \lambda_k)^2 | H'$ for $j \neq k$. We deduce that H belongs to $\mathbb{C}[x]\{\lambda_1, \ldots, \lambda_N\}$. We can express H in the form

$$H = H_0(\lambda_1, \ldots, \lambda_N) + \ldots + H_{N-1}(\lambda_1, \ldots, \lambda_N) x^{N-1}.$$

We have $\nu(H_j) \geq \nu(h(x) - x) - j$ for all $0 \leq j < N$. Since $\tilde{\nu}(X(\varphi_1)) > N$ then $H_j \in (\lambda_1, \ldots, \lambda_N)$ for all $0 \leq j < N$. We define

$$\tau = (h(x) - H(x, g_1(y^{1/k}), \ldots, g_N(y^{1/k})), y).$$

By construction we obtain $\tau_{|y=0} \equiv h$ whereas τ is the identity over the fixed points. Moreover $H(x, g_1(y^{1/k}), \ldots, g_N(y^{1/k})) \in \mathbb{C}\{x, y\} \cap (y)$ since H is symmetric in $(\lambda_1, \ldots, \lambda_N)$ and $H \in (\lambda_1, \ldots, \lambda_N)$. $\square$

LEMMA 9.1.1. *Let φ_1, φ_2 be elements of $\mathcal{D}_f$ such that $SP(\varphi_1) = SP(\varphi_2)$ and $\varphi_{1,0}(1) \equiv \varphi_{2,0}(1)$. Then, we can choose $X(\varphi_j) = u_j f\partial/\partial x$ for j in $\{1, 2\}$ such that $u_1 - u_2 \in (y)$.*

PROOF. We denote $\varphi_j = \exp(\hat{u}_j f\partial/\partial x)$ for $j \in \{1, 2\}$. By hypothesis we have $\hat{u}_1 - \hat{u}_2 \in (y)$. We choose $X(\varphi_j) = v_j f\partial/\partial x$ for $j \in \{1, 2\}$. Since $\hat{u}_j - v_j \in (f^2)$ for $1 \leq j \leq 2$ we define $u_1 = v_1$ and

$$u_2(x, y) = v_2(x, y) + h^2(x, y)\left[\frac{\hat{u}_2 - v_2}{h^2} - \frac{\hat{u}_1 - v_1}{h^2}\right](x, 0)$$

where $h = f/y^m$. It is clear that $u_2 - v_2 \in (f^2)$ and then we obtain $\hat{u}_2 - u_2 \in (f^2)$. Moreover $u_2(x, 0) \equiv u_1(x, 0)$ as we wanted to prove. $\square$

9.2. Shaping the domains

9.2.1. Prerequisites. We will construct a tg-sp conjugation between a diffeomorphism φ and its normal form $X(\varphi)$. At first we will solve the problem in the neighborhood of $y = y_0$ for $y_0 \in B(0, \delta) \setminus \{0\}$; then we will use a partition of the unity to obtain the tg-sp conjugation.

Let X be a (NSD) vector field defined in $U_{\epsilon', \delta}$. Fix $0 < \mu < 1$. We can suppose that $0 < \epsilon' < 1$ and $\delta > 0$ satisfy that whether

$$\{\alpha_\varphi^{(0)}(P), \ldots, \alpha_\varphi^{(j)}(P)\} \subset U_{\epsilon', \delta}$$

then $|\Delta_j(P)| = |\psi_{X(\varphi)}(\varphi^{(j)}(P)) - (\psi_{X(\varphi)}(P) + j)| \leq \mu$. We consider $0 < \epsilon_1 < \epsilon'$ such that $\exp([-3, 3]X(\varphi))(U_{\epsilon_1}) \subset U_{\epsilon'}$.

We fix a number $M > 32$ from now on. We remind the reader that N_T is equal to $2(\tilde{\nu}(X) - 1)$. Let $\epsilon < \epsilon' < 1$; consider a section $T_X^{\epsilon, j}(r, \theta)$. We define

$$Tr^{\epsilon, j}(r, \theta, H) = \exp([-H, H]ir^m X(e^{im\theta}))(T_X^{\epsilon, j}(r, \theta)).$$

There exists $0 < \epsilon_0 < \epsilon_1$ such that for $\kappa = 6(2M + 1)N_T + 3$ the transversal $Tr^{\epsilon, j}(r, \theta, \kappa)$ is well-defined and it is contained in $U_{\epsilon'}$ for all $1 \leq j \leq N_T$ and all $\epsilon \leq \epsilon_0$. Moreover, we choose $\epsilon_0 > 0$ small enough such that $Tr^{\epsilon_0, j}(0, \theta, \kappa)$ is contained in

$$(\alpha_{\xi(X(e^{im\theta}))}, \omega_{\xi(X(e^{im\theta}))})^{-1}_{|x| < \epsilon'}((0, 0), (0, 0))$$

for all $1 \leq j \leq N_T$ and $\theta \in \mathbb{R}$.

9.2.2. Eared domains. Fix $y_0 \in B(0, \delta) \setminus \{0\}$. The construction of the tg-sp conjugation between α_φ and φ relies in dynamical study of $Re(X(\varphi))$. The construction is simpler if $y_0 \notin UN_{X(\varphi)}^{\epsilon}$ since the vector field $\xi(X(\varphi), y, \epsilon)$ is locally trivial in the neighborhood of y_0. Otherwise, we will add some "ears" in order to break the bi-tangent cords.

Let $\epsilon \leq \epsilon_0$; there exists $a > 0$ and $b > 0$ such that

$$\exp(\{-a, b\}X(e^{i\theta m}))(Tr^{\epsilon, j}(0, \theta, \kappa)) \subset U_\epsilon$$

for all $1 \le j \le N_T$ and $\theta \in \mathbb{R}$. Let $0 \le D \le \kappa$; we define

$$O_j^D(r,\theta) = \exp([-a,b]X(e^{i\theta m}))(Tr^{\epsilon,j}(r,\theta,D)) \setminus U_\epsilon.$$

By definition the set $O_j^D(r,\theta)$ is an "ear" of width D over the tangent point $T^{\epsilon,j}_{X(e^{im\theta})}(r,\theta)$. The set $Tr^{\epsilon,j}(r,\theta,D)$ has exactly one end which does not belong to U_ϵ; we denote it by $v_j^D(r,\theta)$. It is the vertex of the ear. For $K=(K_1,\ldots,K_{N_T}) \in [0,\kappa)^{N_T}$ we define $U_\epsilon(K)$ such that $U_\epsilon(K) \cap [(r,\theta)=(r_0,\theta_0)]$ is the interior of

$$[|x| \le \epsilon] \cup O_1^{K_1}(r_0,\theta_0) \cup \ldots \cup O_{N_T}^{K_{N_T}}(r_0,\theta_0).$$

We define $U_{\epsilon,\delta}(K) = U_\epsilon(K) \cap [y \in B(0,\delta)]$. The set U_ϵ is a domain with zero width ears. The topological behavior of $Re(X)$ in domains of type $U_\epsilon(K)$ and U_ϵ is totally analogous. Let

$$U'_\epsilon(K) = \overline{U_\epsilon(K)} \setminus \cup_{(j,r,\theta)\in\{1,\ldots,N_T\}\times[0,\delta)\times\mathbb{R}}\{v_j^{K_j}(r,\theta)\};$$

we define the positive critical trajectory passing through $v_k^{K_k}(r,\theta)$ as

$$\overline{\Gamma^{U'_\epsilon(K)\cup\{v_k^{K_k}(r,\theta)\}}_{\xi(X(e^{im\theta})),+}[v_k^{K_k}(r,\theta)]}.$$

Analogously we define negative critical trajectories. The critical tangent cords are still the critical trajectories not containing singular points and the bi-tangent cords are the critical trajectories containing two vertexes. The bi-tangent cords can be removed by adding ears.

Lemma 9.2.1. *Let X be a (NSD) vector field. Fix r_0 in $[0,\delta)$ and θ_0 in $\mathbb{R}$. For all $\upsilon > 0$ there exists $\eta = (\eta_1,\ldots,\eta_{N_T}) \in [0,\upsilon)^{N_T}$ such that $Re(X(e^{im\theta_0}))$ does not have bi-tangent cords in $U_\epsilon(\eta) \cap [y = r_0 e^{i\theta_0}]$.*

Proof. Let $\zeta \in [0,\kappa)^{N_T}$. We define $H(\zeta) \subset \{1,\ldots,N_T\}^2$ as the set such that $(j,k) \in H(\zeta)$ if $j \neq k$ and there exists a bi-tangent cord joining $v_j^{\zeta_j}(r_0,\theta_0)$ and $v_k^{\zeta_k}(r_0,\theta_0)$ in $U_\epsilon(\zeta)$. It is enough to prove that for all $\upsilon > 0$ there exists $\xi \in [0,\upsilon)^{N_T}$ such that $\sharp H(\zeta+\xi) \le \max(\sharp H(\zeta)-1,0)$; this property implies the lemma by an induction process.

Consider $(j,k) \in H(\zeta)$. We define $\xi_l = 0$ for $l \neq j$. We claim that (j,k) does not belong to $H(\zeta+\xi)$ if $0 < \xi_j << 1$; otherwise there would be a trajectory of $Re(X(e^{im\theta_0}))$ cutting twice $Tr^{\epsilon,j}(r_0,\theta_0,\kappa)$. Moreover, we have that $(j',k') \notin H(\zeta)$ implies $(j',k') \notin H(\zeta+\xi)$ for $\xi_j << 1$ by continuity of the flow. Now, we just choose $\xi_j < \upsilon$ small enough. $\square$

Remark 9.2.1. *From now on we will always consider sets $U_\epsilon(\eta)$ such that $0 \le \eta_j < 1$ for all $1 \le j \le N_T$.*

9.2.3. Changing the boundary of $U_\epsilon(\eta)$. Consider two consecutive sections $T_X^{\epsilon,j}(r,\theta)$ and $T_X^{\epsilon,j+1}(r,\theta)$. We denote by $S_j(r,\theta)$ the closed circular segment between $T_X^{\epsilon,j}(r,\theta)$ and $T_X^{\epsilon,j+1}(r,\theta)$. We define $c_j = 1$ if $Re(X)$ points towards the interior of U_ϵ in $S_j(r,\theta)$; otherwise we have $c_j = -1$. Let ψ_j/y^m be an integral of the time form of X defined in a neighborhood of $S_j(r,\theta)$. We define $h_k(r,\theta) = \psi_j(T_X^{\epsilon,k}(r,\theta))e^{-im\theta}$. We consider the set $Tr_S^{\epsilon,j}(r,\theta,0,0)$ whose image by $\psi_j e^{-im\theta}$ is

$$c_j K + Re\frac{h_j + h_{j+1}}{2} + i[\min(Imh_j, Imh_{j+1}), \max(Imh_j, Imh_{j+1})].$$

We define $Tr_S^{\epsilon,j}(r,\theta,a,b) = \exp(i[-a,b]X)(Tr_S^{\epsilon,j}(r,\theta,0,0))$. We have

LEMMA 9.2.2. *There exists $K > 0$ such that $Tr_S^{\epsilon,j}(r,\theta,\kappa,\kappa)$ is contained in U_ϵ for all $1 \leq j \leq N_T$ and $(r,\theta) \in [0,\delta) \times \mathbb{R}$.*

PROOF. Since $T_X^{\epsilon,k}(r,\theta) = T_X^{\epsilon,k}(r,\theta + \pi N_T)$ we can suppose that θ belongs to $[0,\pi N_T]$. As a consequence

$$K = 1 + 2 \sup_{(j,r,\theta)\in\{1,\ldots,N_T\}\times[0,\delta)\times[0,\pi N_T],\ P\in S_j(r,\theta)} |Re[\psi_j(P)e^{-im\theta} - h_j(r,\theta)]|$$

satisfies $K < \infty$. The choice of K guarantees that

$$K > c_j[Re(\psi_j e^{-im\theta})(P) - Re((h_j(r,\theta) + h_{j+1}(r,\theta))/2)] \tag{9.1}$$

for all $(j,r,\theta) \in \{1,\ldots,N_T\} \times [0,\delta) \times \mathbb{R}$ and every $P \in S_j(r,\theta)$. The situation for $c_j = 1$ is represented in picture 1. Proposition 3.2.2 implies that $\Gamma_{\xi(X(e^{im\theta}))}^{|x|<\epsilon'}[P](c_j w)$

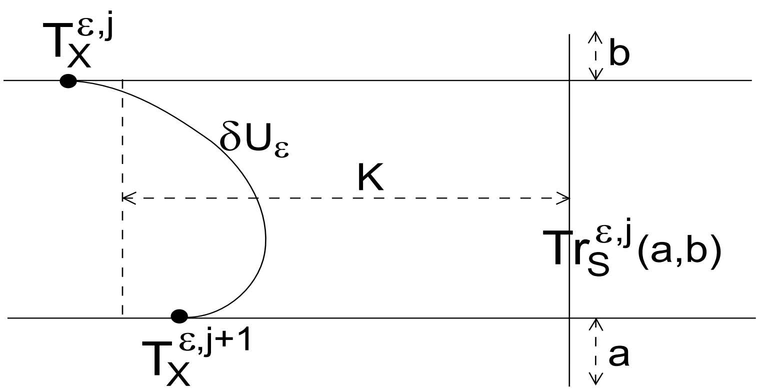

FIGURE 1. Picture of $Tr_S^{\epsilon,j}(r,\theta,a,b)$ in coordinates $\psi_j e^{-im\theta}$

belongs to U_ϵ for all $P \in S_j(0,\theta)$ and all (j,θ,w) in $\{1,\ldots,N_T\} \times \mathbb{R} \times \mathbb{R}^+$. Therefore, equation 9.1 implies that $Tr_S^{\epsilon,j}(r,\theta,0,0)$ is contained in U_ϵ for all (j,r,θ) in $\{1,\ldots,N_T\} \times [0,\delta) \times \mathbb{R}$ and $\delta > 0$ small enough.

We have that $Tr_S^{\epsilon,j}(0,\theta,\kappa,\kappa) \setminus Tr_S^{\epsilon,j}(0,\theta,0,0)$ is contained in

$$(\alpha_{\xi(X(e^{im\theta}))}, \omega_{\xi(X(e^{im\theta}))})_{|x|<\epsilon}^{-1}((0,0),(0,0)).$$

Hence $Tr_S^{\epsilon,j}(r,\theta,\kappa,\kappa) \subset U_\epsilon$ for all $(j,r,\theta) \in \{1,\ldots,N_T\} \times [0,\delta) \times \mathbb{R}$ and $\delta > 0$ small enough. $\square$

Fix $y_0 \in B(0,\delta) \setminus \{0\}$. We can just change $U_\epsilon(\eta)$ by a domain with very similar properties. We consider $L(s) = \cup_{1\leq j\leq N_T} Tr_S^{\epsilon,j}(s,a_j,b_j)$ where $(a_j,b_j) = (-\eta_j,-\eta_{j+1})$ if $Im(h_j(y_0) < Im(h_{j+1}(y_0))$; otherwise we have $(a_j,b_j) = (-\eta_{j+1},-\eta_j)$. The set $L(s)$ is not connected for s in a neighborhood of y_0; indeed $L(s)$ has N_T connected components. Anyway, for $1 \leq j \leq N_T$ there exists $c_j(s) > 0$ and $d_j(s) > 0$ such that $\exp(tX)(v_j(s))$ does not belong to $L(s)$ for $t \in (-c_j(s), d_j(s))$ but it does for $t \in \{-c_j(s), d_j(s)\}$. We consider the domain $W_\epsilon(\eta)$ whose boundary is equal to

$$\cup_{s\in V}[L(s) \cup_{1\leq j\leq N_T} \exp([-c_j(s), d_j(s)]X)(v_j(s))]$$

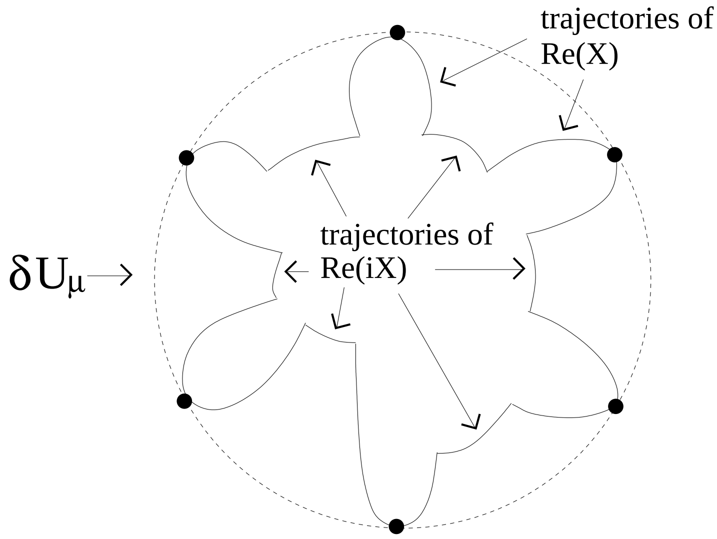

FIGURE 2. Picture of a domain $W_\mu(0)$

for some neighborhood V of y_0. The domain $W_\epsilon(\eta)$ has a very simple boundary; it is composed by a union of trajectories of $Re(X)$ and $Re(iX)$. We define

$$I_\epsilon^R(\eta, s) = [I_\epsilon(\eta) \cap [y=s]] \setminus \cup_{1 \leq j \leq N_T} \Gamma_{\xi(X)}^{\overline{U_\epsilon(\eta)}}[v_j^{\eta_j}(s)]$$

for $I \in \{U, W\}$. The mapping $(\alpha_{\xi(X,s)}, \omega_{\xi(X,s)})_{I_\epsilon(\eta)}$ is constant in the connected components of $I_\epsilon^R(\eta, s)$ for $I \in \{U, W\}$. We call these components regions as usual. There is a bijection between the regions in $W_\epsilon^R(\eta, s)$ and the regions in $U_\epsilon^R(\eta, s)$. Moreover, for every region $Z_W(s)$ in $W_\epsilon^R(\eta, s)$ there exists a unique region $Z_U(s)$ in $U_\epsilon^R(\eta, s)$ such that $Z_W(s) \cap Z_U(s) \neq \emptyset$ for s in a neighborhood of 0. Indeed those regions satisfy $Z_W(s) \subset Z_U(s)$ for all s in a neighborhood of 0. As a consequence the dynamical properties of $U_\epsilon(\eta)$ and $W_\epsilon(\eta)$ are the same.

