ON THE VOLUME OF SINGULAR-HYPERBOLIC SETS

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Abstract. An attractor Λ for a 3-vector field X is singular-hyperbolic if all its singularities are hyperbolic and it is partially hyperbolic with volume expanding central direction. We prove that $C^{1+\alpha}$ singular-hyperbolic attractors, for some $\alpha > 0$, always have zero volume, thus extending an analogous result for uniformly hyperbolic attractors. The same result holds for a class of higher dimensional singular attractors. Moreover, we prove that if Λ is a singular-hyperbolic attractor for X then either it has zero volume or X is an Anosov flow. We also present examples of C^1 singular-hyperbolic attractors with positive volume. In addition, we show that C^1 generically we have volume zero for C^1 robust classes of singular-hyperbolic attractors.

Contents

1. Introduction

The uniform hyperbolic theory of Dynamical Systems was introduced in the 60's by Smale [25] and nowadays the understanding of uniformly hyperbolic systems is fairly complete. However, open sets of systems, such as perturbations of the Lorenz flow [12] or perturbations of the geometric Lorenz flows [1, 10] fail to be uniformly hyperbolic. These flows are not uniformly hyperbolic because they present equilibria accumulated in a robust

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way by regular orbits. The study of the dynamical properties of this kind of flows has been the object of many works in the field and lead to the concept of a *singular-hyperbolic set*, which is a compact invariant set Λ for a 3-flow such that all its singularities are hyperbolic and it is partially hyperbolic with volume expanding central direction [19, 17]; a precise definition of all these concepts will be given below.

A *singular-hyperbolic attractor* is an attracting set which is singular-hyperbolic and contains a dense orbit. This new class of attractors contains the Lorenz attractor as well the geometric Lorenz attractors [18, 26]. Since then, the strategy adopted was the search of properties of this kind of sets that resemble the properties displayed by the uniformly hyperbolic systems. From the topological point of view this has been very successful [18, 8, 14, 16, 19, 13, 5].

Our goal here is to complement the previous topological results from the measure theoretical point of view. Early work in this direction was obtained in [11], where it was shown that the geometric Lorenz model from [10] has an SRB measure which is stochastically stable. More recently, the existence of SRB measures for C^2 singular-hyperbolic attractors was proved in [9, 4]. Furthermore, it was shown in [4] that this measure has a disintegration into conditional measures along the central direction that are absolutely continuous with respect to the Lesbegue measure, and moreover the support of the SRB measure is the whole attractor.

Here we show that singular-hyperbolic attractors of $C^{1+\alpha}$ flows, for some $\alpha > 0$, always have zero volume, thus extending both an analogous result for uniformly hyperbolic attractors in [7] and a generalization for partially hyperbolic sets in [3]. We point out that since the divergence of both the geometric Lorenz attractor and the flow of the Lorenz equations is negative, it is true for these particular systems that the attractors have zero volume [23]. From our proofs we further deduce that the same zero volume property holds for a class of higher dimensional singular-attractors, as the so-called multidimensional Lorenz attractors [6].

We also prove a dichotomy related with the zero volume property: if Λ is a $C^{1+\alpha}$ singular-hyperbolic attractor for a 3-dimensional vector field X, then either it has zero volume or X induces a transitive Anosov flow. In addition, we show that C^1 generically we have zero volume for C^1 robust classes of singular-hyperbolic attractors. We also present examples of C^1 singular-hyperbolic attractors with positive volume.

1.1. **Partial hyperbolicity and singular-hyperbolicity.** Let M be a compact boundaryless d-dimensional manifold, for some $d \geq 3$, and let $\mathfrak{X}^r(M)$ be the set of C^r vector fields on M, endowed with the C^r topology, for some $r \geq 1$. From now on we fix some smooth Riemannian structure on M and an induced normalized volume form m that we call Lebesgue measure. We write also dist for the induced distance on M. Given $X \in \mathfrak{X}^1(M)$, we denote by $(X_t)_{t\in\mathbb{R}}$ the flow induced by X.

We also denote by $\mathfrak{X}^{1+}(M)$ the set of all C^1 vector fields X whose derivative DX is Hölder continuous with respect to the given Riemannian norm, and we say that $X \in \mathfrak{X}^{1+}(M)$ is of class C^{1+}. We obviously have

$$\mathfrak{X}^1(M) \supset \mathfrak{X}^{1+}(M) \supset \mathfrak{X}^r(M), \quad \text{for every } r \geq 2.$$

Let Λ be a compact invariant set of $X \in \mathfrak{X}^1(M)$. We say that Λ is *isolated* if there exists an open set $U \supset \Lambda$ such that

$$\Lambda = \bigcap_{t\in\mathbb{R}} X_t(U).$$

If U above can be chosen such that $X_t(U) \subset U$ for every $t > 0$, then we say that Λ is an *attracting set*. We say that an attracting set Λ is *transitive* if it coincides with the ω-limit set of a regular orbit. An *attractor* is a transitive attracting set, and a *repeller* is an attractor for the reversed vector field $-X$. An attractor, or repeller, is *proper* if it does not coincide with the whole manifold. An invariant set of X is *non-trivial* if it is neither a periodic orbit nor a singularity.

Definition 1. Let Λ be a compact invariant set for $X \in \mathfrak{X}^1(M)$. Given $0 < \lambda < 1$, we say that Λ has a *dominated splitting* if the tangent bundle over Λ can be written as a continuous DX_t-invariant sum of sub-bundles

$$T_\Lambda M = E \oplus F,$$

such that, for some choice of a Riemannian metric $\| \quad \|$, we have for every $t > 0$ and every $x \in \Lambda$

$$\|DX_t \mid E_x\| \cdot \|DX_{-t} \mid F_{X_t(x)}\| < \lambda^t. \tag{1}$$

We say that Λ is *partially hyperbolic* if it has a dominated splitting as in (1) for which E is *uniformly contracting*, i.e. for every $t > 0$ and every $x \in \Lambda$ we have $\|DX_t \mid E_x\| < \lambda^t$.

For $x \in \Lambda$ and $t \in \mathbb{R}$ we let $J_t(x)$ be the absolute value of the determinant of the linear map

$$DX_t \mid F_x : F_x \to F_{X_t(x)}.$$

We say that the sub-bundle F of the partially hyperbolic invariant set Λ is *volume expanding* if $J_t(x) \geq e^{\lambda t}$ for every $x \in \Lambda$ and every $t \geq 0$.

