On the semistability of instanton sheaves over certain projective varieties

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February 10, 2005

Abstract

We show that instanton bundles of rank $r \leq 2n-1$, defined as the cohomology of certain monads, on an n-dimensional projective variety with cyclic Picard group are semistable in the sense of Mumford-Takemoto. Furthermore, we show that rank $r \leq n$ linear bundles with nonzero first Chern class over such varieties are stable. We also show that these bounds are sharp.

2000 MSC: 14J60; 14F05
Keywords: Monads, semistable sheaves

1 Introduction

Let X be a nonsingular projective variety over an algebraically closed field $\mathbb{F}$ of characteristic zero of dimension n, and let $\mathcal{L}$ denote a very ample invertible sheaf; let $\mathcal{L}^{-1}$ denote its inverse.

Given (finite-dimensional) $\mathbb{F}$-vector spaces V, W and U, a *linear monad* on

X is the short sequence of sheaves

$$M_\bullet \ : \ 0 \to V \otimes \mathcal{L}^{-1} \xrightarrow{\alpha} W \otimes \mathcal{O}_X \xrightarrow{\beta} U \otimes \mathcal{L} \to 0 \tag{1}$$

which is exact on the first and last terms, i.e. $\alpha \in \mathrm{Hom}(V, W) \otimes \mathcal{L}$ is injective while $\beta \in \mathrm{Hom}(W, U) \otimes \mathcal{L}$ is surjective. The coherent sheaf $E = \ker \beta / \mathrm{Im}\alpha$ is called the cohomology of the monad $M_\bullet$. The set:

$$S = \{x \in X \mid \alpha(x) \in \mathrm{Hom}(V, W) \text{ is not injective}\}$$

is a subvariety called the *degeneration locus* of the monad $M_\bullet$.

A torsion-free sheaf E on X is said to be a *linear sheaf* on X if it can be represented as the cohomology of a linear monad and it is said to be an *instanton sheaf* on X if in addition it has $c_1(E) = 0$.

Linear monads and instanton sheaves have been extensively studied for the case $X = \mathbb{P}^n$ during the past 30 years, see for instance [5, 6] and the references therein. Buchdahl has studied monads over arbitrary blow-ups of $\mathbb{P}^2$ [1]. In a recent preprint, Costa and Miró-Roig have initiated the study of linear monads and locally-free instanton sheaves over smooth quadric hypersurfaces Q_n within $\mathbb{P}^{n+1}$ ($n \geq 3$) [2]. They have asked whether every such locally free sheaf of rank $n - 1$ is stable (in the sense of Mumford-Takemoto) [2, Question 5.1].

The main goal of this paper is to give a partial answer to their question in a more general context, showing that locally-free instanton sheaves of rank $r \leq 2n - 1$ on an n-dimensional smooth projective variety with cyclic Picard group are semistable, while locally-free linear sheaves of rank $r \leq n$ and $c_1 \neq 0$ on such varieties are stable. Furthermore, we also show that the bounds on the rank are sharp by providing examples of rank $2n$ instanton sheaves and rank $n + 1$ linear sheaves on $\mathbb{P}^n$ which are not semistable.

We conclude the paper by studying the semistability of special sheaves on Q_n, as introduced by Costa and Miró-Roig. Theorem 17 provides a partial answer to Question 5.2 in [2], showing that every rank $r \leq 2n - 1$ locally-free special sheaf E on Q_n with $c_1 = 0$ is semistable, while every rank $r \leq n$ locally-free special sheaf on Q_n with $c_1 \neq 0$ is stable.

Acknowledgment. The first author is partially supported by the FAEPEX grant number 1433/04. The second author is partially supported by the grant MTM2004-00666.