REMARK 9.2.2. *Clearly* $Tr_S^{\epsilon,j}(r, \theta, \kappa, \kappa) \subset \overline{W_\epsilon(\eta)}$ *for all* $1 \leq j \leq N_T$ *and* $(r, \theta) \in [0, \delta) \times \mathbb{R}$.

9.3. Base transversals

For constructing a special conjugation between α_φ and φ in a neighborhood of $y = y_0$ there are two basic steps.

For the first step we choose a trajectory

$$Tr(s) \subset \overline{W_\epsilon(\eta + \kappa - 3)} \cap [y = s]$$

of $Re(iX)$ and we construct a special conjugation σ_{Tr} between α_φ and φ; it is defined in

$$\cup_{s \in V} D_{Tr}(s) = \cup_{s \in V} \exp([-1,1]X) \left(\Gamma_{\xi(X)}^{\overline{W_\epsilon(\eta+\kappa-3)}}[Tr(s)] \cap \overline{W_\epsilon(\eta)} \right)$$

for some neighborhood V of $y = y_0$. The second step is a process of interpolation for conjugations obtained by considering different base transversals. In this section we focus on the first step.

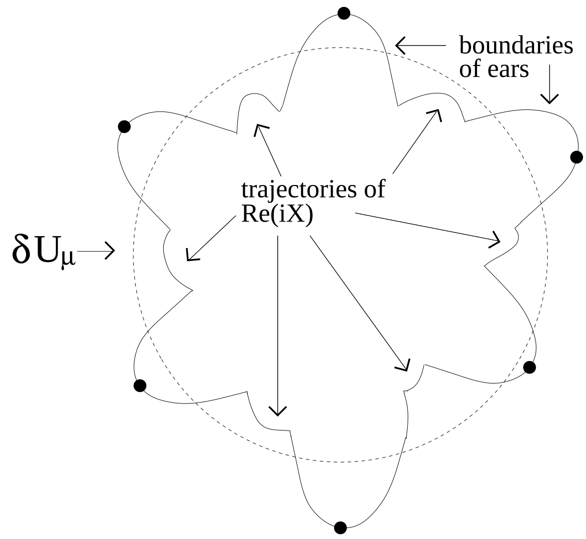

FIGURE 3. Picture of a domain $W_\mu(C_1, \ldots, C_{t_a})$

The set $\psi_{X(\varphi)}(Tr(s))$ is of the form $z \in c(s) + i[d(s), e(s)]$; thus $\psi_{X(\varphi)}(D_{Tr}(s))$ can be expressed as

$$[Imz \in [d(s), e(s)]] \cap [Rez \in [c(s) - q_1(Imz, s), c(s) + q_2(Imz, s)]]$$

where q_j is upper semi-continuous and defined in $\cup_{s \in V}[d(s), e(s)] \times \{s\}$ for $j \in \{1, 2\}$. We will define σ_{Tr} in

$$\cup_{s \in V}([Im(\psi_{X(\varphi)}) \in [d(s), e(s)]] \cap [Re(\psi_{X(\varphi)}) \in (c(s) - 1/3, c(s) + 4/3)])$$

and then we will extend to $D_{Tr}(s)$ by using $\sigma_{Tr} \circ \alpha_\varphi = \varphi \circ \sigma_{Tr}$. In order to assure that such a extension is well-defined it is enough to prove the following lemma:

LEMMA 9.3.1. *Let* $(x_0, s), (x_1, s) \in Tr(s)$. *Suppose*

$$t_j - c(s) \in [-q_1(Img(\psi_{X(\varphi)}(x_j, s)), s), q_2(Img(\psi_{X(\varphi)}(x_j, s)), s)]$$

for $j \in \{1, 2\}$. *Then* $\exp(t_0 X)(x_0, s) = \exp(t_1 X)(x_1, s)$ *implies* $x_0 = x_1$ *and* $t_0 = t_1$.

PROOF. If $Img(\psi_{X(\varphi)}(x_0, s)) \neq Img(\psi_{X(\varphi)}(x_1, s))$ then the trajectory of $Re(X)$ passing trough $\exp(t_0 X)(x_0, s)$ cuts twice $Tr(s)$. That is impossible by the Rolle property. Hence $x_0 = x_1$; moreover $t_0 = t_1$ since otherwise there is a cycle and that violates the Rolle property. $\square$

Next we explain how to construct σ_{Tr}. Last lemma justifies the use of $\psi_{X(\varphi)}$ as a coordinate in $D_{Tr}(s)$. Therefore, we consider the system of coordinates given by $(z, s) = (\psi_{X(\varphi)}(x, s), s)$. We want $(\sigma_{Tr})_{|Tr(s)} \equiv Id$, i.e. $\sigma(z, s) = (z, s)$ for $z \in c(s) + i[d(s), e(s)]$. That choice implies

$$\sigma(c(s) + i\xi + 1, s) = (c(s) + i\xi + 1 + \Delta \circ (\psi_{X(\varphi)}, s)^{(-1)}(c(s) + i\xi, s), s)$$

for all $\xi \in [d(s), e(s)]$. We denote $A = (\psi_{X(\varphi)}, y) \circ \varphi \circ \alpha_\varphi^{(-1)} \circ (\psi_{X(\varphi)}, y)^{(-1)}$. Since $\epsilon \leq \epsilon_0$ (see subsection 9.2.1) then $\exp([-2, 2]X)(\overline{W_\epsilon(\eta)}) \subset U_{\epsilon'}$. We deduce that

$A(.,s)$ is defined in

$$z \in c(s) + [-1/3, 4/3] + i[d(s), e(s)].$$

Consider the partition $I_1 = (-1/3, 2/3), I_2 = (1/3, 4/3)$ of $(-1/3, 4/3)$. Let h_1, h_2 be a partition of the unity associated to the covering $I_1 \cup I_2$. We define

$$B(c + i\xi, s) = h_1(c - c(s))(c + i\xi, s) + h_2(c - c(s))A(c + i\xi, s)$$

for $c + i\xi \in c(s) + (-1/3, 4/3) + i[d(s), e(s)]$. By choice B is the identity in the neighborhood of $Tr(s)$ whereas $B = A$ in the neighborhood of $\sigma_{Tr}(Tr(s))$. We define $\sigma_{Tr} = (\psi_{X(\varphi)}, y)^{(-1)} \circ B \circ (\psi_{X(\varphi)}, y)$. We obtain

$$|\psi_{X(\varphi)} \circ \sigma_{Tr} - \psi_{X(\varphi)}| = |z \circ B \circ (\psi_{X(\varphi)}, y) - \psi_{X(\varphi)}| \leq \mu.$$

The inequality is a consequence of $|\Delta(x, y)| \leq \mu$ in $U_{\epsilon'}$. The conjugation σ_{Tr} can be extended to $D_{Tr}(s)$ by applying the formula $\sigma \circ \alpha_\varphi = \varphi \circ \sigma$. We define $D\sigma_{Tr}(x_0, s)$ the jacobian matrix of $(\sigma_{Tr})_{|y=s}$ at $x = x_0$. Then $D\sigma_{Tr}(x_0, s)$ is a 2×2 real-valued matrix.

PROPOSITION 9.3.1. *For $\mu < 1$ there exists a universal $\mu_{uv} > 0$ such that*
- *σ_{Tr} is C^∞ in the interior of $\cup_{s \in V} D_{Tr}(s)$.*
- *$|\psi_{X(\varphi)} \circ \sigma_{Tr} - \psi_{X(\varphi)}| \leq 2\mu$ in $\cup_{s \in V} D_{Tr}(s)$.*
- *$||D((\psi_{X(\varphi)}, y) \circ \sigma_{Tr} \circ (\psi_{X(\varphi)}, y)^{(-1)}) - Id|| \leq \mu_{uv}\mu$.*

The last inequality holds in $\cup_{y \in V}[\psi_{X(\varphi)}(D_{Tr}(y))] \times \{y\}$. The latter properties express that $\sigma_{Tr} \sim Id$ and $D\sigma_{Tr} \sim Id$.

PROOF. The mapping σ_{Tr} is C^∞ by construction. Suppose that $\alpha_\varphi^{(j)}(x_0, s)$ belongs to $\exp([0,1]X)(Tr(s))$ for some $j \in \mathbb{Z}$. We have

$$\sigma_{Tr}(x_0, s) = \varphi^{(-j)} \circ \sigma_{Tr} \circ (\alpha_\varphi^{(j)}(x_0, s)).$$

Since $|\psi_{X(\varphi)} \circ \sigma_{Tr} - \psi_{X(\varphi)}| < 1$ in $\exp([0,1]X)(Tr(s))$ then the point $\sigma_{Tr} \circ (\alpha_\varphi^{(j)}(x_0, s))$ belongs to $\exp([-2,2]X)(\overline{W_\epsilon(\eta)}) \cup W_\epsilon(\eta + \kappa - 2) \subset U_{\epsilon'}$. As a consequence we obtain

$$|\psi_{X(\varphi)}(\sigma_{Tr}(x_0, s)) - \psi_{X(\varphi)}(x_0, s)| \leq \mu + |\Delta_{-j} \circ \sigma_{Tr} \circ \alpha_\varphi^{(j)}(x_0, s)| \leq 2\mu.$$

Let $h(x, y) = (\psi_{X(\varphi)}(x, y), y)$; we have

$$\frac{\partial(\Delta_j \circ h^{(-1)})}{\partial \overline{z}}(z_0, s) = 0$$

and

$$\frac{\partial(\Delta_j \circ h^{(-1)})}{\partial z}(z_0, s) = \left| \frac{1}{2\pi i} \int_{|z - z_0| = 1} \frac{\Delta_j \circ h^{(-1)}}{(z - z_0)^2} dz \right| \leq \mu.$$

for $(z_0, s) \in D_{Tr}(s)$ and $j \in \mathbb{Z}$. For the second inequality we need Δ_j defined in $\exp(B(0,1)X)(z_0, s)$. Such a property can be fulfilled by requiring

$$\exp([-3,3]X)(\overline{W_\epsilon(\eta)}) \cup W_\epsilon(\eta + \kappa - 1) \subset U_{\epsilon'}.$$

That is the case, since $W_\epsilon(\eta + \kappa - 1) \subset W_\epsilon(\kappa) \subset U_{\epsilon'}$ and $\exp([-3,3]X)(U_\epsilon) \subset U_{\epsilon'}$ (see subsection 9.2.1). By making $j = 1$ we deduce that $||DA - Id|| \leq \mu$. As a consequence we obtain

$$||DB - Id|| \leq ||DA - Id|| + \mu \sup_{\upsilon \in \mathbb{R}} \left| \frac{\partial h_2}{\partial \upsilon}(\upsilon) \right| \leq \mu\mu_1$$

for $\mu_1 = 1 + \sup_{v\in\mathbb{R}} |\partial(h_2)/\partial v|$ and $(z,s) \in h(\exp([0,1]X(\varphi))(Tr(s)))$. If $(z+j,s)$ belongs to the latter domain then

$$B(z,s) = h \circ \varphi^{(-j)} \circ h^{(-1)} \circ B(z+j,s).$$

By simplifying we obtain

$$z \circ B(z,s) - z = (z \circ B(z+j,s) - (z+j)) + \Delta_{-j} \circ h^{(-1)} \circ B(z+j,s).$$

That leads us to

$$||DB - Id||(z,s) \leq \mu\mu_1 + ||D(\Delta_{-j} \circ h^{(-1)} \circ B(z+j,s))||.$$

We develop the previous expression to obtain

$$||DB - Id||(z,s) \leq \mu\mu_1 + ||D(\Delta_{-j} \circ h^{(-1)})||(1 + ||DB(z+j,s)) - Id||);$$

we can still simplify to have

$$||DB - Id||(z,s) \leq \mu\mu_1 + \mu(1+\mu_1\mu) \leq \mu_{uv}\mu$$

for $(z,s) \in \psi_{X(\varphi)}(D_{Tr}(s)) \times \{s\}$ and $\mu_{uv} = 1 + 2\mu_1$. Therefore

$$||D((\psi_{X(\varphi)}, y) \circ \sigma_{Tr} \circ (\psi_{X(\varphi)}, y)^{(-1)})(z,y) - Id|| \leq \mu_{uv}\mu$$

for $(z,y) \in \psi_{X(\varphi)}(D_{Tr}(y)) \times \{y\}$. □

9.4. The M-interpolation process

Since a single transversal can not intersect all the trajectories of $Re(X)$ then somehow we have to interpolate conjugations obtained by taking different transversals. Throughout this section we consider strips $\cup_{s\in V} B_\zeta(s)$ such that

$$\psi_{X(\varphi)}(B_\zeta(s)) = [z \in [a_\leftarrow(s) - \zeta, a_\rightarrow(s) + \zeta] + i[c_\downarrow(s), c_\uparrow(s)]]$$

where $c_\uparrow - c_\downarrow \equiv M$. The functions $a_\leftarrow$, $a_\rightarrow$, $c_\downarrow$ and $c_\uparrow$ are continuous in V. These functions are real-valued but we allow $a_\leftarrow \equiv -\infty$ and $a_\rightarrow \equiv \infty$. We denote the curve $B_\zeta(s) \cap [Img(\psi_{X(\varphi)}) = c_j(s)]$ by $\gamma_j^\zeta(s)$ for $j \in \{\uparrow,\downarrow\}$. Let $\sigma_\downarrow$ and $\sigma_\uparrow$ be special mappings defined in the neighborhood of $\cup_{s\in V} B_1(s)$ and conjugating α_φ and φ. Let $h = (\psi_{X(\varphi)}(x,y), y)$; suppose that the inequalities $|z \circ h \circ \sigma_j - z \circ h| \leq 2\mu$ and

$$||D(h \circ \sigma_j \circ h^{(-1)}) - Id||(h(x,s)) \leq \mu^j \mu$$

are fulfilled in the neighborhood of $\cup_{s\in V} B_1(s)$ for some $\mu^j > 0$ and every $j \in \{\uparrow,\downarrow\}$. Let g be a mapping defined in the neighborhood of a curve γ; we denote by (g,γ) the germ of g in the neighborhood of γ. We want to prove

Proposition 9.4.1. *For some $C(\mu^\uparrow,\mu^\downarrow) > 0$ and all $0 < \mu < C(\mu^\uparrow,\mu^\downarrow)$ there exists a C^∞ special diffeomorphism $\sigma_\updownarrow$ defined in $\cup_{s\in V} B_0(s)$ such that we have $\sigma_\updownarrow \circ \alpha_\varphi = \varphi \circ \sigma_\updownarrow$ and $(\sigma_\updownarrow, \gamma_j^0(s)) = (\sigma_j, \gamma_j^0(s))$ for $(s,j) \in V \times \{\uparrow,\downarrow\}$. Moreover, we obtain*

- $|\psi_{X(\varphi)} \circ \sigma_\updownarrow - \psi_{X(\varphi)}|(x,y) \leq 2\mu$
- $||D((\psi_{X(\varphi)}, y) \circ \sigma_\updownarrow \circ (\psi_{X(\varphi)}, y)^{(-1)}) - Id||(x,y) \leq \mu^\updownarrow \mu$

for all $(x,y) \in \cup_{s\in V} B_0(s)$. Moreover $\mu^\updownarrow$ depends only on $\mu^\uparrow$ and $\mu^\downarrow$.

Let $h = (\psi_{X(\varphi)}(x,y), y)$; we define $A_j^\zeta(s) = \sigma_j(\gamma_j^\zeta(s))$. Let $\mu > 0$ small enough; since $||D(h \circ \sigma_j \circ h^{(-1)}) - Id|| \leq \mu^j \mu$ we have that $h(A_j^\zeta(s))$ is parameterized by $Re(z)$ for (j, ζ, s) in the set $\{\uparrow, \downarrow\} \times [0,1] \times V$. We obtain that $Re(z \circ h(A_j^1(s)))$ contains $[a_\leftarrow(s) - 1/2, a_\rightarrow(s) + 1/2]$ for $j \in \{\uparrow, \downarrow\}$ and $s \in V$ by considering $\mu < 1/4$. We denote

$$\tau_j(s) = A_j^1(s) \cap [Re(\psi_{X(\varphi)}) \in [a_\leftarrow(s) - 1/2, a_\rightarrow(s) + 1/2]]).$$

Let $u(\leftarrow) = -1$, $u(\rightarrow) = 1$, $v(\uparrow) = 1$ and $v(\downarrow) = -1$; we define

$$P_{j,k}(s) = A_j^1(s) \cap [Re(\psi_{X(\varphi)}) = a_k(s) + u(k)/2]$$

for $(j,k) \in \{\uparrow, \downarrow\} \times \{\leftarrow, \rightarrow\}$. Consider the curve $\tau_k(s)$ such that

$$\psi_X(\tau_k(s)) = a_k(s) + u(k)/2 + i[Img(\psi_X(P_{\downarrow,k}(s))), Img(\psi_X(P_{\uparrow,k}(s)))]$$

for $k \in \{\leftarrow, \rightarrow\}$ and $s \in V$. We define

$$\tau(s) = \tau_\leftarrow(s) \cup \tau_\uparrow(s) \cup \tau_\rightarrow(s) \cup \tau_\downarrow(s);$$

it is a Jordan curve. We denote by $D(s)$ the closure of the bounded component of $[y = s] \setminus \tau(s)$. We define

$$B_j(s) = B_1(s) \cap [Img(\psi_{X(\varphi)}) \in [c_\downarrow + (1 + v(j))M/8, c_\uparrow - (1 - v(j))M/8]]$$

for $j \in \{\uparrow, \downarrow\}$.

