Insecurity for lattice translation surfaces of small genus, with applications to polygonal billiards

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Abstract

A configuration (i. e., a pair of points) in a riemannian manifold X is secure if all connecting geodesics can be blocked by a finite subset of X. A space is insecure if it has insecure configurations. If X is insecure, what is the size of the set of insecure configurations? We study this problem for translation surfaces.

We show that for lattice translation surfaces of genus two almost all configurations are insecure. This implies the insecurity of almost all configurations in certain polygons.

Introduction and motivation

Let X be a riemannian manifold (possibly with corners and other singularities, see below). Unless explicitly stated otherwise, we will consider only finite geodesics $\gamma = \{\gamma(t),\, a \leq t \leq b\} \subset X$. If $\gamma(c) = z$ for $c \in (a, b)$, we say that γ *passes through* z;[1] the points $x = \gamma(a), y = \gamma(b)$ are the *endpoints* of γ. We will refer to pairs of points $\{x, y\} \subset X$ as the *configurations*. A geodesic γ is a *connecting geodesic* for $\{x, y\}$ if x, y are its endpoints and γ does not pass through either x or y.

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[1] Geodesics do not pass through singular points.

Let Γ be a collection of geodesics in X. A finite set $B \subset X$ is a *blocking set* for Γ if every geodesic $\gamma \in \Gamma$ passes through a point $z \in B$. Let $\{x, y\}$ be a configuration, and let $\Gamma(x, y)$ be the set of connecting geodesics. We say that $\{x, y\}$ is a *secure* (resp. *insecure*) *configuration* if there exists (resp. does not exist) a blocking set for $\Gamma(x, y)$. The *space X is secure* if every configuration in X is secure.

Let X be a secure space. It is *uniformly secure* if there exists $N \in \mathbb{N}$ s.t. every $\Gamma(x, y)$ can be blocked by a set $B = B(x, y)$ of at most N points. The minimal value of N is the *security threshold* of X. A flat torus of n dimensions is uniformly secure; its security threshold is 2^n [3]. Security seems a rare property. Flat tori are the only examples of smooth secure spaces known to the author. This paper continues the study of security of *rational polygons* and *translation surfaces* started in [4].

A translation surface is a closed surface S with a finite singular set Σ and a special atlas on $S \setminus \Sigma$. It comes with a riemannian metric which is flat on $S \setminus \Sigma$ and has *cone singularities* of a particular kind on Σ. The subject of translation surfaces arose in the study of the billiard ball problem in rational polygons [17]. A rational polygon $P \subset \mathbb{R}^2$ defines a translation surface $S = S(P)$; the study of billiard orbits in P is equivalent to the study of geodesics in S.[2]

Translation surfaces S with a very large group $\mathrm{Aff}(S)$ of affine symmetries are special in that their geodesics behave in a strikingly regular way [15]. The *Veech group* $\mathrm{SL}(S) \subset \mathrm{SL}(2, \mathbb{R})$ of a translation surface consists of differentials of $\varphi \in \mathrm{Aff}(S)$. If $\mathrm{SL}(S)$ is a (necessarily nonuniform) lattice, then S is a *lattice translation surface*. If $S = S(P)$, we will say that P is a *lattice polygon*. A lattice $G \subset \mathrm{SL}(2, \mathbb{R})$ is arithmetic (resp. nonarithmetic) if G is commensurable (resp. not commensurable) with $\mathrm{SL}(2, \mathbb{Z})$. A lattice translation surface is *arithmetic* or *nonarithmetic*, depending on whether the group $\mathrm{SL}(S)$ is arithmetic or not.

Let S (resp. P) be a lattice translation surface (resp. lattice polygon). Then S (resp. P) is secure iff it is arithmetic [4]. Thus, if P is an arithmetic polygon, then for any configuration $\{x, y\}$ all billiard trajectories in $\Gamma(x, y)$ can be blocked by a finite set $B(x, y) \subset P$.[3] If P is a nonarithmetic lattice

[2]In fact, the construction $P \to S(P)$ considerably predates [17]. See [2] and the references there. The work [17] was the first to obtain unexpected applications of this correspondence to the billiard dynamics in general polygons.

[3]A blocking set is explicitely constructed in [4].

polygon, then by [4], there exists a configuration $\{x, y\}$ such that no finite set $B \subset P$ blocks all of the geodesics in $\Gamma(x, y)$.

The argument in [4] exhibits a finite collection of special configurations such that at least one of them is insecure; it does not yield a criterion for a configuration to be secure or insecure. This raises natural questions pertaining to the security of configurations in nonarithmetic polygons. More generally, let X be an insecure riemannian space. What is the nature of the set of insecure configurations? A few examples outline the possibilities. For instance, it is possible that every configuration in X is insecure. This is the case when X is a negatively curved surface.[4] It is also possible that almost all configurations are secure, but certain, special configurations are not. This is the case for the round sphere of any dimension greater than one.

In this paper we investigate from this viewpoint the space of configurations in a nonarithmetic lattice translation surface. We show that the set of insecure configurations has positive measure. Moreover, if the surface has genus two, then the set of insecure configurations has full measure. As an application of these results, we prove the insecurity of almost all configurations in specific polygons.

The exposition is organized as follows. In Section 1 we briefly recall the relevant statements about geodesics in lattice translation surfaces, and outline the main ideas of the paper. Section 2 is the technical basis for our approach. There we investigate the security of certain collections of connecting geodesics inside basic building blocks of a translation surface. In Section 3 we apply this material to translation surfaces of genus two; we also use it to show that almost every configuration in certain polygons is insecure.

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1 Preliminaries and outline

For convenience of the reader, we collect here the material on translation surfaces pertaining to the paper. We refer the reader to [2, 5] and to the survey [8] for more detailed information.

[4]Viktor Schroeder, private communication.