2 Instanton sheaves on cyclic varieties

Note that if E is the cohomology of a linear monad as in (1), then:

$$\mathrm{rk}(E) = w - v - u \quad \text{and} \quad c_1(E) = (v-u)\cdot \ell$$

where $w = \dim W$, $v = \dim V$, $u = \dim U$ and $\ell = c_1(\mathcal{L})$. Thus any instanton sheaf E can be represented as the cohomology of a monad of the following type:

$$0 \to (\mathcal{L}^{-1})^{\oplus c} \xrightarrow{\alpha} \mathcal{O}_X^{\oplus r+2c} \xrightarrow{\beta} \mathcal{L}^{\oplus c} \to 0 \tag{2}$$

where r is the rank and c is called the charge of E. It also follows that the total Chern class of E is given by, in the case $u = v$:

$$c(E) = \frac{1}{(1-\ell^2)^c} = (1 + \ell^2 + \ell^4 + \cdots)^c \quad .$$

Remark 1. For $X = \mathbb{P}^n$, instanton sheaves exist for $r \geq n-1$ and all c [5]. For X being a smooth quadric hypersurface of dimension $n \geq 3$, instanton sheaves exist for $r \geq n-1$ and all c [2]. It would be very interesting to obtain existence results for a wider class of varieties.

A smooth projective variety X is said to be *cyclic* if $\mathrm{Pic}(X) = \mathbb{Z}$. Examples of cyclic varieties are projective spaces, smooth quadric hypersurfaces Q_n within $\mathbb{P}^{n+1}$ $(n \geq 3)$, Grassmannians and general smooth projective surfaces $X \subset \mathbb{P}^3$ of degree $d \geq 4$.

From now on, we denote $E(k) = E \otimes \mathcal{L}^{\otimes k}$ if k is positive and $E(k) = E \otimes (\mathcal{L}^{-1})^{\otimes k}$ if k is negative. Of course, in this notation, $\mathcal{L} = \mathcal{O}_X(1)$ and $\mathcal{L}^{-1} = \mathcal{O}_X(-1)$.

Proposition 2. *Let X be a smooth projective cyclic variety of dimension n and let E be a linear sheaf on X. Assume that $\omega_X \cong \mathcal{O}_X(\lambda)$ for some integer $\lambda > 0$. Then, we have:*

(1) $H^0(E(k)) = H^0(E^*(k)) = 0$ *for all* $k \leq -1$*,*

(2) $H^1(E(k)) = 0$ *for all* $k \leq -2$*,*

(3) $H^i(E(k)) = 0$ *for all* k *and* $2 \leq i \leq n-2$*,*

(4) $H^{n-1}(E(k)) = 0$ *for all* $k \geq \lambda + 2$*,*

(5) $H^n(E(k)) = 0$ *for all* $k \geq \lambda + 1$*,*

and if E *is locally-free:*

(6) $H^n(E^*(k)) = 0$ *for all* $k \geq \lambda + 1$*.*

Proof. The crucial observation is that by Kodaira Vanishing Theorem we have

$$H^i(\mathcal{O}_X(k)) = 0 \text{ for all } i < n \text{ and } k \leq -1; \text{ and}$$

$$H^i(\mathcal{O}_X(k) \otimes \omega_X) = 0 \text{ for all } i > 0 \text{ and } k \geq 1.$$

By Serre's duality $H^i(X, \mathcal{O}_X(k) \otimes \omega_X) \cong H^{n-i}(X, \mathcal{O}_X(-k))$. So, we conclude that

$$H^0(\mathcal{O}_X(k)) = 0 \text{ for all } k \leq -1,$$

$$H^i(\mathcal{O}_X(k)) = 0 \text{ for all } k \text{ and } 1 \leq i \leq n-1, \text{ and}$$

$$H^n(\mathcal{O}_X(k)) = 0 \text{ for all } k \geq \lambda + 1.$$

Assuming that E is the cohomology of a linear monad

$$0 \to \mathcal{O}_X(-1)^{\oplus a} \xrightarrow{\alpha} \mathcal{O}_X^{\oplus b} \xrightarrow{\beta} \mathcal{O}_X(1)^{\oplus c} \to 0 \quad , \tag{3}$$

let $K = \ker \beta$; it is a locally-free sheaf of rank $b - c$ fitting into the sequences:

$$0 \to K(k) \to \mathcal{O}_X(k)^{\oplus b} \xrightarrow{\beta} \mathcal{O}_X(k+1)^{\oplus c} \to 0 \text{ and} \tag{4}$$

$$0 \to \mathcal{O}_X(k-1)^{\oplus a} \xrightarrow{\alpha} K(k) \to E(k) \to 0 \quad . \tag{5}$$

