STOCHASTIC STABILITY OF NON-UNIFORMLY HYPERBOLIC DIFFEOMORPHISMS

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Abstract. We prove that the statistical properties of random perturbations of a diffeomorphism with dominated splitting having mostly contracting center-stable direction and non-uniformly expanding center-unstable direction are described by a finite number of stationary measures. We also give necessary and sufficient conditions for the stochastic stability of such dynamical systems. We show that a certain C^2-open class of non-uniformly hyperbolic diffeomorphisms introduced by Alves, Bonatti and Viana in [ABV] are stochastically stable. Our setting encompasses that of partially hyperbolic diffeomorphisms as well as uniformly hyperbolic diffeomorphisms.

Contents

1. Introduction

Dynamical Systems Theory focuses mainly on two problems: what is the behavior of a given system, and how this behavior changes under small modifications of the law that governs the system. Properties which are shared by the original system and its perturbations are called "stable". This work is about the stability of certain dynamical systems from a probabilistic point of view.

Date: September 12, 2005.

2000 *Mathematics Subject Classification.* Primary: 37D25. Secondary: 37C40, 37H15.

Key words and phrases. Dominated splitting, non-uniform hyperbolicity, SRB measure, random perturbation, stochastic stability.

J.F.A. and V.A. were partially supported by FCT through CMUP, POCI/MAT/61237/2004, and grants of FCT. Part of this work was done while V.A. enjoyed a leave at IMPA and C.V enjoyed a leave at CMUP.

Much of the recent progress in Dynamics is a consequence of a probabilistic approach to the understanding of complicated dynamical systems, where one focuses on the statistical properties of "typical orbits", in the sense of large volume in the ambient space. We deal here with diffeomorphisms $f : M \to M$ on compact manifolds. The most basic statistical data are the time averages

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j(z)}$$

where δ_w represents the Dirac measure at a point w. The Ergodic Theorem asserts that time averages do exist at almost every point for any invariant probability. Moreover, if the measure is ergodic then the time average coincides with the space average, that is, the invariant probability measure itself. However, many invariant measures are singular with respect to volume in general, and so the Ergodic Theorem is not enough to understand the behavior of positive volume (Lebesgue measure) sets of orbits.

An *SRB measure* is an invariant probability measure for which time averages exist and coincide with the space average, for a set of initial conditions with positive Lebesgue measure. This set is called the *basin* of the measure. Sinai, Ruelle and Bowen introduced this notion more than thirty years ago, and proved that for uniformly hyperbolic (Axiom A) diffeomorphisms and flows time averages exist for Lebesgue almost every point and coincide with one of finitely many SRB measures; see [Si, BR, Bo, Ru].

The problem of existence and finiteness of SRB measures and their stability under perturbations of the system, beyond the Axiom A setting, still remains as a main goal of Dynamics. The construction of the so called Gibbs u-states, by Pesin and Sinai in [PS] was the beginning of the extension of the Sinai, Ruelle and Bowen ideas to partially hyperbolic systems, a fruitful generalization of the notion of uniform hyperbolicity, which more recently was shown to encompass Lorenz-like flows [Tu, MPP] and to be a consequence of robust transitivity [BDP]. We refer the reader to [BDV, Vi2, Yo2] for surveys on much of the progress obtained so far.

The papers of Alves, Bonatti and Viana [ABV, BoV], and Dolgopyat [Do] are of special interest to us here since they prove existence and finiteness of SRB measures for partially hyperbolic diffeomorphisms, under the assumption that the central direction is either "mostly contracting" [BoV, Do] or "non-uniformly expanding" [ABV]. Indeed, we are going to prove that a large class of systems in this setting, having weak hyperbolicity properties, namely an invariant dominated splitting with weak expansion and contraction, are stochastically stable. This kind of diffeomorphisms has been intensely studied in recent years. See [BDV, Vi2] and references therein for a global view of the state of the art in this setting.

It was conjectured by Palis [Pa] that every dynamical system can be approximated by another one having only finitely many SRB measures, whose basins cover Lebesgue almost every point of the ambient manifold. Moreover, these SRB measures should be stable under perturbations of the system, in a stochastic sense.

Stochastic stability means that time averages do not change much under small random perturbations – the precise definition will be given in the next section. This concept was introduced by Kolmogorov and Sinai and much developed by the pioneering work of Kifer; see [Ki1, Ki2] and references therein. The reader can also see [Li] for a recent survey on random dynamical systems.

Stochastic stability is well-known for uniformly expanding maps and for uniformly hyperbolic diffeomorphisms [Ki1, Yo1, Vi1] restricted to the basin of their attractors. More recently, [AA] proved that the same is true for an open class of non-uniformly expanding maps.

In this paper we prove stochastic stability for a large class of diffeomorphisms admitting an invariant dominated splitting with non-uniformly expanding center-unstable direction and mostly contracting center-stable direction. Precise definitions and statements of our results follow.

1.1. **Non-uniformly hyperbolic diffeomorphisms.** Let $f : M \to M$ be a smooth map defined on a compact Riemannian manifold M. We write $\|\cdot\|$ for the induced norm on TM and fix some normalized Riemannian volume form m on M that we call *Lebesgue measure*. As explained in the Introduction, we say that an f-invariant Borel probability measure μ on M is an *SRB measure* if for a positive Lebesgue measure set of points $x \in M$

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi\big(f^n x\big) = \int \varphi \, d\mu, \tag{1}$$

for every continuous map $\varphi : M \to \mathbb{R}$. We define the *basin* $B(\mu)$ of μ as the set of those points x in M for which (1) holds for all continuous φ.

Let $f : M \to M$ be a diffeomorphism for which there is a strictly forward f-invariant open set $U \subset M$, that is $f(\overline{U}) \subset U$, and there is a continuous Df-invariant splitting $E^{cs} \oplus E^{cu}$ of $T_U M$, the tangent bundle over U. The bundles E^{cs} and E^{cu} will be called *center-stable* and *center-unstable* and have dimensions u and s, respectively, with $u, s \geq 1$ and $s + u = \dim(M)$. We will assume several conditions on the splitting of $T_U M$:

(a) *Dominated decomposition:* there exists a constant $0 < \lambda < 1$ for which

$$\|Df|E^{cs}_x\| \cdot \|Df^{-1}|E^{cu}_{fx}\| \leq \lambda \quad \text{for all } x \in U.$$

(b) *Nonuniform expansion along the central-unstable direction:* there is $c_u > 0$ such that for Lebesgue almost all $x \in U$

$$\limsup_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Df^{-1}|E^{cu}_{f^j x}\| \leq -c_u.$$

Theorem 6.3 of [ABV] shows that a diffeomorphism f with a dominated splitting non-uniformly expanding along the center-unstable direction has some ergodic *Gibbs cu-state* μ supported in the attractor $\Lambda = \bigcap_{j=0}^{\infty} f^j(U)$, that is, μ is an invariant probability measure whose $\dim E^{cu}$ larger Lyapunov exponents are positive and whose conditional measures along the corresponding local unstable manifolds are almost everywhere absolutely continuous with respect to Lebesgue measure on such manifolds. This notion is a non-uniform version of the Gibbs u-states introduced by Pesin and Sinai [PS]. Moreover if the derivative of f has uniform behavior over the sub-bundle E^{cs}, then μ is an SRB measure. This is a well known consequence of the absolute continuity of the conditional measures of μ and absolute continuity of the stable lamination [Pe]: the union of the stable manifolds through the points whose time averages are given by μ is a positive Lebesgue measure subset of the basin of μ.

As shown in [BoV, Do] there are some cases where non-uniform behavior on the direction of E^{cs} is enough for ensuring the existence of SRB measures for f. A sufficient condition which we assume here is that on every disk at any local unstable manifold there exists a positive Lebesgue measure subset of points of that disk having negative Lyapunov exponents along the central-stable direction:

(c) *Mostly contracting central-stable direction:* every disk D contained in an unstable local manifold satisfies

$$\limsup_{n\to+\infty}\frac{1}{n}\log\|Df^n|E^{cs}_x\|<0$$

for a positive Lebesgue measure set of points $x\in D$.

Conditions (a)-(c) above are the main conditions we assume on the C^2 diffeomorphism f. Condition (a) is a classical one and was already considered in [HPS] as a weakening of the uniform hyperbolicity conditions. Condition (b) was first considered in [ABV] and together with condition (a) it ensures the existence of Gibbs cu-states. Condition (c) was considered in [BoV, Do] as a means to ensure that Gibbs cu-states are SRB measures. These conditions are satisfied by C^2 open classes of diffeomorphisms beyond uniformly hyperbolic systems, as shown in the last section.

1.2. **Statement of results.** In this work we are interested in studying random perturbations of a C^2 diffeomorphism $f\colon M\to M$ satisfying conditions (a)-(c) from Subsection 1.1. For that, we take a continuous map

$$\begin{array}{rccc}\Phi: & T & \longrightarrow & \mathrm{Diff}^2(M,M)\\ & t & \longmapsto & f_t\end{array}$$

from a metric space T into the space of C^2 diffeomorphisms of M, with $f=f_{t^*}$ for some fixed $t^*\in T$. Given $x\in M$, a *random orbit* of x will be a sequence $(f^n_{\underline{t}}x)_{n\ge1}$ where $\underline{t}$ denotes an element $(t_1,t_2,t_3,\dots)$ in the product space $T^{\mathbb{N}}$ and

$$f^n_{\underline{t}}=f_{t_n}\circ\cdots\circ f_{t_1}\quad\text{for}\quad n\ge1.$$

We also take a family $(\theta_\varepsilon)_{\varepsilon>0}$ of probability measures on T such that their supports $\mathrm{supp}\,(\theta_\varepsilon)$ form a nested family of connected compact sets and $\mathrm{supp}\,(\theta_\varepsilon)\to\{t^*\}$ when $\varepsilon\to0$. We assume some quite general non-degeneracy conditions on the map Φ and the measures θ_ε (see the beginning of Section 2) and refer to $\{\Phi,(\theta_\varepsilon)_{\varepsilon>0}\}$ as a *random perturbation* of f. It is known [Ar1] that every smooth map $f:M\to M$ of a compact manifold always admits a random perturbation satisfying these non-degeneracy conditions.

In the setting of random perturbations of a map, a Borel probability measure μ^ε on M is said to be a *physical measure* if for a positive Lebesgue measure set of points $x \in M$ one has

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi\big(f_{\underline{t}}^j x\big) = \int \varphi \, d\mu^\varepsilon, \tag{2}$$

for all continuous $\varphi\colon M \to \mathbb{R}$ and $\theta_\varepsilon^{\mathbb{N}}$ almost every $\underline{t} \in T^{\mathbb{N}}$. We denote the set of points $x \in M$ for which (2) holds for every φ and $\theta_\varepsilon^{\mathbb{N}}$ almost every $\underline{t} \in T^{\mathbb{N}}$ by $B(\mu^\varepsilon)$ and call it the *basin of* μ^ε.

The map $f\colon M \to M$ is said to be *stochastically stable* if the weak* accumulation points (when $\varepsilon > 0$ goes to zero) of the physical measures of the random perturbation are convex linear combinations of the SRB measures of f (for this notion of stochastic stability, see [Ar2]).

Theorem A. *Let $\{\Phi, (\theta_\varepsilon)_{\varepsilon>0}\}$ be a random perturbation of a C^2 diffeomorphism f admitting a strictly forward invariant open set U with a dominated splitting $T_U M = E^{cs} \oplus E^{cu}$, where the center-stable direction is mostly contracting and the center-unstable direction is non-uniformly expanding. Then there is $l \in \mathbb{N}$ such that for small enough $\varepsilon > 0$ there exist physical measures $\mu_1^\varepsilon, \dots, \mu_l^\varepsilon$ with pairwise disjoint supports with* $\operatorname{supp}(\mu_i^\varepsilon) \subset B(\mu_i^\varepsilon)$ *for $i = 1, \dots, l$, satisfying*

(1) *for each $x \in M$ and $\theta_\varepsilon^{\mathbb{N}}$ almost every $\underline{t} \in T^{\mathbb{N}}$ there is $i = i(x, \underline{t}) \in \{1, \dots, l\}$ such that*
$$\mu_i^\varepsilon = \lim_{n\to+\infty} \frac{1}{n} \sum_{j=1}^{n-1} \delta_{f_{\underline{t}}^j x};$$

(2) *for $1 \le i \le l$ and any probability measure η with support contained in $B(\mu_i^\varepsilon)$ we have*
$$\mu_i^\varepsilon = \lim_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} (f_{\underline{t}}^j)_* \eta \quad \textit{for } \theta_\varepsilon^{\mathbb{N}} \textit{ almost every } \underline{t} \in T^{\mathbb{N}};$$

(3) *the number of physical measures is at most the number of SRB measures.*

The first item above means that almost all random orbits of every point of M have time averages given by a physical measure out of $\mu_1^\varepsilon, \dots, \mu_l^\varepsilon$, a property akin to the one obtained in [ABV] for points in the topological basin of partially hyperbolic sets.

Now we present a notion that will play an important role in the study of the stochastic stability of a diffeomorphism with a dominated splitting. We say that f is *non-uniformly expanding along the center-unstable direction for random orbits* if there is $c > 0$ such that

$$\limsup_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Df^{-1}|E^{cu}_{f_{\underline{t}}^j x}\| \le -c \tag{3}$$

for Lebesgue almost every $x \in U$ and $\theta_\varepsilon^{\mathbb{N}}$ almost every $\underline{t} \in T^{\mathbb{N}}$, at least for small $\varepsilon > 0$.

The main result below gives a characterization of the stochastically stable non-uniformly hyperbolic diffeomorphisms. To the best of our knowledge this is the only result pointing in the direction of general stochastic stability for diffeomorphisms with a dominated splitting.