Let S be a translation surface, and let $\Sigma = \{c_1, \dots, c_p\}$ be its cone set. The cone angles satisfy $\alpha(c_k) = 2m_k\pi$, $m_k = 2, 3, \dots$. They determine the genus of the surface via $2g(S) - 2 = \sum_{k=1}^{p}(m_k - 1)$. The moduli space $\mathcal{M}_g$ of translation surfaces of a given genus is stratified according to these data:

$$\mathcal{M}_g = \cup_{\{m_1,\dots,m_p\}}\mathcal{M}_g(m_1, \dots, m_p).$$

The genus of the translation surface $S(P)$ corresponding to a rational polygon is determined by the topology and by the angles of P. When P is simply connected, $g(S(P))$[5] is given by equation 11. The surface $S(P)$ is tiled by an even number, $2N(P)$, of copies of the polygon, yielding the decomposition $S(P) = \cup_{i=1}^{2n} P_i$. The canonical projection $q : S(P) \to P$ is an isometry on every P_i. This projection sends geodesics in $S(P)$ into billiard orbits in P; it reduces the questions pertaining to the security of configurations in P to the corresponding questions in $S(P)$. The most important translation surfaces[6] are the surfaces $S(P)$; however, they form a set of positive codimension in the moduli space of translation surfaces.

Points of $S \setminus \Sigma$ are the regular points of the translation surface. The *direction* of a tangent vector based at a regular point yields a mapping of the unit tangent bundle into $\mathbb{R}/2\pi\mathbb{Z}$. It is invariant under the geodesic flow, hence we can talk about the geodesics in a given direction. Geodesics whose endpoints belong to Σ are the *saddle connections*. Let $\Delta = \Delta(S) \subset [0, 2\pi)$ be the set of their directions.

A periodic geodesic, $\gamma \subset S$, is contained in a unique (maximal, open) flat cylinder $C \subset S$. The isometry $C \simeq \mathbb{R}/l\mathbb{Z} \times (0, w)$ determines the *length* $l = l(C)$ and the *width* $w = w(C)$ of the cylinder. By the *direction of* C we will mean the direction of γ (or any other periodic geodesic contained in C). We will call distinct cylinders with the same direction *parallel*. Any flat cylinder has a particular affine diffeomorphism, called the *Dehn twist*.

A direction θ is *periodic* (resp. *fully periodic*) if S has a cylinder with direction θ (resp. if S is the (disjoint) union of parallel cylinders and saddle connections with direction θ). Let θ be a fully periodic direction, and let $C_1, \dots, C_p$ be the cylinders with direction θ. Slightly abusing notation, we write $S = \cup_{i=1}^{p} C_i$, and call this the *cylinder decomposition* in direction θ. A diffeomorphism $\varphi \in \mathrm{Aff}(S)$ is *parabolic* if the matrix $d\varphi \in \mathrm{SL}(2, \mathbb{R})$ is parabolic. A fully periodic direction θ is parabolic if there exists a (nontrivial)

[5]We will call it the genus of the polygon P.

[6]From this author's viewpoint.

parabolic diffeomorphism $\varphi \in \mathrm{Aff}(S)$ preserving the cylinder decomposition in direction θ.

There are obvious inclusions between the sets of directions introduced above, but for a typical $S \in \mathcal{M}_g$, $g > 1$, these sets differ. If S is a lattice translation surface, then they coincide. We will use the notation $\Delta = \Delta(S)$ for this set, and call it the *set of parabolic directions*. To any $\theta \in \Delta$ corresponds the *basic parabolic diffeomorphism* $\varphi_\theta \in \mathrm{Aff}(S)$ [5, 6]. Acting on S by diffeomorphisms φ_θ corresponding to various parabolic directions, we bring points of S to a "canonical form".

This technique was used in [6] to investigate periodic points in nonarithmetic lattice translation surfaces. Here we use it to study the set of insecure configurations in such surfaces. The observation that makes this approach work is that the set of insecure configurations is invariant under the action of the group $\mathrm{Aff}(S)$ [3, 4]. The Lebesgue measure on the space of configurations allows us to compare the sizes of its subsets.

Along these lines, we prove that the set of insecure configurations in any nonarithmetic lattice translation surface has positive measure. The result is probably not optimal: This set has full measure, provided that the surface has genus 2. See Theorem 1. It is plausible that the latter holds for all nonarithmetic lattice translation surfaces. On the other hand, as recent studies indicate, translation surfaces of genus two are special from several viewpoints [1, 10, 11, 12, 13].

2 Adjacent cylinders

Let S be a translation surface, and let $C \subset S$ be a (maximal, open) cylinder. We will use the terminology and notation of [5]. A cylinder has a direction, $\theta = \theta(C)$, length, $l = l(C)$, and width, $w = w(C)$. If $X \subset S$, we will denote by $\mathrm{clos}(X) \subset S$ its closure. Slightly abusing the language, we will say that $\mathrm{clos}(C) \subset S$ is a *closed cylinder*. We call distinct cylinders in S with the same direction *parallel*. If the intersection of closed parallel cylinders $\mathrm{clos}(C_1), \mathrm{clos}(C_2)$ contains a saddle connection, we say that C_1, C_2 are *adjacent*; the saddle connections contained in $\mathrm{clos}(C_1) \cap \mathrm{clos}(C_2)$ are the *common saddle connections* of C_1, C_2.

Let $C_1, C_2 \subset S$ be 2 adjacent cylinders, and let $m_i \in C_i, i = 1, 2$, be arbitrary points. Recall that $\Gamma(m_1, m_2)$ is the set of geodesics in S connecting these points. Let $G = G(m_1, m_2) \subset \Gamma(m_1, m_2)$ denote the subset of connecting geodesics γ satisfying $\gamma \subset \mathrm{clos}(C_1 \cup C_2)$, and such that γ crosses from C_1 to C_2 exactly once; the crossing point belongs to the interior of a common saddle connection, $\sigma = \sigma(\gamma)$. We will denote by $G^\sigma = G^\sigma(m_1, m_2) \subset G(m_1, m_2)$ the subset of geodesics that cross through σ.

Let now $q_1, q_2 \geq 0$ be integers. There is at most one geodesic $\gamma(q_1, q_2) \in G^\sigma$ that starts from m_1, winds $1 + q_1$ times around C_1, crosses into C_2, winds $1 + q_2$ times around C_2, and finally reaches m_2. We denote by $Q \subset \mathbb{Z}^2$ the set of pairs (q_1, q_2) such that the geodesic $\gamma(q_1, q_2)$ exists. Let $G_\sigma \subset G^\sigma$ be the subset formed by the geodesics $\gamma(q_1, q_2)\,, (q_1, q_2) \in Q$.