Definition 2. Let Λ be a compact invariant set for $X \in \mathfrak{X}^r(M)$, $r \geq 1$. We say that Λ is a *singular-hyperbolic set* for X if all the singularities of Λ are hyperbolic, and Λ is partially hyperbolic with volume expanding central direction either for X or for the reverse flow $-X$.

We emphasize that *a compact invariant set Λ without singularities for X on a 3-manifold which is partially hyperbolic with volume expanding central direction*, either for X or for $-X$ is a *uniformly hyperbolic set of saddle-type*, by [19, Proposition 1.8]. In the 3-dimensional setting this means that E is one-dimensional and F can be written as a sum of two invariant one-dimensional sub-bundles $F = [X] \oplus G$, where $[X]$ is the flow direction and

G is uniformly expanding. In general, a compact invariant set Λ for X is uniformly hyperbolic of saddle-type if

$$T_\Lambda M = E \oplus [X] \oplus G$$

is a continuous DX_t-invariant splitting with the sub-bundle $E \neq 0$ uniformly contracted and the sub-bundle $G \neq 0$ uniformly expanded by DX_t for $t > 0$.

An embedded disk $\gamma \subset M$ is a (local) *strong-unstable manifold*, or a *strong-unstable disk*, if $\operatorname{dist}(X_{-t}(x), X_{-t}(y))$ tends to zero exponentially fast as $t \to +\infty$, for every $x, y \in \gamma$. Similarly, γ is called a (local) *strong-stable manifold*, or a *strong-stable disk*, if $\operatorname{dist}(X_t(x), X_t(y)) \to 0$ exponentially fast as $n \to +\infty$, for every $x, y \in \gamma$. It is well-known that every point in a uniformly hyperbolic set possesses a local strong-stable manifold $W^{ss}_{loc}(x)$ and a local strong-unstable manifold $W^{uu}_{loc}(x)$ which are disks tangent to E_x and G_x at x respectively with topological dimensions $d_E = \dim(E)$ and $d_G = \dim(G)$ respectively. Considering the action of the flow we get the (global) *strong-stable manifold*

$$W^{ss}(x) = \bigcup_{t>0} X_{-t}\Big(W^{ss}_{loc}\big(X_t(x)\big)\Big)$$

and the (global) *strong-unstable manifold*

$$W^{uu}(x) = \bigcup_{t>0} X_{t}\Big(W^{uu}_{loc}\big(X_{-t}(x)\big)\Big)$$

for every point x of a uniformly hyperbolic set. Similar notions are defined in a straightforward way for diffeomorphisms. These are immersed submanifolds with the same differentiability of the flow or the diffeomorphism. In the case of a flow we also consider what we call the *stable manifold* $W^s(x) = \cup_{t\in\mathbb{R}} X_t\big(W^{ss}(x)\big)$ and *unstable manifold* $W^u(x) = \cup_{t\in\mathbb{R}} X_t\big(W^{uu}(x)\big)$ for x in a uniformly hyperbolic set, which are flow invariant.

1.2. **Statement of the results.** Our first result broadly generalizes the results of Bowen-Ruelle [7] which show that a uniformly hyperbolic transitive subset of saddle-type for a C^{1+} flow has zero volume.

Theorem A. *Let $X \in \mathfrak{X}^{1+}(M)$ be a vector field on a 3-dimensional manifold M. Then any proper singular-hyperbolic attractor or repeller for X has zero volume.*

Moreover, we obtain the following dichotomy extending a similar result obtained by two of the authors in [3] to the continuous time setting.

Theorem B. *Let Λ be a transitive isolated uniformly hyperbolic set of saddle type for $X \in \mathfrak{X}^{1+}(M)$ on a d-dimensional manifold M, for some $d \geq 3$. Then either Λ has zero volume or X is a transitive Anosov vector field.*

A *transitive Anosov* vector field X is a vector field without singularities such that the entire manifold M is a uniformly hyperbolic set of saddle-type. Using these results we can deduce the following statement which generalizes to singular-hyperbolic attractors (a class which includes all transitive uniformly hyperbolic invariant subsets of saddle-type) a similar one obtained in [3] for transitive uniformly hyperbolic sets of C^{1+} diffeomorphisms.

Theorem C. *Let Λ be a singular-hyperbolic attractor for $X \in \mathfrak{X}^{1+}(M)$ where M is a 3-manifold. Then either Λ has zero volume or X is a transitive Anosov vector field.*

An interesting consequence of the zero volume property of C^{1+} singular-hyperbolic attractors and uniformly hyperbolic sets of saddle-type coupled with an extension of the notion of Axiom A system is as follows.

Recall that a vector field $X \in \mathfrak{X}^1(M)$ is *Axiom A* if the non-wandering set $\Omega(X)$ is both hyperbolic and the closure of its periodic orbits and singularities. The *spectral decomposition theorem* (see e.g. [24]) asserts that if X is Axiom A, then there is a disjoint decomposition

$$\Omega(X) = H_1 \cup \cdots \cup H_n,$$

where each H_i is a hyperbolic basic set for X for $i = 1, \ldots, n$. Following [18] we say that a vector field $X \in \mathfrak{X}^1(M)$ is *singular Axiom A* if there is a finite disjoint decomposition

$$\Omega(X) = \Lambda_1 \cup \cdots \cup \Lambda_n,$$

where each Λ_i is a hyperbolic basic set, a singular-hyperbolic attractor or a singular-hyperbolic repeller for $i = 1, \ldots, n$. We remark that singular Axiom A systems play an important part in the generic description of C^1-flows on 3-manifolds as proved in [16]. The *topological basin* of an attracting set Λ is the set

$$W^s(\Lambda) = \big\{x \in M : \lim_{t\to+\infty} \operatorname{dist}\big(X_t(x), \Lambda\big) = 0\big\}.$$

A straightforward consequence of Theorems A and B and the well-known pairwise disjoint decomposition

$$M = W^s(\Lambda_1) \cup \cdots \cup W^s(\Lambda_n)$$

from [24, Chpt. 2, Lemma 2.2] is the following.