Lemma 9.4.1. *We have* $D(s) \subset \sigma_\downarrow(B_\downarrow(s)) \cup \sigma_\uparrow(B_\uparrow(s))$ *for all* $s \in V$.

Proof. Let $j \in \{\uparrow, \downarrow\}$; we define

$$\tau_j''(s) = \sigma(B_1(s) \cap [Im(\psi_{X(\varphi)}) = c_j - v(j)3M/4]).$$

We consider

$$\tau_j'(s) = \tau_j''(s) \cap [Re(\psi_{X(\varphi)}) \in [a_\leftarrow(s) - 1/2, a_\rightarrow(s) + 1/2]]$$

for $j \in \{\uparrow, \downarrow\}$. As $\tau_\uparrow(s)$ and $\tau_\downarrow(s)$ the curve $\psi_{X(\varphi)}(\tau_j'(s))$ is parameterized by $Re z \in [a_\leftarrow - 1/2, a_\rightarrow + 1/2]$. We denote by $D_j(s)$ the closure of the only bounded connected component in

$$[y = s] \setminus (\tau_\leftarrow(s) \cup \tau_\rightarrow(s) \cup \tau_j(s) \cup \tau_j'(s))$$

for $j \in \{\uparrow, \downarrow\}$.

We claim that $D_j(s) \subset \sigma_j(B_j(s))$ for $j \in \{\uparrow, \downarrow\}$. That is a consequence of

$$\partial D_j(s) \subset \sigma_j(B_j(s))$$

which we obtain by construction since $\sigma_j \sim Id$ and $D\sigma_j \sim Id$. As a consequence it is enough to prove that $D(s) = D_\downarrow(s) \cup D_\uparrow(s)$ for $s \in V$. Then the inequality $|\psi_{X(\varphi)} \circ \sigma_j - \psi_{X(\varphi)}| < 1/2$ for $j \in \{\uparrow, \downarrow\}$ implies

$$\inf Im[\psi_{X(\varphi)}(\tau_\downarrow'(s))] \geq c_\downarrow(s) + 3M/4 - 1/2$$

and

$$\sup Im[\psi_{X(\varphi)}(\tau_\uparrow'(s))] \leq c_\downarrow(s) + M/4 + 1/2.$$

Since $M > 32$ by choice then $3M/4 - 1/2 > M/4 + 1/2$. That implies

$$\psi_{X(\varphi)}(D(s)) = \psi_{X(\varphi)}(D_\uparrow(s)) \cup \psi_{X(\varphi)}(D_\downarrow(s))$$

which is equivalent to $D(s) = D_\downarrow(s) \cup D_\uparrow(s)$ for $s \in V$. $\square$

We want to define a cut-off function in $D(s)$. Let $\eta : \mathbb{C} \mapsto [0,1]$ be a C^∞ function such that

- $\eta(z) = \eta(iImgz)$, i.e. η only depends in the imaginary part.
- $\eta(ib) = 1$ for $b \in \mathbb{R}$ and $b \leq M/4 + 2$.
- $\eta(ib) = 0$ for $b \in \mathbb{R}$ and $b \geq 3M/4 - 2$.

We define $\eta_D : \cup_{s \in V} D(s) \to [0,1]$ such that

- $\eta_D(x,s) = \eta((\psi_{X(\varphi)} \circ \sigma_\downarrow^{(-1)})(x,s) - ic_\downarrow(s))$ if $(x,s) \in \sigma_\downarrow(B_\downarrow(s))$.
- $\eta_D(x,s) \equiv 0$ in $D(s) \setminus \sigma_\downarrow(B_\downarrow(s))$.

Since η_D is 0 in the neighborhood of $\tau'_\downarrow(s)$ then the function η_D is C^∞ in the interior of $\cup_{s \in V} D(s)$. Let us define an integral $\psi_\updownarrow$ of the time form of φ in $\cup_{s \in V} D(s)$ as follows:

- $\psi_\updownarrow(x,s) = (\eta_D(\psi_{X(\varphi)} \circ \sigma_\downarrow^{(-1)}) + (1-\eta_D)(\psi_{X(\varphi)} \circ \sigma_\uparrow^{(-1)}))(x,s)$ for (x,s) in $D(s) \cap \sigma_\downarrow(B_\downarrow(s)) \cap \sigma_\uparrow(B_\uparrow(s))$.
- $\psi_\updownarrow(x,s) = \psi_{X(\varphi)} \circ \sigma_\uparrow^{(-1)}(x,s)$ if $\eta_D(x,s) = 0$.
- $\psi_\updownarrow(x,s) = \psi_{X(\varphi)} \circ \sigma_\downarrow^{(-1)}(x,s)$ if $\eta_D(x,s) = 1$.

Lemma 9.4.2. *The function $\psi_\updownarrow$ is defined in $\cup_{s \in V} D(s)$ and it is C^∞ in the interior. Moreover, it satisfies $\psi_\updownarrow \circ \varphi = \psi_\updownarrow + 1$.*

Proof. The second property is an immediate consequence of the construction. Since $|\psi_{X(\varphi)} \circ \sigma_j - \psi_{X(\varphi)}| < 1/2$ for $j \in \{\uparrow, \downarrow\}$ and $\mu < 1/4$ then the set $\psi_{X(\varphi)}(D(s))$ contains

$$z \in [a_\leftarrow(s) - 1/2, a_\rightarrow(s) + 1/2] + i[c_\downarrow(s) + 1/2, c_\uparrow(s) - 1/2].$$

We have that

$$Img(\psi_{X(\varphi)}(x,s)) \leq c_\downarrow(s) + M/4 + 2 - 1/2 \implies \eta_D(x,s) = 1$$

and

$$Img(\psi_{X(\varphi)}(x,s)) \geq c_\downarrow(s) + 3M/4 - 2 + 1/2 \implies \eta_D(x,s) = 0$$

by $|\psi_{X(\varphi)} \circ \sigma_j - \psi_{X(\varphi)}| < 1/2$. As a consequence $\psi_\updownarrow$ is well-defined and C^∞ in the interior of $\cup_{s \in V} D(s)$. $\square$

Lemma 9.4.3. *We have $\psi_{X(\varphi)}(B_0(s)) \subset \psi_\updownarrow(D(s))$ for all $s \in V$.*

Proof. Since $|\psi_{X(\varphi)} \circ \sigma_j^{(-1)} - \psi_{X(\varphi)}| < 1/2$ in $\sigma_j(B_j(s))$ then

$$[z \in [a_\leftarrow(s), a_\rightarrow(s)] + ic_j(s)]] \subset \psi_\updownarrow(\tau_j(s)) \text{ for } j \in \{\uparrow, \downarrow\}.$$

Then $|\psi_{X(\varphi)} \circ \sigma_j^{(-1)} - \psi_{X(\varphi)}| < 1/2$ ($j \in \{\uparrow, \downarrow\}$) implies $|\psi_\updownarrow - \psi_{X(\varphi)}| < 1/2$ in $D(s)$. Hence, we obtain $\psi_{X(\varphi)}(B_0(s)) \subset \psi_\updownarrow(D(s))$. $\square$

Lemma 9.4.4. *There exists $C'(\mu_\uparrow, \mu_\downarrow) > 0$ such that $0 < \mu < C'(\mu_\uparrow, \mu_\downarrow)$ implies*

- $|\psi_\updownarrow - \psi_{X(\varphi)}| \leq 2\mu$
- $||D((\psi_\updownarrow, y) \circ (\psi_{X(\varphi)}, y)^{(-1)}) - Id|| \circ (\psi_{X(\varphi)}, y) \leq \mu_0 \mu$

in $\cup_{s \in V} D(s)$. The constant $\mu_0 > 0$ depends only on $\mu^\uparrow$ and $\mu^\downarrow$. The mapping $\psi_\updownarrow$ is injective in $D(s)$ for all $s \in V$.

PROOF. Since $|\psi_{X(\varphi)} \circ \sigma_j^{(-1)} - \psi_{X(\varphi)}| \leq \mu$ for $j \in \{\uparrow, \downarrow\}$ and $\psi_{\updownarrow}$ is a convex combination of $\psi_{X(\varphi)} \circ \sigma_{\downarrow}^{(-1)}$ and $\psi_{X(\varphi)} \circ \sigma_{\uparrow}^{(-1)}$ then $|\psi_{\updownarrow} - \psi_{X(\varphi)}| \leq 2\mu$ in $\cup_{s \in V} D(s)$.

We want to estimate $||D((\psi_{\updownarrow}, y) \circ (\psi_{X(\varphi)}, y)^{(-1)}) - Id||$. If $\eta_D \equiv 0$ in the neighborhood of $P \in \cup_{s \in V} (\psi_{X(\varphi)}, y)(D(s))$ then

$$D((\psi_{\updownarrow}, y) \circ (\psi_{X(\varphi)}, y)^{(-1)}) = D((\psi_{X(\varphi)}, y) \circ \sigma_{\uparrow}^{(-1)} \circ (\psi_{X(\varphi)}, y)^{(-1)})$$

in the neighborhood of P. Since

$$A^{-1} = Id - (A - Id) + (A - Id)^2 - (A - Id)^3 + \dots$$

for real squared matrices such that $||A - Id|| < 1$ then we deduce that

$$||D((\psi_{\updownarrow}, y) \circ (\psi_{X(\varphi)}, y)^{(-1)}) - Id|| \leq 2\mu^{\uparrow}\mu$$

in a neighborhood of P supposed $\mu^{\uparrow}\mu < 1/2$. Analogously, if $\mu^{\downarrow}\mu < 1/2$ and $\eta_D \equiv 1$ in a neighborhood of P then

$$||D((\psi_{\updownarrow}, y) \circ (\psi_{X(\varphi)}, y)^{(-1)}) - Id|| \leq 2\mu^{\downarrow}\mu$$

in a neighborhood of P.

Now, we focus on the interior of $D(s) \cap \sigma_{\downarrow}(B_{\downarrow}(s)) \cap \sigma_{\uparrow}(B_{\uparrow}(s))$. We denote $h = (\psi_{X(\varphi)}, y)$ and $H = (\psi_{\updownarrow}, y) \circ (\psi_{X(\varphi)}, y)^{(-1)}$; we have

$$H = (\eta_D \circ h^{(-1)}) h \circ \sigma_{\downarrow}^{(-1)} \circ h^{(-1)} + (1 - \eta_D \circ h^{(-1)}) h \circ \sigma_{\uparrow}^{(-1)} \circ h^{(-1)}.$$

For $\mu > 0$ small enough we obtain

$$||DH - Id|| \leq 2(\mu^{\uparrow} + \mu^{\downarrow})\mu + ||J||$$

where J^T is equal to

$$\begin{pmatrix} \frac{\partial(\eta_D \circ h^{(-1)})}{\partial Rez} [\psi_{X(\varphi)} \circ \sigma_{\downarrow}^{(-1)} \circ h^{(-1)} - \psi_{X(\varphi)} \circ \sigma_{\uparrow}^{(-1)} \circ h^{(-1)}] \\ \frac{\partial(\eta_D \circ h^{(-1)})}{\partial Imz} [\psi_{X(\varphi)} \circ \sigma_{\downarrow}^{(-1)} \circ h^{(-1)} - \psi_{X(\varphi)} \circ \sigma_{\uparrow}^{(-1)} \circ h^{(-1)}] \end{pmatrix}.$$

Let $K = \sup_{b' \in \mathbb{R}} |(\partial \eta(ib)/\partial b)(b')|$; we have

$$\left| \frac{\partial(\eta_D \circ h^{(-1)})}{\partial Rez} \right| \leq K \left| \frac{\partial(Img[\psi_{X(\varphi)} \circ \sigma_{\downarrow}^{(-1)} \circ h^{(-1)}])}{\partial Rez} \right|.$$

Therefore, we obtain $|\partial(\eta_D \circ h^{(-1)})/\partial Rez| \leq 2K\mu^{\downarrow}\mu$. In a similar way we have $|\partial(\eta_D \circ h^{(-1)})/\partial Imgz| \leq K(1 + 2\mu^{\downarrow}\mu)$. All the previous calculations lead us to

$$||DH - Id|| \leq 2(\mu^{\uparrow} + \mu^{\downarrow})\mu + 4\mu\sqrt{2}K(1 + 2\mu^{\downarrow}\mu).$$

By plugging $\mu^{\downarrow}\mu < 1/2$ into the previous inequality we obtain

$$||DH - Id|| \circ (\psi_{X(\varphi)}, y) \leq 2(\mu^{\downarrow} + \mu^{\uparrow} + 4\sqrt{2}K)\mu$$

in $\cup_{s \in V} D(s)$. We define $\mu_0 = 2\mu^{\downarrow} + 2\mu^{\uparrow} + 8\sqrt{2}K$.

We denote $D'(s) = \psi_{X(\varphi)}(D(s))$. Suppose $\mu_0\mu < 1/4$. The foliations $Rez = cte$ and $Img(H) = cte$ are transversal in $D'(s)$ since $\partial Img(H)/\partial Im(z) > 1 - 1/4 = 3/4$. Moreover $Img(H) = c_k$ contains $\psi_{X(\varphi)}(\tau_k(s))$ and $Rez = a_j(s) + u(j)1/2$ contains $\psi_{X(\varphi)}(\tau_j(s))$ for $j \in \{\leftarrow, \rightarrow\}$ and $k \in \{\uparrow, \downarrow\}$. As a consequence $(Rez, Img(H))$ is injective in $D'(s)$. Suppose $H(z_0, s) = H(z_1, s)$ and $z_0 \neq z_1$; we deduce that $Re(z_0) \neq Re(z_1)$. We consider the connected curve

$$\gamma \equiv [Img(H) = Img(H(z_0, s)) = Img(H(z_1, s))].$$

The tangent vector to γ at any point belongs to $1 + i(-1/3, 1/3)$. Since we also have $\partial Re(H)/\partial Rez > 3/4$ and $\partial Re(H)/\partial Im(z) < 1/4$ then

$$2|Re(z_1 - z_0)|/3 \leq |Re(H)(z_1, s) - Re(H)(z_0, s)| \neq 0.$$

We deduce that H is injective in $\cup_{s \in V} D'(s)$. Thus $\psi_{\updownarrow}$ is injective in $D(s)$ for all $s \in V$. □

PROOF OF PROPOSITION 9.4.1. We define

$$\sigma_{\updownarrow} = (\psi_{\updownarrow}(x, y), y)^{(-1)} \circ (\psi_{X(\varphi)}(x, y), y).$$

Thus $\sigma_{\updownarrow}$ is C^∞ by lemma 9.4.2. By lemmas 9.4.3 and 9.4.4 the mapping $\sigma_{\updownarrow}$ is well-defined in $\cup_{s \in V} B_0(s)$. Moreover, it is injective. By extending $\sigma_{\updownarrow}$ by σ_j in the neighborhood of $\gamma_j^0(s)$ we obtain

$$(\sigma_{\updownarrow}, \gamma_j^0(s)) = (\sigma_j, \gamma_j^0(s))$$

for all $(s, j) \in V \times \{\uparrow, \downarrow\}$. We have

$$\psi_{\updownarrow} \circ (\sigma_{\updownarrow} \circ \alpha_\varphi) = \psi_{X(\varphi)} \circ \alpha_\varphi = \psi_{X(\varphi)} + 1 = \psi_{\updownarrow} \circ (\varphi \circ \sigma_{\updownarrow}).$$

That implies $\sigma_{\updownarrow} \circ \alpha_\varphi = \varphi \circ \sigma_{\updownarrow}$ in $\cup_{s \in V} B_0(s) \cap \cup_{s \in V} \alpha_\varphi^{(-1)}(B_0(s))$. The inequality $|\psi_{\updownarrow} - \psi_{X(\varphi)}| \leq 2\mu$ is equivalent to $|\psi_{X(\varphi)} \circ \sigma_{\updownarrow}^{(-1)} - \psi_{X(\varphi)}| \leq 2\mu$. Therefore, we obtain $|\psi_{X(\varphi)} \circ \sigma_{\updownarrow} - \psi_{X(\varphi)}| \leq 2\mu$. Since

$$(\psi_{X(\varphi)}, y) \circ \sigma_{\updownarrow}^{(-1)} \circ (\psi_{X(\varphi)}, y)^{(-1)} = (\psi_{\updownarrow}, y) \circ (\psi_{X(\varphi)}, y)^{(-1)}$$

then we deduce that

$$||D((\psi_{X(\varphi)}, y) \circ \sigma_{\updownarrow}^{(-1)} \circ (\psi_{X(\varphi)}, y)^{(-1)}) - Id|| \leq \mu_0 \mu$$

by lemma 9.4.4. By considering $\mu_0 \mu < 1/2$ we have

$$||D((\psi_{X(\varphi)}, y) \circ \sigma_{\updownarrow} \circ (\psi_{X(\varphi)}, y)^{(-1)}) - Id|| \leq \mu_{\updownarrow} \mu$$

for $\mu^{\updownarrow} = 2\mu_0$. We are done since μ_0 just depends on $\mu^{\uparrow}$ and $\mu^{\downarrow}$. □