Theorem B. *Let $f : M \to M$ be a C^2 diffeomorphism admitting a strictly forward invariant open set U with a dominated splitting $T_U M = E^{cs} \oplus E^{cu}$, where the center-stable direction is mostly contracting and the center-unstable direction is non-uniformly expanding. Then f is stochastically stable if, and only if, f is non-uniformly expanding along the center-unstable direction for random orbits.*

We exhibit open classes of examples of diffeomorphisms satisfying the conditions of Theorem B which are stochastically stable. The ones we have in mind were presented in [ABV, Appendix A]. These form an open class $\mathcal{D}$ of diffeomorphisms defined on the d-dimensional torus $M = \mathbb{T}^d$, $d \geq 2$, having the whole M as an attractor, $TM = E^{cs} \oplus E^{cu}$ with mostly contracting center-stable direction and exhibiting non-uniform expansion along the E^{cu} direction, see Section 7 for a complete description. Here we prove their stochastic stability by showing that they satisfy the condition in Theorem B above.

Theorem C. *Every $f \in \mathcal{D}$ is non-uniformly expanding for random orbits along the center-unstable direction.*

Further developments in [Ta] show that these diffeomorphisms have a unique SRB measure, and in [Vz] it was already proved that these maps are also statistically stable.

Moreover, the conditions needed to get both Theorems A and B are general enough to enable us to obtain as corollaries of our method the stochastic stability of some other families of diffeomorphisms with dominated splittings.

Obviously uniform expansion along the center-unstable bundle and mostly contractive center-stable bundle fit in our conditions. This setting was studied in [BoV, Do], where it was shown that these conditions are enough to obtain SRB measures for f. Since uniform expansion along the center-unstable direction and dominated splitting are robust in a whole C^1-neighborhood of f, we get non-uniform expansion for random orbits along the center-unstable direction for free.

Corollary D. *Let $f : M \to M$ be a C^2 diffeomorphism admitting a strictly forward invariant open set U with a dominated splitting $T_U M = E^{cs} \oplus E^{u}$, where the center-stable direction is mostly contracting and the center-unstable direction is uniformly expanding. Then f is stochastically stable.*

Another easy remark is that uniform contraction along the center-stable bundle with nonuniform expansion along the center-unstable direction, together with nonuniform expansion for random orbits along this direction, also fit in our setting.

Corollary E. *Let $f : M \to M$ be a C^2 diffeomorphism admitting a strictly forward invariant open set U with a dominated splitting $T_U M = E^{s} \oplus E^{cu}$, where the center-stable direction is uniformly contracting and the center-unstable direction is non-uniformly expanding. If f is non-uniformly expanding on random orbits along the center-unstable direction, then f is stochastically stable.*

In each of these corollaries the non-uniform bundle may admit a further dominated splitting into a uniformly behaved bundle and a central one, where the non-uniform expanding or contracting conditions will apply. This is the setting of *partially hyperbolic diffeomorphisms* (see [BDV] for definitions and main features), where the tangent bundle splits into three sub-bundles

$$T_U M = E^s \oplus E^c \oplus E^u$$

such that both $(E^s \oplus E^c) \oplus E^u$ and $E^s \oplus (E^c \oplus E^u)$ are dominated splittings, E^s is uniformly contracted and E^u is uniformly expanding. Finally, the uniformly hyperbolic case (when $E^c = \{0\}$) is also clearly in our setting (this result was first proved in [Ki1] but our approach is closer to [Yo1]).

Corollary F. *Let $f : M \to M$ be a C^2 diffeomorphism admitting a strictly forward invariant open set U with a (uniformly) hyperbolic splitting $T_U M = E^s \oplus E^u$. Then f is stochastically stable.*

We observe finally that every statement is also true if the region U coincides with the entire manifold, as the examples in Section 7 and in [BoV, ABV]. In addition, the C^2 smoothness is not strictly necessary for our arguments: $C^{1+\alpha}$ for some $\alpha > 0$ is all that is really needed in our arguments and in our references.

The paper is organized as follows. We start the proofs by studying random perturbations in Section 2, where the proof of Theorem A is explained. Then we prove the necessary condition of Theorem B in Section 3, where we also outline the main steps of the proof of sufficiency. The proof of the sufficient condition of Theorem B starts at Section 4 where we state some preliminary results about hyperbolic times and distortion control, extends to Section 5 where we study physical measures under random perturbations with non-uniform expansion, and is completed in Section 6 where we consider the limit when the noise level tends to zero. Finally, in Section 7 we describe the construction of an open class of non-uniformly hyperbolic diffeomorphisms fitting the conditions of our theorems.

2. Random perturbations

To begin our study of random perturbations $\{\Phi, (\theta_\varepsilon)_{\varepsilon>0}\}$ of a diffeomorphism f admitting a partially hyperbolic set U, we observe that if we take $\varepsilon > 0$ small enough, then all random orbits $(f^n_{\underline{t}} x)_{n\geq 1}$ for every $x \in U$ are contained in a compact subset K of U when $\underline{t} \in \operatorname{supp}(\theta_\varepsilon^{\mathbb{N}})$, just by continuity of Φ and because U is strictly forward invariant.

Moreover if we introduce the skew-product map $F : T^{\mathbb{N}} \times M \to T^{\mathbb{N}} \times M$ given by $(\underline{t}, z) \mapsto (\sigma(\underline{t}), f_{t_1}(z))$, where σ is the left shift on sequences $\underline{t} = (t_1, t_2, \dots) \in T^{\mathbb{N}}$, then we have that

$$\hat\Lambda_\varepsilon = \bigcap_{n\geq 0} F^n(\operatorname{supp}(\theta_\varepsilon^{\mathbb{N}}) \times U) \quad \text{with} \quad \Lambda_\varepsilon = \pi(\hat\Lambda_\varepsilon) \subset K, \tag{4}$$

where $\pi : T^{\mathbb{N}} \times M \to M$ is the natural projection. $\hat\Lambda_\varepsilon$ is a forward F invariant set, the *ε-random attractor*.

As mentioned before, we will assume that the random perturbations of the partially hyperbolic map f satisfy some *non-degeneracy conditions*: there is $0 < \varepsilon_0 < 1$ such that for every $0 < \varepsilon < \varepsilon_0$ we may find $n_0 = n_0(\varepsilon) \in \mathbb{N}$ satisfying the following conditions for all $x \in M$ and all $n \geq n_0$:

(1) there is $\xi = \xi(\varepsilon) > 0$ such that $\left\{ f_{\underline{t}}^n x \colon \underline{t} \in (\operatorname{supp} \theta_\varepsilon)^{\mathbb{N}} \right\} \supset B(f^n x, \xi)$;
(2) defining $f_{\odot}^n x : T^{\mathbb{N}} \to M$ as $f_{\odot}^n x(\underline{t}) = f_{\underline{t}}^n(x)$, then $(f_{\odot}^n x)_* \theta_\varepsilon^{\mathbb{N}} \ll \mathrm{m}$.

Condition 1 says that the set of perturbed iterates of any given point covers a full neighborhood of the unperturbed iterates after a threshold for all sufficiently small noise levels. Condition 2 means that each set of perturbation vectors having positive $\theta_\varepsilon^{\mathbb{N}}$ measure must send each point $x \in M$ onto positive Lebesgue measure subsets of M, after a certain number of iterates.

Examples 1 and 2 in [Ar1] show that *every smooth map $f : M \to M$ of a compact manifold always has a random perturbation satisfying the non-degeneracy conditions 1 and 2*, as long as we take $T = \mathbb{R}^p$, $t^* = 0$ and also θ_ε equal to Lebesgue measure restricted to the ball of radius ε around 0 (normalized to become a probability measure), for a sufficiently big number $p \in \mathbb{N}$ of parameters. For manifolds whose tangent bundle is trivial (parallelizable manifolds) the random perturbations which consist in adding at each step a random noise to the unperturbed dynamics clearly satisfy non-degeneracy conditions 1 and 2 for $n_0 = 1$. We observe that this is precisely the kind of perturbations considered by Baladi, Benedicks, Viana and Young in [BaV, BeV, BeY] for quadratic and Hénon maps.

The attractor $\cap_{n\geq 0} f^n(U)$ for f will be denoted by Λ. The conditions above imply that for small $\varepsilon > 0$ every f_t is C^2-close to $f \equiv f_{t^*}$. Then for every neighborhood V of Λ we have $\Lambda_\varepsilon \subset V$ for all small enough ε. Thus the compact set K containing Λ_ε may be taken as a neighborhood of Λ. We assume this in the rest of the paper (this is important in Subsection 3.2).

2.1. **The number of physical measures.** In the setting of random perturbations of a map, we say that a set $A \subset M$ is *forward invariant* if for some given small $\varepsilon > 0$ and for all $t \in \operatorname{supp}(\theta_\varepsilon)$ we have $f_t A \subset A$. A probability measure μ is said to be *stationary*, if for every continuous $\varphi : M \to \mathbb{R}$ it holds the following relation, similar to the non-random setting of invariant measures:

$$\int \varphi \, d\mu = \int \int \varphi\big(f_t x\big) \, d\mu(x) \, d\theta_\varepsilon(t). \tag{5}$$

Remark 2.1. If $(\mu^\varepsilon)_{\varepsilon>0}$ is a family of stationary measures having μ_0 as a weak* accumulation point when ε goes to 0, then it follows from (5) and the convergence of $\operatorname{supp}(\theta_\varepsilon)$ to $\{t^*\}$ that μ_0 must be invariant by $f = f_{t^*}$.

Condition (5) is equivalent to saying that $F_*(\theta_\varepsilon^{\mathbb{N}} \times \mu) = \theta_\varepsilon^{\mathbb{N}} \times \mu$ and it is easy to see that a stationary measure μ satisfies

$$x \in \operatorname{supp}(\mu) \quad \Rightarrow \quad f_t(x) \in \operatorname{supp}(\mu) \text{ for all } t \in \operatorname{supp}(\theta_\varepsilon),$$

just by continuity of Φ. This means that if μ is a stationary measure, then $\operatorname{supp}(\mu)$ is a forward invariant set. Then the interior of $\operatorname{supp}(\mu)$ is nonempty, by non-degeneracy condition 1, and forward invariant by the continuity of the maps f_t. Obviously a physical measure is stationary. In our setting the physical measures μ we deal with are such that $\theta_\varepsilon^{\mathbb{N}} \times \mu$ is F ergodic.

The following is a general result from [Ar1] (see also [BK, Section 2] for an alternative approach) for random perturbations of a diffeomorphism satisfying the non-degeneracy conditions 1 and 2 above, which implies the first item of Theorem A.

Theorem 2.2. *Assume that $\{\Phi, (\theta_\varepsilon)_{\varepsilon>0}\}$ is a random perturbation of a C^1 diffeomorphism f satisfying non-degeneracy conditions 1 and 2. Then for each $\varepsilon > 0$ there are physical measures $\mu_1^\varepsilon, \ldots \mu_{l(\varepsilon)}^\varepsilon$, and for each $x \in M$ there is a $\theta_\varepsilon^{\mathbb{N}}$ mod 0 partition $T_1(x), \ldots, T_{l(\varepsilon)}(x)$ of $T^{\mathbb{N}}$ such that for $1 \le i \le l(\varepsilon)$*

$$\mu_i^\varepsilon = \lim_{n\to+\infty} \frac{1}{n} \sum_{j=1}^{n-1} \delta_{f_{\underline{t}}^j x} \quad \text{for} \quad \underline{t} \in T_i(x).$$

Moreover the supports of the physical measures are pairwise disjoint, have nonempty interior and are contained in their basins: $\operatorname{supp}(\mu_i^\varepsilon) \subset B(\mu_i^\varepsilon), i = 1, \ldots, l(\varepsilon)$.

Now we prove the second item of Theorem A. First we observe that since the basin of each physical measure contains its support, then it also has nonempty interior. Let $\mu^\varepsilon = \mu_i^\varepsilon$ be a physical measure and take any probability measure η with support contained in $B(\mu_i^\varepsilon)$. Given any continuous function $\varphi\colon M \to \mathbb{R}$, we have for each $x \in \operatorname{supp}(\eta)$ and θ_ε almost every $\underline{t} \in T^{\mathbb{N}}$

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f_{\underline{t}}^j x) = \int \varphi \, d\mu^\varepsilon$$

by definition of $B(\mu^\varepsilon)$. Taking integrals over $\operatorname{supp}(\eta)$ with respect to η on both sides of the equality, the Dominated Convergence Theorem gives

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \int \varphi(f_{\underline{t}}^j x) \, d\eta(x) = \int \varphi \, d\mu^\varepsilon.$$

(Recall that we are taking integrals over the support of the probability measure η). Since

$$\int \varphi \circ f_{\underline{t}}^j \, d\eta = \int \varphi \, d(f_{\underline{t}}^j)_* \eta$$

for every integer $j \ge 0$, we have proved item 2 of Theorem A.

To conclude the proof of Theorem A all we need is to show that $l(\varepsilon)$ is constant for every sufficiently small $\varepsilon > 0$ and that l is at most the number of SRB measures of f. We start by recalling that, by assumption, $\{\operatorname{supp}(\theta_\varepsilon)\}_{\varepsilon>0}$ is a nested family of sets. This implies that if μ^ε is a physical measure, then $\operatorname{supp}(\mu^\varepsilon)$ is forward invariant with respect to f_t for all $t \in \operatorname{supp}(\theta_{\varepsilon'})$, whenever $0 < \varepsilon' < \varepsilon$. Since non-degeneracy conditions are satisfied in $\operatorname{supp}(\mu^\epsilon)$ by the probability measure $\theta_{\varepsilon'}$, then Theorem 2.2 ensures that there exists at least one physical measure $\mu^{\varepsilon'}$ with $\operatorname{supp}(\mu^{\varepsilon'}) \subset \operatorname{supp}(\mu^\varepsilon)$ for $0 < \varepsilon' < \varepsilon$. This shows that the number of physical measures is a non-decreasing integer function of ε when $\varepsilon \to 0$.