Corollary D. *If $X \in \mathfrak{X}^{1+}(M)$ is a singular Axiom A vector field on a 3-manifold M, then*

$$\cup\big\{W^s(\Lambda) : \Lambda \text{ is a hyperbolic attractor or a singular-hyperbolic attractor}\big\}$$

has full Lebesgue measure in M.

A construction in [6] shows that there exist C^1 open sets of vector fields exhibiting proper robust attractors containing hyperbolic singularities with any number $k \geq 2$ of expanding directions on d-manifolds M with $d = k + 3$. Moreover as will be shown in Section 5 these attractors satisfy the conclusions of Theorem A. We say that an attractor Λ for $X \in \mathfrak{X}^1(M)$ with isolating neighborhood U is *robust* if there is a C^1 neighborhood $\mathcal{U}$ of X in $\mathcal{X}^1(M)$ such that the maximal invariant set

$$\Lambda_Y(U) = \bigcap_{t\in\mathbb{R}} \overline{Y_t(U)}$$

is transitive for all $Y \in \mathcal{U}$. In the case of 3-manifolds this implies that the set $\Lambda_Y(U)$ is a singular-hyperbolic attractor [19].

Corollary E. *A multidimensional proper singular attractor for the class constructed in* [6] *has volume zero.*

The properties stated above are not valid in general in the C^1 setting: we present in Section 4 an explicit construction of a geometric Lorenz attractor for a flow of class C^1 on any 3-manifold with positive Lebesgue measure.

However we are able to extend the previous results to some locally generic subsets in the C^1 topology. Recall that a subset $\mathcal{G}$ of a set $\mathcal{U}$ is *generic* if it may be written as a countable intersection of open and dense subsets of $\mathcal{U}$. Since $\mathcal{X}^1(M)$ is a Baire space, generic subsets of a given open subset $\mathcal{U}$ of $\mathcal{X}^1(M)$ are dense in $\mathcal{U}$.

Concrete examples of such open sets on 3-manifolds are given by robust singular-hyperbolic attractors, see e.g. [15, 19] which comprise the Lorenz attractor [26] and the geometric Lorenz attractors [1, 10].

Theorem F. *Let Λ be a robust attractor for $X \in \mathcal{X}^1(M)$ on a 3-manifold M with isolating neighborhood U. Then there is a C^1-neighborhood $\mathcal{U}$ of X and a C^1-generic set $\mathcal{G} \subset \mathcal{U}$ such that $\Lambda_Y(U)$ has volume zero for all $Y \in \mathcal{G}$.*

As mentioned above the C^1-open sets of singular-attractors on d-manifolds described in [6] are also in the setting of Theorem 6.1.

Theorem G. *Let Λ be a robust multidimensional singular attractor for a vector field X with isolating neighborhood U as in* [6]*. Then there exists a C^1-neighborhood $\mathcal{U}$ of X and a C^1-generic set $\mathcal{G} \subset \mathcal{U}$ such that $\Lambda_Y(U)$ has volume zero for all $Y \in \mathcal{G}$.*

This paper is organized as follows. Theorem A is a consequence of Theorem 2.1 proved in Section 2 together with Lemma 3.3 proved in Subsection 3.2. Both Theorems B and C are proved in Section 3. Then in Section 4 we present an example of a positive volume C^1 singular-hyperbolic attractor and in Section 5 we show that certain classes of multidimensional robust singular-attractors are in the setting of Theorem 2.1. Finally we show that Theorems F and G are consequences of Theorem 6.1 proved in Section 6.

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2. Partial hyperbolicity and zero volume on C^{1+} flows

The following result plays a crucial role in the proof of Theorem A.

Theorem 2.1. *Let X be a C^{1+} flow on a d-dimensional manifold with $d \geq 3$ and Λ be a partially hyperbolic invariant subset such that*

($\star$) *$\Lambda \cap \gamma$ does not contain d_E-disks for any strong-stable disk γ.*

Then Λ has zero volume.

The proof of the above theorem is a consequence of the following result for compact invariant subsets of C^{1+} diffeomorphisms f with dominated decomposition, whose proof we present in Subsections 2.1 and 2.2.

Before we state the result let us recall the notion of *dominated splitting for a diffeomorphism* f over a compact f-invariant set Λ, which is very similar to the one given by Definition 1 since we need only exchange (1) by

$$\|Df \mid E_x\| \cdot \|Df^{-1} \mid F_x\| < \lambda \tag{2}$$

for all $x \in \Lambda$. Analogously *partial hyperbolicity for a diffeomorphism* f is given by a dominated decomposition $E \oplus F$ over a compact invariant subset Λ with uniform contraction along the direction E.

Theorem 2.2. *Let $f : M \to M$ be a C^{1+} diffeomorphism and let $\Lambda \subset M$ be a partially hyperbolic set with positive volume. Then Λ contains a strong-stable disk.*

Let us now prove Theorem 2.1 using Theorem 2.2. Let Λ be a partially hyperbolic compact invariant set for a flow $X \in \mathfrak{X}^{1+}(M)$ where M is a d-manifold with $d \geq 3$. Assume that condition ($\star$) is satisfied by Λ.

Arguing by contradiction, if $m(\Lambda) > 0$ then setting $f = X_1$, the time-one diffeomorphism induced by the vector field X, we see that Λ is in the setting of Theorem 2.2.

Hence there exists some strong-stable disk γ for f contained in Λ with dimension d_E, which is a strong-stable disk for the flow $X_t, t > 0$ and contradicts property ($\star$). This contradiction shows that $m(\Lambda) = 0$ and proves Theorem 2.1.