9.5. Regions and their limiting curves

Fix $y_0 \in B(0, \delta) \setminus \{0\}$. Consider a region $Z(s) \subset W_\epsilon^R(\eta, s)$ associated to $Re(X(\varphi))$. The number of connected components of $\partial Z(s) \setminus SingX(\varphi)$ is either 1 or 2. Moreover, it is equal to 1 if and only if

$$\alpha_{\xi(X), W_\epsilon(\eta)}(Z(s)) = \omega_{\xi(X), W_\epsilon(\eta)}(Z(s)) \in SingX(\varphi).$$

Every connected component of $\partial Z(s) \setminus SingX(\varphi)$ is contained in a trajectory $\gamma(s) = \Gamma_{\xi(X)}^{\overline{W_\epsilon(\eta)}}[x', s]$. We say that $\gamma(s)$ is a limiting trajectory of $Z(s)$. We denote by $LZ(s)$ the set of limiting trajectories of $Z(s)$. We have $LZ(s) = \{\gamma_0^Z(s), \gamma_1^Z(s)\}$ where $\gamma_j^Z(s)$ depends continuously on $s \in V$ for $j \in \{0, 1\}$ since $Z(s)$ and $\partial Z(s)$ do so. Each curve in $LZ(s)$ contains exactly one vertex of $W_\epsilon(\eta)$. A curve $\Gamma_{\xi(X)}^{\overline{W_\epsilon(\eta)}}[v_j^{\eta_j}(s)]$ limits exactly three regions (see picture 4). Let $\gamma(s) \in LZ(s)$. Either we have

$$Img[\psi_{X(\varphi)}(\gamma(s))] = \inf_{(x,s) \in Z(s)} Img[\psi_{X(\varphi)}(x, s)]$$

or

$$Img[\psi_{X(\varphi)}(\gamma(s))] = \sup_{(x,s) \in Z(s)} Img[\psi_{X(\varphi)}(x, s)].$$

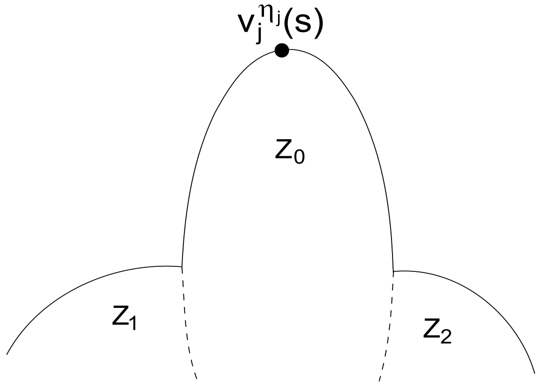

FIGURE 4.

In the former case we define

$$B_Z^\gamma(s)' = \overline{Z(s)} \cap [Img(\psi_{X(\varphi)}) \leq Img(\psi_{X(\varphi)}(\gamma(s))) + M]$$

whereas the definition is

$$B_Z^\gamma(s)' = \overline{Z(s)} \cap [Img(\psi_{X(\varphi)}) \geq Img(\psi_{X(\varphi)}(\gamma(s))) - M]$$

in the latter case. We define $B_Z^\gamma(s) = \exp([-1,1]X)(B_Z^\gamma(s)')$ for both cases. By construction $\psi_{X(\varphi)}(B_Z^\gamma(s))$ is of the form

$$[a_{\leftarrow}(s) - 1, a_{\rightarrow}(s) + 1] + i[c_{\downarrow}(s), c_{\uparrow}(s)]$$

for some functions $a_{\leftarrow}$, $a_{\rightarrow}$, $c_{\uparrow}$ and $c_{\downarrow}$ depending on Z and γ. Moreover we have $c_{\uparrow} - c_{\downarrow} \equiv M$. We define the width $WZ(s)$ of a region $Z(s)$ by the formula

$$WZ(s) = \sup_{(x,s) \in Z(s)} Img[\psi_{X(\varphi)}(x,s)] - \inf_{(x,s) \in Z(s)} Img[\psi_{X(\varphi)}(x,s)].$$

The width WZ is either a positive function in V or $WZ \equiv \infty$ in V. The latter case corresponds to $\sharp LZ \equiv 1$.

9.5.1. The game. Here we define a game; the goal is building a special homeomorphism σ conjugating α_φ and φ in $U_\epsilon \cap [y \in V]$. There are several steps in this game. For a step j and a region $Z \subset W_\epsilon^R(\eta)$ we attach a label $lab_j(Z) \subset LZ \cup \{\Xi\}$. The labels satisfy

- $lab_0(Z) = \emptyset$ for all region $Z \subset W_\epsilon^R(\eta)$.
- If $\Xi \in lab_j(Z)$ then $lab_j(Z) = LZ \cup \{\Xi\}$.

The meaning of the labels is related to the existence of conjugating mappings.

- If $\gamma \in lab_j(Z) \cap LZ$ there exists a special continuous conjugation σ_Z^γ defined in B_Z^γ.
- If $\Xi \in lab_j(Z)$ then there exists a special continuous conjugation σ_Z defined in $\overline{Z}$.
- If $\Xi \in lab_j(Z)$ and $\gamma \in LZ$ then $\sigma_Z = \sigma_Z^\gamma$ in a neighborhood of γ in $\overline{Z}$.

The mappings σ_Z^γ and σ_Z do not depend on j. For a region Z in $W_\epsilon^R(\eta)$ and a curve $\gamma \in LZ$ we denote by $Z_1(Z,\gamma)$ and $Z_2(Z,\gamma)$ the other regions of $W_\epsilon^R(\eta)$ limiting with γ. Next, we introduce some compatibility conditions that the conjugations have to fulfill.

- If $\gamma \in lab_j(Z)$ then $\gamma \in lab_j(Z_1(Z,\gamma)) \cap lab_j(Z_2(Z,\gamma))$.
- If $\gamma \in lab_j(Z)$ then $\sigma_Z^\gamma = \sigma_{Z_k}^\gamma$ in $\partial Z \cap \partial Z_k$ for all $k \in \{1,2\}$.
- If $\gamma \in lab_j(Z)$ the mapping defined by gluing σ_Z^γ and $\sigma_{Z_k}^\gamma$ is C^∞ in the neighborhood of $\partial Z \cap \partial Z_k \cap W_\epsilon(\eta)$.

There is also a technical condition regarding the M-interpolation process.

- If $WZ(y_0) \leq 2M$ then either $lab_j(Z) = \emptyset$ or $\Xi \in lab_j(Z)$.

We define $\mu^{uv} = \max(\mu_{uv}, \mu^{\updownarrow}(\mu_{uv}, \mu_{uv}))$. The next set of conditions assures that $\sigma_Z \sim Id$ and $D\sigma_Z \sim Id$.

- If $\gamma \in lab_j(Z)$ then $|\psi_{X(\varphi)} \circ \sigma_Z^\gamma - \psi_{X(\varphi)}| \leq 2\mu$ in B_Z^γ.
- If $\Xi \in lab_j(Z)$ then $|\psi_{X(\varphi)} \circ \sigma_Z - \psi_{X(\varphi)}| \leq 2\mu$ in Z.
- $||D(\psi_{X(\varphi)} \circ \sigma_Z^\gamma \circ (\psi_{X(\varphi)}, y)^{(-1)}) - Id|| \leq \mu_{uv}\mu$ in B_Z^γ if $\gamma \in lab_j(Z)$.
- $||D(\psi_{X(\varphi)} \circ \sigma_Z \circ (\psi_{X(\varphi)}, y)^{(-1)}) - Id|| \leq \mu^{uv}\mu$ in Z for $\Xi \in lab_j(Z)$.

We introduce a condition making explicit the goal of the game.

- There exists $j \in \mathbb{N}$ such that $\Xi \in lab_j(Z)$ for all $Z \subset W_\epsilon^R(\eta)$.

The numbers ϵ, δ, μ and the domain V can be interpreted as the initial data of the game. We ask these objects to fulfill some prerequisites that we introduce next. We fix $0 < \mu < \min(1, C(\mu_{uv}, \mu_{uv}))$. Let $\epsilon_0 > 0$ as described in subsection 9.2.1; we choose $0 < \epsilon \leq \epsilon_0$ and a small enough $\delta > 0$. The choice of (ϵ, δ, μ) is independent of y_0.

The success in solving the game will imply

PROPOSITION 9.5.1. *Let φ be a (NSD) diffeomorphism. Consider a 3-uple $(\mu, \epsilon, \delta) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$ fulfilling the prerequisites of the game. Then, for all $y_0 \in B(0,\delta) \setminus \{0\}$ there exists a neighborhood $V \subset \mathbb{C}$ of y_0 and a special mapping σ_V defined in $W_{\epsilon,\delta} \cap [y \in V]$ such that*

- *σ_V is C^∞ in $(W_{\epsilon,\delta} \setminus [f=0]) \cap [y \in V]$*
- *$\sigma_V \circ \alpha_\varphi = \varphi \circ \sigma_V$*
- *$|\psi_{X(\varphi)} \circ \sigma_V - \psi_{X(\varphi)}| \leq 2\mu$*
- *$||D((\psi_{X(\varphi)}, y) \circ \sigma_V \circ (\psi_{X(\varphi)}, y)^{(-1)}) - Id||(\psi_{X(\varphi)}, y) \leq \mu^{uv}\mu$*

in $W_{\epsilon,\delta} \cap [y \in V]$.

Roughly speaking the proof goes as follows: since the goal of the game is achieved then we obtain a conjugation σ_Z for each region Z and all of them paste together by the compatibility conditions.

It looks like difficult to achieve the thirteen properties (plus the goal property) we ask the game for. In despite of this we will introduce a process to solve the game such that most of the properties can be trivially checked out.

9.5.2. The algorithm solving the game. The algorithm has several steps. In each step of the game exactly one step of the algorithm is applied. The steps of the algorithm are ranked in a priority list. If the correspondent condition is satisfied then we apply the first step; otherwise we try to apply the second step and so on.

Prerequisites: Fix $y_0 \in B(0,\delta) \setminus \{0\}$. We select $\eta \in [0,1)^{N_T}$ such that there are no bi-tangent cords in $U_\epsilon(\eta) \cap [y = y_0]$. We have to choose a neighborhood V in $B(0,\delta) \setminus \{0\}$ of y_0. We suppose that there are no bi-tangent cords in $W_\epsilon(\eta) \cap [y \in V]$. Moreover, we can also suppose that $WZ(s) > 2M$ for all $s \in V$ if $WZ(y_0) > 2M$ whereas otherwise $WZ(s) \leq 2M + 1$ for all $s \in V$. That choice is possible since $WZ(s)$ is a continuous function.

First step: This step is applied if there exists a region $Z \subset W_\epsilon^R(\eta)$ such that $LZ \subset lab_j(Z)$ but $\Xi \notin lab_j(Z)$. The M-interpolation process condition implies that $WZ(s) > 2M$ for all $s \in V$. Let us denote $(\alpha_{\xi(X,s)}, \omega_{\xi(X,s)})_{W_\epsilon(\eta)}$ by (α, ω). Next, we choose a transversal Tr to $Z(s)$. If $\alpha(Z) = \infty$ we choose $Tr(s) = \overline{end_-(Z(s))}$. If $\alpha(Z) \neq \infty$ and $\omega(Z) = \infty$ we define $Tr(s) = \overline{end_+(Z(s))}$. For the remaining case let us consider a vertex $v_k^{\eta_k}(s)$ in $\overline{Z(s)}$. We choose

$$Tr(s) = \exp(i[0, WZ(s)]X)(v_k^{\eta_k}(s))$$

if $Re(iX)$ points towards Z at $v_k^{\eta_k}$. Otherwise we define

$$Tr(s) = \exp(i[-WZ(s), 0]X)(v_k^{\eta_k}(s)).$$

By the choice of the domains $W_\epsilon(\eta)$ the transversal Tr is a sub-trajectory of $Re(iX)$. We obtain σ_{Tr} by proposition 9.3.1. Let $\gamma_l \in LZ$ ($l \in \{1,2\}$); we denote by γ_l' the curve $\overline{\partial B_Z^{\gamma_l} \cap W_\epsilon(\eta)} \setminus \gamma_l$. We interpolate σ_{Tr} and $\sigma_Z^{\gamma_1}$ in $B_Z^{\gamma_1}$ to obtain σ' such that

$$(\sigma', \gamma_1 \cap \partial Z) = (\sigma_Z^{\gamma_1}, \gamma_1 \cap \partial Z) \text{ and } (\sigma', \gamma_1') = (\sigma_{Tr}, \gamma_1').$$

If $\sharp LZ = 1$ we define $\sigma_Z = \sigma'$. Otherwise we interpolate σ' and $\sigma_Z^{\gamma_2}$ in $B_Z^{\gamma_2}$ to obtain σ_Z such that

$$(\sigma_Z, \gamma_2 \cap \partial Z) = (\sigma_Z^{\gamma_2}, \gamma_2 \cap \partial Z) \text{ and } (\sigma_Z, \gamma_2') = (\sigma', \gamma_2').$$

Let us remark that $(\sigma', \gamma_2') = (\sigma_{Tr}, \gamma_2')$ since $WZ > 2M$. By applying proposition 9.4.1 at most twice we obtain that $|\psi_{X(\varphi)} \circ \sigma_Z - \psi_{X(\varphi)}| \leq 2\mu$ and

$$||D((\psi_{X(\varphi)}, y) \circ \sigma_Z \circ (\psi_{X(\varphi)}, y)^{(-1)}) - Id||(\psi_{X(\varphi)}, y) \leq \mu^{uv}\mu$$

in $\cup_{s \in V} Z(s)$.

Finally, we define $lab_{j+1}(Y) = lab_j(Y) \cup \{\Xi\}$ for all region Y in $W_\epsilon^R(\eta)$ such that $LZ \subset lab_j(Y)$ and $\Xi \notin lab_j(Y)$. Otherwise we define $lab_{j+1}(Y) = lab_j(Y)$. By construction all the properties (except the one regarding the goal) are preserved for lab_{j+1}.

Second step: Suppose there exists a region Z such that $\gamma_0^Z \in lab_j(Z)$ but $\gamma_1^Z \notin lab_j(Z)$. We fix Z; let us consider a sequence

$$(Z, \gamma_0^Z) = (Z_0, \gamma_0) - (Z_1, \gamma_1) - \ldots - (Z_k, \gamma_k)$$

satisfying

- $\gamma_l \in LZ_l$ and $\gamma_l \in LZ_{l-1}$ for all $0 < l \leq k$.
- $Z_l \neq Z_{l+1}$ and $\gamma_l \neq \gamma_{l+1}$ for all $0 \leq l < k$.
- $\gamma_{l+1} \notin lab_j(Z_l)$ for $0 \leq l < k$.
- $WZ_l(y_0) \leq 2M$ for all $0 < l < k$.

Such a sequence will be called a generating sequence. The element (Z, γ_0^Z) is called the root of the sequence. Consider the vertex $v_1^{\eta_1}$ in γ_1; we define

$$Tr(s) = \exp(i[-(\kappa - 3), \kappa - 3]X)(v_1^{\eta_1}(s)).$$

The conjugation σ_{Tr} satisfies the claim in proposition 9.3.1 in the set

$$\cup_{s \in V} D_{Tr}(s) = \cup_{s \in V} \exp([-1,1]X)\left(\Gamma_{\xi(X)}^{\overline{W_\epsilon(\eta+\kappa-3)}}[Tr(s)] \cap \overline{W_\epsilon(\eta)}\right).$$

We claim that

PROPOSITION 9.5.2. *The mapping σ_{Tr} is defined*

- *in a neighborhood of* $\cup_{s \in V} B_Z^{\gamma_1^Z}(s)$ *in* $\overline{W_\epsilon(\eta)}$.

- *in a neighborhood of* $\cup_{s \in V} Z_l(s)$ *in* $\overline{W_\epsilon(\eta)}$ *if* $WZ_l(y_0) \leq 2M$.
- *in a neighborhood of* $\cup_{s \in V} B_{Z_k}^{\gamma_k}(s)$ *in* $\overline{W_\epsilon(\eta)}$ *if* $WZ_k(y_0) > 2M$.

To prove the proposition we require the following lemma

LEMMA 9.5.1. *The number of regions in* $W_\epsilon^R(\eta)$ *is at most* $3N_T$.

PROOF. Every region has at least one limiting curve. The regions limited by a limiting curve are exactly 3. □

PROOF OF PROPOSITION 9.5.2. Since $\kappa - 3 = 6(2M+1)N_T > 2M$ the result is clear for $\cup_{s \in V} B_Z^{\gamma_1^Z}(s)$. By splitting the original generating sequence in several ones we can suppose $(Z_l, \gamma_l) \neq (Z_{l'}, \gamma_{l'})$ for $0 \leq l < l' \leq k$ without lack of generality. Since $\sharp LY \leq 2$ for all region $Y \subset W_\epsilon^R(\eta)$ then $k+1 \leq 6N_T$. Let $v_l^{\eta_l}$ be the vertex in γ_l. For $1 \leq l \leq k$ we define $\kappa_l = \kappa - 3 - (2M+1)(l-1)$ and

$$Tr_l(s) = \exp(i[-\kappa_l, \kappa_l]X)(v_l^{\eta_l}(s)).$$

We claim that $\cup_{s \in V} Tr_l(s)$ is in the interior of $\cup_{s \in V} \left(\Gamma_{\xi(X)}^{\overline{W_\epsilon(\eta+\kappa-3)}}[Tr(s)] \right)$ in the set $\overline{W_\epsilon(\eta+\kappa-3)}$. Since $\kappa - 3 - (2M+1)(6N_T - 2) > 2M+1$ the proposition is a consequence of the claim.