Passing to cohomology, the exact sequence (4) yields:

$$H^i(X, K(t)) = 0 \text{ for all } t \text{ and } 2 \leq i \leq n-1 \ ,$$

$$H^n(X, K(k)) = 0 \text{ for } t \geq \lambda + 1 \ ,$$

$$H^0(X, K(k)) = 0 \text{ for } t \leq -1 \ ,$$

$$H^1(X,K(t)) = 0 \text{ for } t \leq -2 \ .$$

Passing to cohomology, the exact sequence (5) yields:

$$H^0(E(k)) = 0 \text{ for all } k \leq -1 \ ,$$

$$H^1(E(k)) = 0 \text{ for all } k \leq -2 \ ,$$

$$H^i(E(k)) = 0 \text{ for all } k \text{ and } 2 \leq i \leq n-2 \ ,$$

$$H^{n-1}(E(k)) = 0 \text{ for all } k \geq \lambda + 2 \ ,$$

$$H^n(E(k)) = 0 \text{ for all } k \geq \lambda + 1 \ .$$

Dualizing sequences (4) and (5), we obtain:

$$0 \to \mathcal{O}_X(-k-1)^{\oplus c} \xrightarrow{\beta^*} \mathcal{O}_X(-k)^{\oplus b} \to K^*(-k) \to 0 \quad \text{and} \tag{6}$$

$$0 \to E^*(-k) \to K^*(-k) \xrightarrow{\alpha^*} \mathcal{O}_X(-k+1)^{\oplus a} \to \mathcal{E}xt^1(E(k),\mathcal{O}_X) \to 0 \quad . \tag{7}$$

Again, passing to cohomology, (7) forces $H^0(E^*(k)) \subseteq H^0(K^*(k))$ for all k, while (6) implies $H^0(K^*(k)) = 0$ for $k \leq -1$.

Finally, if E is locally-free, we have $H^n(E^*(k)) = 0$ for all $k \geq \lambda+1$, by Serre's duality. □

Remark 3. It follows from (7) that $\mathcal{E}xt^1(E,\mathcal{O}_X) = \text{coker}\alpha^*$, i.e. the degeneration locus of the monad (2) coincides with the support of $\mathcal{E}xt^1(E,\mathcal{O}_X)$.

Proposition 4. *Let E be a linear sheaf on a smooth projective variety X (not necessarily cyclic).*

1. *E is locally-free if and only if its degeneration locus is empty;*
2. *E is reflexive if and only if its degeneration locus is a subvariety of codimension at least 3;*
3. *E is torsion-free if and only if its degeneration locus is a subvariety of codimension at least 2.*

Proof. Let S be the degeneration locus of the linear monad associated to the linear sheaf E. From previous remark, we know that $\mathcal{E}xt^p(E, \mathcal{O}_X) = 0$ for $p \geq 2$ and

$$S = \text{supp } \mathcal{E}xt^1(E, \mathcal{O}_X) = \{x \in X \mid \alpha(x) \text{ is not injective }\}.$$

The first statement is clear; so it is now enough to argue that E is torsion-free if and only if S has codimension at least 2 and that E is reflexive if and only if S has codimension at least 3.

Recall that the m^{th}-singularity set of a coherent sheaf $\mathcal{F}$ on X is given by:

$$S_m(\mathcal{F}) = \{x \in X \mid dh(\mathcal{F}_x) \geq n - m\}$$

where $dh(\mathcal{F}_x)$ stands for the homological dimension of $\mathcal{F}_x$ as an $\mathcal{O}_x$-module:

$$dh(\mathcal{F}_x) = d \quad \Longleftrightarrow \quad \begin{cases} \text{Ext}^d_{\mathcal{O}_x}(\mathcal{F}_x, \mathcal{O}_x) \neq 0 \\ \text{Ext}^p_{\mathcal{O}_x}(\mathcal{F}_x, \mathcal{O}_x) = 0 \ \ \forall p > d. \end{cases}$$

In the case at hand, we have that $dh(E_x) = 1$ if $x \in S$, and $dh(E_x) = 0$ if $x \notin S$. Therefore $S_0(E) = \cdots = S_{n-2}(E) = \emptyset$, while $S_{n-1}(E) = S$. It follows that [7, Proposition 1.20]:

- if codim $S \geq 2$, then $\dim S_m(E) \leq m - 1$ for all $m < n$, hence E is a locally 1^{st}-syzygy sheaf;
- if codim $S \geq 3$, then $\dim S_m(E) \leq m - 2$ for all $m < n$, hence E is a locally 2^{nd}-syzygy sheaf.