2.1. **Pre-balls and bounded distortion.** Here we give some preparatory results for the proof of Theorem 2.2. We fix continuous extensions of the two bundles E and F to some neighborhood U of Λ, that we denote by $\tilde{E}$ and $\tilde{F}$. We do not require these extensions to be invariant under Df. Given $0 < a < 1$, we define the *center-unstable cone field* $\big(C_a^F(x)\big)_{x \in U}$ *of width* a by

$$C_a^F(x) = \big\{v_1 + v_2 \in \tilde{E}_x \oplus \tilde{F}_x \text{ such that } \|v_1\| \leq a \cdot \|v_2\|\big\}. \tag{3}$$

We define the *stable cone field* $\big(C_a^E(x)\big)_{x \in U}$ *of width* a in a similar way, just reversing the roles of the bundles in (3). We fix $a > 0$ and U small enough so that, up to slightly increasing $\lambda < 1$, the domination condition (2) remains valid for any pair of vectors in the two cone fields:

$$\|Df(x)u\| \cdot \|Df^{-1}(f(x))v\| \leq \lambda \cdot \|u\| \cdot \|v\| \tag{4}$$

for every $u \in C_a^E(x)$, $v \in C_a^F(f(x))$, and any point $x \in U \cap f^{-1}(U)$. Note that the unstable cone field is positively invariant:

$$Df(x)C_a^F(x) \subset C_a^F(f(x)),$$

whenever $x, f(x) \in U$. Indeed, the domination (4) together with the invariance of $F = \tilde{F} \mid \Lambda$ imply that

$$Df(x)C_a^F(x) \subset C_{\lambda a}^F(f(x)) \subset C_a^F(f(x)),$$

for every $x \in \Lambda$. This extends to any $x \in U \cap f^{-1}(U)$ just by continuity. Analogously the stable cone field is negatively invariant:

$$Df^{-1}(x)C_a^E(x) \subset C_a^E(f^{-1}(x)),$$

whenever $x, f(x) \in U$.

If $a > 0$ is taken sufficiently small in the definition of the cone fields, and we choose $\delta_1 > 0$ also small so that the δ_1-neighborhood of Λ should be contained in U, then by continuity

$$\|Df(y)u\| \leq \lambda^{-1/2} \cdot \|Df \mid E_x\| \cdot \|u\|, \tag{5}$$

whenever $x \in \Lambda$, $\operatorname{dist}(x, y) \leq \delta_1$, and $u \in C_a^E(y)$.

We say that an embedded C^1 submanifold $N \subset U$ is *tangent to the stable cone field* if the tangent subspace to N at each point $x \in N$ is contained in $C_a^E(x)$. Then, by the domination property (2), $f^{-1}(N)$ is also tangent to the stable cone field, if it is contained in U. In particular, if $N, f^{-1}(N), \ldots, f^{-k}(N) \subset U$, then $Df^k \mid T_{f^{-k}(x)N}$ is a $\lambda^{k/2}$-contraction by (5), since $\|Df \mid E_x\| < 1$ by partial hyperbolicity. Thus, denoting by dist_N the *distance along* N given by the length of the shortest smooth curve connecting two given points inside N, we obtain

Lemma 2.3. *Let $\Delta \subset U$ be a C^1 disk of radius $\delta < \delta_1$ tangent to the stable cone field. There exists $n_0 \geq 1$ such that for $n \geq n_0$ and $x \in \Delta$ with $\operatorname{dist}_\Delta(x, \partial\Delta) \geq \delta/2$ there is a neighborhood V_n of x in Δ such that f^{-n} maps V_n diffeomorphically onto a disk of radius δ_1 around $f^{-n}(x)$. Moreover,*

$$\operatorname{dist}_{f^{-n+k}(V_n)}(f^{-n+k}(y), f^{-n+k}(z)) \leq \lambda^{k/2} \cdot \operatorname{dist}_{f^{-n}(V_n)}(f^{-n}(y), f^{-n}(z))$$

for every $1 \leq k \leq n$ and every $y, z \in V_n$.

We shall sometimes refer to the sets V_n as *pre-balls*. The next corollary is a consequence of the contraction given by the previous lemma, together with some Hölder control of the tangent direction which can be found in [2, Corollary 2.4, Proposition 2.8]

Corollary 2.4. *There exists $C > 1$ such that given Δ as in Lemma 2.3 and given any pre-ball $V_n \subset \Delta$ with $n \geq n_0$, then for all $y, z \in V_n$*

$$\frac{1}{C} \leq \frac{|\det Df^{-n} \mid T_y\Delta|}{|\det Df^{-n} \mid T_z\Delta|} \leq C.$$

2.2. **A local unstable disk inside Λ.** Assume that Λ has positive volume and given an embedded disk Δ in M denote by m_Δ the measure naturally induced by the volume form m on Δ. Choosing a m density point of Λ, we laminate a neighborhood of that point into disks tangent to the stable cone field. Since the relative Lebesgue measure of the intersections of these disks with Λ cannot be all equal to zero, we obtain some disk Δ intersecting Λ in a positive m_Δ subset. Hence, in the setting of Theorem 2.2, we assure that there is a disk Δ tangent to the stable cone field intersecting Λ in a positive m_Δ subset. Let $H = \Delta \cap \Lambda$.

Lemma 2.5. *There exist an infinite sequence of integers $1 \leq k_1 < k_2 < \cdots$ and, for each $n \in \mathbb{N}$, a disk Δ_n of radius $\delta_1/4$ tangent to the stable cone field such that the relative Lebesgue measure of $f^{-k_n}(H)$ in Δ_n converges to 1 as $n \to \infty$.*

Proof. Let $\varepsilon > 0$ be some small number. Let K be a compact subset of H and A be an open neighborhood of H in Δ such that

$$m_\Delta(A \setminus K) < \varepsilon m_\Delta(K).$$

Choose n sufficiently large so that for each $x \in K$ we have $V_x \subset A$, where V_x is the pre-ball associated to n. This pre-ball is mapped diffeomorphically by f^{-n} onto a ball $B_{\delta_1}(f^{-n}(x))$ of radius δ_1 around $f^{-n}(x)$ tangent to the stable cone field. Let $W_x \subset V_x$ be the pre-image of the ball $B_{\delta_1/4}(f^{-n}(x))$ of radius $\delta_1/4$ under this diffeomorphism. By compactness we have

$$K \subset W_{x_1} \cup ... \cup W_{x_m},$$

for some $x_1, ..., x_m \in K$. Let I be a maximal set of $\{1, ..., m\}$ such that for $i, j \in I$ with $i \neq j$ we have $W_{x_i} \cap W_{x_j} = \emptyset$. By maximality, each W_{x_j}, $1 \leq j \leq m$, intersects some W_{x_i} with $i \in \mathcal{U}$. Hence $\{V_{x_i}\}_{i \in I}$ is a covering of K. By Corollary 2.4 there is a uniform constant $\theta > 0$ such that

$$\frac{m_\Delta(W_{x_i})}{m_\Delta(V_{x_i})} \geq \theta, \quad \text{for every } i \in I.$$