The claim is true for $l = 1$. Suppose it is true for $l = l_0 < k$. We have $WZ_{l_0} < 2M+1$; as a consequence for all $s \in V$ there exists a unique point $(x_0, s) \in Tr_{l_0}(s)$ such that

$$v_{l_0+1}^{\eta_{l_0+1}}(s) \in \Gamma_{\xi(X)}^{\overline{W_\epsilon(\eta+\kappa-3)}}[x_0, s].$$

Moreover $(x_0, s) = \exp(i\iota_{l_0}(s)X)(v_{l_0}^{\eta_{l_0}}(s))$ for some $\iota_{l_0}(s)$ in $(-2M-1, 2M+1)$. We deduce that

$$\exp(i[-\kappa_{l_0} + |\iota_{l_0}(s)|, \kappa_{l_0} - |\iota_{l_0}(s)|]X)(v_{l_0+1}^{\eta_{l_0+1}}(s)) \subset \Gamma_{\xi(X)}^{\overline{W_\epsilon(\eta+\kappa-3)}}[Tr(s)]$$

for all $s \in V$. Since $|\iota_{l_0}(s)| < 2M+1$ and $\kappa_{l_0+1} = \kappa_{l_0} - (2M+1)$ we are done. □

The assignment of the labels is natural. If $Y \subset W_\epsilon^R(\eta)$ is not in any generating sequence then $lab_{j+1}(Y) = lab_j(Y)$. If (Y, γ) is in a generating sequence then $lab_j(Y) = \{\gamma_0^Y, \gamma_1^Y, \Xi\}$ for $WY(y_0) \leq 2M$; otherwise we include γ in $lab_{j+1}(Y)$. We also define $lab_{j+1}(Z) = \{\gamma_0^Z, \gamma_1^Z\}$.

We have to prove two things. The first one is that we are not redefining any σ_Y^γ or σ_Y for any Y or γ because we claimed that these data do not depend on j. The second one is that the conditions are fulfilled; all of them are trivial except the compatibility conditions.

LEMMA 9.5.2. *Consider a region* $Y \subset W_\epsilon^R(\eta)$ *and* $\gamma_0^Y \in lab_j Y$ *such that* $(Y, \gamma_0^Y) \neq (Z, \gamma_0^Z)$. *Then* (Y, γ_0^Y) *does not belong to any generating sequence whose root is* (Z, γ_0^Z).

PROOF. Suppose we have a generating sequence

$$(Z, \gamma_0^Z) = (Z_0, \gamma_0) - (Z_1, \gamma_1) - \ldots - (Z_k, \gamma_k)$$

such that $(Z_k, \gamma_k) = (Y, \gamma_0^Y)$ for $k > 0$. The curve γ_k belongs to LZ_{k-1} but not to $lab_j(Z_{k-1})$ by the definition of generating sequence. On the other hand since $\gamma_k \in lab_j(Z_k)$ then we obtain $\gamma_k \in lab_j(Z_{k-1})$ by the compatibility conditions for step j. That is a contradiction. □

Lemma 9.5.3. *The compatibility conditions are fulfilled for the step $j+1$.*

Proof. Let $Y \subset W_\epsilon^R(\eta)$ be a region. If $\gamma_0^Y \in lab_j(Y)$ the compatibility conditions for (Y, γ_0^Y) in the step j and $j+1$ are the same. Therefore, we can suppose $\gamma_0^Y \in lab_{j+1}(Y) \setminus lab_j(Y)$. The compatibility conditions for the step j imply that $\gamma_0^Y \notin lab_j(Z_1(Y, \gamma_0^Y)) \cup lab_j(Z_2(Y, \gamma_0^Y))$.

Suppose $(Y, \gamma_0^Y) = (Z_k, \gamma_k)$ for a generating sequence

$$(Z, \gamma_0^Z) = (Z_0, \gamma_0) - (Z_1, \gamma_1) - \ldots - (Z_k, \gamma_k).$$

The region Z_{k-1} is equal to $Z_{l_0}(Y, \gamma_0^Y)$ for some $l_0 \in \{1, 2\}$. Suppose $l_0 = 1$ without lack of generality. By construction we obtain that $\gamma_0^Y \in lab_{j+1}(Z_1(Y, \gamma_0^Y))$. Moreover, we can replace (Y, γ_0^Y) with $(Z_2(Y, \gamma_0^Y), \gamma_0^Y)$ in the generating sequence. As a consequence γ_0^Y is in $lab_{j+1}(Z_2(Y, \gamma_0^Y))$.

Now, suppose $(Y, \gamma_1^Y) = (Z_k, \gamma_k)$ but (Y, γ_0^Y) does not belong to any generating sequence whose root is (Z, γ_0^Z). In this case $\gamma_0^Y \in lab_{j+1}(Y)$ implies $WY(y_0) \leq 2M$. We can append $(Z_j(Y, \gamma_0^Y), \gamma_0^Y)$ at the end of the series and we still have a generating sequence. Therefore that leads us to $\gamma_0^Y \in lab_{j+1}(Z_1(Y, \gamma_0^Y)) \cap lab_{j+1}(Z_2(Y, \gamma_0^Y))$.

The remaining compatibility conditions are obvious because all the $\sigma_{Y'}$ or $\sigma_{Y'}^\gamma$ that we define are just restrictions of σ_{Tr}. □

Third step: Suppose $j = 0$. We choose Z such that

$$\alpha(Z) = \omega(Z) \in SingX(\varphi).$$

We consider the generating sequences of the form

$$Z = Z_0 - (Z_1, \gamma_1) - \ldots - (Z_k, \gamma_k)$$

where $\gamma_1 = \gamma_0^Z$. The root of the sequence is Z_0. The conditions we require to the generating sequence are the same than in the second step; we just remove the conditions involving γ_0.

The process for constructing a special conjugation between α_φ and φ and the assignment of the labels $lab_1(Y)$ are analogous to the ones in the second step.

The goal of the game:

Lemma 9.5.4. *The goal of the game is achieved.*

Proof. Suppose that no step of the algorithm is applicable to the step j of the game; hence $j > 0$. For every region $Y \subset W_\epsilon^R(\eta)$ we have that either $lab_j(Y) = \emptyset$ or $\Xi \in lab_j(Y)$. We claim that $\Xi \in lab_j(Y)$ for all region $Y \subset W_\epsilon^R(\eta)$. Otherwise there exist $Y_0, Y_1 \subset W_\epsilon^R(\eta)$ such that $LY_0 \cap LY_1 \neq \emptyset$, $\Xi \in lab_j(Y_0)$ and $lab_j(Y_1) = \emptyset$. Let γ be an element of $LY_0 \cap LY_1$; it satisfies $\gamma \in lab_j(Y_1)$ by the compatibility conditions. That is a contradiction.

If for a step j of the game we apply the second step of the algorithm then for step $j+1$ we apply the first step. Since the number of regions is at most $3N_T$ then we have that there exists $j_0 \leq 6N_T$ such that $\Xi \in lab_{j_0}(Y)$ for all region $Y \subset W_\epsilon^R(\eta)$. □

proof of proposition 9.5.1. Let $(\mu, \epsilon, \delta) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$ fulfilling all the prerequisites. For every $y_0 \in B(0, \delta) \setminus \{0\}$ we choose $V_{y_0} \subset B(0, \delta) \setminus \{0\}$ satisfying the corresponding prerequisites for a neighborhood of y_0. By applying the game we find σ_V defined in $(\overline{W_\epsilon(\eta)} \setminus [f=0]) \cap [y \in V_{y_0}]$ for some $\eta(y_0) \in [0,1)^{N_T}$. The properties in proposition 9.5.1 for the domain

$$(W_\epsilon \setminus [f=0]) \cap [y \in V_{y_0}] \subset (W_\epsilon(\eta) \setminus [f=0]) \cap [y \in V_{y_0}]$$

are deduced from the properties of the game. Moreover, by defining $\sigma_{V|f=0} \equiv Id$ we extend σ_V continuously to $f=0$ since $|\psi_{X(\varphi)} \circ \sigma_V - \psi_{X(\varphi)}| \leq 2\mu$. □

9.6. Conjugating a diffeomorphism and its normal form

For each $y_0 \in B(0,\delta) \setminus \{0\}$ there exists a neighborhood V_{y_0} where the claim in proposition 9.5.1 holds. It is evident that $\cup_{s \in B(0,\delta)} V_s = B(0,\delta) \setminus \{0\}$. Let $B(0,\delta) \setminus \{0\} = \cup_{j \in J} V_j$ be a locally finite refinement of $\cup_{s \in B(0,\delta)} V_s$. We choose a partition of the unity h_j ($j \in J$) associated to the covering $\cup_{j \in J} V_j$. The function

$$\psi_\varphi = \sum_{j \in J} h_j(y)(\psi_{X(\varphi)} \circ \sigma_{V_j}^{(-1)})$$

is a candidate to be an integral of the time form of φ defined in a neighborhood of $(0,0)$ deprived of the line $y=0$. We have to explain the meaning of the previous formula. So far we were dealing with simply connected sets like $\cup_{s \in V} D_{Tr}(s)$ or $\cup_{s \in V} B_1(s)$. Now we want to define ψ_φ in a domain $U_{\epsilon,\delta} \setminus [f=0]$ whose intersection with the fibers is not simply connected. Anyway, we have

$$\psi_{X(\varphi)} \circ \sigma_{V_j}^{(-1)}(P) - \psi_{X(\varphi)}(P) = t \Leftrightarrow \sigma_{V_j}^{(-1)}(P) = \exp(tX(\varphi))(P).$$

Hence the function $\psi_{X(\varphi)} \circ \sigma_{V_j}^{(-1)} - \psi_{X(\varphi)}$ is single valued and so $\psi_\varphi - \psi_{X(\varphi)}$ is a single valued function such that $|\psi_\varphi - \psi_{X(\varphi)}| \leq 2\mu$ in its domain of definition.

Proposition 9.6.1. *Consider $(\mu, \epsilon_2, \delta_2) \in (\mathbb{R}^+)^3$ fulfilling the prerequisites of the game. Suppose $\max(\mu, \mu\mu^{uv}) < 1/4$. There exist $\epsilon > 0$ and $\delta > 0$ such that for all $y_0 \in B(0,\delta) \setminus \{0\}$ the map σ_V provided by proposition 9.5.1 satisfies that $\sigma_V^{(-1)}$ is well-defined in $U_{\epsilon,\delta} \cap [y \in V]$.*

Proof. Since $|\psi_{X(\varphi)} \circ \sigma_V - \psi_{X(\varphi)}| \leq 2\mu < 1/2$ then

$$\sigma_V(P) \in \exp(B(0,1/2)X(\varphi))(P)$$

for all $P \in W_{\epsilon_2,\delta_2}$. Thus $\sigma_V(P) = \sigma_V(Q)$ implies $Q \in \exp(t_0 X(\varphi))(P)$ for some $t_0 \in B(0,1)$. We consider $U_{\epsilon,\delta}$ such that $\exp(B(0,2)X(\varphi))(U_{\epsilon,\delta})$ is contained in W_{ϵ_2,δ_2}. Since $D\sigma_V \sim Id$ we obtain

$$[\psi_{X(\varphi)} \circ \sigma_V(Q) - \psi_{X(\varphi)} \circ \sigma_V(P)] \cdot t_0 \geq |t_0|^2/2$$

supposed $P \in \exp(B(0,1)X(\varphi))(U_{\epsilon,\delta}) \setminus [f=0]$. The $\cdot$ stands for the scalar product in $\mathbb{R}^2$. Then

$$\sigma_V(P) = \sigma_V(Q) \implies t_0 = 0 \implies P = Q.$$

Thus $\sigma_{V(y_0)}$ is injective in $\exp(B(0,1)X(\varphi))(U_{\epsilon,\delta})$ for $y_0 \in B(0,\delta) \setminus \{0\}$. Fix $y_0 \in B(0,\delta) \setminus \{0\}$ and consider $P \in (U_{\epsilon,\delta} \setminus [f=0]) \cap [y \in V]$. We define the path $\gamma : \mathbb{S}^1 \to \exp(B(0,1)X(\varphi))(U_{\epsilon,\delta})$ such that

$$\gamma(\lambda) = \sigma_V(\exp(\lambda X(\varphi))(P)).$$

Since $|\psi_{X(\varphi)} \circ \sigma_V - \psi_{X(\varphi)}| \leq 2\mu < 1/2$ then γ is not homotopic to a trivial loop in $[y = y(P)] \setminus \{P\}$. But clearly γ is homotopically trivial in $\sigma_V(\exp(\overline{B(0,1)}X(\varphi))(P))$; we deduce that

$$P \in \sigma_V(\exp(\overline{B(0,1)}X(\varphi))(P)) \subset \sigma_V(\exp(B(0,1)X(\varphi))(U_{\epsilon,\delta}))$$

and then $\sigma_V^{(-1)}$ is well-defined in $U_{\epsilon,\delta} \cap [y \in V]$. □

Last lemma implies the existence of an integral of the time form of φ in a neighborhood of $(0,0)$ deprived of $y = 0$.

Proposition 9.6.2. *Let φ be a (NSD) diffeomorphism. There exists $(\mu, \epsilon, \delta) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$ such that there exists a tg-sp mapping σ satisfying*

- *σ and $\sigma^{(-1)}$ are C^∞ in $U_{\epsilon,\delta} \setminus [yf = 0]$.*
- *$\sigma \circ \alpha_\varphi = \varphi \circ \sigma$.*
- *$|\psi_{X(\varphi)} \circ \sigma^{(j)} - \psi_{X(\varphi)}| \leq 2\mu$ for $j \in \{-1, 1\}$ in $U_{\epsilon,\delta} \setminus [yf = 0]$.*
- *$||D((\psi_{X(\varphi)}, y) \circ \sigma \circ (\psi_{X(\varphi)}, y)^{(-1)}) - Id||(\psi_{X(\varphi)}, y) \leq 4\mu^{uv}\mu$.*

Proof. Suppose $\max(\mu, \mu\mu^{uv}) < 1/4$. Let $U_{\epsilon_3,\delta}$ be the domain provided by the previous proposition; the function ψ_φ is defined in $U_{\epsilon_3,\delta}$. We consider $U_{\epsilon,\delta}$ such that $\exp(B(0,1)X(\varphi))(U_{\epsilon,\delta}) \subset U_{\epsilon_3,\delta}$. We define

$$\sigma = (\psi_\varphi, y)^{(-1)} \circ (\psi_{X(\varphi)}, y) \quad \text{and} \quad \sigma^{(-1)} = (\psi_{X(\varphi)}, y)^{(-1)} \circ (\psi_\varphi, y).$$

By the definition of ψ_φ we have $|\psi_\varphi - \psi_{X(\varphi)}| \leq 2\mu$. Thus $\sigma^{(-1)}(P)$ belongs to $\exp(\overline{B}(0, 2\mu)X(\varphi))(P)$ for all $P \in U_{\epsilon_3,\delta}$. That implies $|\psi_{X(\varphi)} \circ \sigma^{(-1)} - \psi_{X(\varphi)}| \leq 2\mu$ in $U_{\epsilon_3,\delta}$. The mappings σ_V provided by proposition 9.5.1 satisfy

$$||D((\psi_{X(\varphi)}, y) \circ \sigma_V^{(-1)} \circ (\psi_{X(\varphi)}, y)^{(-1)}) - Id||(\psi_{X(\varphi)}, y) \leq 2\mu^{uv}\mu$$

in $U_{\epsilon_3,\delta} \setminus [y = 0]$. That leads us to

$$||D((\psi_\varphi, y) \circ (\psi_{X(\varphi)}, y)^{(-1)}) - Id||(\psi_{X(\varphi)}, y) \leq 2\mu^{uv}\mu \tag{9.2}$$

in the domain $U_{\epsilon_3,\delta} \setminus [y = 0]$. Let $P \in U_{\epsilon,\delta} \setminus [y = 0]$; proceeding like in proposition 9.6.1 we find a unique $Q \in \exp(B(0,1)X(\varphi))(P)$ such that $\psi_\varphi(Q) = \psi_{X(\varphi)}(P)$. Since $\sigma^{(-1)}(Q) = P$ we deduce that

$$|\psi_{X(\varphi)} \circ \sigma - \psi_{X(\varphi)}| \leq 2\mu$$

in $U_{\epsilon,\delta} \setminus [yf = 0]$. The mappings σ, $\sigma^{(-1)}$ are well-defined C^∞ local diffeomorphisms in $U_{\epsilon,\delta} \setminus [yf = 0]$. Moreover, since $\sigma(P)$, $\sigma^{(-1)}(P)$ belong to $\exp(B(0,1)X(\varphi))(P)$ then σ and $\sigma^{(-1)}$ can be extended continuously to $[f/y^m = 0]$ as the identity mapping. Finally, the inequality 9.2 and $2\mu^{uv}\mu < 1/2$ imply

$$|||D((\psi_{X(\varphi)}, y) \circ \sigma \circ (\psi_{X(\varphi)}, y)^{(-1)} - Id||(\psi_{X(\varphi)}, y) \leq 4\mu^{uv}\mu$$

in $U_{\epsilon,\delta}$. □

Corollary 9.6.1. *Suppose $m > 0$. Let φ be a (NSD) diffeomorphism. Consider the tg-sp mapping σ conjugating α_φ and φ and provided by proposition 9.6.2. Then σ and $\sigma^{(-1)}$ admit a continuous extension to $y = 0$ such that $\sigma_{|y=0} \equiv Id$.*

Proof. We define $\sigma_{U_{\epsilon,\delta}\cap[y=0]} = \sigma^{(-1)}_{U_{\epsilon,\delta}\cap[y=0]} \equiv Id$. By prop. 9.6.2 we have

$$\{\sigma(P), \sigma^{(-1)}(P)\} \subset \exp(\overline{B}(0, 2\mu)X(\varphi))(P)$$

for all $P \in U_{\epsilon,\delta} \setminus [y = 0]$. Since $\exp(tX(\varphi))(Q)$ is continuous in t and Q then the mappings σ and $\sigma^{(-1)}$ are continuous in $U_{\epsilon,\delta} \cap [y = 0]$. $\square$

Remark 9.6.1. *When $(N, m) = (1, 0)$ we can choose $y_0 = 0$ and the result in proposition 9.5.1 is still true for some V neighborhood of 0. We can proceed as in proposition 9.6.2 to obtain that σ_V is a germ of homeomorphism such that it is C^∞ outside of $f = 0$.*

9.6.1. Proof of theorem 8.1 for $m > 0$ and $(N, m) = (1, 0)$. We already proved the sufficient condition. Since $SP(\varphi_1) = SP(\varphi_2)$ then we obtain $SP(X(\varphi_1)) = SP(X(\varphi_2))$. We denote by σ_j $(j \in \{1, 2\})$ the germ of homeomorphism conjugating α_{φ_j} and φ_j (see proposition 9.6.2, corollary 9.6.1 and remark 9.6.1). Since $Re(X(\varphi_1))$ and $Re(X(\varphi_2))$ are conjugated by a germ of homeomorphism σ' by theorem 6.1 then we define

$$\sigma = \sigma_2 \circ \sigma' \circ \sigma_1^{(-1)}.$$

The mapping σ is a germ of homeomorphism (corollary 9.6.1 and remark 9.6.1) conjugating φ_1 and φ_2. Since σ_j $(j \in \{1, 2\})$ and σ' are C^∞ outside of $[yf = 0]$ then the same property is satisfied by σ. For $(N, m) = (1, 0)$ the mapping σ is C^∞ in $U_{\epsilon,\delta} \setminus [f = 0]$.