In order to analyze the geodesics $\gamma(q_1, q_2)$, we represent $\mathrm{clos}(C_1 \cup C_2) \subset S$ by the union of two rectangles, $R_1 \cup R_2 \subset \mathbb{R}^2$. See figure 1. Then $\mathrm{clos}(C_1 \cup C_2) \subset S$ is obtained by identifying the vertical sides of R_1, R_2, and a part of their horizontal sides, by parallel translations. The horizontal interval $I \subset R_1 \cup R_2$ represents the saddle connection σ. By our assumption, the interior of I is free of identifications. The curve $\tilde{\gamma}(q_1, q_2) \subset R_1 \cup R_2$ representing $\gamma(q_1, q_2)$ connects the points $\tilde{m}_i = (x_i, y_i), i = 1, 2$ in $R_1 \cup R_2$. See figure 1. This curve consists of $1 + q_1 + q_2 + 1$ parallel segments; we describe them below. The first segment goes from $\tilde{m}_1$ to the right vertical side of R_1; then there are q_1 segments going across R_1 all the way; then there are q_2 parallel segments joining the left vertical boundary of $R_1 \cup R_2$ to the right vertical boundary of R_2; the first one of them crosses I; and the last segment of $\tilde{\gamma}(q_1, q_2)$ goes from the left vertical side of R_2 to $\tilde{m}_2 = (x_2, y_2)$. We will refer to the first and the last segments of $\tilde{\gamma}(q_1, q_2)$ as *incomplete*; the remaining $q_1 + q_2$ are the *complete segments*.

Let $\xi(q_1, q_2) \in \sigma$ be the crossing point of $\gamma(q_1, q_2) \in G_\sigma$; let $\Xi \subset \sigma$ be the range of the mapping $(q_1, q_2) \mapsto \xi(q_1, q_2)$. The following lemma will be useful.

Lemma 1 *Set*
$$\tau = \frac{y_1 l_2}{y_2 l_1}. \tag{1}$$
i) Let $\tau \in \mathbb{R} \setminus \mathbb{Q}$. Then the mapping $(q_1, q_2) \mapsto \xi(q_1, q_2)$ is one-to-one, and Ξ is a dense subset of σ.
ii) If $\tau \in \mathbb{Q}$ then the following dichotomy holds:
a) The set Q is empty; b) The set Q is infinite but the set Ξ is finite.

Proof. We assume without loss of generality that the interval representing the saddle connection σ in $R_1 \cup R_2$ satisfies $I = [0, \ell]$ where $\ell = |\sigma|$. See

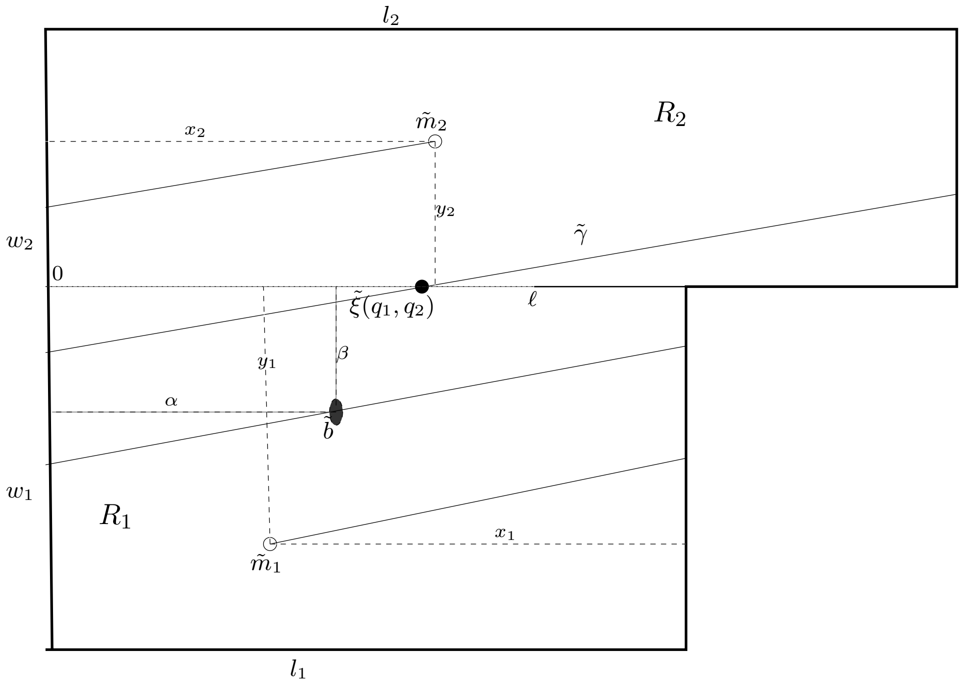

Figure 1: Connecting geodesics for adjacent cylinders

figure 1. For any pair $q_1, q_2 \geq 0$ there is a planar curve $\tilde{\gamma}(q_1, q_2)$ consisting of $2 + q_1 + q_2$ parallel segments and satisfying the descriprion above; let $0 \leq \tilde{\xi}(q_1, q_2) < l_2$ be the point where $\tilde{\gamma}(q_1, q_2)$ crosses the base of R_2. Then the connecting geodesic $\gamma(q_1, q_2)$ exists iff $0 < \tilde{\xi}(q_1, q_2) < \ell$.