Hence

$$m_\Delta\big(\cup_{i \in I} W_{x_i}\big) = \sum_{i \in I} m_\Delta(W_{x_i}) \geq \sum_{i \in I} \theta m_\Delta(V_{x_i}) \geq \theta m_\Delta(K).$$

Setting

$$\rho = \min\left\{\frac{m_\Delta(W_{x_i} \setminus K)}{m_\Delta(W_{x_i})} : i \in I\right\},$$

we have

$$\begin{aligned} \varepsilon m_\Delta(K) &\geq m_\Delta(A \setminus K) \\ &= m_\Delta\big(\cup_{i \in I} W_{x_i} \setminus K\big) \\ &= \rho m_\Delta\big(\cup_{i \in I} W_{x_i}\big) \\ &= \rho\theta m_\Delta(K). \end{aligned}$$

This implies that $\rho < \varepsilon/\theta$. Since $\varepsilon > 0$ can be taken arbitrarily small, increasing n we may take W_{x_i} such that the relative Lebesgue measure of K in W_{x_i} is arbitrarily close to 1. Then, by the bounded distortion provided by Corollary 2.4, the relative Lebesgue measure of $f^{-n}(H) \supset f^{-n}(K)$ in $f^{-n}(W_{x_i})$, which is a disk of radius $\delta_1/4$ around $f^{-n}(x_i)$ tangent to unstable cone field, can be made arbitrarily close to 1. □

Let us now prove that there is a strong-stable disk of radius $\delta_1/4$ inside Λ. Let $(\Delta_n)_n$ be the sequence of disks given by Lemma 2.5, and consider $(x_n)_n$ the sequence of points at which these disks are centered. Up to taking subsequences, we may assume that the centers of the disks converge to some point x. By Ascoli-Arzela, these disks converge to some disk Δ_∞ centered at x. By construction, every point in Δ_∞ is accumulated by some iterate of a point in $H \subset \Lambda$, and so $\Delta_\infty \subset \Lambda$.

Note that each Δ_n is contained in the k_n-iterate of Δ, which is a disk tangent to the stable cone field. The domination property implies that the angle between Δ_n and E goes to zero as $n \to \infty$, uniformly on Λ. In particular, Δ_∞ is tangent to E at every point in $\Delta_\infty \subset \Lambda$. By Lemma 2.3, given any $k \geq 1$, then f^k is a $\sigma^{k/2}$-contraction on Δ_n for every large n. Passing to the limit, we get that f^k is a $\sigma^{k/2}$-contraction on Δ_∞ for every $k \geq 1$. In particular, we have shown that the subspace E_x is uniformly contracting for Df. The fact that $T_\Lambda M = E \oplus F$ is a dominated splitting implies that any contraction Df may exhibit along the complementary direction F_x is weaker than the contraction in the E_x direction. Then, by [22], there exists a unique strong-stable manifold $W^{ss}_{loc}(x)$ tangent to E and which is contracted by the positive iterates of f. Since Δ_∞ is contracted by every f^k, and all its positive iterates are tangent to stable cone field, then Δ_∞ is contained in $W^{ss}_{loc}(x)$.

This completes the proof of Theorem 2.2.

3. Positive volume versus transitive Anosov flows

In this section we prove Theorems A, B and C.

3.1. Positive volume transitive hyperbolic sets and Anosov flows.

We start by proving Theorem B. Let Λ be a transitive uniformly hyperbolic set for $X \in \mathfrak{X}^{1+}(M)$ such that $m(\Lambda) > 0$, where M is a d-manifold, for some $d \geq 3$.

Lemma 3.1. *If there exists a point $x \in \Lambda$ in the interior of $W^{ss}_{loc}(x) \cap \Lambda$, then $\Lambda \supset W^{ss}(y)$ for all $y \in \Lambda$. Moreover, the set $W^u(\Lambda)$ formed by the union of all unstable manifolds through points of Λ is an open neighborhood of Λ.*

Here the interior of $W^{ss}_{loc}(x) \cap \Lambda$ is taken with respect to the topology of the disk $W^{ss}_{loc}(x)$. The proof follows [19, Lemma 2.16] and [4, Lemma 2.8] closely.

Proof. Let $x \in \Lambda$ be such that x is in the interior of $W^{ss}(x) \cap \Lambda$. Let $\alpha(x) \subset \Lambda$ be its α-limit set. Then

$$W^{ss}(z) \subset \Lambda \quad \text{for every } z \in \alpha(x), \tag{6}$$

since any compact part of the strong-stable manifold of z is accumulated by backward iterates of any small neighborhood of x inside $W^{ss}(x)$. Here we are using that the contraction along the strong-stable manifold, which becomes an expansion for negative time, is uniform.

Clearly the invariant set $\alpha(x) \subset \Lambda$ is uniformly hyperbolic.

It also follows from (6) that the union

$$S = \bigcup_{y \in \alpha(x)} W^{ss}(y) \quad \text{or} \quad S = W^{ss}(\alpha(x))$$

of the strong-stable manifolds through the points of $\alpha(x)$ is contained in Λ. By continuity of the strong-stable manifolds and the fact that $\alpha(x)$ is a closed set, we get that S is also closed. Again S is a uniformly hyperbolic set.

We claim that $W^u(S)$, the union of the unstable manifolds of the points of S, is an open set. To prove this, we note that S contains the whole stable manifold $W^s(z)$ of every $z \in S$: this is because S is invariant and contains the strong-stable manifold of z. Now the union of the strong-unstable manifolds through the points of $W^s(z)$ contains a neighborhood of z. This proves that $W^u(S)$ is a neighborhood of S. Thus the backward orbit of any point in $W^u(S)$ must enter the interior of $W^u(S)$. Since the interior is, clearly, an invariant set, this proves that $W^u(S)$ is open, as claimed.