9.7. Comparing tg-sp conjugations

We suppose from now on that $N > 1$ and $m = 0$. We already proved the existence of a tg-sp conjugation between α_φ and φ. Moreover, such a conjugation does not extend continuously to $y = 0$ since that would imply that $\varphi_{|y=0}$ is analytically trivial.

Suppose $SP(\varphi_1) = SP(\varphi_2)$; we can suppose that $\varphi_{1,|y=0} \equiv \varphi_{2,|y=0}$ up to an analytic change of coordinates (see proposition 9.1.1). We denote $X(\varphi_j)$ and $\psi_{X(\varphi_j)}$ by X_j and ψ_j respectively for $j \in \{1, 2\}$. We denote α_{φ_j} by α_j. We can choose $X_{1,|y=0} = X_{2,|y=0}$ by lemma 9.1.1. Let $k \in \mathbb{N}$ such that $f(x, y^k) = 0$ is the union of N curves $x = g_j(y)$ for $1 \leq j \leq N$. For $1 \leq j \leq N$ we define $Res^j_{1,2}(y) = (Res_{X_2} - Res_{X_1})(g_j(y), y)$. Let $(x - g_1(y))^{c_1} \ldots (x - g_N(y)^{c_N}$ be the decomposition of $f(x, y^k)$ in irreducible factors.

Lemma 9.7.1. *There is a choice of ψ_1 and ψ_2 such that $(\psi_2 - \psi_1)(x, y^k)$ is of the form*

$$\frac{\beta}{\prod_{1\leq j\leq N} (x - g_j(y))^{c_j - 1}} + \sum_{j=1}^{N} R^j_{1,2}(y) \ln(x - g_j(y))$$

for some $\beta \in \mathbb{C}\{x, y\} \cap (y)$.

Proof. The function β satisfies

$$\frac{\partial}{\partial x}\left(\frac{\beta}{\prod_{1\leq j\leq N} (x - g_j(y))^{c_j - 1}}\right) = \left(\frac{u_1 - u_2}{u_1 u_2 f}\right)(x, y^k) - \sum_{j=1}^{N} \frac{R^j_{1,2}(y)}{x - g_j(y)}.$$

Then $X_{1,|y=0} \equiv X_{2,|y=0}$ implies $u_1 - u_2 \in (y)$. Moreover $R_{1,2}^j \in (y)$ for all $1 \leq j \leq N$ since $SP(X_1) = SP(X_2)$. As a consequence the right-hand side of the equation is of the form $h(x,y)/f(x,y^k)$ where $h \in (y)$. The equation

$$\frac{\partial}{\partial x}\left(\frac{\beta'}{\prod_{1 \leq j \leq N}(x - g_j(y))^{c_j - 1}}\right) = \frac{h(x,y)/y}{f(x,y^k)}$$

is free of residues and then it admits a solution $\beta' \in \mathbb{C}\{x,y\}$. We define $\beta = y\beta'$. $\square$

As a consequence of the lemma we have $\psi_2 - \psi_1 = O(y^{1/k})$ in every compact simply connected set contained in the universal covering of $U_{\epsilon,\delta} \setminus [f = 0]$. Let σ' be the special homeomorphism conjugating $Re(X_1)$ and $Re(X_2)$ and constructed in chapter 6. We have

$$\sigma' = (\psi_2, y)^{(-1)} \circ (\psi_1', y)$$

where ψ_1' is a modification of ψ_1. Let $f = f_1^{n_1} \dots f_p^{n_p}$ be the decomposition in irreducible factors of f. We claim that

LEMMA 9.7.2. *There is a choice of ψ_1 and ψ_2 such that the function $f(\psi_2 - \psi_1')$ is a $O(f_1 \dots f_p y^{1/k})$ for some $k > 0$.*

PROOF. It is enough to prove the lemma for the modifications attached to the strips since a relation like $f(\psi_2 - \psi_1') = O(f_1 \dots f_p y^{1/k})$ is preserved by the partition of the unity process we use to paste them. Consider the notations in lemma 6.3.2. Let $k > 0$ such that $f(x,y^k) = 0$ is the union of N curves $x = g_j(y)$ for $1 \leq j \leq N$. By the proof of lemma 6.3.2 we have

$$f[\psi_2 - \psi_1'](x,y^k) - \beta(x,y)(x - g_1(y)) \dots (x - g_N(y))$$

is a $O(y(x - g_1(y)) \dots (x - g_N(y)))$. The function β is the one we obtained in the previous lemma. Therefore

$$f[\psi_2 - \psi_1'](x,y^k) = O(y(x - g_1(y)) \dots (x - g_N(y)))$$

and then $f[\psi_2 - \psi_1'] = O(y^{1/k} f_1 \dots f_p)$. $\square$

COROLLARY 9.7.1. *There exists a special germ of homeomorphism σ' conjugating $Re(X_1)$ and $Re(X_2)$ and such that $\sigma'_{|y=0} \equiv Id$.*

Let $Tr_2(s)$ be a trajectory of $Re(iX_2)$; we use $Tr_2(s)$ as a base transversal to construct a conjugation σ_{Tr}^2 between α_{φ_2} and φ_2. The curve $\sigma'^{(-1)}(Tr_2(s))$ is transversal to $Re(X_1)$; it is contained in a level set $Re(\psi_1') = h(s)$. The idea is replacing ψ_1 with ψ_1' and the function $\Delta_j^1 = \psi_1 \circ \varphi_1^{(j)} - (\psi_1 + j)$ with the function $\Delta_j' = \psi_1' \circ \varphi_1^{(j)} - (\psi_1' + j)$. We define $\Delta_j^2 = \psi_2 \circ \varphi_2^{(j)} - (\psi_2 + j)$ and $r = \psi_1' - \psi_1$.

We notice that we required the function ψ_1 only to fulfill three properties, namely $|\psi_1 \circ \varphi_1^{(j)} - (\psi_1 + j)| \leq \mu$,

$$||D(\Delta^1 \circ \alpha_1^{(-1)} \circ (\psi_1, s)^{(-1)})|| \leq \mu \text{ and } ||D(\Delta_j^1 \circ (\psi_1, s)^{(-1)})|| \leq \mu.$$

Analogous properties are also satisfied for ψ_1' and Δ_j'.

LEMMA 9.7.3. *Let $\mu > 0$. There exist $0 < \upsilon_0 < \upsilon_1$ and $\delta > 0$ such that*

$$\{\alpha_{\varphi_1}^{(0)}(x_0,y), \dots, \alpha_{\varphi_1}^{(j)}(x_0,y)\} \subset U_{\upsilon_0} \Rightarrow \{\varphi_1^{(0)}(x_0,y), \dots, \varphi_1^{(j)}(x_0,y)\} \subset U_{\upsilon_1}$$

for $j \in \mathbb{Z}$ and $(x_0,y) \in U_{\upsilon_0,\delta}$. Moreover, we can obtain

- $|\Delta_j'(x_0,y)| \leq \mu$.

- $||D(r \circ (\psi_1, y)^{(-1)}, y)|| \circ (\psi_1, y) = O(y^{1/k})$ *in* $U_{\upsilon_0,\delta}$.
- $||D(\Delta' \circ \alpha_1^{(-1)} \circ (\psi_1', y)^{(-1)}, y)|| \circ (\psi_1', y) \leq \mu$ *in* $U_{\upsilon_0,\delta}$.
- $|\Delta_j' - \Delta_j^1|(x_0, y) \leq \iota(y)$.
- $||D(\Delta_j' \circ (\psi_1', y)^{(-1)})||(\psi_1'(x_0, y), y) \leq \mu$.

where $\iota(y) = O(y^{1/k})$ *does not depend on* (x_0, y) *or* $j \in \mathbb{Z}$.

PROOF. By theorem 7.1 we can choose υ_0 and υ_1 such that $|\Delta_j^1| \leq \mu/2$ in $\exp(\overline{B}(0,2)X_1)(U_{\upsilon_0,\delta})$. By Cauchy's formula we deduce that

$$||D(\Delta_j^1 \circ (\psi_1, y)^{(-1)})|| \leq \mu/2$$

and

$$||D(\Delta^1 \circ \alpha_1^{(-1)} \circ (\psi_1, y)^{(-1)})|| \leq \mu/2$$

in $\exp(\overline{B}(0,1)X_1)(U_{\upsilon_0,\delta})$. We remind the reader that $\Re X_1(\psi_1' - \psi_1) = 0$ whereas $\Im X_1(\psi_1' - \psi_1) = O(y^{1/k})$. As a consequence

$$\frac{\partial(r \circ (\psi_1, y)^{(-1)})}{\partial x_1} = \Re X_1(\psi_1' - \psi_1) = 0$$

and

$$\frac{\partial(r \circ (\psi_1, y)^{(-1)})}{\partial x_2} = \Im X_1(\psi_1' - \psi_1) = O(y^{1/k}) = \iota'(y)$$

for $(x, y) = (x_1 + ix_2, y)$ in $U_{\upsilon_0,\delta}$. Since $|(\psi_1 \circ \varphi_1^{(j)} - \psi_1) - j| \leq \mu/2$ then

$$|r \circ \varphi_1^{(j)} - r| \leq (\mu/2)\iota'(y) = O(y^{1/k}).$$

The equation $\Delta_j' - \Delta_j^1 = r \circ \varphi_1^{(j)} - r$ implies

$$|\Delta_j' - \Delta_j^1|(x_0, y) \leq (\mu/2)\iota'(y) = O(y^{1/k}).$$

For $\delta > 0$ small enough we obtain $|\Delta_j'(x_0, y)| \leq \mu$. Since

$$\alpha_1^{(-1)} \circ (\psi_1', y)^{(-1)} = (\psi_1', y)^{(-1)} \circ (z - 1, y)$$

then to conclude the proof is enough to bound $||D(\Delta_j' \circ (\psi_1', y)^{(-1)})||$ in the set $\exp(\overline{B}(0,1)X_1)(U_{\upsilon_0,\delta})$. We have

$$\Delta_j' \circ (\psi_1', y)^{(-1)} = [(\Delta_j^1 + r \circ \varphi_1^{(j)} - r) \circ (\psi_1, y)^{(-1)}] \circ [(\psi_1, y) \circ (\psi_1', y)^{(-1)}].$$

Since

$$(\psi_1', y) \circ (\psi_1, y)^{(-1)}(z, y) = (z + r \circ (\psi_1, y)^{(-1)}(z, y), y)$$

then $||D((\psi_1', y) \circ (\psi_1, y)^{(-1)} - Id)|| = ||D(r \circ (\psi_1, y)^{(-1)})||$. We have

$$r \circ \varphi_1^{(j)} \circ (\psi_1, y)^{(-1)} = (r \circ (\psi_1, y)^{(-1)}) \circ ((\psi_1, y) \circ \varphi_1^{(j)} \circ (\psi_1, y)^{(-1)}).$$

We can develop the previous expression to obtain

$$r \circ \varphi_1^{(j)} \circ (\psi_1, y)^{(-1)} = (r \circ (\psi_1, y)^{(-1)}) \circ (z + j + \Delta_j^1 \circ (\psi_1, y)^{(-1)}, y).$$

All the previous work lead us to

$$||D(\Delta_j' \circ (\psi_1', y)^{(-1)})|| \leq [\mu/2 + (1 + \mu/2)O(y^{1/k}) + O(y^{1/k})](1 + O(y^{1/k}))$$

and then we obtain $||D(\Delta_j' \circ (\psi_1', y)^{(-1)})|| \leq \mu$ for $\delta > 0$ small enough. □

9.7.1. Setup. We can suppose that the domains U_{υ_0} and U_{υ_1} provided by lemma 9.7.3 satisfy $|\psi_2 \circ \varphi_2^{(j)} - (\psi_2 + j)| \leq \mu$,

$$||D(\Delta^2 \circ \alpha_2^{(-1)} \circ (\psi_2, y)^{(-1)})|| \leq \mu \text{ and } ||D(\Delta_j^2 \circ (\psi_2, y)^{(-1)})|| \leq \mu$$

in $U_{\upsilon_0,\delta}$ by shrinking these domains if necessary.

There exists $0 < \epsilon' < \upsilon_0$ such that

$$\tilde{U}_{\epsilon',\delta} \cup \sigma'(\tilde{U}_{\epsilon',\delta}) \cup \sigma'^{(-1)}(\tilde{U}_{\epsilon',\delta}) \subset U_{\upsilon_0,\delta}$$

for $\tilde{U}_{\epsilon,\delta} = \exp(B(0,4)X(\varphi_2))(U_{\epsilon,\delta})$. We want to choose some $0 < \epsilon_0 < \epsilon'$ satisfying the conditions in subsection 9.2.1 with respect to the vector field X_2. We will consider domains $W_\epsilon^2(\eta)$ for $\epsilon \leq \epsilon_0$ and $0 \leq \eta_j < 1$ for all $1 \leq j \leq N_T$. Hence $\partial W_\epsilon^2(\eta) \cap [y = s]$ is the union of sub-trajectories of $Re(X_2)$ and $Re(iX_2) = 0$. We define $W_\epsilon^1(\eta) = \sigma'^{(-1)}(W_\epsilon^2(\eta))$.

Given a sub-trajectory $Tr_2(s)$ of $Re(iX_2)$ the definition of $D_{Tr}^2(s)$ is the usual one, namely

$$D_{Tr}^2(s) = \exp([-1,1]X_2)\left(\Gamma_{\xi(X_2)}^{\overline{W_\epsilon^2(\eta+\kappa-3)}}[Tr^2(s)] \cap \overline{W_\epsilon^2(\eta)}\right).$$

Then $Tr_1(s) = \sigma'^{(-1)}(Tr_2(s))$ is transversal to $Re(X_1)$ even if it is not anymore a sub-trajectory of $Re(iX_1)$. We define $D_{Tr}^1(s) = \sigma'^{(-1)}(D_{Tr}^2(s))$. For a choice of a transversal $\cup_{s\in V} Tr_2(s)$ we obtain that proposition 9.3.1 can be applied to obtain conjugations σ_{Tr}^1 and σ_{Tr}^2 defined in $\cup_{s\in V} D_{Tr}^1(s)$ and $\cup_{s\in V} D_{Tr}^2(s)$ respectively.

9.7.2. Approaching $y = 0$. Next lemma is the key tool to prove that we can find σ_1 and σ_2 behaving in a similar way when $y \to 0$ and such that σ_j is a tg-sp mapping conjugating α_j and φ_j for $j \in \{1,2\}$.

LEMMA 9.7.4. *Let $\tau > 0$. There exists $\zeta > 0$ and $c_0 > 0$ such that for $(x_2, y_0) \in U_{\epsilon',c_0}$ and $j \in \mathbb{Z}$ satisfying*

$$\{\alpha_2^{(0)}(x_2,y_0), \ldots, \alpha_2^{(j)}(x_2,y_0)\} \subset U_{\epsilon'}$$

then $|\Delta_j^2(x_2,y_0) - \Delta_j'(x_1,y_0)| < \tau$ if $\sigma'(x_1,y_0) \in \exp(B(0,\zeta)X_2)(x_2,y_0)$. Moreover, we have

$$\sigma' \circ \varphi_1^{(j)}(x_1,y_0) \in \exp(B(0, |\psi_2(x_2,y_0) - \psi_1'(x_1,y_0)| + \tau)X_2)(\varphi_2^{(j)}(x_2,y_0)).$$

PROOF. Since

$$\begin{aligned}\psi_2 \circ \sigma' \circ \varphi_1^{(j)}(x_1,y_0) - \psi_2 \circ \varphi_2^{(j)}(x_2,y_0) &= \psi_1' \circ \varphi_1^{(j)}(x_1,y_0) - \psi_2 \circ \varphi_2^{(j)}(x_2,y_0) \\ &= (\psi_1'(x_1,y_0) - \psi_2(x_2,y_0)) + (\Delta_j'(x_1,y_0) - \Delta_j^2(x_2,y_0))\end{aligned}$$

then it is enough to prove $|\Delta_j^2(x_0,y_0) - \Delta_j'(x_1,y_0)| < \tau$.