A material point traveling along $\tilde{\gamma}(q_1, q_2)$ experiences the horizontal displacement of $x_1 + q_1 l_1 + q_2 l_2 + x_2$ and the vertical displacement of $y_1 + y_2$. Hence the slope of $\tilde{\gamma}(q_1, q_2)$ satisfies

$$k = k(q_1, q_2) = \frac{y_1 + y_2}{x_1 + q_1 l_1 + q_2 l_2 + x_2}. \tag{2}$$

While traveling along the crossing segment of $\tilde{\gamma}(q_1, q_2)$, the material point starts from the left boundary of $R_1 \cup R_2$ at the height $k(x_1 + q_1 l_1)$; by the time it crosses into R_2, it increases its height by $k\tilde{\xi}(q_1, q_2)$. Thus

$$k(x_1 + q_1 l_1 + \tilde{\xi}(q_1, q_2)) = y_1. \tag{3}$$

From equations 2, 3 we obtain (after elementary transformations)

$$\tilde{\xi}(q_1, q_2) = \frac{(y_1 x_2 - y_2 x_1)}{y_1 + y_2} + q_2 \frac{y_1 l_2}{y_1 + y_2} - q_1 \frac{y_2 l_1}{y_1 + y_2}. \tag{4}$$

Let $z \mapsto \hat{z}$ be the affine transformation of $\mathbb{R}$ given by

$$\hat{z} = \frac{z(y_1 + y_2) - (y_1 x_2 - y_2 x_1)}{y_2 l_1},$$

and set $\hat{\xi}(q_1, q_2)$ be the image of $\tilde{\xi}(q_1, q_2)$. By equation 4, $\hat{\xi}(q_1, q_2) = q_2 \tau - q_1$. By preceding remarks, the connecting geodesic $\gamma(q_1, q_2)$ exists iff $q_2 \tau - q_1 \in (\hat{0}, \hat{\ell})$.

To simplify the exposition, we assume now that $(\hat{0}, \hat{\ell}) \subset [0, 1]$ and that $\tau < 1$. (We leave the reduction to this case to the reader.) Then our claims immediately follow from the classical properties of circular rotations. ∎

We will now study the situation when several connecting geodesics pass through the same point in $\text{clos}(C_1 \cup C_2)$. Let $\gamma \in G$ be a connecting geodesic, and let $b \in \text{clos}(C_1 \cup C_2) \cap \text{interior}(\gamma)$. Let $\tilde{b} \in R_1 \cup R_2$ be the point representing b. We say that γ passes through b *in a regular way* if $\tilde{b}$ belongs to a complete segment of $\tilde{\gamma}$. Let the setting be as in Lemma 1. We will need the following proposition.

Lemma 2 *Let $b \in C_1$ be a point such that there are 3 distinct geodesics, $\gamma, \gamma', \gamma''$ in G passing through b in a regular way. Let $(q_1, q_2), (q_1', q_2'), (q_1'', q_2'') \in \mathbb{N}^2$ be their parameters.*
Then either the cylinder lengths l_1, l_2 are commensurate or the three points $\vec{q} = (q_1, q_2), \vec{q}' = (q_1', q_2'), \vec{q}'' = (q_1'', q_2'')$ are colinear.

Proof. Let $\tilde{b} = (\alpha, \beta)$. See figure 1. Let $\tilde{\gamma}$ run into $\tilde{b}$ after p complete segments, where $0 \leq p \leq q_1$. Let p', p'' be the corresponding numbers for $\tilde{\gamma}', \tilde{\gamma}''$ respectively. Consider the material point moving along γ from m_1 to b. Its horizontal (resp. vertical) displacement is $x_1 + pl_1 + \alpha$ (resp. $y_1 - \beta$). See figure 1. This determines the slope of γ, and equation 2 yields

$$\frac{y_1 - \beta}{x_1 + pl_1 + \alpha} = \frac{y_1 + y_2}{x_1 + q_1 l_1 + q_2 l_2 + x_2}.$$

After elementary algebra, we obtain

$$p(y_1 + y_2)l_1 - q_1(y_1 - \beta)l_1 - q_2(y_1 - \beta)l_2 =$$

$$-x_1(y_1 + y_2) - \alpha(y_1 + y_2) + y_1(x_1 + x_2) - \beta(x_1 + x_2). \tag{5}$$

The right hand sides of equation 5 and its counterparts corresponding to $\tilde{\gamma}', \tilde{\gamma}''$ are the same; subtracting equation 5 from the other two, we obtain, by elementary transformations

$$(q_1' - q_1)l_1 + (q_2' - q_2)l_2 = (p' - p)\frac{y_1 + y_2}{y_1 - \beta} l_1, \tag{6}$$

$$(q_1'' - q_1)l_1 + (q_2'' - q_2)l_2 = (p'' - p)\frac{y_1 + y_2}{y_1 - \beta} l_1, \tag{7}$$

If $p' = p$ or $p'' = p$, then one of the equations above implies that l_1, l_2 are commensurate. Otherwise we divide the latter by the former, and obtain

$$\frac{(q_2'' - q_2)l_2 + (q_1'' - q_1)l_1}{(q_2' - q_2)l_2 + (q_1' - q_1)l_1} = \frac{(p'' - p)}{(p' - p)}. \tag{8}$$

If

$$\begin{vmatrix} q_2'' - q_2 & q_1'' - q_1 \\ q_2' - q_2 & q_1' - q_1 \end{vmatrix} \neq 0 \tag{9}$$

then, by equation 8, $\frac{l_2}{l_1} \in \mathbb{Q}$. The determinant in equation 9 vanishes iff the integer vectors $\vec{q}'' - \vec{q}, \vec{q}' - \vec{q}$ are linearly dependent over $\mathbb{Q}$, equivalently iff the integer points $\vec{q}, \vec{q}', \vec{q}''$ are colinear. ∎

We will now formulate and prove the main result of this section.

Proposition 1 *Let S be a translation surface, and let $C_1, C_2 \subset S$ be adjacent cylinders. Let l_1, w_1 and l_2, w_2 be the respective parameters. Assume that l_1, l_2 are incommensurate.*
Let $m_i \in C_i, i = 1, 2$ be arbitrary points and let y_1, y_2 be the respective vertical coordinates. See figure 1. Let $G = G(m_1, m_2) \subset \Gamma(m_1, m_2)$ be the set of connecting geodesics γ such that $\gamma \subset clos(C_1 \cap C_2)$, and γ crosses from C_1 to C_2 exactly once.

Let $\frac{y_1 l_2}{y_2 l_1}$ be the number introduced in Lemma 1. If this number is irrational, then a point in $clos(C_1 \cap C_2)$ can block at most a finite number of geodesics in G.