Hence if we show that l is bounded from above, we conclude that l is constant for all small enough $\varepsilon > 0$.

We observe that $\operatorname{supp}(\mu^\varepsilon)$ is forward invariant under $f = f_{t^*}$ and, moreover, conditions (a)-(c) hold on $f \mid \operatorname{supp}(\mu^\varepsilon)$ because they hold Lebesgue almost everywhere in U (by assumption) and $\operatorname{supp}(\mu^\varepsilon)$ has nonempty interior. Thus [Vz, Theorem C] guarantees the existence of at least one SRB measure μ with $\operatorname{supp}(\mu) \subset \operatorname{supp}(\mu^\varepsilon)$.

We have seen that each support of a physical measure μ^ε must contain at least the support of one SRB measure for the unperturbed map f. Since the number of SRB measures is finite we have $l \le p$, where p is the number of those measures. This concludes the proof of Theorem A.

3. Stochastic stability

In this section we start the proof of Theorem B. Let $f : M \to M$ be a C^2 diffeomorphism admitting a strictly forward invariant open set U with a dominated splitting $T_U M = E^{cs} \oplus E^{cu}$, where the center-stable direction is mostly contracting and the center-unstable direction is non-uniformly expanding. We first prove that non-uniform expansion along the center-unstable direction for random orbits is a necessary condition for stochastic stability.

3.1. **Stochastic stability implies non-uniform hyperbolicity.** Taking $\varepsilon > 0$ small we know from Theorem A that there is a finite number of physical measures $\mu_1^\varepsilon, \dots \mu_l^\varepsilon$ and, for each $x \in U$, there is a $\theta_\varepsilon^{\mathbb{N}}$ mod 0 partition $T_1(x), \dots, T_l(x)$ of $T^{\mathbb{N}}$ for which

$$\mu_i^\varepsilon = \lim_{n\to+\infty} \frac{1}{n} \sum_{j=1}^{n-1} \delta_{f_{\underline{t}}^j(x)} \quad \text{for each } \underline{t} \in T_i(x).$$

Furthermore, since $\log \|Df^{-1}|E_x^{cu}\|$ is a continuous map, then we have for every $x \in U$ and $\theta_\varepsilon^{\mathbb{N}}$ almost every $\underline{t} \in T^{\mathbb{N}}$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Df^{-1}|E^{cu}_{f_{\underline{t}}^j x}\| = \int \log \|Df^{-1}|E_x^{cu}\| d\mu_i^\varepsilon,$$

for some physical measure μ_i^ε with $1 \le i \le l$. Hence, for proving that f is non-uniformly expanding along the center-unstable direction for random orbits it suffices to show that there is $c_0 > 0$ such that if $1 \le i \le l$ then, for small $\varepsilon > 0$,

$$\int \log \|Df^{-1}|E_x^{cu}\| d\mu_i^\varepsilon < -c_0.$$

The next result is proved in [AA, Lemma 5.1] for endomorphisms, but the proof still works in our case, since it only uses weak* convergence and the definition of physical and SRB measures.

Lemma 3.1. *Let $\varphi\colon M \to \mathbb{R}$ be continuous. Given $\delta > 0$ there is $\varepsilon_0 > 0$ such that if $\varepsilon \le \varepsilon_0$, then*

$$\left| \int \varphi d\mu^\varepsilon - \int \varphi d\mu_\varepsilon \right| < \delta,$$

for some linear convex combination μ_ε of the physical measures of f.

Therefore there are real numbers $w_1(\varepsilon),\dots,w_p(\varepsilon)\ge 0$ such that $w_1(\varepsilon)+\dots+w_p(\varepsilon)=1$ and $\mu_\varepsilon = w_1(\varepsilon)\mu_1+\dots+w_p(\varepsilon)\mu_p$. Since μ_i is a physical measure for $1\le i\le p$, we have for Lebesgue almost every $x\in B(\mu_i)$

$$\int \log\|Df^{-1}|E^{cu}_x\|d\mu_i = \lim_{n\to+\infty}\frac{1}{n}\sum_{j=0}^{n-1}\log\|Df^{-1}|E^{cu}_{f^j(x)}\| \le -c<0.$$

Hence

$$\int \log\|Df^{-1}|E^{cu}_x\|d\mu_\varepsilon \le -c, \quad \text{so} \quad \int \log\|Df^{-1}|E^{cu}_x\|d\mu_i^\varepsilon \le -c/2,$$

applying Lemma 3.1 to $\varphi=\log\|Df^{-1}|E^{cu}\|$, obtaining that f is non-uniformly expanding along the center-unstable direction for random orbits, as we want.

3.2. **Non-uniform hyperbolicity implies stochastic stability.** Let us explain why in our setting non-uniform expansion along the center-unstable direction for random orbits is a sufficient condition for stochastic stability.

In order to prove that $f=f_{t^*}$ is stochastically stable, it suffices to show that every weak* accumulation point μ of any family $(\mu^\varepsilon)_{\varepsilon>0}$, where each μ^ε is a physical measure of level ε, is absolutely continuous with respect to the Lebesgue measure along local center-unstable disks and that the Lyapunov exponents along the tangent directions to these disks are all strictly positive, i.e., that every such μ is a Gibbs cu-state for f. This follows from a combination of results. The first one is a consequence of the techniques developed in [BoV, ABV] and is detailed in [Vz, Theorem C].

Theorem 3.2. *Let f be a C^2 diffeomorphism having a topological attractor exhibiting a dominated splitting which is non-uniformly expanding along the center-unstable direction and mostly contracting along the center-stable direction.*

Then every ergodic Gibbs cu-state is an SRB measure for f. Moreover there are finitely many SRB measures whose basins cover a full Lebesgue measure subset of the topological basin U of the attractor.

The proof of the next result can be found in [Vz, Corollary 4.1].

Theorem 3.3. *Let f be a C^2 diffeomorphism having a topological attractor with dominated splitting and let μ be a Gibbs cu-state for f. Then every ergodic component of μ is a Gibbs cu-state.*

Assuming further that f is non-uniformly expanding along the center-unstable direction, then every ergodic SRB measure is a Gibbs cu-state.

These results ensure that every Gibbs cu-state μ for f has finitely many ergodic components which are SRB measures for f and also that μ can be written as a linear convex combination of these SRB measures. Hence it is enough to prove the following.

Theorem 3.4. *Let $(\Phi, (\theta_\varepsilon)_{\varepsilon>0})$ is a random perturbation of the C^2 diffeomorphism f with a topological attractor having a dominated splitting which is non-uniformly expanding along the center-unstable direction and mostly contracting along the center-stable direction.*

If f is non-uniformly expanding along the central direction for random orbits, then every weak* accumulation point μ^0 of $(\mu^\varepsilon)_{\varepsilon>0}$, when $\varepsilon \to 0$, is a Gibbs cu-state *for f.*

If this holds, then every weak* accumulation point of the stationary measures is in the convex hull of the finitely many SRB measures of f. This means that f is stochastically stable.

In the following section we prove Theorem 3.4. The strategy is to adapt notions from [ABV] and [Vz] to deduce that in our setting stationary ergodic measures μ^ε admit *cylinders with mass uniformly bounded away from zero* having leaves which are *uniformly contracted backward* by a sequence of perturbations. This is the content of Section 5.

Having this, it is not difficult to show that these cylinders accumulate, when $\varepsilon \to 0$, to cylinders having the same properties with respect to any weak* accumulation point of $(\mu^\varepsilon)_{\varepsilon>0}$. This is the purpose of Subsections 6.1 and 6.2.

Now we just have to show that every f-invariant measure which is a limit of stationary measures admitting a cylinder as above must be a Gibbs cu-state, which is proved in Subsection 6.3. Combining these steps we prove Theorem 3.4 which together with Theorems 3.2 and 3.3 complete the proof of the sufficient condition on Theorem B.

4. Curvature and distortion control

Here we outline some local geometrical and dynamical consequences of the dominated decomposition and non-uniformly expanding assumptions, referring mainly to previous works [ABV] and [AA] for proofs.

4.1. **Curvature of center-unstable disks.** It was shown in [ABV] that f satisfies a bounded curvature property over disks having the tangent space at each point contained in a cone field around the center-unstable direction. Here we present the "random version" of the main results in [ABV, Section 2].

Given $0 < a < 1$ we define the *center-unstable cone field* $C_a^{cu} = (C_a^{cu}(x))_{x\in U}$ *of width* a by

$$C_a^{cu}(x) = \{v_1 + v_2 \in E_x^{cs} \oplus E_x^{cu} \colon \|v_1\| \le a\|v_2\|\} \tag{6}$$

and the *center-stable cone field* $C_a^{cs} = (C_a^{cs}(x))_{x\in U}$ *of width* a in the same manner but reversing the roles of the bundles in (6).

Up to increasing λ slightly we may fix a and ε small enough so that condition (a) of Subsection 1.1 (dominated decomposition) extends to vectors in the cone fields for all maps nearby f, i.e.

$$\|Df_t(x)v^{cs}\| \cdot \|Df_t^{-1}(f_t x)v^{cu}\| \le \lambda \|v^{cs}\| \cdot \|v^{cu}\| \tag{7}$$

for all $v^{cs} \in C_a^{cs}(x)$, $v^{cu} \in C_a^{cu}(f_t x)$, $x \in U$ and $t \in \operatorname{supp}(\theta_\varepsilon)$. Moreover, the domination property above together with the continuity of Φ and the closeness of t to t^* imply that $Df_t C_a^{cu}(x)$ is contained in a cone of width λa centered around $Df_t E_x^{cu}$, defined as above with respect to the splitting $Df_t E_x^{cs} \oplus Df_t E_x^{cu}$. Since the subspaces $Df_t E_x^{cs}, Df_t E_x^{cu}$ are close to $E^{cs}_{f_t x}, E^{cu}_{f_t x}$ respectively, then $Df_t C^{cu}_a(x) \subset C^{cu}_a(f_t x)$ if $\varepsilon > 0$ is small enough. By analogous arguments we get $D(f_t)^{-1} C^{cs}_a(x) \subset C^{cs}_a(f_t^{-1} x)$.

Given an embedded sub-manifold $S \subset U$ we say that S is *tangent to the center-unstable cone field* if $T_x S \subset C^{cu}_a(x)$ for all $x \in S$. Hence $f_t(S)$ is also tangent to the center-unstable cone field. The curvature of these sub-manifolds and their iterates will be approximated in local coordinates by the notion of Hölder variation of the tangent bundle as follows.

Let us take δ_0 sufficiently small so that if we take $V_x = B(x, \delta_0)$, then the exponential map $\exp_x : V_x \to T_x M$ is a diffeomorphism onto its image for all $x \in M$. We are going to identify V_x through the local chart $\exp_x^{-1}$ with the neighborhood $U_x = \exp_x V_x$ of the origin in $T_x M$. Identifying x with the origin in $T_x M$ we get that E^{cu}_x is contained in $C^{cu}_a(y)$ for all $y \in U_x$, reducing δ_0 if needed. Then the intersection of E^{cs}_x with $C^{cu}_a(y)$ is the zero vector. So if $x \in S$ then $T_y S$ is the graph of a linear map $A_x(y) : E^{cu}_x \to E^{cs}_x$ for $y \in U_x \cap S$.

For $C > 0$ and $\zeta \in (0,1)$ we say that the *tangent bundle of S is (C,ζ)-Hölder* if

$$\|A_x(y)\| \le C \operatorname{dist}_S(x,y)^\zeta \quad \text{for all} \quad y \in U_x \cap S \quad \text{and} \quad x \in U, \tag{8}$$

where $\operatorname{dist}_S(x,y)$ is *the distance along* S defined by the length of the shortest smooth curve from x to y inside S.

Up to choosing smaller $a > 0$ and $\varepsilon > 0$ we may assume that there are $\lambda < \lambda_1 < 1$ and $0 < \zeta < 1$ such that for all norm one vectors $v^{cs} \in C^{cs}_a(x), v^{cu} \in C^{cu}_a(x)$, $x \in U$ it holds

$$\|Df_t(x) v^{cs}\| \cdot \|Df_t^{-1}(f_t x) v^{cu}\|^{1+\zeta} \le \lambda_1.$$

For these values of λ_1 and ζ, given a C^1 sub-manifold $S \subset U$ tangent to the center-unstable cone field we define

$$\kappa(S) = \inf\{C > 0 : TS \text{ is } (C,\zeta)\text{-Hölder}\}. \tag{9}$$

The proofs of the results that we present below may be obtained by mimicking the proofs of the corresponding ones in [ABV], and we leave it as an exercise to the reader. The basic ingredients in those proofs are the cone invariance and dominated decomposition properties that we have already extended for nearby perturbations f_t of the diffeomorphism f.

Proposition 4.1. *There is $C_1 > 0$ such that for every C^1 sub-manifold $S \subset U$ tangent to the center-unstable cone field and every $\underline{t} \in T^{\mathbb{N}}$*

(1) *there exists n_1 such that $\kappa(f^n_{\underline{t}} S) \le C_1$ for all $n \ge n_1$ with $f^k_{\underline{t}} S \subset U$ for all $1 \le k \le n$;*
(2) *if $\kappa(S) \le C_1$ then $\kappa(f^n_{\underline{t}} S) \le C_1$ for all $n \ge 1$ such that $f^k_{\underline{t}} S \subset U$ for all $1 \le k \le n$;*
(3) *in particular, if S is as in the previous item, then*

$$J_n : f^n_{\underline{t}} S \ni x \mapsto \log |\det(Df | T_x f^n_{\underline{t}} S)|$$

is (L_1, ζ)-Hölder continuous with $L_1 > 0$ depending only on C_1 and f, for every $n \ge 1$.

The bounds provided by Proposition 4.1 may be seen as bounds on the curvature of embedded disks tangent to the center-unstable cone field.