The desired statements follow from the observation that E is torsion-free if and only if it is a locally 1^{st}-syzygy sheaf, while E is reflexive if and only if it is a locally 2^{nd}-syzygy sheaf [6, p. 148-149]. □

Remark 5. Note that if E is a locally-free linear sheaf on X, which is represented as the cohomology of the linear monad

$$M_\bullet \ : \ 0 \to \mathcal{O}_X(-1)^{\oplus a} \xrightarrow{\alpha} \mathcal{O}_X^{\oplus b} \xrightarrow{\beta} \mathcal{O}_X(1)^{\oplus c} \to 0 \ \ ,$$

its dual E^* is also a linear sheaf, being represented as the cohomology of the dual linear monad

$$M_\bullet^* \ : \quad 0 \to \mathcal{O}_X(-1)^{\oplus c} \xrightarrow{\beta^*} \mathcal{O}_X^{\oplus b} \xrightarrow{\alpha^*} \mathcal{O}_X(1)^{\oplus a} \to 0 \ \ .$$

In particular, if E is a locally-free instanton sheaf on X then its dual E^* is also an instanton.

3 Semistability of instanton sheaves

Recall that a torsion-free sheaf E on a cyclic variety X is semistable if for every coherent sheaf $0 \neq F \hookrightarrow E$ we have

$$\mu(F) := \frac{c_1(F)\ell^{n-1}}{rk(F)} \leq \frac{c_1(E)\ell^{n-1}}{rk(E)} := \mu(E).$$

Furthermore, if for every coherent sheaf $0 \neq F \hookrightarrow E$ with $0 < rk(F) < rk(E)$ we have

$$\mu(F) := \frac{c_1(F)\ell^{n-1}}{rk(F)} < \frac{c_1(E)\ell^{n-1}}{rk(E)} := \mu(E)$$

then E is said to be stable. A sheaf E is said to be properly semistable if it is semistable but not stable. It is also important to recall that E is (semi)stable if and only if E^* is (semi)stable if and only if $E \otimes \mathcal{L}^{\otimes k}$ is (semi)stable.

The goal of this section is to study the (semi)stability of instanton bundles.

Proposition 6. *Every rank 2 torsion-free instanton sheaf on a cyclic variety is semistable.*

Proof. Let us first consider a rank 2 reflexive sheaf F on X such that $H^0(F(-1)) = 0$; we argue that F is semistable. Indeed, if F is not semistable, then any destabilizing sheaf $L \hookrightarrow F$ with torsion-free quotient F/L must be reflexive (see [6, p. 158]). But every rank 1 reflexive sheaf is locally-free, thus $L = \mathcal{O}_X(d)$ with $d = c_1(L) > 0$ since $\mathrm{Pic}(X) = \mathbb{Z}$. It follows that $H^0(F(-d)) \neq 0$, hence $H^0(F(-1)) \neq 0$ as well.

Now if E is a rank 2 torsion-free sheaf with $H^0(E^*(-1)) = 0$, then $F = E^*$ is a rank 2 reflexive sheaf with $H^0(F(-1)) = 0$. But we've seen that such F is semistable, hence E is also semistable. Together with the first statement in Proposition 2, the desired result follows. □

For instanton sheaves of higher rank, we have our first main result:

Theorem 7. *Let E be a rank r instanton sheaf on a cyclic variety X of dimension n.*

- *If E is reflexive and $r \leq n$, then E is semistable;*
- *if E is locally-free and $r \leq 2n-1$, then E is semistable.*

Since smooth quadric hypersurfaces are cyclic, the above statement provides in particular a partial answer to the questions raised in [2, Questions 5.1 and 5.2].

The proof of Theorem 7 is based on a very useful criterion, due to Hoppe [4], to decide whether a reflexive sheaf on cyclic variety is (semi)stable.