Proof. Assume the opposite, and let $b \in \mathrm{clos}(C_1 \cup C_2)$ block infinitely many geodesics in G. Since they don't pass through the boundary of $\mathrm{clos}(C_1 \cup C_2)$, the point b belongs to the interior of $\mathrm{clos}(C_1 \cup C_2)$. Since $\mathrm{clos}(C_1 \cup C_2)$ contains finitely many common saddle connections, there is a common saddle connection σ such that b blocks infinitely many geodesics in G^σ.

We will use the notation of Lemma 1 and Lemma 2. Since all but a finite number of geodesics in G^σ belong to G_σ, the point b blocks infinitely many geodesics in G_σ. We denote this collection of geodesics by G_0, and let $Q_0 \subset Q$ be the corresponding infinite set of parameters. Thus, $G_0 = \{\gamma(q_1, q_2) : (q_1, q_2) \in Q_0\}$.

Note that $b \in \mathrm{interior}(\sigma) \cup C_1 \cup C_2$. By symmetry of our setting, we assume wlg that $b \in \mathrm{interior}(\sigma) \cup C_1$. If $b \in \mathrm{interior}(\sigma)$ then, by Lemma 1, it can block at most one geodesic. Hence, $b \in C_1$.

If $\gamma \in G_0$ passes through b not in a regular way, then the first segment of $\tilde{\gamma}$ contains $\tilde{b}$. This determines the slope of $\tilde{\gamma}$, and hence all geodesics in G_0 but at most one pass through b in a regular way. Let $G_1 \subset G_0$ be the corresponding infinite collection, and let $Q_1 \subset Q_0$ be their parameter set.

By Lemma 2, we have $Q_1 \subset L \cap \mathbb{N}^2$, where L is a line (actually, a ray) in $\mathbb{R}^2$. The limit of the ratios q_1/q_2, as $(q_1, q_2) \to \infty$ and $(q_1, q_2) \in L$, is the inverse slope of L. Since L contains more than one integer point, its slope is rational.

Let $\tau = y_1 l_2 / y_2 l_1$. By the proof of Lemma 1, there is a finite interval $I \subset \mathbb{R}$, $|I| = |\sigma|$, s.t. $q_2\tau - q_1 \in I$ for all $(q_1, q_2) \in Q_1$. Hence

$\tau^{-1} = \lim(q_1/q_2)$, as $(q_1, q_2) \to \infty$. We have obtained $\tau^{-1} \in \mathbb{Q}$, which is a contradiction. ■

3 Insecure configurations for lattice translation surfaces

Let S be a lattice translation surface, let $\mathrm{Aff}(S)$ be the group of (orientation preserving) affine diffeomorphisms of S, let $SL(S) \subset \mathrm{SL}(2, \mathbb{R})$ be the Veech group of S, and let $\Delta = \Delta(S)$ be the set of parabolic directions. See the Introduction and Section 1. For $\theta \in \Delta$ and $z \in S$ we denote by $0 \leq \rho_\theta(z) < 1$ the corresponding *rotation number* [6, 4]. Suppose that z is a regular point. Then z is contained in a unique geodesic, γ, with direction θ; the parabolic diffeomorphism φ_θ induces the rotation of γ with the rotation number $\rho_\theta(z)$.

Proposition 2 *Let S be a lattice translation surface, and let $\theta \in \Delta$. Let $C', C'' \subset S$ be adjacent cylinders with direction θ. Suppose that their lengths are incommensurate.*
Let $z' \in C', z'' \in C''$ be arbitrary points. If $\rho_\theta(z')/\rho_\theta(z'')$ is irrational, then the configuration (z', z'') is insecure.

Proof. Let C be a flat cylinder, and let l, w be its length and width. Let $0 \leq x < l, 0 < y < w$ be the natural coordinates in C. The Dehn twist $\varphi = \varphi_C$ acts on C by

$$\varphi(x, y) = (x + \frac{y}{w} l, y).$$

The diffeomorphism φ induces a rotation on every closed geodesic $y = \mathrm{const}$ in C; the corresponding rotation number is y/w.

Let now l', w' and l'', w'' be the length and the width of C', C'' respectively. The ratios $\mu' = l'/w', \mu'' = l''/w''$ are the *moduli* of C', C'' respectively. Let $(x', y'), (x'', y'')$ be the coordinate systems in C', C'' respectively. For a cylinder C with direction θ there is a positive integer $m(C)$ such that for any $z \in C$ the rotation number $\rho_\theta(z)$ satisfies $\rho_\theta(z) = m(C) y/w$.

Set $m' = m(C'), m'' = m(C'')$. By Proposition 1, it suffices to show that the number $y' l''/y'' l'$ is irrational. We have, by elementary algebra

$$\frac{y' l''}{y'' l'} = \frac{\frac{m' y'}{w'}}{\frac{m'' y''}{w''}} \frac{m'' \mu''}{m' \mu'} = \frac{\rho_\theta(z')}{\rho_\theta(z'')} \frac{m'' \mu''}{m' \mu'}. \tag{10}$$

By [15], μ'/μ'' is rational. ∎

We will work with *ordered configurations*, i. e., with the space $S \times S$. The group Aff(S) naturally acts on $S \times S$, and the normalized Lebesgue measure $\lambda \times \lambda$ is invariant under this action. We will refer to the induced measure on the space of configurations as the Lebesgue measure as well. We will use the same notation for it. Speaking of a set of positive measure, or about a property that holds almost everywhere, we will always mean the Lebesgue measure.

The following technical proposition is needed in connection with the material about insecure configurations in lattice translation surfaces of genus two.

Lemma 3 *Let S be a nonarithmetic translation surface, and suppose that the following conditions are satisfied:*
1) There is a parabolic direction α decomposing S into 2 cylinders;
2) There is a transversal parabolic direction β decomposing S into at least 2 cylinders.
Then almost every configuration in S is insecure.

Proof. We assume without loss of generality that α, β are the horizontal and the the vertical directions respectively [6]. For $z \in S$ we set for brevity $\rho_\alpha(z) = \rho_0(z), \rho_\beta(z) = \rho_1(z)$. Denote by H_1, H_2 (resp. $V_1, \dots, V_t$) the cylinders of the horizontal (resp. vertical) decomposition. For any parabolic direction, the cylinder lengths are incommensurate. (If they are, then S is arithmetic [5, 6].[7])

Denote by F the set of configurations. The subset of $\{x, y\} \in F$ such that both points belong to $H_1 \cup H_2$ has full measure. We restrict our attention to this subset, and denote it by F as well. Let $F_{\mathrm{ins}} \subset F$ be the set of insecure configurations. We will show that almost every $\{x, y\}$ belongs to F_{ins}. We will use the obvious fact that the action of Aff(S) preserves the (in)security of configurations.