Finally, consider any backward dense orbit in Λ of a point that we call w. On the one hand $\alpha(w) = \Lambda$. On the other hand, $X_{-t}(w)$ must belong to $W^u(S)$ for some $t > 0$, and so $\alpha(w) \subset S$ by invariance. This implies that $\Lambda \subset S$ and since $S \subset \Lambda$ by construction, we see that $\Lambda = S$. □

Proof of Theorem B. Assume that $m(\Lambda) > 0$. Then Theorem 2.2 applied to the map $f = X_1$ and to the set Λ with dominated decomposition given by the splitting

$$E \oplus \big([X] \oplus F\big),$$

ensures that there exists a strong-stable disk γ contained in Λ. Analogously applying Theorem 2.2 to $f = X_{-1}$ and to the set Λ with dominated decomposition given by the splitting

$$\big(E \oplus [X]\big) \oplus F,$$

we get a strong-unstable disk δ contained in Λ.

Now the existence of γ enables us to use Lemma 3.1 and deduce that Λ contains the strong-stable manifolds of each of its points and that $W^u(\Lambda)$ is an open neighborhood of Λ. In the same way, using Lemma 3.1 for the flow generated by $-X$, from the existence of δ we deduce that Λ contains the strong-unstable manifolds of all of its points, that is $W^u(\Lambda) \subset \Lambda$.

But since $W^u(\Lambda)$ is an open neighborhood of Λ, we conclude that Λ is simultaneously open and closed in M. Hence $\Lambda = M$ by connectedness.

This shows that the whole of M is a transitive uniformly hyperbolic set for X and completes the proof of Theorem B. $\square$

3.2. **Positive volume singular-hyperbolic sets and Anosov flows.** Now we prove Theorems A and C. For that we need some preliminary results which show in particular that transitive singular-hyperbolic sets satisfy condition $(\star)$.

In what follows X is a vector field in $\mathfrak{X}^{1+}(M)$ and M is a 3-manifold.

Lemma 3.2. *Let Λ be a transitive partially hyperbolic invariant set for X with volume expanding central direction. Then*

- *either $W^{ss}(x) \cap \Lambda$ contains no strong stable disks for all $x \in \Lambda$,*
- *or Λ is a uniformly hyperbolic set (and in particular Λ does not contain singularities).*

Proof. Let us suppose that there exists $x \in \Lambda$ such that x is in the interior of $W^{ss}(x) \cap \Lambda$. Let $\alpha(x) \subset \Lambda$ be its α-limit set. Then we have (6). It follows that $\alpha(x)$ does not contain any singularity. Indeed, [19, Theorem B] proves that the strong-stable manifold of each singularity meets Λ only at the singularity. Therefore by [19, Proposition 1.8] the invariant set $\alpha(x) \subset \Lambda$ is uniformly hyperbolic.

As in the proof of Lemma 3.1 we have that

$$S = W^{ss}(\alpha(x)) \subset \Lambda$$

and that S is closed. Again we see that S does not contain any singularity σ, for otherwise we would have $W^{ss}(\sigma) \supset W^{ss}(z)$ for some $z \in \alpha(x)$ which by (6) would contradict [19, Theorem B]. Thus S is a uniformly hyperbolic set.

Then $W^u(S)$ is also an open set as in the proof of Lemma 3.1. Since we are assumming that Λ is transitive, again by the same arguments in the proof of Lemma 3.1 we get that $\Lambda = S$. This shows that Λ is uniformly hyperbolic and in particular it does not contain any singularity of X. $\square$

Proof of Theorem A. Note that if Λ is a transitive singular-hyperbolic set for X, then since Λ contains singularities, by Lemma 3.2 we have that Λ satisfies property $(\star)$ in the statement of Theorem 2.1. Hence Λ has zero volume, thus concluding the proof of Theorem A since a singular-hyperbolic attractor is transitive by definition. $\square$

We can obtain a stronger conclusion if we further assume that Λ is an attractor.

Lemma 3.3. *Let Λ be a singular-hyperbolic attractor for X. Then*

- *either $W^{ss}(x) \cap \Lambda$ contains no strong stable disks for all $x \in \Lambda$,*
- *or $\Lambda = M$ is a uniformly hyperbolic set and X is a transitive Anosov vector field.*

Proof. Assume that there exists $x \in \Lambda$ such that x is in the interior of $W^{ss}(x) \cap \Lambda$. From Lemma 3.2 we know that there exists a uniformly hyperbolic set $S \subset \Lambda$ such that $W^u(S)$ is an open neighborhood of S. Moreover we also have that $\Lambda = S$.

However if Λ is an attractor, then $W^u(S) \subset \Lambda$ and so we get $\Lambda = W^u(S)$. Hence Λ is closed and also open. The connectedness of M implies that $\Lambda = S = M$. In particular X has no singularities and the whole of M admits a uniformly hyperbolic structure with a dense orbit, thus X is a transitive Anosov vector field. □

Proof of Theorem C. Let Λ be a singular-hyperbolic attractor for a C^{1+} vector field X on a 3-manifold. If $m(\Lambda) > 0$ then according to Theorem 2.2 we get that there exists some strong-stable disk γ contained in Λ.

Hence since Λ does not satisfy the first alternative of Lemma 3.3 we conclude that $\Lambda = M$ and so X is an Anosov vector field. This concludes the proof of Theorem C. □

4. A positive volume singular-hyperbolic attractor

To construct an example of a 3-dimensional C^1 flow exhibiting a singular-hyperbolic attractor with positive volume we start with the construction of a *dynamically defined Cantor set with positive one-dimensional Lebesgue measure.*

This construction is carried out in detail in [21, Section 4.2] giving (after a trivial change of coordinates) a C^1 2-to-1 surjective map

$$\varphi : [-1/2, a] \cup [b, 1/2] \to [-1/2, 1/2],$$

such that both $\varphi \mid [-1/2, a]$ and $\varphi \mid [b, 1/2]$ are diffeomorphisms onto $[-1/2, 1/2]$, and $-1/2 < a < 0 < b < 1/2$ are fixed. The map φ is such that φ' is *not of bounded variation nor Hölder continuous.* Moreover, the construction is performed in such a way that the maximal positive invariant set (dynamically defined Cantor set)

$$K = \bigcap_{n \geq 0} \varphi^{-n}([-1/2, 1/2]) \quad \text{satisfies} \quad \lambda(K) > 0,$$

where we denote by λ the standard Lebesgue measure on the real line.