4.2. **Hyperbolic times.** From the condition of non-uniform expansion along the center-unstable direction we will be able to deduce some uniform expansion at certain times which are precisely defined through the following notion.

Definition 4.2. Given $0<\alpha<1$ we say that $n\ge 1$ is a α-hyperbolic time for $(\underline{t},x)\in T^{\mathbb{N}}\times U$ if

$$\prod_{j=n-k+1}^{n}\|Df^{-1}|E^{cu}_{f^j_{\underline{t}}x}\|\le\alpha^k\quad\text{for all}\quad k=1,\dots,n.$$

The main technical result ensuring the existence of hyperbolic times is due to Pliss [Pl], whose proof can be found in [ABV, Lemma 3.1] or [Ma2, Section 2].

Lemma 4.3. *Let $H\ge c_2>c_1$ and $\zeta=(c_2-c_1)/(H-c_1)$. Given real numbers $a_1,\dots,a_N$ satisfying*

$$\sum_{j=1}^{N}a_j\ge c_2N\quad\textit{and}\quad a_j\le H\ \ \textit{for all}\ \ 1\le j\le N,$$

there are $l>\zeta N$ and $1<n_1<\ldots<n_l\le N$ such that

$$\sum_{j=n+1}^{n_i}a_j\ge c_1\cdot(n_i-n)\ \ \textit{for each}\ \ 0\le n<n_i,\ i=1,\dots,l.$$

Using Lemma 4.3 it is not difficult to show that the condition of non-uniform expansion for random orbits along the center-unstable direction is enough to ensure that almost all points have infinitely many hyperbolic times according to the following result whose proof can be easily adapted from [ABV, Corollary 3.2].

Proposition 4.4. *There exist $\gamma,\alpha>0$ depending only on f such that for $\theta^{\mathbb{N}}\times \mathrm{m}$ almost all $(\underline{t},x)\in T^{\mathbb{N}}\times U$ and a sufficiently big integer $N\ge 1$, there exist $1\le n_1<\cdots<n_k\le N$, with $k\ge\gamma N$, which are α-hyperbolic times for $(\underline{t},x)$.*

Let n be a α-hyperbolic time for $(\underline{t},x)\in T^{\mathbb{N}}\times U$. This implies that $Df^{-k}|E^{cu}_{f^n_{\underline{t}}x}$ is a contraction for all $k=1,\dots,n$. In addition, if $a>0$ and $\varepsilon>0$ are taken small enough in the definition of the cone fields and the random perturbations, then taking $\delta_1>0$ also small, we have by continuity

$$\|Df_t^{-1}|E^{cu}_{f_ty}\|\le\alpha^{-1/2}\|Df^{-1}|E^{cu}_{fx}\| \tag{10}$$

for all $t\in\operatorname{supp}(\theta_\varepsilon^{\mathbb{N}}), x\in\overline{fU}$ and $y\in U$ with $\operatorname{dist}(x,y)<\delta_1$. As a consequence of this the next result is obtained following [ABV, Lemma 2.7].

Lemma 4.5. *Given any C^1 disk $\Delta\subset U$ tangent to center-unstable cone field, $x\in\Delta$ and $n\ge 1$ a α-hyperbolic time for $(\underline{t},x)$, we have*

$$\operatorname{dist}_{f^{n-k}_{\underline{t}}\Delta}(f^{n-k}_{\underline{t}}(y),f^{n-k}_{\underline{t}}(x))\le\alpha^{k/2}\operatorname{dist}_{f^n_{\underline{t}}\Delta}(f^n_{\underline{t}}y,f^n_{\underline{t}}x),\quad k=1,\dots,n,$$

for every point $y\in\Delta$ such that $\operatorname{dist}_{f^n_{\underline{t}}(\Delta)}(f^n_{\underline{t}}(y),f^n_{\underline{t}}(x))\le\delta_1$.

Using the previous lemma and the Hölder continuity property given by Proposition 4.1 the following bounded distortion result can be deduced as in [ABV, Proposition 2.8].

Proposition 4.6. *There exists $C_2 > 1$ such that, given any C^1 disk Δ tangent to the center-unstable cone field with $\kappa(\Delta) \le C_1$, and given any $x \in \Delta$ and $n \ge 1$ a α-hyperbolic time for $(\underline{t}, x)$, then*

$$\frac{1}{C_2} \le \frac{|\det Df^n_{\underline{t}} | T_y\Delta|}{|\det Df^n_{\underline{t}} | T_x\Delta|} \le C_2$$

for every $y \in \Delta$ such that $\mathrm{dist}_{f^n_{\underline{t}}(\Delta)}(f^n_{\underline{t}}(y), f^n_{\underline{t}}(x)) \le \delta_1$.

5. Center-unstable cylinders

Now we show that μ^ε admits a cylinder with very specific properties in the setting of maps with dominated splitting which are non-uniformly expanding along the center-unstable direction for random orbits.

Let μ^ε be a physical measure of level ε for some small $\varepsilon > 0$ and take m_D the normalized Lebesgue measure on some C^1 disk D tangent to the center-unstable cone field such that m_D-almost every point of D is in $B(\mu^\varepsilon)$ and satisfies (3). It is possible to choose such a disk, because $B(\mu^\varepsilon)$ has nonempty interior in U. Now define for each $n \ge 1$

$$\mu^{\underline{t}}_n = \frac{1}{n}\sum_{j=0}^{n-1} (f^j_{\underline{t}})_* \, \mathrm{m}_D \, . \tag{11}$$

We know from Theorem A that each μ^ε is the weak* limit of the sequence $(\mu^{\underline{t}}_n)_n$ for a $\theta_\varepsilon^{\mathbb{N}}$ generic $\underline{t}$ by item (2) of Theorem A. We fix a $\theta_\varepsilon^{\mathbb{N}}$ generic $\underline{t}$ in everything that follows within this section.

A cylinder $\mathcal{C} \subset M$ is the image of a C^1 diffeomorphism $\phi : B^u \times B^s \hookrightarrow M$ where B^k is the k-dimensional unit ball of $\mathbb{R}^k, k = s, u$. We will say that a C^1 disk D *crosses* $\mathcal{C}$ if $D \cap \mathcal{C}$ is a graph over B^u: there exists $g : B^u \to B^s$ such that $D \cap \mathcal{C} = \{\phi(w, g(w)) : w \in B^u\}$.

The following is the main result of this subsection.

Proposition 5.1. *Let μ^ε be a stationary probability measure for $\{\Phi, (\theta_\varepsilon)_{\varepsilon>0}\}$ where f is a non-uniformly hyperbolic C^2 diffeomorphism. Then there are $\beta = \beta(f, c_u) > 0, \rho = \rho(f, c_u) > 0, d = d(f, c_u) > 0$, a cylinder $\mathcal{C} = \phi(\mathcal{B}^u \times B^s)$ and a family $\mathcal{K}$ of disks tangent to the center-unstable cone field which cross $\mathcal{C}$ and whose union is the set K such that*

1. *$\mu^\varepsilon(K \cap \mathcal{C}) \ge \beta$ and both $\phi(B^u \times 0)$ and $\phi(u \times B^s)$ are disks containing a sub-disk with radius $\ge \rho$ for all $u \in B^u$;*
2. *for every disk $\gamma \in \mathcal{K}$ there exists a sequence $\underline{s} \in \mathrm{supp}\, \theta_\varepsilon^{\mathbb{N}}$ such that $(f^n_{\underline{s}})^{-1} \mid \gamma$ is a $\alpha^{n/2}$-contraction: for $w, z \in \gamma$*

$$\mathrm{dist}_{(f^n_{\underline{s}})^{-1}\gamma}\left((f^n_{\underline{s}})^{-1}(w), (f^n_{\underline{s}})^{-1}(z)\right) \le \alpha^{n/2}\, \mathrm{dist}_\gamma(w, z)$$

where dist_γ is the induced distance on γ by the Riemannian metric on M.

(3) *there exists a component ν of μ^ϵ with mass uniformly bounded from below by β such that the disintegration $\{\nu_\gamma\}_\gamma$ of $\nu \mid \mathcal{C}$ along the disks $\gamma \in \mathcal{K}$ has densities with respect to the Lebesgue induced measure m_γ on γ uniformly bounded from above and below: $d^{-1} \le (d\nu_\gamma/dm_\gamma) \le d$, ν_γ almost everywhere and for almost every $\gamma \in \mathcal{K}$.*

The proof follows an idea in [ABV]: to consider a component of the average $\mu_n^{\underline{t}}$ calculated at hyperbolic times and its weak* accumulation points.

Proof. To control the densities of the push-forwards at hyperbolic times we set

$$A = \{x \in D : \mathrm{dist}_D(x, \partial D) \ge \delta_1\}$$

where dist_D is the distance along D, and take δ_1 small enough so that $m_D(A) > 0$. Then we define for each $n \ge 1$ (we recall that $\underline{t}$ is $\theta_\varepsilon^{\mathbb{N}}$ generic fixed from the beginning)

$$H_n = \{x \in A : n \text{ is a simultaneous hyperbolic time for } (\underline{t}, x)\}.$$

We note that Lemma 4.5 ensures that $\mathrm{dist}_{f^n_{\underline{t}}(D)}(f^n_{\underline{t}}(x), \partial f^n_{\underline{t}}(D)) \ge \delta_1$ for every $x \in H_n$.

Let $D_n(x, \delta_1)$ be the δ_1-neighborhood of $f^n_{\underline{t}}(x)$ inside $f^n_{\underline{t}}(D)$. Then Proposition 4.6 ensures that the density of $((f^n_{\underline{t}})_* m_D) \mid D_n(x, \delta_1)$ with respect to $m_{D_n(x,\delta_1)}$ is uniformly bounded from above and from below if we normalize both measures.

To extend this control of the density to a significant portion of D we use the following result proved in [ABV, Proposition 3.3 and Lemma 3.4].

Lemma 5.2. *There is $\omega > 0$ (depending only on M, the curvature of center-unstable disks and on the dimension u of the center-unstable bundle) such that for all $n \ge 1$ we can find a finite subset $\hat{H}_n$ of H_n satisfying*

(1) *$\hat{B}_n = \{D_n(x, \delta_1/4), x \in \hat{H}_n\}$ is a pairwise disjoint collection;*
(2) *the union $B_n = \cup \hat{B}_n$ is such that $((f^n_{\underline{t}})_* m_D)(B_n) \ge \omega \cdot m_D(H_n)$.*

5.1. **A special component of the time average.** We define a component of the average measure $\mu_n^{\underline{t}}$ defined in (11)

$$\nu_n = \frac{1}{n} \sum_{j=0}^{n-1} ((f^j_{\underline{t}})_* m_D) \mid B_j. \tag{12}$$

Lemma 5.3. *There is $\beta_0 > 0$ such that $\nu_n(\cup_{j=0}^{n-1} f^j_{\underline{t}}(D)) \ge \beta_0$ for all big enough $n \ge 1$.*

Proof. We note that

$$\frac{1}{n} \sum_{j=0}^{n-1} m_D(H_j) = \int \int \chi_{H_j}(x)\, dm_D(x)\, d\#_n(j) = \int \left(\int \chi_{H_j}(x)\, d\#_n(j) \right) dm_D(x)$$

where $\#_n$ is the uniform distribution on $\{0, \dots, n-1\}$. By Proposition 4.4 for big n we must have that the inner integral is bounded from below by $\gamma > 0$. By Lemma 5.2 the mass of ν_n is bounded from below by $\omega \cdot n^{-1} \sum_{j=0}^{n-1} m_D(H_j) \ge \omega\gamma m_D(D)$ for big enough n. We just have to take $\beta_0 = \omega\gamma m_D(D)$ since $\mathrm{supp}\,(\nu_n) \subset \cup_{j=0}^{n-1} f^j_{\underline{t}}(D)$. □

With these settings the support of ν_n is a finite union $\cup_{j=0}^{n-1} B_j$ of disks having size bounded from above and below. Let ν be an accumulation point of $(\nu_n)_{n\geq 1}$ in the weak* topology: $\nu = \lim_k \nu_{n_k}$. Then the support of ν is contained in $B_\infty = \cap_{n\geq 1}\overline{\cup_{j>n} B_j}$.

This construction shows that for $y \in B_\infty$ there are sequences $k_j \to \infty$ of integers and disks $D_j = D_{k_j}(x_{k_j}, \delta_1/4)$ and points $y_j \in D_j$ such that $y_j \to y$ when $j \to \infty$. We know from subsections 4.1 and 4.2 that D_j are C^1 center-unstable disks containing a inner δ_1-ball. Moreover the sequence $(D_j)_j$ is relatively compact by the Ascoli-Arzela Theorem, hence up to taking subsequences we have $x_{k_j} \to x$ and $D_j \to D_x$ in the C^1 topology when $j \to \infty$ for some $x \in B_\infty$ and a disk D_x centered at x with radius $\delta_1/4$. Thus $y \in \overline{D_x} \subset B_\infty$.

5.2. **Special sequence of backward contracting parameters.** Now we obtain the special sequence of parameters for which we have uniform backward contraction. Let $(j(n))_{n\geq 1}$ be the subsequence of indexes such that $D_n = D_{k_{j(n)}} \to D_x$ as above when $n \to \infty$. Then $(t_{k_{j(n)}})_n$ admits a convergent subsequence to some $s_1 \in \operatorname{supp} \theta_\varepsilon$.

To avoid too many subscripts we let that subsequence be indexed by k_n^0 with $n \geq 1$. This is a subsequence of $(k_j)_j$. By definition of hyperbolic times we know that $(f_{t_{k_n^0}})^{-1}$ is a $\alpha^{1/2}$-contraction on $D_{k_n^0}$ for all $n \geq 1$. Hence by the C^1 convergence of the disks and the C^2 continuity of the family Φ, we must have that $(f_{s_1})^{-1}$ is a $\alpha^{1/2}$-contraction on D_x.