Denote by $F_{12} \subset F$ the set of configurations $\{x, y\}$ such that the points belong to different cylinders. Let F_{11}, F_{22} be defined analogously. Then $F = F_{12} \cup F_{11} \cup F_{22}$, a disjoint union. By Proposition 2, almost every configuration in F_{12} is insecure. In view of symmetry, it remains to show that almost every configuration in F_{11} is insecure.

[7] This is a special case of a general criterion for arithmeticity.

Let $\varphi, \psi \in \text{Aff}(S)$ be the basic parabolic diffeomorphisms in the horizontal, vertical directions respectively. The set F_{11} is invariant under φ. Let $G_{11} \subset F_{11}$ be the set of $\{x, y\}$ such that $\rho_0(x), \rho_0(y)$ and $\rho_0(x)/\rho_0(y)$ are irrational. Then G_{11} is a subset of full measure, invariant under φ; it suffices to show that almost every $\{x, y\} \in G_{11}$ is insecure.

Let γ be a periodic geodesic in H_1, and let $l = l(H_1)$ be its length. The vertical cylinders decompose γ into consecutive closed intervals $I_1, \dots, I_N$ so that I_k, I_{k+1} have a common endpoint, s_k. For every $1 \leq k \leq N$ there is a vertical cylinder $V(k)$ s.t. interior$(I_k) \subset V(k)$; at s_k the geodesic γ crosses from $V(k)$ to $V(k+1)$.

Figure 2 presents an example that illustrates the discussion. In figure 2 the cylinders H_1, H_2 are represented by the lower, upper horizontal rectangles respectively. There are 3 vertical cylinders: V', V'', V'''. They partition the geodesic γ into 5 intervals: $I_1, \dots, I_5$. We have $I_1 \subset V'$, $I_2 \subset V''$, $I_3 \subset V'$, $I_4 \subset V''$, $I_5 \subset V'''$. The cylinders V' and V'' (resp. V'' and V''', resp. V''' and V') are adjacent and have incommensurable lengths.

The decomposition above does not depend on the choice of γ. We set $\rho_0(\gamma) = \rho_0(z)$, where $z \in \gamma$ is an arbitrary point. We have $\varphi^n(z) = z + n\rho_0(\gamma) \cdot l \mod l$. As z runs through γ, the rotation number $\rho_1(z)$ varies piecewise linearly and continuously; in particular, $\rho_1(z)$ is irrational for almost all $z \in \gamma$.

We assume without loss of generality that the lengths of the adjacent vertical cylinders $V(1), V(2)$ are incommensurate. Let $\{x, y\} \in G_{11}$ and let α, β be the horizontal geodesics containing x, y respectively. By our assumptions on $\rho_0(x), \rho_0(y)$, there are infinitely many $n \in \mathbb{N}$ s.t.

$$\varphi^n(x) = x + n\rho_0(\alpha) \cdot l \mod l \ \in \text{interior}(I_1),$$

and

$$\varphi^n(y) = y + n\rho_0(\beta) \cdot l \mod l \ \in \text{interior}(I_2).$$

Let $n \in \mathbb{N}$ be such that the inclusions above are satisfied. By preceding remarks, the condition that $\rho_1(\varphi^n(x))/\rho_1(\varphi^n(y))$ is irrational defines a set of full measure. By Proposition 2, the configuration $\{\varphi^n(x), \varphi^n(y)\}$ is insecure.

Thus, for almost every $\{x, y\} \in F_{11}$ there exists a diffeomorphism $g \in \text{Aff}(S)$ such that $g \cdot \{x, y\}$ is an insecure configuration. Therefore, almost every configuration $\{x, y\} \in F_{11}$ is insecure. ∎

We will now prove the main result of this section.

Theorem 1 *The set of insecure configurations of a nonarithmetic lattice surface has positive measure. If the surface has genus 2, then almost every configuration is insecure.*

Proof. Let S be a nonarithmetic lattice surface. Then it has a parabolic direction, θ, with at least 2 cylinders.[8] Let C_1, C_2 be adjacent cylinders in direction θ with incommensurate lengths. The set F_{12} of configurations $\{x, y\}$ such that the points x, y belong to different cylinders in $(C_1 \cup C_2) \subset S$ has measure $2\lambda(C_1)\lambda(C_2) > 0$. By Proposition 2, almost every configuration in F_{12} is insecure.

Assume now that S has genus 2. Then S has infinitely many mutually *transversal* parabolic decompositions with 2 cylinders [1, 9], and Lemma 3 applies. ∎

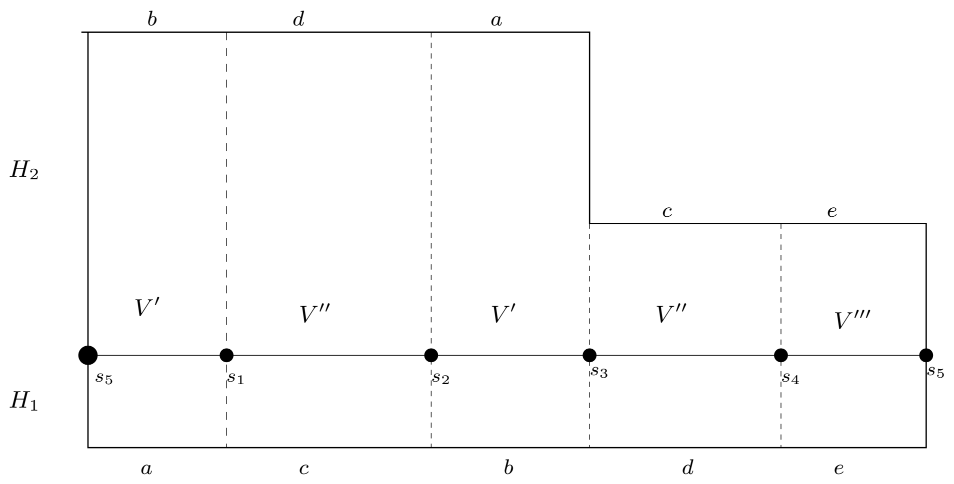

Figure 2: A pair of transversal parabolic decompositions

4 Applications to polygonal billiards

Let $P \in \mathbb{R}^2$ be an arbitrary polygon. A configuration $\{x, y\}$ is a pair of points in P; it is secure if all billiard orbits connecting x, y can be blocked by a finite set $B \subset P$. A polygon is secure if every configuration in it is secure.