Proposition 8. *Let E be a rank r reflexive sheaf on a cyclic variety X. If $H^0((\wedge^q E)_{\mathrm{norm}}) = 0$ for $1 \leq q \leq r-1$, then E is stable. If $H^0((\wedge^q E)_{\mathrm{norm}}(-1)) = 0$ for $1 \leq q \leq r-1$, then E is semistable.*

Proof. For a contradiction, assume that E is not stable, and let F be the destabilizing reflexive sheaf of rank q, $1 \leq q \leq r-1$, with torsion-free quotient G. So, we have an exact sequence

$$0 \longrightarrow F \longrightarrow E \longrightarrow G \longrightarrow 0$$

and, moreover, $\mu(F) = \frac{c_1(F)}{rk(F)} \geq \frac{c_1(E)}{rk(E)} = \mu(E)$. The injective map $F \to E$ induces an injective map $\mathcal{O}_X(c_1(F)) = det(F) = \wedge^q F \hookrightarrow \wedge^q E$, determining a non-zero section in $H^0(\wedge^q E(-c_1(F)))$. Since $\mu(F) = \frac{c_1(F)}{rk(F)} \geq \frac{c_1(E)}{rk(E)} = \mu(E)$, it follows that $H^0(\wedge^q E_{norm}) \neq 0$, as desired.

The second statement regarding stability is proved in exactly the same way. □

Proof of Theorem 7. We argue that every instanton sheaf on an n-dimensional cyclic variety X satisfying the conditions of the theorem fulfill Hoppe's criterion (see Proposition 8).

Indeed, let E be a rank r reflexive instanton sheaf on X. Assume that E can be represented as the cohomology of the linear monad as in (2).

Considering the long exact sequence of exterior powers associated to the sheaf sequence (4), twisted by $\mathcal{L}^{-1}$, we have:

$$0 \to \wedge^q K(-1) \to \wedge^q(\mathcal{O}_X^{\oplus r+2c})(-1) \to \cdots \quad .$$

Thus $H^0(\wedge^q K(-1)) = 0$ for $1 \leq q \leq r+c$.

Now consider the long exact sequence of symmetric powers associated to the sheaf sequence (5), twisted by $\mathcal{L}^{-1}$:

$$0 \to \mathcal{O}_X(-q-1)^{\binom{c+q-1}{q}} \to K \otimes \mathcal{O}_X(-q)^{\binom{c+q-2}{q-1}} \to \cdots$$

$$\to \wedge^{q-1} K \otimes \mathcal{O}_X(-2)^c \to \wedge^q K(-1) \to \wedge^q E(-1) \to 0 \quad .$$

Cutting into short exact sequences and passing to cohomology, we obtain

$$H^0(\wedge^p E(-1)) = 0 \ \text{ for } \ 1 \leq p \leq n-1 \quad , \tag{8}$$

and this proves the first statement.

Now if E is locally-free, then the dual E^* is also an instanton sheaf on X, so

$$H^0(\wedge^q(E^*)(-1)) = 0 \ \text{ for } \ 1 \leq q \leq n-1 \quad . \tag{9}$$

But $\wedge^p(E) \simeq \wedge^{r-p}(E^*)$, since $\det(E) = \mathcal{O}_X$; it follows that:

$$H^0(\wedge^p E(-1)) = H^0(\wedge^{r-p}(E^*)(-1)) = 0 \quad \text{for} \quad 1 \leq r-p \leq n-1$$
$$\implies \quad r-n+1 \leq p \leq r-1 \tag{10}$$

Together, (9) and (10) imply that if E is a rank $r \leq 2n-1$ locally-free instanton sheaf, then:

$$H^0(\wedge^p E(-1)) = 0 \ \text{ for } \ 1 \leq p \leq 2n-2$$

hence E is semistable by Hoppe's criterion. □

On the other hand, we have:

Proposition 9. *Let $H = h^0(\mathcal{L})$. For $r > (H-2)c$, there are no stable rank r instanton sheaves of charge c on X.*

In particular, for $X = \mathbb{P}^n$ and $\mathcal{L} = \mathcal{O}_{\mathbb{P}^n}(1)$, it follows that every locally-free instanton sheaf on $\mathbb{P}^n$ of charge 1 and rank r with $n \leq r \leq 2n-1$ must be properly semistable; for $X = Q_n$ and $\mathcal{L} = \mathcal{O}_{Q_n}(1)$, every locally-free instanton sheaf on Q_n of charge 1 and rank r with $n+1 \leq r \leq 2n-1$ must be properly semistable.