We also have that $(t_{k_n^0-1})_n$ admits a subsequence tending to some $s_2 \in \operatorname{supp} \theta_\varepsilon$ indexed by $(k_n^1)_n$, which is a subsequence of $(k_n^0)_n$. In general we have that $t_{k_n^{\ell-1}-\ell} \to s_\ell$ when $n \to \infty$ where $(k_n^\ell)_n$ is a subsequence of $(k_n^{\ell-1})_n$ for every $\ell \geq 0$. The same continuity arguments as above ensure that $(f_{s_\ell} \circ \cdots \circ f_{s_1})^{-1}$ is a $\alpha^{\ell/2}$-contraction on D_x.

This shows that for every accumulation disk $D_x \in B_\infty$ as above there exists a subsequence $\underline{s} \in \operatorname{supp} \theta_\varepsilon^{\mathbb{N}}$ such that $(f_{\underline{s}}^j)^{-1} \mid D_x$ is a α^j-contraction for every $j \geq 1$. We have proved item (2) in the statement of Proposition 5.1.

In what follows, we denote by $\mathcal{B}$ the family of center-unstable disks in B_∞ obtained through this limit process.

5.3. **Construction of the cylinder.** Now we start the construction of the cylinder. Given any disk $D \in \mathcal{B}$, the compactness of B_∞ and the uniformity of δ_0 (the radius of invertibility of the exponential map of M defined in Subsection 4.1) enables us to construct a (open) cylinder $\mathcal{C}$ over any sub-disk D_0 of D with radius $\rho \in (0, \min\{\delta_0, \delta_2\})$ by considering the images under the exponential map of vectors in $T_z M$ orthogonal to $T_z D_0$ and with norm less than ρ. We assume that the connected components υ of every center-unstable disk γ that crosses $\mathcal{C}$ have diameter smaller than 2ρ inside γ. We call $\mathcal{C}$ a ρ-cylinder.

We assume that $\rho < \delta_1/100$. Write $B_j(\rho)$ for the disks obtained from B_j removing the ρ-neighborhood of the boundary of every disk in B_j and let $\hat{B}_j(\rho)$ be the union of the points in $B_j(\delta)$. Then setting

$$\nu_{n,\rho} = \nu_n \mid \cup_{j=0}^{n-1} \hat{B}_j(\delta)$$

we see that $((f_{\underline{t}}^j)_* m_D) \mid \hat{B}_j(\rho) \geq (1-\delta) \cdot ((f_{\underline{t}}^j)_* m_D) \mid \hat{B}_j$ for some $\delta = \delta(\rho) > 0$. The value of $\delta > 0$ may be taken independently of j because the bounded distortion property at hyperbolic times (Proposition 4.6) implies that the relative mass removed from the disks is comparable for all iterates.

Hence for a sufficiently small $\rho > 0$ as above we may assume that $\nu_{n,\rho}(M) \geq \beta_0/2$ for all n big enough. We fix this value of $\rho > 0$ from now on. Letting ν_ρ be an accumulation point of $\nu_{n,\rho}$ for a subsequence of $(\nu_{n_k})_k$ we have $\nu_\rho \leq \nu$.

The ρ-cylinders $\mathcal{C}$ constructed as above have uniform size (depending on ρ only), meaning that they contain a ball of radius ρ. Observe that $B_\infty \subseteq \Lambda_\epsilon \subseteq B(\Lambda, \overline{\epsilon})$ for some $\overline{\epsilon} > 0$, where we write $B(\Lambda, \overline{\epsilon})$ for $\cup_{x\in\Lambda} B(x, \overline{\epsilon})$. Then the cover of B_∞ by the family of all cylinders admits a minimal cover $\mathcal{C}_1, \dots, \mathcal{C}_k$.

We claim that k is bounded above uniformly independently of ϵ. Indeed, any cover of B_∞ by such cylinders is part of a cover of Λ_ϵ by ρ-balls. Let N be the minimum number of ρ-balls needed to cover Λ_ϵ. Since Λ_ϵ is in a $\overline{\epsilon}$-neighborhood of Λ, N is a constant equal to the minimum number of ρ-balls needed to cover Λ, for small enough $\epsilon > 0$. Hence $k \leq N$.

This shows that for some cylinder $\mathcal{C} \in \{\mathcal{C}_1, \dots, \mathcal{C}_k\}$ we must have

$$\nu(\mathcal{C}) \geq \nu_\rho(\mathcal{C}) \geq \frac{\nu_\rho(B_\infty)}{N} = \frac{\nu_\rho(M)}{N} \geq \frac{\beta_0}{2N}.$$

According to the construction of the ρ-cylinders, for every disk $D(\rho) \in \hat{B}_j(\rho)$ such that $D(\rho) \cap \mathcal{C} \neq \emptyset$, then the components of $D \cap \mathcal{C}$ cross $\mathcal{C}$, where D is the corresponding disk in $\hat{B}_j$ whose truncation gives $D(\rho)$, $j \geq 1$. Moreover by an arbitrarily small change in ρ we may assume that $\nu(\partial\mathcal{C}) = 0$.

Let us denote by $\mathcal{K}_n$ the components of the intersection $D \cap \mathcal{C}$ that cross $\mathcal{C}$, for all $D \in B_n$ and $n \geq 1$, and let $K_n = \cup\mathcal{K}_n$ be the union of the points in $\mathcal{K}_n$. In addition let $\mathcal{K}$ be the set of disks from $\mathcal{B}$ that cross $\mathcal{C}$ and K the set of all points in $\mathcal{K}$. Then for all $n \geq 1$

$$\nu_{n,\rho}(\mathcal{C}) = \nu_{n,\rho}(\mathcal{C} \cap \cup_{j=0}^{n-1} K_j) \leq \nu_n(\mathcal{C} \cap \cup_{j=0}^{n-1} K_j).$$

Hence taking limits of subsequences (recall that $\nu(\partial\mathcal{C}) = 0$ and $\nu_\rho \leq \nu$) we arrive at

$$\frac{\beta_0}{2N} \leq \nu_\rho(\mathcal{C}) \leq \limsup_{n\to\infty} \nu_n(\mathcal{C} \cap \cup_{j=0}^{n-1} K_j).$$

But since K contains the set of accumulation points of $(\cup_{j=0}^{n-1} K_j)_{n\geq 1}$ and ν_n is defined by the average (12), we have that

$$\limsup_{n\to\infty} \nu_n(\mathcal{C} \cap \cup_{j=0}^{n-1} K_j) \leq \nu(\mathcal{C} \cap K)$$

and so $\nu(\mathcal{C} \cap K) \geq \beta_0/(2N)$.

However $\mu^\varepsilon \geq \nu$ by construction, hence $\mu^\varepsilon(\mathcal{C} \cap K) \geq \beta_0/(2N)$ also. We stress that either β_0, N or ρ do not depend on the choice of ε nor of $\underline{t}$.

This proves the statement of item (1) of Proposition 5.1.

5.4. **Densities along center-unstable disks.** Let $\mathcal{C}$ be the ρ-cylinder constructed before and let $D_0 \in \mathcal{K}$ be the base disk of $\mathcal{C}$. Write $\mathcal{K}_j$ and K_j as before and set $\mathcal{K}_\infty = \mathcal{K}$. We take a sequence $(\mathcal{P}_n)_{n\geq 1}$ of increasing partitions of the family $\mathcal{D} = \cup_{0\leq k\leq\infty} K_j$ as follows.

The cylinder $\mathcal{C}$ is endowed with the orthogonal projection onto the base disk $p : \mathcal{C} \to D_0$ and the disks in $\mathcal{D}$ define a projection $\pi : \mathcal{D} \to \xi_0$ where $\xi_0 = p^{-1}(\{x_0\})$ for some fixed $x_0 \in D_0$.

To define $\mathcal{P}_k$, first take a sequence $(\mathcal{V}_k)_{k\geq 1}$ of increasing partitions of ξ_0 whose diameter tends to zero. Next introduce the space $\hat{\mathcal{D}} = \cup_{0\leq j\leq\infty} K_j \times \{j\}$. Then for any given k we say that two elements $(x, i) \in K_i \times \{i\}$ and $(y, j) \in K_j \times \{j\}$ of $\hat{\mathcal{D}}$ are in the same atom of $\mathcal{P}_k$ if both x, y project under π into the same atom of $\mathcal{V}_k$ and either $i, j \geq k$ or $i = j < k$.

Observe that since ξ_0 is diffeomorphic to a ball of some Euclidean space and we may identify each disk $\gamma \in \mathcal{K}_j$ with $\pi(\gamma)$ for all $0 \leq j \leq \infty$, then we may assume that the union $\partial\mathcal{P}_k$ of the boundaries of the elements of $\mathcal{P}_k$ satisfies $\mu(\partial\mathcal{P}_k) = 0$ by suitably choosing the sequence $\mathcal{V}_k$, i.e., the boundaries of the elements of $\mathcal{V}_k$ should have zero measure with respect to $\hat{\mu} = \pi_*(\mu)$.

Given $x \in \hat{\mathcal{D}}$ and writing $\mathcal{P}_k(x)$ for the atom of $\mathcal{P}_k$ which contains x, it is clear that $\mathcal{P}_k(x) \supset \mathcal{P}_{k+1}(x)$ for all $k \geq 1$ and also that $\cap_{k\geq 1}\mathcal{P}_k(x)$ equals $\pi^{-1}(\{x\}) \cap \mathcal{C}$.

Let A be a Borel subset of D_0 and $\zeta \in \mathcal{K}_j$. Since the projection p sends ζ diffeomorphically on D_0 and in $\mathcal{C}$ the angles involved in the projection are uniformly bounded, we may find a constant $C > 0$ such that

$$\frac{1}{C} \cdot \frac{m_0(A)}{m_0(D_0)} \leq \frac{m_\zeta\big(p^{-1}(A) \cap \zeta\big)}{m_\zeta(\zeta)} \leq C \cdot \frac{m_0(A)}{m_0(D_0)}, \tag{13}$$

where we have written m_0 and m_ζ for the Lebesgue induced measures on D_0 and ζ by m, respectively.

Proposition 4.6 ensures that the density of $\big(f^j_{\underline{t}}\big)_*(m_D)$ with respect to Lebesgue measure on each disk $\gamma \in \mathcal{K}_j$ is bounded from above and from below, thus

$$\frac{1}{C_2} \leq \frac{\big(f^j_{\underline{t}}\big)_*(m_D)\big(p^{-1}(A) \cap \zeta\big)}{m_\zeta\big(p^{-1}(A) \cap \zeta\big)} \leq C_2 \quad \text{and} \quad \frac{1}{C_2} \leq \frac{\big(f^j_{\underline{t}}\big)_*(m_D)\big(\zeta\big)}{m_\zeta(\zeta)} \leq C_2. \tag{14}$$

Combining (13) and (14) we get

$$\frac{1}{C_2^2 C} \cdot \frac{m_0(A)}{m_0(D_0)} \leq \frac{\big(f^j_{\underline{t}}\big)_*(m_D)\big(p^{-1}(A) \cap \zeta\big)}{\big(f^j_{\underline{t}}\big)_*(m_D)(\zeta)} \leq C_2^2 C \cdot \frac{m_0(A)}{m_0(D_0)}, \tag{15}$$

for all big enough values of j.

Now we define a sequence $(\hat{\nu}_n)_{n\geq 1}$ of measures on $\hat{D}$ by

$$\hat{\nu}_n\big(E_0 \times \{0\} \cup \cdots \cup E_{n-1} \times \{n-1\}\big) = \frac{1}{n}\sum_{j=0}^{n-1}(f^j_{\underline{t}})_*(m_D)(E_j)$$

where $E_i \subset K_i$ for $i = 0, \ldots, n-1$, and $\hat{\nu}_n(E) = 0$ for all $E \subset \cup_{n\leq j\leq\infty} K_j$. Observe that given $k \geq 1$ and $x \in \hat{\mathcal{D}}$ the atom $\mathcal{P}_k(x)$ is formed by a union of disks in $\cup_{0\leq j\leq k-1}\mathcal{K}_j$. Hence by the definition of $\hat{\nu}_n$ and by (15) we conclude that

$$\frac{1}{d} \cdot \hat{\nu}_n(\mathcal{P}_k(x)) \cdot m_0(A) \leq \hat{\nu}_n\big(p^{-1}(A) \cap \mathcal{P}_k(x)\big) \leq d \cdot \hat{\nu}_n(\mathcal{P}_k(x)) \cdot m_0(A), \tag{16}$$

where $d = C_2^2 C/m_0(D_0)$.

It is easy to see that any weak* accumulation point of $\hat\nu_n$ is supported in $K \times \{\infty\}$. Moreover if we choose a sequence n_k such that $\nu_{n_k} \to \nu$, then this just means that $\hat\nu_{n_k}$ tends to a measure $\hat\nu$ such that $\hat\nu(E \times \{\infty\}) = \nu(E)$ for all $E \subset K$. Since we may assume without loss that $\nu\Big(\partial\big(\mathcal{P}_k(D) \cap p^{-1}(A)\big)\Big) = 0$ for all $k \ge 0$, by the choice of $\mathcal{V}_k$ during the construction above, the inequalities (16) also hold in the limit, i.e.

$$\frac{1}{d} \cdot \hat\nu(\mathcal{P}_k(x)) \cdot m_0(A) \le \hat\nu\big(p^{-1}(A) \cap \mathcal{P}_k(x)\big) \le d \cdot \hat\nu(\mathcal{P}_k(x)) \cdot m_0(A). \tag{17}$$

By the theorem of Radon-Nikodym (17) means that the density of ν along the disks $\gamma \in \mathcal{K}$ is bounded above and below, as stated in item (3) of Proposition 5.1. This concludes the proof of this proposition. □

6. Accumulation cylinders

In what follows we fix a decreasing sequence $\varepsilon_k \to 0$ when $k \to \infty$ and a sequence $\mu_k = \mu^{\varepsilon_k}$ of ergodic stationary measures. We write also ν_k for the component of each μ_k for $k \ge 1$ with well behaved disintegrations given by Proposition 5.1.