[8]If every parabolic direction has 1 cylinder then S is a torus.

A polygon P is rational if the angles between its sides are π-rational.[9] A classical construction associates with a rational polygon P a translation surface $S(P)$. We say that P is a lattice polygon if $S(P)$ is a lattice translation surface. Billiard orbits in lattice polygons satisfy the Veech dichotomy [15]. A lattice polygon P is arithmetic (resp. nonarithmetic) if the lattice translation surface $S(P)$ is arithmetic (resp. nonarithmetic).

A lattice polygon is secure iff it is arithmetic [4]. Denote by μ the Lebesgue measure in $P \subset \mathbb{R}^2$; let $\mu \times \mu$ be the induced measure on the space $F = F(P)$ of configurations. By the genus of a rational polygon we will mean the genus of its translation surface.

Theorem 2 *Let P be a nonarithmetic lattice polygon of genus 2. Then almost every configuration in P is insecure.*

Proof. Let P be any rational polygon, and let $N = N(P)$ be the least common denominator of the angles of P. The cardinality of fiber $q^{-1}(x)$ of the canonical projection $q : S(P) \to P$ is at most $2N$; it is equal to $2N$ for almost all $x \in P$. Denote by $F(S)$ the space of configurations in $S(P)$, and let $q : F(S) \to F(P)$ be the induced mapping. A configuration $\{x, y\} \in F(P)$ is secure iff for any $\tilde{x} \in q^{-1}(x), \tilde{y} \in q^{-1}(y)$ the configuration $\{\tilde{x}, \tilde{y}\} \in F(S)$ is secure [4]. See also [3].

Recall that $F(S)$ is endowed with the measure $\lambda \times \lambda$ where λ is the Lebesgue measure in S. With the normalization $\mu(P) = 1, \lambda(S) = 2N$, we have $q_*(\lambda) = \mu, q_*(\lambda \times \lambda) = \mu \times \mu$. Hence, almost every configuration in P is secure iff almost every configuration in $S(P)$ is secure. The claim now follows from the second statement of Theorem 1. ∎

Before applying Theorem 2, we discuss the genus of rational polygons, and then briefly survey the subject of lattice polygons. Let P be a simply connected rational p-gon, let $v_1, \dots, v_p$ be the corners of P, and let $\frac{m_i}{n_i}\pi, 1 \leq i \leq p$, be the corner angles.[10] Let $N = N(P)$ be the least common denominator of these fractions. A corner v_i is *essential* if $m_i > 1$. (A polygon without essential corners has genus one, and hence is arithmetic.)

Lemma 4 *Let P be a simply connected rational polygon of genus 2. Then one of the following possibilities holds.*

[9] A simply connected polygon is rational iff all its vertex angles are π-rational.

[10] All fractions are in lowest terms.

a) The polygon has at most 4 corners, and exactly one of them is essential; its angle is $2\pi/n$ where n is odd, and $N = 2n$.
b) The polygon has exactly one essential corner; its angle is $3\pi/n$, where n is even and not divisible by 3, and $N = n$.
c) The polygon has exactly two essential corners; both corner angles are $2\pi/n$ where n is odd, and $N = n$.

Proof. The genus of P satisfies [2]

$$g = 1 + \frac{N}{2} \sum_{i=1}^{p} \frac{m_i - 1}{n_i}. \tag{11}$$

(Thus, a polygon without essential corners has genus one, and hence is arithmetic.) We rewrite equation 11 as

$$\sum \frac{N}{n_i}(m_i - 1) = 2(g - 1), \tag{12}$$

where the summation is over essential corners of P. In our case, equation 12 implies

$$\sum \frac{N}{n_i}(m_i - 1) = 2.$$

The contribution of each essential corner to the left hand side is a positive integer. This immediately limits the possibilities to the 3 cases in the statement. ■

What rational polygons are nonarithmetic lattice polygons? This question generated a considerable literature. Although it is still open, there are many partial results, and many examples. We present some of them below. Let T_n be the right triangle whose smallest angle is π/n, where $n = 4, 5, 6, \dots$. For $n \geq 3$ let I_n (resp. J_n) be the isosceles triangle with the base (resp. apex) angle π/n. Let R_n, $n \geq 3$, be the regular n-gon. In [15, 16] W. Veech showed that I_n, R_n are nonarithmetic lattice polygons for $n \neq 3, 4, 6$.[11]

For positive numbers p, q, r such that $p + q + r = 1$ let $T(p, q, r)$ be the unique (up to scaling) triangle with angles $\pi p, \pi q, \pi r$. By [7, 14], the only acute non-isosceles triangles which are lattice polygons are $T(1/3, 1/4, 5/12)$

[11] The remaining cases are arithmetic. See, e.g., [5].

and $T(1/3, 1/5, 7/15)$, and $T(1/3, 2/9, 4/9)$.[12] A right (resp. acute isosceles) triangle T is a lattice polygon iff $T = T_n$ (resp. $T = J_n$) [7, 14]. Besides, by [15] and [5], these triangles are arithmetic iff $n = 3, 4, 6$.

Lemma 5 *The only triangles of genus* 2 *are the right triangles* $T(1/2, 1/5, 3/10)$, $T(1/2, 1/8, 3/8)$, $T(1/2, 1/10, 2/5)$ *and the isosceles triangles* $T(1/5, 2/5, 2/5)$, $T(1/6, 1/6, 2/3)$.

Proof. We use Lemma 4 and elementary number theory. For natural numbers a, b we will say that a *divides* b if b/a is an integer. We will denote this by $a|b$. In particular, $1|b$ and $b|b$.