Proof. For the second part, note that if E is a stable torsion-free sheaf with $c_1(E) = 0$, then $H^0(E) = 0$. Indeed, if $H^0(E) \neq 0$, then there is a map $\mathcal{O}_X \to E$, which contradicts stability.

It follows from the sequences (4) and (5) for $k = 0$ that:

$$H^0(E) \simeq H^0(K) \simeq \ker\{ \; H^0\beta \; : \; H^0(\mathcal{O}_X^{\oplus r+2c}) \to H^0(\mathcal{L}^{\oplus c}) \; \} \quad .$$

If $r > (H-2)c$, then the map $H^0\beta$ cannot be injective, $H^0(E) \neq 0$ and E cannot be stable. □

Now dropping the $c_1(E) = 0$ condition, we obtain:

Theorem 10. *Let E be a rank $r \leq n$ linear sheaf on a cyclic variety X of dimension n.*

- *If E is reflexive and $c_1(E) > 0$, then E is stable;*
- *if E is locally-free and $c_1(E) \neq 0$, then E is stable.*

Proof. Since E is a linear sheaf, it is represented as the cohomology of a linear monad

$$0 \to \mathcal{O}_X(-1)^{\oplus a} \xrightarrow{\alpha} \mathcal{O}_X^{\oplus b} \xrightarrow{\beta} \mathcal{O}_X(1)^{\oplus c} \to 0 \quad ,$$

so that $c_1(E) = (c-a)\ell$.

Assuming $c - a > 0$, we have $\mu(\wedge^q E) = q(c-a)/r > 0$, hence $(\wedge^q E)_{\rm norm} = (\wedge^q E)(t)$ for some $t \leq -1$.

On the other hand, arguing as in the proof of Theorem 7 we get

$$H^0((\wedge^q E)(-1)) = 0 \text{ for all } q \leq n-1 \quad . \tag{11}$$

Therefore, if E is a rank $r \leq n$ reflexive sheaf represented as the cohomology of a linear monad and $c_1(E) > 0$, then:

$$H^0((\wedge^p E)_{norm}) = 0 \ \text{ for } \ 1 \leq p \leq r-1.$$

Hence E is stable by Hoppe's criterion.

For the second statement, note that if E is a locally-free linear sheaf with $c_1(E) < 0$, then E^* is a locally-free linear sheaf with $c_1(E^*) > 0$. By the argument above, E^* is stable; hence E is stable whenever $c_1(E) \neq 0$, as desired. □

We will end this section with an example which illustrates that the upper bound in the rank given in Theorem 7 is sharp, in the sense that, for each n, there are rank $2n$ locally-free instanton sheaves on certain n-dimensional cyclic varieties which are not semistable. To prove it we first need to provide the following useful cohomological characterization of linear sheaves on projective spaces.

Proposition 11. *Let F be a torsion-free sheaf on $\mathbb{P}^n$. F is a linear sheaf if and only if the following cohomological conditions hold:*

- *for $n \geq 2$, $H^0(F(-1)) = 0$ and $H^n(F(-n)) = 0$;*
- *for $n \geq 3$, $H^1(F(k)) = 0$ for $k \leq -2$ and $H^{n-1}(F(k)) = 0$ for $k \geq -n+1$;*
- *for $n \geq 4$, $H^p(F(k)) = 0$ for $2 \leq p \leq n-2$ and all k.*

Proof. The fact that linear sheaves satisfy the cohomological conditions above is a consequence of Proposition 2.

For the converse statement, first note that $H^0(F(-1)) = 0$ implies that $H^0(F(k)) = 0$ for $k \leq -1$, while $H^n(F(-n)) = 0$ implies that $H^n(F(k)) = 0$ for $k \geq -n$. Moreover, we claim that ($q = 0, \ldots, n$ and $p = 0, -1, \ldots, -n$):

$$H^q(F(-1) \otimes \Omega^{-p}_{\mathbb{P}^n}(-p)) = 0 \ \text{ for } q \neq 1 \ \text{ and for } q = 1, \ p \leq -3 \ . \qquad (12)$$

Now the key ingredient is the *Beilinson spectral sequence* [6]: for any coherent sheaf F on $\mathbb{P}^n$, there exists a spectral sequence $\{E_r^{p,q}\}$ whose E_1-term is given by ($q = 0, \ldots, n$ and $p = 0, -1, \ldots, -n$):

$$E_1^{p,q} = H^q(F \otimes \Omega_{\mathbb{P}^n}^{-p}(-p)) \otimes \mathcal{O}_{\mathbb{P}^n}(p)$$

which converges to

$$E^i = \begin{cases} F \ , \ \text{if } p+q=0 \\ 0 \text{ otherwise} \end{cases} .$$