We observe that since $(\operatorname{supp} \theta_\varepsilon)_{\varepsilon>0}$ is a nested family of connected compact subsets shrinking to $\{t^*\}$ when $\varepsilon \to 0$, and for each $\varepsilon > 0$ and any stationary measure μ^ε the set $\operatorname{supp} \mu^\varepsilon$ is f_t-invariant for all $t \in \operatorname{supp} \theta_\varepsilon$, we may choose the sequence μ_k so that $(\operatorname{supp} \mu_k)_k$ is a nested family of f-invariant compact subsets.

Proposition 6.1. *Let μ be a weak* accumulation point of $(\mu_k)_k$. Then there exists $d_0 > 0$, a cylinder $\mathcal{C}$, a family $\mathcal{K}$ of disks tangent to the center-unstable cone field which cross $\mathcal{C}$, whose union is the set K, and a component ν of μ such that*

1. *$\nu(K \cap \mathcal{C}) \ge \beta$;*
2. *$(f^n)^{-1} \mid \gamma$ is a $\alpha^{n/2}$-contraction on every disk $\gamma \in \mathcal{K}$, where $f = f_{t^*}$.*
3. *the disintegration $\{\nu_\gamma\}_\gamma$ of $\nu \mid \mathcal{C}$ along the disks $\gamma \in \mathcal{K}$ has densities with respect to the Lebesgue induced measure m_γ on γ uniformly bounded from above and below: $d_0^{-1} \le (d\nu_\gamma/dm_\gamma) \le d_0$;*

The value of β above is the same from the statement of Proposition 5.1. The value of d_0 depends only on d from Proposition 5.1.

Item (2) above shows that every $\gamma \in \mathcal{K}$ is a center-unstable disk in U. This means that E^{cu}_x is uniformly expanded by Df for every $x \in \gamma$. The domination property for the splitting $E^{cu} \oplus E^{cs}$ guarantees that any eventual expansion along the complementary direction is weaker than this. Thus γ is contained in the unique local strong-unstable manifold $W^u_{loc}(x)$ tangent to E^{cu}_x, see [Pe].

In item (3) we can also say that the disintegration of μ along the disks of $\mathcal{K}$ has densitities bouded from above and from below, since ν is a component of μ.

Proof. Let μ_k be as stated in the beginning of the subsection and let $\mathcal{C}_k, \mathcal{K}_n$ and K_n be the corresponding cylinders, families of disks and sets from Proposition 5.1. We assume that $\mu_k \to \mu$ in the weak* topology when $k \to \infty$. Then μ is an f-invariant probability measure

(Remark 2.1). We take also ν a limit point of ν_k in the weak* topology. Note that since $\nu_n \le \mu_n$ for all $n \ge 1$ then $\nu \le \mu$ also.

The compactness of M ensures that for some subsequence k_n the cylinder $\mathcal{C}_{k_n}$ tends to a cylinder $\mathcal{C}$. In fact, each cylinder $\mathcal{C}_k$ is a diffeomorphic image of $\phi_k : B^u \times B^s \hookrightarrow M$, with B^ℓ the ℓ-dimensional unit ball of $\mathbb{R}^\ell, \ell = s, u$. By the Ascoli-Arzela Theorem there is a subsequence $(k_n)_{n\ge1}$ such that $\phi_{k_n}(B^u \times 0)$ converges in the C^1-topology to a disk $D_0 = \phi(B^u \times 0)$ in M. Since the diameters of $\phi_{k_n}(B^u \times 0)$ and $\phi_{k_n}(0 \times B^s)$ are uniformly bounded from below by $\rho > 0$ (by Proposition 5.1) and by the construction of $\mathcal{C}_k$, defining $\mathcal{C}$ as the set of images under the exponential map of vectors in T_zM orthogonal to T_zD_0 and with norm less than ρ, then $\overline{\mathcal{C}}_{k_n}$ tends to $\overline{\mathcal{C}}$ in the Hausdorff topology.

Let $\mathcal{K}$ be the family of disks D in $\mathcal{C}$ which are accumulated by sequences of disks D_n in $\mathcal{K}_{k_n}$ for $n \ge 1$. Since every disk D_n is tangent to the center-unstable cone field of f, the continuity of the cone field on U assures that every disk $D \in \mathcal{K}$ is also a center-unstable disk.

Remark 6.2. It will be useful to note that up to taking a slightly smaller base disk D_0 we may assume without loss that the disks in K_{k_n} cross $\mathcal{C}$ for all big enough k.

For any fixed $\gamma \in \mathcal{K}$ let $x, y \in D$ and take $(x_n)_n, (y_n)_n$ sequences in $\gamma_n \in \mathcal{K}_{k_n}$ such that $x_n \to x$ and $y_n \to y$ when $n \to \infty$. From item (3) of Proposition 5.1 we know that there are sequences of parameters $(\underline{s}(n))_{n\ge1}$ such that $\underline{s}(n) \in \operatorname{supp} \theta^{\mathbb{N}}_{\varepsilon_{k_n}}$ and

$$\operatorname{dist}_{(f^j_{\underline{s}(n)})^{-1}(\gamma_n)} \left((f^j_{\underline{s}(n)})^{-1}(x_n), (f^j_{\underline{s}(n)})^{-1}(y_n)\right) \le \alpha^{j/2} \operatorname{dist}_{\gamma_n}(x_n, y_n)$$

for every $j \ge 1$ and for every given $n \ge 1$. Fixing $j \ge 1$ we get

$$(s_1(n), \dots, s_j(n)) \to (t^*, \dots, t^*) \quad \text{when} \quad n \to \infty,$$

because $\operatorname{supp}(\theta^{\mathbb{N}}_{\varepsilon_{k_n}}) \to \{t^*\}$. The continuity of $f_t(x)$ with respect to $(t, x) \in T \times M$ implies that

$$\operatorname{dist}_{f^{-j}(\gamma)} \left(f^{-j}(x), f^{-j}(y)\right) \le \alpha^{j/2} \operatorname{dist}_\gamma(x, y)$$

for every given $j \ge 1$. Hence f^{-j} is an $\alpha^{j/2}$-contraction on every $\gamma \in \mathcal{K}$, which proves item (2) of Proposition 6.1.

Now since $\nu_k(\mathcal{C}_k \cap \mathcal{K}_k) \ge \beta > 0$ for all $k \ge 1$ from Proposition 5.1, if we fix $\delta > 0$ then for all n big enough we get $K_{k_n} \subset B(K, \delta)$. Letting $\delta > 0$ be such that $\mu(\partial B(K, \delta)) = 0$ and so also $\nu(\partial B(K, \delta)) = 0$ (this holds except for an at most countable set of values of δ), then

$$\nu(B(K, \delta)) = \lim_{n\to\infty} \nu_{k_n}(B(K, \delta)) \ge \beta > 0.$$

Moreover $K = \cap_{\delta>0} B(K, \delta)$ thus $\nu(K) = \inf_{\delta>0} \nu(B(K, \delta)) \ge \beta$. This proves item (1) of the statement of Proposition 6.1.

6.1. **Absolute continuity of limit measure on accumulation cylinder.** Here we prove item (3) of Proposition 6.1. We recall that the limit cylinder $\mathcal{C}$ has base $D_0 \in \mathcal{K}$.

We take a sequence $(\mathcal{P}_n)_{n\ge1}$ of increasing partitions of the family $\hat{\mathcal{D}} \subset \cup_{1\le j\le\infty} K_j \times \{j\}$ of all disks which cross $\mathcal{C}$ (by Remark 6.2 this family contains disks from infinitely many distinct $\mathcal{K}_k$), defined in the same fashion as in the proof of Proposition 5.1. For the rest of the proof we write $\mathcal{K}_\infty$ for $\mathcal{K}$.

Let $p : \mathcal{C} \to D_0$ be the orthogonal projection on the base disk and $\pi : \mathcal{D} \to \xi_0$ be the projection along the leaves of $\cup_j \mathcal{K}_j$ where $\xi_0 = p^{-1}(\{x_0\})$ for some fixed $x_0 \in D_0$. Take a sequence $(\mathcal{V}_k)_{k\geq 1}$ of increasing partitions of ξ_0 with diameter tending to zero and define $\mathcal{P}_k$ in the same way as in Subsection 5.4.

Exactly as in Subsection 5.4, we may assume without loss that the union $\partial\mathcal{P}_k$ of the boundaries of the elements of $\mathcal{P}_k$ have zero measure with respect to $\hat\mu = \pi_*(\mu)$, by an adequate choice of the sequence of partitions. Note that given $x \in \mathcal{D}$, then $\mathcal{P}_k(x) \supset \mathcal{P}_{k+1}(x)$ for all $k \geq 1$ and $\cap_{k\geq 1}\mathcal{P}_k(x)$ equals $\pi^{-1}(\{x\}) \cap \mathcal{C}$.

Let A be a Borel subset of D_0 and $\zeta \in \mathcal{K}_j$. Then we have (13) by the same reasons. Moreover item (3) of Proposition 5.1 ensures that the density of ν_j with respect to Lebesgue measure on ζ is bounded from above and from below, thus

$$\frac{1}{d} \leq \frac{\nu_j\big(p^{-1}(A)\cap\zeta\big)}{m_\zeta\big(p^{-1}(A)\cap\zeta\big)} \leq d \quad \text{and} \quad \frac{1}{d} \leq \frac{\nu_j(\zeta_i)}{m_\zeta(\zeta)} \leq d. \tag{18}$$

Combining (13) with (18) we get for all $\zeta \in \mathcal{K}_j$ and all j

$$\frac{1}{Cd^2}\cdot\frac{m_0(A)}{m_0(D_0)} \leq \frac{\nu_j\big(\pi^{-1}(A)\cap\zeta\big)}{\nu_j(\zeta)} \leq Cd^2\cdot\frac{m_0(A)}{m_0(D_0)}. \tag{19}$$

Likewise the argument in Subsection 5.4 we define a sequence $(\hat\nu_n)_{n\geq 1}$ of measures on $\hat{\mathcal{D}}$ by

$$\hat\nu_n\big(E_0\times\{0\}\cup\cdots\cup E_{n-1}\times\{n-1\}\big) = \frac{1}{n}\sum_{j=0}^{n-1}\nu_j(E_j)$$

where $E_i \subset K_i$ for $i = 0,\dots,n-1$, and $\hat\nu_n(E) = 0$ for all $E \subset \cup_{n\leq j\leq\infty}K_j$. Note that for $k \geq 1$ and $x \in \hat{\mathcal{D}}$ the atom $\mathcal{P}_k(x)$ is a union of disks in $\cup_{0\leq j\leq k-1}\mathcal{K}_j$. Hence by the definition of $\hat\nu_n$ and by (19) we deduce

$$\frac{1}{d_0}\cdot\hat\nu_n(\mathcal{P}_k(x))\cdot m_0(A) \leq \hat\nu_n\big(p^{-1}(A)\cap\mathcal{P}_k(x)\big) \leq d_0\cdot\hat\nu_n(\mathcal{P}_k(x))\cdot m_0(A), \tag{20}$$

where $d_0 = Cd^2/m_0(D_0)$. Note that D_0 is an accumulation disk and so its size depends only on the value of d, hence d_0 depends only on d.

As in the proof of item (3) of Proposition 5.1, any weak* accumulation point of $\hat\nu_n$ is supported in $K\times\{\infty\}$ and $\nu_n \to \nu$ means that $\hat\nu_n$ tends to a measure $\hat\nu$ such that $\hat\nu(E\times\{\infty\}) = \nu(E)$ for all $E \subset K$. Then the inequalities (20) also hold in the limit

$$\frac{1}{d_0}\cdot\hat\nu(\mathcal{P}_k(x))\cdot m_0(A) \leq \hat\nu\big(p^{-1}(A)\cap\mathcal{P}_k(x)\big) \leq d_0\cdot\hat\nu(\mathcal{P}_k(x))\cdot m_0(A),$$

which, by the theorem of Radon-Nikodym means that the density of ν along the disks $\gamma \in \mathcal{K}$ is bounded above and below, since A, k and x are arbitrary.

This proves item (3) of Proposition 6.1 and concludes the proof. □

6.2. **Almost every ergodic component is a Gibbs *cu*-state.** Here and in the next subsection we conclude the proof of Theorem 3.4 by showing that *every f-invariant probability measure μ, obtained as a limit measure of the measures μ_k, and admitting a component ν satisfying properties (1)-(3) of Proposition 3.4 must be a Gibbs cu-state.*

Let $(\mu_x)_{x\in M}$ be the Ergodic Decomposition of μ (see e.g. [Ma2]), i.e. the unique family of measures up to μ-null sets which satisfies: μ_x is a f-invariant ergodic probability measure for μ-a.e. x and for every bounded measurable function $\varphi : M \to \mathbb{R}$ we have $\int\big(\int \varphi\, d\mu_x\big)\, d\mu(x) = \int \varphi\, d\mu$.

Lemma 6.3. *The measure μ_x is a cu-Gibbs state for μ-a.e. $x \in K$.*

Proof. Let Σ be the full μ measure subset of M where the family (μ_x) giving the Ergodic Decomposition of μ is defined. Let R be the subset of *Oseledets regular points* with respect to f (see e.g. [Ma2]), that is, the full μ-measure subset of M on whose points Lyapunov exponents are well defined. Then $K \cap \Sigma \cap R$ has full μ-measure on K, since $\mu(K) > 0$ (recall the statement of Proposition 3.4).