We can suppose $j > 0$ without lack of generality. We suppose that $\tau < 1$ since it is enough to prove the result for $\tau > 0$ small. We denote $|\psi_2(x_2,y_0) - \psi_1'(x_1,y_0)|$ by d; we suppose $d < 1/2$. We obtain that

$$\{\alpha_2^{(0)} \circ \sigma'(x_1,y_0), \ldots, \alpha_2^{(j)} \circ \sigma'(x_1,y_0)\} \subset \exp(B(0,1/2)X_2)(U_{\epsilon',\delta}).$$

That leads us to

$$\{\alpha_1^{(0)}(x_1,y_0), \ldots, \alpha_1^{(j)}(x_1,y_0)\} \subset \sigma'^{(-1)}(\exp(B(0,1/2)X_2)(U_{\epsilon',\delta})) \subset U_{\upsilon_0}.$$

As a consequence $|\Delta_j^2(x_0,y_0) - \Delta_j'(x_1,y_0)|$ is well defined for $d < 1/2$.

To prove the lemma we split U_{υ_0} in two sets $\overline{U_\upsilon}$ and $U_{\upsilon_0} \setminus U_\upsilon$. The value of $\upsilon > 0$ will be determined later on. Our idea is splitting $\exp([0,j]X_2)(x_2,y_0)$ in pieces contained in either $\overline{U_\upsilon}$ or $U_{\upsilon_0} \setminus U_\upsilon$. Depending on the set we will use different methods in order to bound $|\Delta_j^2(x_2, y_0) - \Delta_j'(x_1, y_0)|$.

Let $\upsilon > 0$ such that $\exp([a,b]X_2)(P) \subset \overline{U_\upsilon}$ for $\{a,b\} \subset [0,j] \cap \mathbb{Z}$ with $a \leq b$ implies

$$\left| \sum_{l=a-1}^{h} \Delta^2 \circ \varphi_2^{(l)}(P) \right| < \frac{\tau}{2C_0} \text{ for } a-1 \leq h \leq b-1.$$

We will choose a precise value for $C_0 > 0$ later on.

For $\exp([a,b]X_2)(x_2, y_0) \subset \overline{U_\upsilon}$ we have

$$\exp([a,b]X_1)(x_1, y_0) \subset \sigma'^{(-1)}[\exp(B(0,1/2)X_2)(U_{\upsilon,\delta})] \subset U_{\upsilon',\delta}$$

where $\lim_{(\upsilon,\delta)\to(0,0)} \upsilon'(\upsilon,\delta) = 0$ since σ' is a homeomorphism. We can choose υ such that $\exp([a',b']X_1)(Q) \subset \overline{U_{\upsilon'}}$ implies

$$|\Delta^1_{h-a'+2}(\varphi_1^{(a'-1)}(Q))| = \left| \sum_{l=a'-1}^{h} \Delta^1 \circ \varphi_1^{(l)}(Q) \right| < \frac{\tau}{4C_0}$$

for $a'-1 \leq h \leq b'-1$. By lemma 9.7.3 we obtain that

$$\left| \sum_{l=a'-1}^{h} \Delta' \circ \varphi_1^{(l)}(Q) \right| < \frac{\tau}{2C_0} \text{ for } a'-1 \leq h \leq b'-1$$

if $y(Q)$ is close to 0.

Now suppose that $\exp([a,b]X_2)(x_2, y_0) \subset U_{\epsilon'} \setminus U_\upsilon$. Such a thing implies that

$$[\exp([a-1,b]X_1)(x_1,y_0) \cup \{\varphi_1^{(a-1)}(x_1,y_0), \ldots, \varphi_1^{(b)}(x_1,y_0)\}] \cap U_{\upsilon_2} = \emptyset$$

for some $\upsilon_2 > 0$ independent of the choices of a, b, (x_0, y_0) and (x_1, y_0).

The sub-trajectory $\exp([0,j]X_2)(x_0,y_0)$ splits in at most $N_T + 1$ trajectories contained in either $\overline{U_\upsilon}$ or $U_{\upsilon_0} \setminus U_\upsilon$ since the number of tangent points between $Re(X_2)_{|y=s}$ and $\partial U_\upsilon \cap [y=s]$ is exactly N_T. The sub-trajectories $\exp([0,l]X_2)(P)$ contained in $U_{\upsilon_0} \setminus U_\upsilon$ satisfy that l is uniformly bounded by a constant $C > 0$ independent of P. We define

$$\tau_h = \frac{\tau}{(1+\mu)^{2(N_T+1-h)(C+1)}}$$

for $1 \leq h \leq N_T + 1$. We choose $C_0 > 0$ such that $\tau_{h+1} - \tau_h > \tau/C_0$ for all $1 \leq h \leq N_T$. Let $a_0 = -1$; we define recursively

$$\gamma_{h+1} = \exp([a_h+1, a_{h+1}]X_2)(x_2,y_0) \quad (\{a_h, a_{h+1}\} \subset \mathbb{Z})$$

such that $\gamma_{h+1} \subset \overline{U_\upsilon}$ or $\gamma_{h+1} \subset U_{\epsilon'} \setminus U_\upsilon$ but the respective condition is not fulfilled for $\exp([a_h+1, a_{h+1}+1]X_2)(x_2,y_0)$. We obtain a curve γ_h for all $1 \leq h \leq L$ and some $L \leq N_T + 1$; we also have $a_L = j$. We define $D_b = \Delta_b^2(x_2,y_0) - \Delta_b'(x_1,y_1)$ and $D_0 = 0$; we have

$$D_b = D_{b-1} + [\Delta^2 \circ \varphi_2^{(b-1)}(x_2,y_0) - \Delta_j' \circ \varphi_1^{(b-1)}(x_1,y_1)].$$

Our goal is proving that for d close to 0 we have

$$|D_1| < \tau_1, \ldots, |D_{a_1}| < \tau_1, \ldots, |D_{a_{L-1}+1}| < \tau_L, \ldots, |D_{a_L}| < \tau_L.$$

That would prove the lemma since $\tau_1 < \ldots < \tau_L \leq \tau$.

We will proceed by induction. Suppose $|D_1| < \tau_1, \ldots, |D_{a_l}| < \tau_l$ for $d < d_l$ and $y_0 \in B(0, c_0^l)$. If $\gamma_{l+1} \subset \overline{U_\upsilon}$ then

$$|D_h| \leq |D_{a_l}| + \left| \sum_{q=a_l}^{h-1} \Delta^2 \circ \varphi_2^{(q)}(x_2, y_0) \right| + \left| \sum_{q=a_l}^{h-1} \Delta' \circ \varphi_1^{(q)}(x_1, y_0) \right|.$$

for all $a_l + 1 \leq h \leq a_{l+1}$. We have $|D_h| < \tau_l + 2\tau/(2C_0) < \tau_{l+1}$ for $d < d_{l+1} < d_l$ and $|y_0| < c_0^{l+1} < c_0^l$ by our choice of $C_0 > 0$. Suppose now $\gamma_{l+1} \subset U_{\epsilon'} \setminus U_\nu$. We have

$$|D_{a_l+h+1}| \leq |D_{a_l+h}| + |\Delta^2 \circ \varphi_2^{(a_l+h)}(x_2, y_0) - \Delta' \circ \varphi_1^{(a_l+h)}(x_1, y_0)|$$

for $0 \leq h \leq a_{l+1} - a_l - 1 \leq C$. The difference $\Delta' - \Delta^1$ is a $O(y^{1/k})$ by lemma 9.7.3. On the other hand $\Delta^1 - \Delta^2$ is a holomorphic function whose value at $y = 0$ is identically 0; therefore $\Delta' - \Delta^2$ is a $O(y^{1/k})$. We obtain

$$|D_{a_l+h+1}| \leq |D_{a_l+h}| + |\Delta_2 \circ \varphi_2^{(a_l+h)}(x_2, y_0) - \Delta_2 \circ \varphi_1^{(a_l+h)}(x_1, y_0)| + O(y_0^{1/k}).$$

We have

$$|\psi_2 \circ \varphi_2^{(a_l+h)}(x_2, y_0) - \psi_2 \circ \sigma' \circ \varphi_1^{(a_l+h)}(x_1, y_0)| \leq d + |D_{a_l+h}|.$$

We also have that $\psi_2 \circ \sigma' - \psi_2 = o(1)$ in the complementary of U_{υ_2} since $\sigma'_{|y=0} \equiv Id$ (the notation $o(1)$ stands for a function tending to 0 when $y \to 0$). That implies

$$|\psi_2 \circ \varphi_2^{(a_l+h)}(x_2, y_0) - \psi_2 \circ \varphi_1^{(a_l+h)}(x_1, y_0)| \leq d + |D_{a_l+h}| + o(1).$$

Since $||D(\Delta^2 \circ (\psi_2, y)^{(-1)})|| \leq \mu$ then

$$|D_{a_l+h+1}| \leq |D_{a_l+h}| + \mu(d + |D_{a_l+h}| + o(1)) + O(y_0^{1/k})$$

for $0 \leq h \leq a_{l+1} - a_l - 1 \leq C$. Now suppose

$$|D_{a_l+h}| \leq \tau_{l+1} \frac{1}{(1+\mu)^{2(C+1-h)}}$$

for $d < d_{l+1}^h \leq d_l$ and $|y_0| < c_0^{l+1,h} \leq c_0^l$; that result is clearly true for $h = 0$, $d_{l+1}^0 = d_l$ and $c_0^{l+1,0} = c_0^l$ by the choice of τ_l and τ_{l+1}. Then

$$|D_{a_l+h+1}| \leq \frac{1}{1+\mu} \tau_{l+1} \frac{1}{(1+\mu)^{2(C+1-(h+1))}} + \mu d + o(1).$$

We obtain

$$|D_{a_l+h+1}| \leq \tau_{l+1} \frac{1}{(1+\mu)^{2(C+1-(h+1))}}.$$

for $d < d_{l+1}^{h+1} \leq d_l^h$ and $|y_0| < c_0^{l+1,h+1} \leq c_0^{l+1,h}$. The proof is complete; we just define $d_{l+1} = \min_{0 \leq h \leq a_{l+1}-a_l} d_{l+1}^h$ and $c_0^{l+1} = \min_{0 \leq h \leq a_{l+1}-a_l} c_0^{l+1,h}$. □

9.7.3. Constructing a special conjugation. Consider $y_0 \in B(0, \delta)$ and a domain $W_\epsilon^2(\eta)$ such that $W_\epsilon^2(\eta) \cap [y = y_0]$ does not have bi-tangent cords. We consider a neighborhood V of y_0 fulfilling the pre-requisites of the algorithm solving the game with respect to X_2. Let $\cup_{s \in V} Tr_2(s)$ one of the transversals we use throughout the game to build a special conjugation σ_{Tr}^2 between α_2 and φ_2 defined in $\cup_{s \in V} D_{Tr}^2(s)$. Then

LEMMA 9.7.5. *We have*

$$|\psi_2 \circ \sigma^2_{Tr} \circ \sigma' \circ \sigma^{1(-1)}_{Tr} - \psi'_1| \leq H(y)$$

in $\cup_{s \in V} \sigma^1_{Tr}(D^1_{Tr}(s))$*. Moreover* $H(y)$ *is a* $o(1)$*; it does not depend on* y_0 *or* V*.*

PROOF. We denote

$$\psi_2(Tr_2(s)) = \psi'_1(Tr_1(s)) = c(s) + i[d(s), e(s)]$$

for $s \in V$. We consider the functions A_2 and B_2 defined as in section 9.3 with respect to X_2 and ψ_2. Analogously we define A_1 and B_1 with respect to X_1 and ψ'_1. We have

$$A_1(z, y) = (z + \Delta' \circ \alpha_1^{(-1)} \circ (\psi'_1, y)^{(-1)}(z, y), y)$$

and

$$A_2(z, y) = (z + \Delta^2 \circ \alpha_2^{(-1)} \circ (\psi_2, y)^{(-1)}(z, y), y);$$

both mappings are defined in $z \in (c(s) + [-1/3, 4/3]) + i[d(s), e(s)]$. We define

$$(w_1, y) = \alpha_1^{(-1)} \circ (\psi'_1, y)^{(-1)}(z, y) \text{ and } (w_2, y) = \alpha_2^{(-1)} \circ (\psi_2, y)^{(-1)}(z, y).$$

The definition implies $\sigma'(w_1, y) = (w_2, y)$. Since $\sigma'_{|y=0} \equiv Id$ then $w_2 - w_1 = o(1)$. That leads us to

$$\Delta^2(w_2, y) - \Delta'(w_1, y) = \Delta^2(w_1, y) - \Delta'(w_1, y) + o(1) = o(1)$$

since $\Delta^2 - \Delta' = (\Delta^2 - \Delta^1) + (\Delta^1 - \Delta') = O(y^{1/k})$. As a consequence we have $z \circ A_1 - z \circ A_2 = o(1)$. Since B_l is obtained by interpolating A_l and Id then $z \circ B_1 - z \circ B_2 = o(1)$ in $z \in (c(s) + (-1/3, 4/3)) + i[d(s), e(s)]$; this is equivalent to

$$|\psi_2 \circ \sigma^2_{Tr} \circ (\psi_2, y)^{(-1)}(z, y) - \psi'_1 \circ \sigma^1_{Tr} \circ (\psi'_1, y)^{(-1)}(z, y)| \leq H^2(y) = o(1)$$

in $\cup_{s \in V}((c(s) + (-1/3, 4/3)) + i[d(s), e(s)])$.

We will extend the result to the remaining part of $\cup_{s \in V} \psi_2(D^2_{Tr}(s))$. Let $(w_2, y) \in D^2_{Tr}(y)$; there exists a number $j \in \mathbb{Z}$ such that

$$\alpha_2^{(j)}(w_2, y) \in \exp((-1/3, 4/3)X_2)(Tr_2(y)).$$

We denote the point $\alpha_2^{(j)}(w_2, y)$ by (w'_2, y). We also denote

$$(w_1, y) = \sigma'^{(-1)}(w_2, y) \text{ and } (w'_1, y) = \sigma'^{(-1)}(w'_2, y).$$

We have that $\psi_2 \circ \sigma^2_{Tr}(w_2, y) - \psi'_1 \circ \sigma^1_{Tr}(w_1, y)$ is equal to

$$(\psi_2 \circ \sigma^2_{Tr}(w'_2, y) - \psi'_1 \circ \sigma^1_{Tr}(w'_1, y)) + (\Delta^2_{-j} \circ \sigma^2_{Tr}(w'_2, y) - \Delta'_j \circ \sigma^1_{Tr}(w'_1, y)).$$

We have $\psi_2 \circ \sigma^2_{Tr}(w'_2, y) - \psi_2 \circ \sigma' \circ \sigma^1_{Tr}(w'_1, y) = o(1)$ by the first part of the proof. Lemma 9.7.4 implies that

$$\psi_2 \circ \sigma^2_{Tr}(w_2, y) - \psi'_1 \circ \sigma^1_{Tr}(w_1, y) = o(1)$$

and then

$$\psi_2 \circ \sigma^2_{Tr} \circ (\psi_2, y)^{(-1)} - \psi'_1 \circ \sigma^1_{Tr} \circ (\psi'_1, y)^{(-1)} = o(1)$$

in $\cup_{s \in V} \psi_2(D^2_{Tr}(s)) \times \{s\}$. □

Now, suppose that we want to paste two conjugations

$$\sigma_{\downarrow} = \sigma^2_{\downarrow} \circ \sigma' \circ \sigma^{1(-1)}_{\downarrow} \text{ and } \sigma_{\downarrow} = \sigma^2_{\uparrow} \circ \sigma' \circ \sigma^{1(-1)}_{\uparrow}.$$

The conjugations σ^2_j are constructed taking base transversals Tr^j_2 for $j \in \{\uparrow, \downarrow\}$ whereas σ^1_j are constructed taking base transversals $\sigma^{'(-1)}(Tr^j_2)$. We suppose that $D^2_{Tr^{\uparrow}}(s) \cap D^2_{Tr^{\downarrow}}(s)$ contains a strip $B^2_1(s)$ for $s \in V$ where

$$\psi_2(B^2_{\zeta}(s)) = [z \in [a_{\leftarrow}(s) - \zeta, a_{\rightarrow}(s) + \zeta] + i[c_{\downarrow}(s), c_{\uparrow}(s)]]$$

and $c_{\uparrow} - c_{\downarrow} \equiv M$. We define $B^1_{\zeta}(s) = \sigma^{'(-1)}(B^2_{\zeta}(s))$. We use the M-interpolation process to conjugate $\sigma^j_{\uparrow}$ and $\sigma^j_{\downarrow}$ to obtain σ^j for $j \in \{1, 2\}$.

Lemma 9.7.6. *Suppose* $|\psi_2 \circ \sigma_l - \psi'_1| \leq H'(y) = o(1)$ *in* $\cup_{s \in V} \sigma^1_l(B^1_1(s))$ *for* $l \in \{\uparrow, \downarrow\}$ *and a function* H' *independent of* l, y_0 *or* V. *Then*

$$|\psi_2 \circ \sigma^2 \circ \sigma' \circ \sigma^{1(-1)} - \psi'_1| \leq J(y)$$

in $\cup_{s \in V} \sigma^1(B^1_0(s))$. *Moreover* $J(y)$ *is a* $o(1)$; *it does not depend on* y_0 *or* V.