Now we adapt this map φ so that it becomes a *Lorenz-like* map, which can be used to define a geometric Lorenz flow in the sense of [1, 10]. We may assume without loss that both $\varphi \mid [-1/2, a]$ and $\varphi \mid [b, 1/2]$ are increasing. We now consider a C^1 extension ϕ of φ to $[-3/4, 3/4] \setminus \{0\}$ satisfying (see Figure 1):

(1) $\phi \mid [-3/4, 0)$ and $\phi \mid (0, 3/4]$ are increasing;
(2) $-3/4 < \phi(-3/4)$ and $\phi(3/4) < 3/4$;
(3) $\lim_{x \to 0^-} \phi(x) = 3/4$ and $\lim_{x \to 0^+} \phi(x) = -3/4$;
(4) $\lim_{x \to 0^-} \phi'(x) = +\infty$ and $\lim_{x \to 0^+} \phi'(x) = -\infty$.

Note also that the maximal positive invariant subset for ϕ in $[-3/4, 3/4]$ is the same set K as before and so has positive one-dimensional Lebesgue measure.

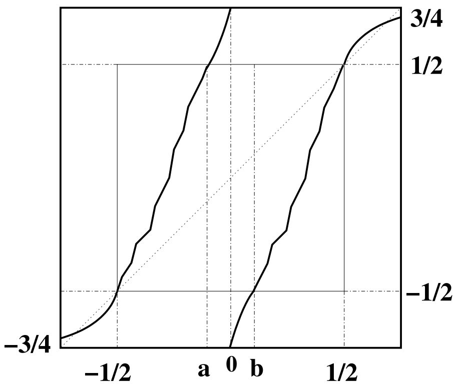

FIGURE 1. The one-dimensional Lorenz-like map.

Now this map can be used as a basis for the standard construction of a flow X exhibiting a *geometric Lorenz attractor* as explicitly described in [23, Chapter 7, Section 3.2] and sketched in Figure 2. If we denote by Λ the

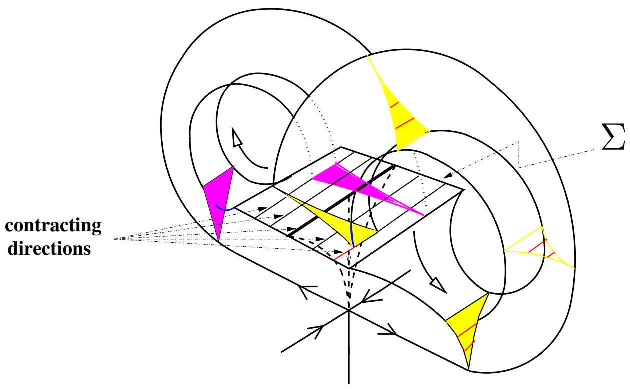

FIGURE 2. The geometric Lorenz attractor with the contracting directions on the cross-section Σ.

attractor obtained for the vector field X just constructed, then the set K can now be interpreted as a projection of the set $\Lambda \cap \Sigma$ through the stable leaves crossing the cross-section Σ (drawn in Figure 2).

Since the projection along this stable leaves is smooth, because it coincides with the usual projection along lines with constant coordinate in Euclidean space, this shows that the two-dimensional Lebesgue measure (area) of $\Lambda \cap \Sigma$ must be positive.

In what follows, given $A \subset M$ and $[a,b] \subset \mathbb{R}$ we shall write

$$X_{[a,b]}(A) = \{X_t(x) : \ a \le t \le b \quad \text{and} \quad x \in A\}.$$

Finally using that Σ is a cross-section for the flow, we consider a small flow box through Σ and the set $\Lambda_\Sigma^\varepsilon = X_{[-\varepsilon,\varepsilon]}(\Lambda \cap \Sigma)$ for small $\varepsilon > 0$ is contained in Λ by flow invariance and has positive volume, since it can be seen as a direct product of $[-\varepsilon,\varepsilon] \times (\Lambda \cap \Sigma)$, because X_t is a diffeomorphism for all $t \in \mathbb{R}$.

Therefore we conclude that $m(\Lambda) \ge m\big(\Lambda_\Sigma^\varepsilon\big) > 0$.

5. Robust attractors in higher dimensions

Here we briefly describe the construction of singular-attractors in [6] with any number of expanding dimensions, and show that this class of attractors satisfies condition $(\star)$.

Consider a "solenoid" constructed over a uniformly expanding map $f : \mathbb{T}^k \to \mathbb{T}^k$ of the k-dimensional torus, for some $k \ge 2$. That is, let $\mathbb{D}$ be the unit disk on $\mathbb{R}^2$ and consider a smooth embedding $F : \mathbb{T}^k \times \mathbb{D} \to \mathbb{T}^k \times \mathbb{D}$ of $N = \mathbb{T}^k \times \mathbb{D}$ into itself, which preserves and contracts the foliation

$$\mathcal{F}^s = \big\{\{z\} \times \mathbb{D} : z \in \mathbb{T}^k\big\},$$

and moreover the natural projection $\pi : N \to \mathbb{T}^k$ on the first factor conjugates F to f: $\pi \circ F = f \circ \pi$.

Now consider the linear flow over defined on $M = N \times [0,1]$ given by the vector field $X = (0,1)$ on $TN \times \mathbb{R}$. Modify the flow on a cylinder $U \times \mathbb{D} \times [0,1]$ around the orbit of a point $p = (z,0) \in N$, where U is a neighborhood of z in $\mathbb{T}^k$, in such a way as to create a hyperbolic singularity σ of saddle-type with k-expanding and 3 contracting eigenvalues, as depicted in Figure 3.

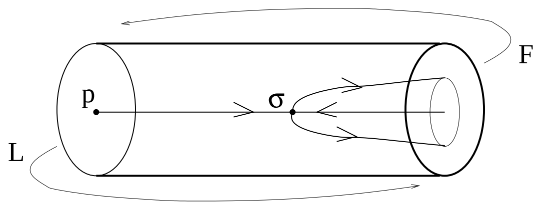

FIGURE 3. A sketch of the construction of a singular-attractor in higher dimensions

This modified flow defines a transition map L from $\Sigma_0 = \mathbb{T}^k \times \{0\}$ to $\Sigma_1 = \mathbb{T}^k \times \{1\}$ which through the identification given by $(1,w) \sim_F (0, F(w))$ defines the return map to the global cross-section Σ_0 of a flow Y on the space $M^F = M/\sim_F$.