Observe that by the uniform backward contraction property of the disks of $\mathcal{K}$ we have for μ-a.e. $x \in K \cap \Sigma \cap R$

$$\lim_{n\to+\infty} \frac{1}{n} \log \|Df^{-n} \mid E^{cu}_x\| \le \frac{1}{2}\log\alpha < 0,$$

which implies that the Lyapunov exponents along the tangent direction E^{cu} to the disks of $\mathcal{K}$ are all positive.

Consider a measurable set B_0 such that

$$m_\gamma(B_0 \cap \gamma) = 0 \quad \text{for all} \quad \gamma \in \mathcal{K}$$

and $\mu(B_0)$ is maximal among all measurable subsets with this property. Since ν is absolutely continuous along the disks of $\mathcal{K}$ we have $\nu(B_0) = 0$. In what follows set $Z_0 = K \cap \Sigma \cap R \setminus B_0$.

By definition of Ergodic Decomposition, for every measurable set $A \subset Z_0$ we have

$$\mu(A) = \int \mu_x(A)\, d\mu(x) \tag{21}$$

and we want to express this as an integral over Z_0 in order to use the properties of ν.

The ergodicity of μ_x ensures that $\mu_x(A) = \lim_{n\to+\infty} n^{-1}\sum_{j=0}^{n-1}\chi_A(f^j(x))$ for μ-a.e. x, where $\chi_A(x) = 1$ if $x \in A$ and 0 otherwise. Hence except for a μ-zero subset of points we see that $\mu_x(A) > 0$ only if x has some iterate in $A \subset Z_0$.

Let $k(z)$ be the smallest positive integer such that $f^{-k(z)}(z) \in Z_0$, which is defined μ-almost everywhere in Z_0. Note that $\mu_z = \mu_{f^i(z)}$ for all $i \in \mathbb{Z}$. Thus we may write

$$\mu(A) = \int_{Z_0} k(z)\, \mu_z(A)\, d\mu(z),$$

since for μ-a.e. z we remove $\mu_z(A) = \mu_{f^{-1}(z)}(A) = \cdots = \mu_{f^{-k(z)+1}(z)}(A)$ from (21).

Now we use the following technical result which can be deduced from §3 of [Rh], and whose proof can also be found in [ABV, Lemma 6.2].

Lemma 6.4. *Let λ be a finite measure on a measure space Z, with $\lambda(Z) > 0$. Let $\mathcal{K}$ be a measurable partition of Z, and $(\lambda_z)_{z\in Z}$ be a family of finite measures on Z such that*

1. *the function $z \mapsto \lambda_z(A)$ is measurable, and it is constant on each element of $\mathcal{K}$, for any measurable set $A \subset Z$*
2. *$\{w : \lambda_z = \lambda_w\}$ is a measurable set with full λ_z-measure, for every $z \in Z$.*

Assume that $\lambda(A) = \int \ell(z)\lambda_z(A)\,d\lambda$ for some measurable function $\ell : Z \to \mathbb{R}_+$ and any measurable subset A of Z. Let $\{\tilde\lambda_\gamma,\, \gamma \in \mathcal{K}\}$, and $\{\tilde\lambda_{z,\gamma},\, \gamma \in \mathcal{K}\}$, be disintegrations of λ and λ_z, respectively, into conditional probability measures along the elements of the partition $\mathcal{K}$. Then $\tilde\lambda_{z,\gamma} = \tilde\lambda_\gamma$ for λ-almost every $z \in Z$ and $\hat\lambda_z$-almost every γ, where $\hat\lambda_z$ is the quotient measure induced by λ_z on $\mathcal{K}$.

We set $Z = Z_0$, $\lambda = (\mu \mid Z_0)$, $\lambda_z = (\mu_z \mid Z_0)$, $\ell(z) = k(z)$ and $\mathcal{K}$ as before, with $z \in Z_0$, and apply Lemma 6.4. We conclude that the disintegration $\mu_{z,\gamma}$ of μ_z along the disks $\gamma \in \mathcal{K}$ coincides almost everywhere with the disintegration μ_γ of $\mu \mid Z_0$ along the same family of disks. Therefore, since we have already shown by Proposition 6.1 and the choice of B_0 that μ_γ is absolutely continuous with respect to m_γ, and also that the Lyapunov exponents of μ_z along the tangent directions to the disks of $\mathcal{K}$ are all positive, we conclude that μ_z is an ergodic *cu*-Gibbs state for f, for $z \in Z_0$. This finishes the proof of the lemma. □

6.3. **The accumulation measure is a Gibbs *cu*-state.** Here we finish the proof of Theorem B.

We fix $\varepsilon_k \to 0$, $\mu_k = \mu^{\varepsilon_k}$ and $\mu = \lim_{k\to\infty} \mu_k$ in the weak* topology as in the previous section. We denote by $\mathcal{C}$ the cylinder and by K the compact subset in the statement of Proposition 6.1 with respect to μ. We also denote by $\mathcal{C}^k$ the cylinder and by K^k the compact subset from the statement of Proposition 5.1 with respect to each μ_k, $k \geq 1$.

Define G to be the set of all points $x \in \Sigma \cap R$ (i.e. Oseledets regular points whose orbit defines an ergodic f-invariant measure, see the proof of Lemma 6.3) such that μ_x is a *cu*-Gibbs state and set

$$\nu_0 = \int_G \mu_x \, d\mu(x).$$

Since $\mu_x = \mu_{f^i(x)}$ for all $i \in \mathbb{Z}$ and μ-a.e. x the measure ν_0 is f-invariant and not identically zero since $G \supset K$. By construction, $\nu_0/\nu_0(G)$ is a *cu*-Gibbs state.

The purpose of this section is to prove the following.

Proposition 6.5. *$\mu = \nu_0$, that is μ is a Gibbs cu-state.*

Before the proof we present some useful lemmas. By the results of [BoV, ABV] and by Theorem 3.2 we know that $\nu_0/\nu_0(G)$ is a physical measure and we can use the following result which corresponds to [Vz, Lemma 3.5]. For each $n \geq 1$ set $K_n = \{x \in M : \tau(x) \leq n\}$, where $\tau(x) = \min\{k \geq 1 : f^{-k}(x) \in K\}$. By the Recurrence Theorem this function is finite μ-almost everywhere.

Lemma 6.6. *Let $R_j = \{x \in K : \tau(x) = j\}$. Then there exists $C > 0$ and $\lambda_0 \in (0,1)$ such that given any physical measure ν for f we have $\nu(R_j) \leq C \cdot \lambda_0^j$ for every $j \geq 1$.*

Now for every $k, n \geq 1$ we set

$$A_n^k = \bigcap_{\underline{t} \in \operatorname{supp}(\theta_{\varepsilon_k}^{\mathbb{N}})} \bigcap_{j=1}^{n} f_{\underline{t}}^j \big(M \setminus K^k\big) \cap K^k$$

and define $K_n^k = M \setminus A_n^k$ and $\mu_k^n = \mu_k \mid K_n^k$.

The following result is a consequence of the assumption that f_t is is C^2-close to $f \equiv f_{t^*}$ when t is close to t^* together with Lemma 6.6.

Lemma 6.7. *There is $C_0 > 0$ and for any given $n \in \mathbb{N}$ there is $\ell \geq 1$ such that*

$$\mu_k(A_n^k) < C_0 \lambda_0^{n+1} \quad \textit{for every} \quad k > \ell.$$

Moreover for every fixed $n \geq 1$ we have $\mu_k^n \to \mu \mid K_n$ when $k \to \infty$ in the weak* topology.

Proof. We start by noting that defining $\eta = \mu - \nu_0$ we have $\eta(K) = 0$ according to Lemma 6.3, that is $\mu \mid K = \nu_0 \mid K$. So from Lemma 6.6 for any given n we have that

$$A_n = \bigcap_{j=1}^{n} f^j(M \setminus K) \cap K$$

satisfies $\mu(A_n) \leq C \sum_{j>n} \lambda_0^j = C' \lambda_0^{n+1}$, where $C' = C/(1-\lambda_0)$.

Since K is closed and μ is a Borel regular measure, fixing the number n of iterates involved we have that for small enough $\zeta_1, \zeta_2 > 0$ the set

$$A_n(\zeta_1, \zeta_2) = \bigcap_{j=1}^{n} f^j \big(B(M \setminus K, \zeta_2)\big) \cap B(K, \zeta_1)$$

also satisfies $\mu(A_n(\zeta_1,\zeta_2)) < 2C'\lambda_0^{n+1}$, where $B(X,\zeta) = \cup_{x \in X} B(x,\zeta)$ is the ζ-neighborhood of X in M for any $\zeta > 0$ and any subset X. Moreover $A_n(\zeta_1,\zeta_2)$ is an open neighborhood of A_n so through an arbitrarily small change in ζ_1, ζ_2 we may assume that $\mu(\partial A_n(\zeta_1,\zeta_2)) = 0$ in what follows.

Note that for fixed n and ζ_1 as above, there exists $\ell_1 \in \mathbb{N}$ such that for small enough $\zeta_2 < \zeta_1$ we have $K^k \subset B(K, \zeta_2/2) \subset B(K, \zeta_1)$ and $M \setminus K^k \subset B(M \setminus K, \zeta_2)$ for all $k > \ell_1$. Since f_t depends continuously on t and the number n of iterates is fixed, we may take ℓ_1 big enough and $\zeta_2 > 0$ small enough such that $A_n^k \subset A_n(\zeta_1, \zeta_2)$ for all $k \geq \ell_1$.

But $\mu_k \to \mu$ in the weak* topology so by the assumption on the boundary of $A_n(\zeta_1,\zeta_2)$ we arrive at $\mu_k(A_n^k) \leq \mu_k(A_n(\zeta_1,\zeta_2)) \leq 4C'\lambda_0^{n+1}$ for all big enough k. The first statement of the lemma is obtained resetting ℓ to a bigger value (if needed) and letting $C_0 = 4C'$.

For small enough $\xi_1, \xi_2 > 0$ we now define

$$A_n^k(\xi_1,\xi_2) = \bigcap_{\underline{t} \in \operatorname{supp}(\theta_{\varepsilon_k}^{\mathbb{N}})} \bigcap_{j=1}^{n} f_{\underline{t}}^j \big(B(M \setminus K^k, \xi_2)\big) \cap B(K^k, \xi_1)$$

which is an open neighborhood of A_n^k. Again for fixed n and $\xi_1 > 0$ there exists $\ell_2 \in \mathbb{N}$ such that for small enough $\xi_2 < \xi_1$ we have $K \subset B(K^k, \xi_2/2) \subset B(K^k, \xi_1)$ for all $k > \ell_2$. Since f_t is C^2 close to $f = f_{t^*}$ for big k we check that $A_n \subset A_n^k(\xi_1,\xi_2)$ for all $k \geq \ell_2$.

Choosing small values of $\zeta_1, \zeta_2 > 0$ and taking ℓ big enough the above arguments ensure that for all $k \geq \ell$ it holds

$$A_n \subset A_n(\zeta_1, \zeta_2) \subset A_n^k(2\zeta_1, 2\zeta_2) \subset A_n(4\zeta_1, 4\zeta_2) \tag{22}$$

and we can simultaneously assume that we have

$$\mu(\partial A_n(\zeta_1, \zeta_2)) = 0 = \mu(\partial A_n(4\zeta_1, 4\zeta_2)). \tag{23}$$

Let us take an open set B such that $\mu(\partial B) = 0$ (the collection of all such sets generates the Borel σ-algebra μ mod 0). Using (22) we get for all $k \geq \ell$

$$\mu_k(B \cap [M \setminus A_n(4\zeta_1, 4\zeta_2)]) \leq \mu_k(B \cap [M \setminus A_n^k(2\zeta_1, 2\zeta_2)]) \leq \mu_k(B \cap [M \setminus A_n(\zeta_1, \zeta_2)])$$

and letting $k \to \infty$ and using (23) we arrive at

$$\begin{aligned} \mu(B \cap [M \setminus A_n(4\zeta_1, 4\zeta_2)]) &\leq \liminf_{k\to\infty} \mu_k(B \cap [M \setminus A_n^k(2\zeta_1, 2\zeta_2)]) \\ &\leq \limsup_{k\to\infty} \mu_k(B \cap [M \setminus A_n^k(2\zeta_1, 2\zeta_2)]) \\ &\leq \mu(B \cap [M \setminus A_n(\zeta_1, \zeta_2)]). \end{aligned}$$

Finally letting $\zeta_2 \to 0$ first and then $\zeta_1 \to 0$ also the compact sets $M \setminus A_n(4\zeta_1, 4\zeta_2)$ and $M \setminus A_n(\zeta_1, \zeta_2)$ both grow to $M \setminus A_n$ which clearly equals K_n. In the same way $M \setminus A_n^k(2\zeta_1, 2\zeta_2)$ grows to $M \setminus A_n^k = K_n^k$.

This together with the last sequence of inequalities shows that $\mu_k^n(B) \to (\mu \mid K_n)(B)$ when $k \to \infty$, finishing the proof. □

Proof of Proposition 6.5. For each $n \geq 1$ we set $\nu_n = \nu_0 \mid K_n$. Note that $\nu_n = \mu \mid K_n$ since $\nu_0 \mid K = \mu \mid K$ and both μ and ν_0 are f-invariant. In the weak* topology $\nu_n \to \nu$ when $n \to \infty$ because of the following simple fact.

Lemma 6.8. *For ν_0-almost every z there exists $j \leq 0$ such that $f^j(z) \in K$.*

Proof. Given B a Borel subset we have that $\mu_x(B) = \lim_{n\to+\infty} n^{-1} \sum_{j=0}^{n-1} \chi_B(f^j(x))$ for μ-almost every x. Since $\mu(K) > 0$, if $\nu_0(B) > 0$ then for some $x \in K$ we have $\mu_x(B) > 0$ and there exists $j \geq 0$ such that $f^j(x) \in B$. □

Moreover Lemma 6.7 implies that $\mu_k^n \to \mu_k$ when $n \to \infty$ in the weak* topology in a uniform way, since $\mu_k(A_n^k) \to 0$ when $n \to \infty$ uniformly in k. In addition, the same lemma ensures that for any fixed $n \in \mathbb{N}$ we have $\mu_k^n \to \nu_n$ when $k \to \infty$ in the weak* topology by the definition of ν_n.