Applying the Beilinson spectral sequence to $F(-1)$, it then follows that it degenerates at the E_2-term, so that the monad

$$\begin{aligned} 0 \quad &\rightarrow \quad H^1(F(-1) \otimes \Omega^2_{\mathbb{P}^n}(2)) \otimes \mathcal{O}_{\mathbb{P}^n}(-2) \rightarrow \\ &\rightarrow \quad H^1(F(-1) \otimes \Omega^1_{\mathbb{P}^n}(1)) \otimes \mathcal{O}_{\mathbb{P}^n}(-1) \rightarrow H^1(F(-1)) \otimes \mathcal{O}_{\mathbb{P}^n} \rightarrow 0 \end{aligned} \tag{13}$$

has $F(-1)$ as its cohomology. Tensoring (13) by $\mathcal{O}_{\mathbb{P}^n}(1)$, we conclude that F is the cohomology of a linear monad, as desired.

The claim (12) follows from repeated use of the exact sequence

$$\begin{aligned} H^q(F(k))^{\oplus m} \rightarrow H^q(F(k+1) \otimes \Omega_{\mathbb{P}^n}^{-p-1}(-p-1)) \rightarrow \\ \rightarrow H^{q+1}(F(k) \otimes \Omega_{\mathbb{P}^n}^{-p}(-p)) \rightarrow H^{q+1}(F(k))^{\oplus m} \end{aligned} \tag{14}$$

associated with Euler sequence for p-forms on $\mathbb{P}^n$ twisted by $F(k)$:

$$0 \rightarrow F(k) \otimes \Omega_{\mathbb{P}^n}^{-p}(-p) \rightarrow F(k)^{\oplus m} \rightarrow F(k) \otimes \Omega_{\mathbb{P}^n}^{-p-1}(-p) \rightarrow 0 \ , \tag{15}$$

where $q = 0, \ldots, n$, $p = 0, -1, \ldots, -n$ and $m = \begin{pmatrix} n+1 \\ -p \end{pmatrix}$. □

We are finally ready to construct rank $2n$ locally-free instanton sheaves on $\mathbb{P}^n$ which are not semistable.

Example 12. Let $X = \mathbb{P}^n$, $n \geq 4$. By Fløystad's theorem [3], there is a linear monad:

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus 2} \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^n}^{\oplus n+3} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^n}(1) \rightarrow 0 \tag{16}$$

whose cohomology F is a locally-free sheaf of rank n on $\mathbb{P}^n$ and $c_1(F) = 1$.

Dualizing we get a linear monad:

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-1) \xrightarrow{\beta^*} \mathcal{O}_{\mathbb{P}^n}^{\oplus n+3} \xrightarrow{\alpha^*} \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus 2} \to 0$$

whose cohomology is F^*, hence it is a locally-free linear sheaf of rank n on $\mathbb{P}^n$ and $c_1(F^*) = -1$.

Take an extension E of F^* by F:

$$0 \to F \to E \to F^* \to 0.$$

Such extensions are classified by $\mathrm{Ext}^1(F^*, F) = H^1(F \otimes F)$. We claim that there are non-trivial extensions of F^* by F. Indeed, we consider the exact sequences

$$0 \to K = \ker(\beta) \to \mathcal{O}_{\mathbb{P}^n}^{\oplus n+3} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^n}(1) \to 0 \ , \tag{17}$$

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus 2} \to K \to F \to 0 \tag{18}$$

associated to the linear monad (16). We apply the exact covariant functor $\cdot \otimes F$ to the exact sequences (17) and (18) and we obtain the exact sequences

$$0 \to K \otimes F \to F^{\oplus n+3} \to F(1) \to 0 \ ,$$

$$0 \to F(-1)^{\oplus 2} \to K \otimes F \to F \otimes F \to 0 \ \ .$$

Using Proposition 2, we obtain $H^i(K \otimes F) = H^i(F \otimes F) = 0$ for all $i \geq 3$. Hence, $\chi(F \otimes F) = h^0((F \otimes F)) - h^1((F \otimes F)) + h^2((F \otimes F))$. On the other hand,

$$\chi(F \otimes F) = \chi(K \otimes F) - 2\chi(F(-1)) =$$
$$(n+3)\chi(F) - \chi(F(1)) - 2\chi(F(-1)) = 8 - \frac{n^2}{2} - \frac{n}{2} < 0 \ \ , \ \ \text{if} \ \ n \geq 4 \ \ .$$

Thus if $n \geq 4$, we must have $h^1((F \otimes F)) > 0$, hence there are non-trivial extensions of F^* by F.