In [6] it is shown that if the expanding rate of f is sufficiently big, then the set

$$\Lambda = \bigcup_{T>0} \overline{\bigcap_{t>T} Y_t(\Sigma_0)}$$

is a robust partially hyperbolic attractor with singularities.

To see that Λ satisfies condition $(\star)$, note first that the intersection $\Lambda\cap\Sigma_0$ is contained in the hyperbolic solenoid $\Lambda_0 = \cap_{n\geq 0}F^n(N)$. Moreover, the strong-stable manifold of every given point $(z,w,t)\in\Lambda$ (where $(z,w,t)\in \mathbb{T}^k\times\mathbb{D}\times[0,1]$) contains the disk $\{z\}\times\mathbb{D}\times\{t\}$ which is a leaf of $\mathcal{F}^s$. Hence

$$W^s_{loc}(z,w,0)\cap\Lambda\subset\big(\{z\}\times\mathbb{D}\cap\Lambda_0\big)\times\{0\},$$

and this last set is a *Cantor set*, since it is the intersection of a strong-stable manifold with a uniformly hyperbolic solenoid. Thus $W^s_{loc}(z,w,0)\cap\Lambda$ does not contain any 2-disk.

Since Σ_0 is a global cross-section, any other strong-stable leaf W is such that $W\cap\Lambda$ does not contain any 2-disk D, for otherwise by the flow invariance of Λ the first return to Σ_0 of the points of D would be contained in a strong-stable leaf and would contain an open set, contradicting the previous paragraph. This shows that this class of robust multidimensional singular attractors is in the setting of Theorem 2.1 and so they have zero volume in the C^{1+} setting.

6. Zero volume in the C^1 generic setting

Here we prove Theorems F and G as a consequence of the following result.

Theorem 6.1. *Let Λ be an isolated partially hyperbolic set satisfying condition $(\star)$ for $X\in\mathfrak{X}^1(M)$ on a d-dimensional manifold M with $d\geq 3$. Given an isolating neighborhood U of Λ, let $\mathcal{U}\subset\mathfrak{X}^1(M)$ be such that $\Lambda_Y(U)$ is partially hyperbolic and also satisfies condition $(\star)$ for all $Y\in\mathcal{U}$.*

If $\mathcal{U}$ is C^1-open, then there exists a generic set $\mathcal{G}\subset\mathcal{U}$ such that $\Lambda_Y(U)$ has volume zero for all $Y\in\mathcal{G}$.

Proof. Let Λ be an isolated partially hyperbolic invariant compact subset for a flow $X\in\mathfrak{X}^r(M)$, for some $r\geq 1$, such that Λ satisfies condition $(\star)$. We always write U for the isolating neighborhood of Λ.

We consider the sets

- $\mathcal{U} = \{Y\in\mathfrak{X}^1(M) : \Lambda_Y(U)$ is partially hyperbolic satisfying $(\star)\}$ which we assume is a C^1 open subset of $\mathfrak{X}^1(M)$;
- $\mathcal{V} = \{Y\in\mathfrak{X}^2(M) : \Lambda_Y(U)$ is partially hyperbolic satisfying $(\star)\}$;
- $\mathcal{U}_\varepsilon = \{Y\in\mathcal{U} : m\big(\Lambda_Y(U)\big)<\varepsilon\}$.

Since every C^1 flow X is arbitrarily close to some C^2 flow Y in the C^1 topology (see e.g. [20]) and we are assumming that $\mathcal{U}$ is C^1-open, we conclude that $\mathcal{V}$ is dense in $\mathcal{U}$ in the C^1 topology.

We claim that $\mathcal{U}_\varepsilon$ is open and dense in $\mathcal{U}$ in the C^1 topology. After proving this claim the proof of Theorem 6.1 finishes by setting $\mathcal{G}=\cap_{n\geq 1}\mathcal{U}_{1/n}$. In what follows we prove this claim.

Let $Y\in\mathcal{U}_\varepsilon$ be given. Then for every fixed $T>0$ we set

$$\Lambda_Y^T=\cap_{t=-T}^T\operatorname{clos}\big(Y_t(U)\big)\subseteq\Lambda_Y(U)\quad\text{and}\quad\varepsilon_1=\varepsilon-m\big(\Lambda_Y^T\big)>0.$$

There exists $\delta > 0$ such that
$$m\Big(B\big(\Lambda_Y^T, \delta\big) \setminus \Lambda_Y^T\Big) < \frac{\varepsilon_1}{2}.$$
Let $B^{C_1}(Y, \zeta)$ denote the C^1-neighborhood of radius ζ around Y. Using that T is finite, by continuity and compactness we find $\zeta > 0$ such that
$$Z \in B^{C_1}(Y, \zeta) \quad \Rightarrow \quad \Lambda_Z^T \subset B\big(\Lambda_Y^T, \delta\big).$$
We have that for all $Z \in \mathcal{U} \cap B^{C_1}(Y, \zeta)$
$$m\big(\Lambda_Z^T\big) \le m\Big(B\big(\Lambda_Y^T, \delta\big)\Big) \le m\big(\Lambda_Y^T\big) + \frac{\varepsilon_1}{2} \le \frac{m\big(\Lambda_Y^T\big) + \varepsilon}{2} < \varepsilon.$$
Since $\Lambda_Z(U) \subset \Lambda_Z^T$, we conclude $m\big(\Lambda_Z(U)\big) < \varepsilon$, for all $z \in \mathcal{U} \cap B^{C_1}(Y, \zeta)$.

This proves that $\mathcal{U}_\varepsilon$ is C^1-open. To prove that $\mathcal{U}_\varepsilon$ is C^1-dense in $\mathcal{U}$, just observe that $\mathcal{V} \cap \mathcal{U} \subset \mathcal{U}_\varepsilon$ by Theorem 2.1. Since $\mathcal{V}$ is C^1-dense in $\mathcal{U}$ this concludes the proof of the claim and ends the proof of Theorem 6.1. □

Proof of Theorem F. This is a straightforward consequence of Theorem 6.1 since a C^1 robust attractor for 3-flows is a singular-hyperbolic attractor as shown in [19]. □

Proof of Theorem G. This is also an immediate consequence of Theorem 6.1 since the class of multidimensional singular-attractors constructed in [6] is in the setting of Theorem 2.1, as shown in Section 5. □

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