Let T be a triangle of genus 2 that falls into the case c) of Lemma 4. Then $T = T(1/p, 2/n, 2/n)$ where p divides n. Set $p = n/q$ where $q|n$. From

$$\frac{1}{p} + \frac{2}{n} + \frac{2}{n} = 1$$

we obtain $q = n - 4$. Since n is odd and $q|n$, we have $q < n/2$, yielding $n < 8$. This leaves us with the possible cases $n = 5, 7$. Examining them further, we find that the only possibility is $n = p = 5$, yielding $T(1/5, 2/5, 2/5)$.

Let T be a triangle of genus 2 falling into the case b) of Lemma 4. Then $T = T(1/p, 1/q, 3/n)$ where p, q divide n. Setting $p = n/i, q = n/k$, we obtain the equation $i + k = n - 3$. The possibility $i = k = n/2$ is out; using that n is not divisible by 3 and the symmetry between i, k, we have without loss of generality, $i \le n/2, k \le n/4$. This and the equation above imply $n \le 12$. Using that n is not divisible by 3, we find that the only solutions are $5 + 2 = 7$ and $4 + 1 = 5$. They yield the triangles $T(1/2, 1/5, 3/10)$ and $T(1/2, 1/8, 3/8)$ respectively.

We leave the analysis of case a) to the reader. It is similar, and yields the two remaining triangles. ∎

The results above have a curious consequence.

Proposition 3 *All triangles of genus* 2 *are lattice triangles. Triangle* $T(1/6, 1/6, 2/3)$ *is arithmetic, and the other four are nonarithmetic.*

Proof. The claim that all triangles of genus 2, except $T(1/6, 1/6, 2/3)$, are lattice triangles is immediate from Lemma 5 and the quoted results of [15]

[12] They are nonarithmetic, and have genus greater than 2.

and [7, 14]. The triangle $T(1/6, 1/6, 2/3)$ is an arithmetic lattice triangle, by [5]. The non-arithmeticity of the other four genus 2 triangles follows, by [5], from the lattice property. ■

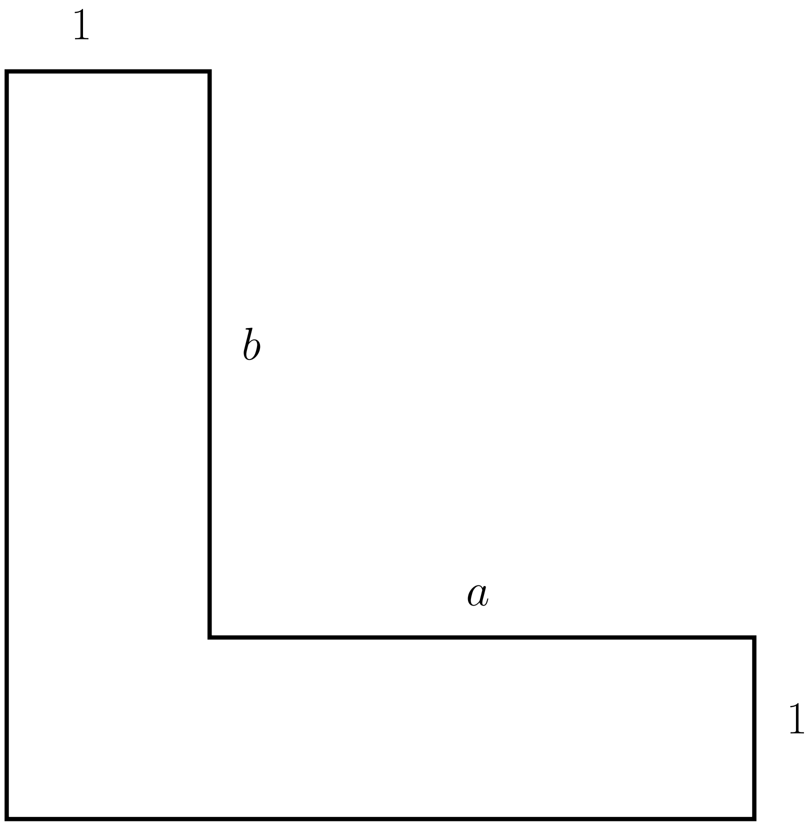

Figure 3: Parameters for L-shaped polygons

We will now study the security/insecurity of configurations in concrete polygons. For arbitrary $a, b > 0$ we denote by $L(a, b)$ the L-shaped hexagon of figure 3. By equation 11, the genus of $L(a, b)$ is 2.[13]

Theorem 3 *1. Almost every configuration in the right triangles* $T(1/2, 3/10, 1/5), T(1/2, 3/8, 1/8), T(1/2, 2/5, 1/10)$ *and the isosceles triangle* $T(2/5, 2/5, 1/5)$ *is insecure. Any configuration in the isosceles triangle*

[13]Polygons $L(a, b)$ fall into the case b) of Lemma 4.

$T(2/3, 1/6, 1/6)$ is secure.
2. Let a, b be positive irrational numbers such that $a - b \in \mathbb{Q}$ and $(a+b+1)^2 \in \mathbb{Q}$. Then almost every configuration in $L(a, b)$ is insecure. If $a, b \in \mathbb{Q}$, then any configuration in $L(a, b)$ is secure.

Proof. 1. The first claim is immediate from Theorem 2 and Proposition 3. By the same proposition, triangle $T(2/5, 2/5, 1/5)$ is arithmetic, hence secure [4]. This proves the second claim.
2. Let a, b be arbitrary, and set $u = u(a, b), v = v(a, b)$ be the solutions of the system

$$2a + 1 = u + \sqrt{v}, \ 2b + 1 = -u + \sqrt{v}. \tag{13}$$

By [5], $L(a, b)$ is arithmetic iff $a, b \in \mathbb{Q}$. By [4], arithmetic polygons are secure, hence the second claim.
By [9], $L(a, b)$ is a lattice polygon iff $u, v \in \mathbb{Q}$. By equation 13, $u = a - b, v = (a + b + 1)^2$. Thus, the first condition holds iff $L(a, b)$ is a nonarithmetic lattice polygon. Theorem 2 implies the first claim. ∎

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