Proof. We use the notations in section 9.4. We choose $\mu > 0$ such that $\max(\mu, \mu\mu^{uv}) < 1/16$. In $D^2(s) \subset \sigma^2_{\downarrow}(B^2_{\downarrow}(s)) \cup \sigma^2_{\uparrow}(B^2_{\uparrow}(s))$ there is an integral of the time form $\psi^2_{\updownarrow}$ of φ_2 such that

$$\psi^2_{\updownarrow} = \eta^2_D(\psi_2 \circ \sigma^{2(-1)}_{\downarrow}) + (1 - \eta^2_D)\psi_2 \circ \sigma^{2(-1)}_{\uparrow}$$

and $\eta^2_D(x, s) = \eta(\psi_2 \circ \sigma^{2(-1)}_{\downarrow}(x, s) - ic_{\downarrow}(s))$. In an analogous way we define

$$\psi^1_{\updownarrow} = \eta^1_D(\psi'_1 \circ \sigma^{1(-1)}_{\downarrow}) + (1 - \eta^1_D)\psi'_1 \circ \sigma^{1(-1)}_{\uparrow}$$

where $\eta^1_D(x, s) = \eta(\psi'_1 \circ \sigma^{1(-1)}_{\downarrow}(x, s) - ic_{\downarrow}(s))$. Then we have $\sigma^2 = (\psi^2_{\updownarrow}, y)^{(-1)} \circ (\psi_2, y)$ whereas $\sigma^1 = (\psi^1_{\updownarrow}, y)^{(-1)} \circ (\psi'_1, y)$. Since

$$\psi_2 \circ \sigma^2 \circ \sigma' \circ \sigma^{1(-1)} - \psi'_1 = \psi_2 \circ (\psi^2_{\updownarrow}, y)^{(-1)} \circ (\psi^1_{\updownarrow}, y) - \psi'_1$$

then it is enough to estimate the right hand side.

Let $E_{\downarrow}(s) = \sigma^1(B^1_0(s)) \cap [Img\psi'_1 \leq c_{\downarrow}(s) + 5]$. Since

$$\sigma^1(B^1_0(s)) \subset \sigma^1_{\downarrow}(B^1_{\downarrow}(s)) \cup \sigma^1_{\uparrow}(B^1_{\uparrow}(s))$$

then $E_{\downarrow}(s) \cap \sigma^1_{\uparrow}(B^1_{\uparrow}(s)) = \emptyset$ implies $E_{\downarrow}(s) \subset \sigma^1_{\downarrow}(B^1_{\downarrow}(s))$. The former propriety is a consequence of

$$\sigma^1_{\uparrow}(B^1_{\uparrow}(s)) \subset [Img\psi'_1 \geq c_{\downarrow}(s) + M/4 - 1/2]$$

and $5 < M/4 - 1/2$. As a consequence we have $\psi^1_{\updownarrow} = \psi'_1 \circ \sigma^{1(-1)}_{\downarrow}$ and $\eta^1_D \equiv 1$ in $\cup_{s \in V} E_{\downarrow}(s)$. By definition we have $\eta^2_D \circ \sigma_{\downarrow} \equiv \eta^1_D$ in $\cup_{s \in V}[\sigma^1_{\downarrow}(B^1_{\downarrow}(s)) \cap \sigma^1_{\downarrow}(B^1_0(s))]$; moreover

$$\psi_2 \circ \sigma^{2(-1)}_{\downarrow} \circ \sigma_{\downarrow} = \psi'_1 \circ \sigma^{1(-1)}_{\downarrow}$$

in $\cup_{s \in V} \sigma^1_{\downarrow}(B^1_{\downarrow}(s))$. We deduce that $\psi^2_{\updownarrow} \circ \sigma_{\downarrow} = \psi^1_{\updownarrow}$ in $\cup_{s \in V} E_{\downarrow}(s)$. As a consequence we obtain $\sigma^2 \circ \sigma' \circ \sigma^{1(-1)} = \sigma_{\downarrow}$ in $\cup_{s \in V} E_{\downarrow}(s)$ and then

$$\psi_2 \circ \sigma^2 \circ \sigma' \circ \sigma^{1(-1)} - \psi'_1 = o(1)$$

in $\cup_{s \in V} E_{\downarrow}(s)$.

Consider the set $E_{\uparrow}(s) = \sigma^1(B^1_0(s)) \cap [Img\psi'_1 \geq c_{\uparrow}(s) - 5]$. We can prove $E_{\uparrow}(s) \cap \sigma^1_{\downarrow}(B^1_{\downarrow}(s)) = \emptyset$ in an analogous way than in the previous paragraph. Hence, we obtain $\eta_D^1 \equiv 0$ in $\cup_{s\in V} E_\uparrow(s)$. We have $\sigma_\uparrow(E_\uparrow(s)) \subset [Img\psi_2 \geq c_\uparrow(s)-5-2(1/2)]$; moreover $\sigma_\uparrow(E_\uparrow(s)) \subset \sigma_\uparrow^2(B_\uparrow^2(s))$ since

$$\sigma_\downarrow^2(B_\downarrow^2(s)) \subset [Img\psi_2 \leq c_\uparrow(s) - M/4 + 1/2]$$

and $-5-1/2-1/2 > -M/4+1/2$. Hence, we obtain $\eta_D^2 \equiv 0$ in $\sigma_\uparrow(E_\uparrow(s))$. Moreover, that implies $\psi_\updownarrow^2 \circ \sigma_\uparrow = \psi_\updownarrow^1$ in $\cup_{s\in V} E_\uparrow(s)$ and then

$$\psi_2 \circ \sigma^2 \circ \sigma' \circ \sigma^{1(-1)} - \psi_1' = o(1) \ \text{ in } \cup_{s\in V} E_\uparrow(s).$$

Finally, consider the set

$$E(s) = \sigma^1(B_0^1(s)) \cap [Img\psi_1' \in [c_\downarrow(s)+4, c_\uparrow(s)-4].$$

The set $E(s)$ is contained in $\sigma_\downarrow^1(B_1^1(s)) \cap \sigma_\uparrow^1(B_1^1(s))$. As a consequence $\sigma_\downarrow$ and $\sigma_\uparrow$ are defined in $E(s)$ for $s \in V$. We have

$$\psi_\updownarrow^2 \circ \sigma_\downarrow - \psi_\updownarrow^1 = (1-\eta_D^1)(\psi_2 \circ \sigma_\uparrow^{2(-1)} \circ \sigma_\downarrow - \psi_1' \circ \sigma_\uparrow^{1(-1)})$$

which can be expressed also as

$$\psi_\updownarrow^2 \circ \sigma_\downarrow - \psi_\updownarrow^1 = (1-\eta_D^1)(\psi_2 \circ \sigma_\uparrow^{2(-1)} \circ \sigma_\downarrow - \psi_2 \circ \sigma_\uparrow^{2(-1)} \circ \sigma_\uparrow).$$

The relations $\psi_2 \circ \sigma_\uparrow - \psi_2 \circ \sigma_\downarrow = \psi_1' - \psi_1' + o(1) = o(1)$ and

$$||D(\psi_2 \circ \sigma_\uparrow^{2(-1)} \circ (\psi_2, y)^{(-1)}) - Id|| < 2\mu\mu^{uv}$$

imply

$$\psi_\updownarrow^2 \circ \sigma_\downarrow - \psi_\updownarrow^1 = o(1).$$

Since $\psi_\updownarrow^2 \circ \sigma^2 \circ \sigma' \circ \sigma^{1(-1)} = \psi_\updownarrow^1$ we deduce that

$$\psi_\updownarrow^2 \circ \sigma^2 \circ \sigma' \circ \sigma^{1(-1)} - \psi_\updownarrow^2 \circ \sigma_\downarrow = o(1).$$

We use $||D(\psi_2 \circ (\psi_\updownarrow^2, y)^{(-1)}) - Id|| \leq \mu\mu^{uv}$ to prove

$$\psi_2 \circ \sigma^2 \circ \sigma' \circ \sigma^{1(-1)} - \psi_2 \circ \sigma_\downarrow = o(1).$$

Since

$$\psi_2 \circ \sigma^2 \circ \sigma' \circ \sigma^{1(-1)} - \psi_1' = (\psi_2 \circ \sigma_\downarrow - \psi_1') + (\psi_2 \circ \sigma^2 \circ \sigma' \circ \sigma^{1(-1)} - \psi_2 \circ \sigma_\downarrow)$$

then we obtain

$$\psi_2 \circ \sigma^2 \circ \sigma' \circ \sigma^{1(-1)} - \psi_1' = o(1) + o(1) = o(1)$$

in $\cup_{s\in V} E(s)$ as we wanted to prove. □

Now we consider the diffeomorphisms σ_V^j conjugating α_j and φ_j in $U_{\epsilon,\delta} \cap [y \in V]$ for $j \in \{1,2\}$. An iterative application of the previous lemma allows to prove

COROLLARY 9.7.2. *Let $\mu > 0$ small enough. We have*

$$|\psi_2 \circ \sigma_V^2 \circ \sigma' \circ \sigma_V^{1(-1)} - \psi_1'| \leq L(y) = o(1)$$

for some function L not depending on V.

Let us define

$$\psi^2 = \sum_{V\in J} h_V(y)(\psi_2 \circ \sigma_V^{2(-1)}) \text{ and } \psi^1 = \sum_{V\in J} h_V(y)(\psi_1' \circ \sigma_V^{1(-1)}).$$

The mapping $\sigma = (\psi^2, y)^{(-1)} \circ (\psi^1, y)$ is a tg-sp conjugation between φ_1 and φ_2.

LEMMA 9.7.7. *The mapping σ extends to a germ of homeomorphism in a neighborhood of $(0,0)$ by defining $\sigma_{|y=0} \equiv Id$.*

PROOF. We define $\sigma_V = \sigma_V^2 \circ \sigma' \circ \sigma_V^{1(-1)}$. We have

$$\psi^2 \circ \sigma' - \psi^1 = \sum_{V \in J} h_V(y)[\psi_2 \circ \sigma_V^{2(-1)} \circ \sigma' - \psi_1' \circ \sigma_V^{1(-1)}].$$

We can express the previous equation in the form

$$\psi^2 \circ \sigma' - \psi^1 = \sum_{V \in J} h_V(y)[\psi_2 \circ \sigma_V^{2(-1)} \circ \sigma' - \psi_2 \circ \sigma_V^{2(-1)} \circ \sigma_V].$$

We consider the expression

$$\psi_2 \circ \sigma_V^{2(-1)} \circ (\psi_2, y)^{(-1)} \circ (\psi_2, y) \circ \sigma' - \psi_2 \circ \sigma_V^{2(-1)} \circ (\psi_2, y)^{(-1)} \circ (\psi_2, y) \circ \sigma_V.$$

We have that $|\psi_2 \circ \sigma_V - \psi_1'| \leq L(y) = o(1)$ by hypothesis whereas $\psi_2 \circ \sigma' - \psi_1' = 0$. As a consequence we obtain

$$|\psi_2 \circ \sigma_V - \psi_2 \circ \sigma'| \leq L(y) = o(1).$$

Since $||D(\psi_2 \circ \sigma_V^{2(-1)} \circ (\psi_2, y)^{(-1)}) - Id|| \leq 2\mu\mu^{uv}$; we deduce that

$$|\psi^2 \circ \sigma' - \psi^1| \leq (1 + 2\mu\mu^{uv})L(y) \sum_{V \in J} h_V(y) = o(1).$$

We remark that $\psi^2 \circ \sigma = \psi^1$, therefore we obtain $\psi^2 \circ \sigma - \psi^2 \circ \sigma' = o(1)$. The mapping $\psi_2 \circ (\psi^2, y)^{(-1)}$ satisfies $||D(\psi_2 \circ (\psi^2, y)^{(-1)}) - Id|| \leq 4\mu\mu^{uv}$ and then

$$\psi_2 \circ \sigma - \psi_1' = \psi_2 \circ \sigma - \psi_2 \circ \sigma' = o(1).$$

The last equation implies that σ and $\sigma^{(-1)}$ can be extended continuously to $y = 0$ by defining $\sigma_{|y=0} \equiv \sigma'_{|y=0} \equiv Id$ and $(\sigma^{(-1)})_{|y=0} \equiv (\sigma'^{(-1)})_{|y=0} \equiv Id$. □

The proof of theorem 8.1 is now complete. Moreover, we also proved the Main Theorem since it is a consequence of theorem 8.1 and propositions 8.1.4 and 8.2.2.

REMARK 9.7.1. *We constructed a germ of special homeomorphism σ conjugating φ_1 and φ_2 such that $SP(\varphi_1) = SP(\varphi_2)$. Since σ is the composition of three tg-sp mappings which are C^∞ at a neighborhood of $(0,0)$ deprived of $yf = 0$ then σ is still C^∞ in the complementary of $yf = 0$.*

COROLLARY 9.7.3. *Let $f \in \mathbb{C}\{x,y\}$ satisfying the (NSD) conditions. Let $\varphi_1, \varphi_2 \in \mathcal{D}_f$. If $SP(\varphi_1) = SP(\varphi_2)$ then φ_1 and φ_2 are conjugated by a germ of special homeomorphism σ such that*

- *σ and $\sigma^{(-1)}$ are C^∞ outside $f = 0$ if $(N,m) = (1,0)$.*
- *σ and $\sigma^{(-1)}$ are C^∞ outside $yf = 0$ if $(N,m) \neq (1,0)$.*

It is well known that a homeomorphism σ conjugating φ_1, φ_2 in Diff$(\mathbb{C},0)$ can not be chosen to be C^∞. Let $\nu = \nu(\varphi_1(x) - x)$; Martinet and Ramis [**MR83**] pointed out that if $\nu = 2$ and σ is C^1 in a neighborhood of the origin then σ is either holomorphic or anti-holomorphic. Afterwards Ahern and Rosay [**AR95**] proved such a property for any order $\nu > 1$ if σ is $C^{3\nu}$. Finally Rey [**Rey96**] improved the previous result to obtain that a C^ν conjugation is either holomorphic or anti-holomorphic, moreover Rey's result is the best possible. As a consequence the conjugation σ provided in corollary 9.7.3 is not in general C^∞ at the points of $f = 0$. But it could be extended in a C^∞ way to $y = 0$? The answer is no in general. The diffeomorphisms φ_1 and φ_2 in $\mathcal{D}_{x^3(y-x)^2}$ such that

$$\varphi_1 = \exp\left(\frac{x^3(y-x)^2}{1+x^2y(y-x)^2}\frac{\partial}{\partial x}\right) \text{ and } \varphi_2 = \exp\left(x^3(y-x)^2\frac{\partial}{\partial x}\right)$$

are conjugated by a special homeomorphism which can not be chosen to be C^1 in $[y = 0] \setminus \{(0,0)\}$.

Bibliography

[AR95] P. Ahern and J.-P. Rosay, *Entire functions in the classification of differentiable germs tangent to the identity, in one or two variables.*, Trans. of the American Math. Soc. **347** (1995), no. 2, 543–572.

[Bru71] A. Bruno, *Analytic form of differential equations i.*, Trudy Moskov. Mat. Obshch. (1971), no. 25, 119–262.

[Bru72] ———, *Analytic form of differential equations ii.*, Trudy Moskov. Mat. Obshch. (1972), no. 26, 199–239.

[Cam78] Cesar Camacho, *On the local structure of conformal mappings and holomorphic vector fields in* $\mathbb{C}^2$*.*, Asterisque (1978), no. 59-60, 83–94.

[DES] A. Douady, F. Estrada, and P. Sentenac, *Champs de vecteurs polynômiaux sur* $\mathbb{C}$*.*, To appear in the Proceedings of Boldifest.

[Eca82] J. Ecalle, *Les fonctions résurgentes et leurs applications*, Publ. Math. d'Orsay (1982).

[I$^+$92] Yu. S. Il'yashenko et al., *Nonlinear stokes phenomena*, Advances in soviet mathematics, vol. 14, American mathematical society, 1992.

[Lea97] Leau, *Étude sur les équations functionelles à une ou plusieurs variables.*, Ann. Fac. Sci. Toulouse (1897), no. 11.

[Lor99] F. Loray, *5 leçons sur la structure transverse d'une singularité de feuilletage holomorphe en dimension 2 complexe.*, Monographies Red TMR Europea Sing. Ec. Dif. Fol. (1999), no. 1, 1–92.

[MR83] J. Martinet and J.-P. Ramis, *Classification analytique des équations differentielles non linéaires résonnantes du premier ordre*, Ann. Sci. Ecole Norm. Sup. **4** (1983), no. 16, 571–621.

[PM97] R. Pérez-Marco, *Fixed points and circle maps*, Acta Math. **2** (1997), no. 179, 243–294.

[Rey96] Jérome Rey, *Difféomorphismes résonnants de* $(\mathbb{C}, 0)$, Thesis. Université Paul Sabatier, 1996.

[Ris99] E. Risler, *Linéarisation des perturbations holomorphes des rotations et applications.*, Mm. Soc. Math. Fr. (N.S.) (1999), no. 77, viii+102 pp.

[Shc82] A. A. Shcherbakov, *Topological classification of germs of conformal mappings with identity linear part*, Vestnik Moskov. Univ. Ser. I Mat. Mekh. **111** (1982), no. 3, 52–57.

[Vor81] S.M. Voronin, *Analytical classification of germs of conformal mappings* $(\mathbb{C}, 0) \to (\mathbb{C}, 0)$*.*, Functional Anal. Appl. **1** (1981), no. 15.

[Yoc95] J.-C. Yoccoz, *Théorème de Siegel, polynômes quadratiques et nombres de Brjuno*, Asterisque (1995), no. 231.