Let $\zeta > 0$ and a continuous $\varphi : M \to \mathbb{R}$ be given. Then we may find a big enough $n \geq 1$ such that

$$|\eta(\varphi) - \nu_n(\varphi)| \leq \zeta \quad \text{and} \quad |\mu_k^n(\varphi) - \mu_k(\varphi)| \leq \zeta$$

for every sufficiently big $k \geq 1$ — we stress that for the second inequality we need the uniform bound on $\mu_k(A_n^k)$ provided by Lemma 6.7.

Having fixed n we may now take k big enough keeping the above inequalities and satisfying also

$$|\mu_k(\varphi) - \mu(\varphi)| \leq \zeta \quad \text{and} \quad |\eta_n(\varphi) - \mu_k^n(\varphi)| \leq \zeta.$$

Finally putting this all together we arrive at

$$\begin{aligned}|\eta(\varphi)-\mu(\varphi)| \;&\le\; |\eta(\varphi)-\eta_n(\varphi)|+|\eta_n(\varphi)-\mu_k^n(\varphi)|+\\ &\quad |\mu_k^n(\varphi)-\mu_k(\varphi)|+|\mu_k(\varphi)-\mu(\varphi)|\le 4\zeta.\end{aligned}$$

This concludes the proof of the proposition. □

As explained in the beginning of Subsection 3.2 this is precisely what is needed to conclude stochastic stability for f. Theorem B is proved.

7. A stochastically stable class

In this section we present a robust class of partially hyperbolic diffeomorphisms satisfying conditions (a)-(c) and also condition (3) for random orbits. Here we take U equal to M. This presentation follows closely [ABV] and we just sketch the main points. The C^1 open classes of transitive non-Anosov diffeomorphisms presented in [BoV, Section 6], as well as other robust examples from [Ma1], are constructed in a similar way.

We start with a linear Anosov diffeomorphism $\hat{f}$ on the d-dimensional torus $M=\mathbb{T}^d$, $d\ge 2$. We write $TM=E^u\oplus E^s$ the corresponding hyperbolic decomposition of the tangent fiber bundle. Let V be a small closed domain in M for which there exist unit open cubes K^0 and K^1 in $\mathbb{R}^d$ such that $V\subset\pi(K^0)$ and $\hat{f}(V)\subset\pi(K^1)$, where $\pi:\mathbb{R}^d\to\mathbb{T}^d$ is the canonical projection. Now, let f be a diffeomorphism on $\mathbb{T}^d$ such that

(A) f admits invariant cone fields C^{cu} and C^{cs}, with small width $a>0$ and containing, respectively, the unstable bundle E^u and the stable bundle E^s of the Anosov diffeomorphism $\hat{f}$;

(B) f is *volume hyperbolic*: there is $\sigma_1>1$ so that
$$|\det(Df|T_x\Delta^{cu})|>\sigma_1 \quad\text{and}\quad |\det(Df|T_x\Delta^{cs})|<\sigma_1^{-1}$$
for any $x\in M$ and any disks Δ^{cu}, Δ^{cs} tangent to C^{cu}, C^{cs}, respectively.

(C) f is C^1-close to $\hat{f}$ in the complement of V, so that there exists $\sigma_2<1$ satisfying
$$\|(Df|T_x\Delta^{cu})^{-1}\|<\sigma_2 \quad\text{and}\quad \|Df|T_x\Delta^{cs}\|<\sigma_2$$
for any $x\in(M\setminus V)$ and any disks Δ^{cu}, Δ^{cs} tangent to C^{cu}, C^{cs}, respectively.

(D) there exist some small $\delta_0>0$ satisfying
$$\|(Df|T_x\Delta^{cu})^{-1}\|<1+\delta_0 \quad\text{and}\quad \|Df|T_x\Delta^{cs}\|<1+\delta_0$$
for any $x\in V$ and any disks Δ^{cu} and Δ^{cs} tangent to C^{cu} and C^{cs}, respectively.

Closeness in (C) should be enough to ensure that $f(V)$ is also contained in the projection of a unit open cube. If $\tilde{f}$ is a torus diffeomorphism satisfying (A), (B), (D), and coinciding with $\hat{f}$ outside V, then any map f in a C^1 neighborhood of $\tilde{f}$ satisfies all the previous conditions. Results in [ABV, Appendix] show in particular that for any f satisfying (A)–(D) there exist $c_u>0$ such that f is non-uniformly expanding along its center-unstable direction as in condition (b) at Subsection 1.1.

7.1. **Behavior over random orbits.** Here we take f as described above and T a small neighborhood of f in the C^2 topology in such a way that conditions (A)–(D) hold for every $g \in T$. We define $\Phi(t) = f_t = t$ and take $(\theta_\varepsilon)_{\varepsilon>0}$ a family of measures on T as before. We are going to show that any such f is non-uniformly expanding along the center-unstable direction on random orbits and, in the process, we will also see that this is a neighborhood of maps with mostly contracting center-stable direction. This will be done by showing that there is $c > 0$ such that for any disk Δ^{cu} tangent to the center-unstable cone field and $\theta_\varepsilon^{\mathbb{N}} \times m$ almost all $(\underline{t}, x) \in T^{\mathbb{N}} \times \Delta^{cu}$

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \left\| \left(Df | T_{f_{\underline{t}}^j x} \Delta_j^{cu}(\underline{t}) \right)^{-1} \right\| \leq -c, \tag{24}$$

where $\Delta_j^{cu}(\underline{t}) = f_{\underline{t}}^j \Delta^{cu}$. To explain this, let $B_1, \ldots, B_p, B_{p+1} = V$ be any partition of $\mathbb{T}^d$ into small domains, in the same sense as before: there exist open unit cubes K_i^0 and K_i^1 in $\mathbb{R}^d$ such that

$$B_i \subset \pi(K_i^0) \quad \text{and} \quad f(B_i) \subset \pi(K_i^1). \tag{25}$$

Let us fix Δ^{cu} any disk tangent to the center-unstable cone field and define m to be Lebesgue measure in Δ^{cu} normalized so that $m(\Delta^{cu}) = 1$.

Lemma 7.1. *There is $\zeta > 0$ such that for $\theta_\varepsilon^{\mathbb{N}} \times m$ almost all $(\underline{t}, x) \in T^{\mathbb{N}} \times \Delta^{cu}$ and large enough $n \geq 1$ we have*

$$\#\{0 \leq j < n : f_{\underline{t}}^j(x) \in B_1 \cup \ldots \cup B_p\} \geq \zeta n. \tag{26}$$

Moreover, there is $0 < \tau < 1$ for which the set I_n of points $(\underline{t}, x) \in T^{\mathbb{N}} \times \Delta^{cu}$ whose orbits do not spend a fraction ζ of the time in $B_1 \cup \ldots \cup B_p$ up to iterate n is such that $(\theta_\varepsilon^{\mathbb{N}} \times m)(I_n) \leq \tau^n$ for large $n \geq 1$.

In particular, if we take a constant $\underline{t} = (f, f, f, \ldots)$ we get the same conclusion for the unperturbed f, that is, there is $\tau \in (0, 1)$ such that the set J_n of points $x \in \Delta^{cu}$ whose orbits do not spend a fraction ζ of the time in $B_1 \cup \ldots \cup B_p$ up to iterate n is such that $m(J_n) \leq \tau^n$ for large $n \geq 1$.

Proof. Let us fix $n \geq 1$ and $\underline{t} \in T^{\mathbb{N}}$. For a sequence $\underline{i} = (i_0, \ldots, i_{n-1}) \in \{1, \ldots, p+1\}^n$ we write

$$[\underline{i}] = \Delta^{cu} \cap B_{i_0} \cap (f_{\underline{t}}^1)^{-1}(B_{i_1}) \cap \cdots \cap (f_{\underline{t}}^{n-1})^{-1}(B_{i_{n-1}})$$

and define $g(\underline{i}) = \#\{0 \leq j < n : i_j \leq p\}$. We start by observing that for $\zeta > 0$ the number of sequences $\underline{i}$ such that $g(\underline{i}) < \zeta n$ is bounded by

$$\sum_{k<\zeta n} \binom{n}{k} p^k \leq \sum_{k \leq \zeta n} \binom{n}{k} p^{\zeta n}.$$

Using Stirling's formula (cf. [BoV, Section 6.3]) the expression on the right hand side is bounded by $(e^\gamma p^\zeta)^n$, where $\gamma > 0$ depends only on ζ and $\gamma(\zeta) \to 0$ when $\zeta \to 0$.

Assumption (B) ensures that $m([\underline{i}]) \le \sigma_1^{-(1-\zeta)n}$ (recall that $m(\Delta^{cu}) = 1$). Hence the measure of the union $I_n(\underline{t})$ of all the sets $[\underline{i}]$ with $g(\underline{i}) < \zeta n$ is bounded by

$$\sigma_1^{-(1-\zeta)n}(e^{\gamma}p^{\zeta})^n.$$

Since $\sigma_1 > 1$ we may choose ζ so small that $e^{\gamma}p^{\zeta} < \sigma_1^{(1-\zeta)}$. Then $m(I_n(\underline{t})) \le \tau^n$ with $\tau = e^{\gamma+\zeta-1}p^{\zeta} < 1$ for big enough $n \ge N$. Note that τ and N do not depend on $\underline{t}$.

Remark 7.2. If $x \in M \setminus I_n(\underline{t})$, then

$$\#\{0 \le j < n : f_{\underline{t}}^j(x) \in B_{p+1}\} \le (1-\zeta)\cdot n$$

by definition of $I_n(\underline{t})$.

Setting

$$I_n = \bigcup_{\underline{t}\in T^{\mathbb{N}}}\big(\{\underline{t}\}\times I_n(\underline{t})\big) \quad \text{we have} \quad (\theta_\varepsilon^{\mathbb{N}}\times m)(I_n) \le \tau^n \tag{27}$$

for big $n \ge N$, by Fubini's Theorem. Since $\sum_n(\theta_\varepsilon^{\mathbb{N}}\times m)(I_n) < \infty$ then Borel-Cantelli's Lemma implies

$$(\theta_\varepsilon^{\mathbb{N}}\times m)\left(\bigcap_{n\ge1}\bigcup_{k\ge n} I_k\right) = 0$$

and this means that $\theta_\varepsilon^{\mathbb{N}}\times m$ almost every $(\underline{t},x) \in T^{\mathbb{N}}\times\Delta^{cu}$ satisfies (26).

For the unperturbed case just observe that the estimate in (27) is uniform in $\underline{t}$, so it also holds for a constant $\underline{t} = (f,f,f,\dots)$ and the rest of the arguments are unchanged. □

Lemma 7.3. *For $0 < \lambda < 1$ there are $\eta > 0$ and $c_0 > 0$ such that, if f_t also satisfies conditions (C) and (D) for all $t \in T$, then we have*

1. $m\big(\{x \in M : \sum_{j=0}^{n-1}\log\|\big(Df|T_{f_{\underline{t}}^j x}\Delta_j^{cu}(\underline{t})\big)^{-1}\| \le -cn\}\big) \ge 1-\tau^n$ *for all $\underline{t}\in T^{\mathbb{N}}$ and for every large n;*
2. $\limsup_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\log\|\big(Df|T_{f_{\underline{t}}^j x}\Delta_j^{cu}(\underline{t})\big)^{-1}\| \le -c$ *for $\theta_\varepsilon^{\mathbb{N}}\times m$ almost all $(\underline{t},x) \in T^{\mathbb{N}}\times\Delta^{cu}$.*

In particular, for a constant $\underline{t} = (f,f,f,\dots)$ item (2) above still holds for this $\underline{t}$ and m-almost every $x \in \Delta^{cu}$.

Proof. Let $\{B_1,\dots,B_p,B_{p+1},\dots,B_{p+1}\}$ be a measurable cover of M as before and $\zeta > 0$ be the constant provided by Lemma 7.1. We fix $\eta > 0$ sufficiently small so that $\lambda^{\zeta}(1+\eta) \le e^{-c}$ holds for some $c > 0$ and take $x \in M\setminus I_n(\underline{t})$ for some $n \ge 1$ and $\underline{t}\in T^{\mathbb{N}}$. Conditions (C) and (D) now imply

$$\prod_{j=0}^{n-1}\|\big(Df|T_{f_{\underline{t}}^j x}\Delta_j^{cu}(\underline{t})\big)^{-1}\| \le \lambda^{\zeta n}(1+\eta)^{(1-\zeta)n} \le e^{-cn}. \tag{28}$$

by Remark 7.2. Hence the set in item 1 is contained in $M\setminus I_n(\underline{t})$, proving the statement of this item by the second part of the statement of Lemma 7.1.

This also means that the second item of the statement holds for $\theta^{\mathbb{N}}\times m$ almost every $(\underline{t},x)\in T^{\mathbb{N}}\times\Delta^{cu}$ by the statement of Lemma 7.1.

For the unperturbed case the arguments are analogous. □

Since Δ^{cu} was an arbitrary disk tangent to the center-unstable cone, we conclude from Lemma 7.3 that f is non-uniformly expanding along the center-unstable direction for random orbits. Moreover the same arguments apply verbatim to $\|Df^{-1} \mid T_{f^{-j}x}f^{-j}(\Delta^{cs})\|$ and any disk Δ^{cs} tangent to the center-stable cone, so that Lemmas 7.1 and 7.3 holds for this cone also.

Since the statements of both Lemmas 7.1 and 7.3 are true in the unperturbed case, this shows that every map $f \in T$ is non-uniformly expanding along the center-unstable direction and mostly contracting along the center-stable direction.

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