Using the cohomological criterion given in Proposition 11, it is easy to see that the extension of linear sheaves is also a linear sheaf. Moreover, $c_1(E) = 0$. So, E is a rank $2n$ locally-free instanton sheaf of charge 3 which is not semistable.

For $X = \mathbb{P}^n$, $2 \leq n \leq 3$, arguing as above, we can construct a rank $2n$ locally-free instanton which is not semistable as a non-trivial extension E of

F^* by F, where F is a linear sheaf represented as the cohomology of the linear monad

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus 4} \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^n}^{\oplus n+7} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus 3} \to 0.$$

In the next example, using the same idea as above, we show that there are rank $n+2$ reflexive instanton sheaves which are not semistable; our argument fails because there are no rank $r \leq n-1$ linear sheaves E on $\mathbb{P}^n$ with $c_1(E) < 0$. We do not know whether there are unstable rank $n+1$ reflexive instanton sheaves, i.e. whether the bound $r \leq n$ in the first part of Theorem 7 is indeed sharp.

Example 13. Let $X = \mathbb{P}^n$, $n \geq 3$. By Fløystad's theorem [3], there is a linear monad:

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus n-2} \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^n}^{\oplus n+1} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^n}(1) \to 0$$

whose cohomology F is a rank 2 reflexive linear sheaf on $\mathbb{P}^n$ and $c_1(F) = n-3$.

Next, consider the rank n locally free linear sheaf G associated to the linear monad:

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus a} \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^n}^{\oplus 2n+2a-3} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus n+a-3} \to 0 \quad (a \geq 1).$$

Note that $c_1(G) = 3-n$.

As in the previous example, an extension of G by F is a rank $n+2$ reflexive instanton sheaf which is not semistable. The choice of a suitable value of the parameter a guarantees the existence of non-trivial extensions.

To conclude this section, we show that the upper bounds in the rank given in both parts of Theorem 10 are also sharp:

Example 14. Let $X = \mathbb{P}^n$, $n \geq 2$. By Fløystad's theorem [3], there is a linear monad:

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus 4} \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^n}^{\oplus n+9} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus 5} \to 0 \tag{19}$$

whose cohomology G is a locally-free sheaf of rank n on $\mathbb{P}^n$ and $c_1(G) = -1$.

Now G^* is the cohomology of the dual monad

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus 5} \xrightarrow{\beta^*} \mathcal{O}_{\mathbb{P}^n}^{\oplus n+9} \xrightarrow{\alpha^*} \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus 4} \to 0 \quad .$$

It follows that:

$$H^1(G^*) = H^1(\ker\alpha^*) = \operatorname{coker}\{H^0\alpha^* : H^0(\mathcal{O}_{\mathbb{P}^n}^{\oplus n+9}) \to H^0(\mathcal{O}_{\mathbb{P}^n}(1)^{\oplus 4})\} \ .$$

Since $n \geq 2$ forces $4n+4 > n+9$, the generic map α will have $\operatorname{coker}(H^0\alpha^*) \neq 0$. In other words, there exists a rank n locally-free linear sheaf G on $\mathbb{P}^n$ with $c_1(G) = -1$ and $H^1(G^*) \neq 0$.

Take an extension E of such a linear sheaf G by $\mathcal{O}_{\mathbb{P}^n}$:

$$0 \to \mathcal{O}_{\mathbb{P}^n} \to E \to G \to 0. \tag{20}$$

Using the cohomological criterion given in Proposition 11, it is easy to see that E is a rank $n+1$ locally-free linear sheaf with $c_1(E) = c_1(G) = -1$. It is not stable, since $H^0(E) \neq 0$.

Note also that there are nontrivial extensions of G by $\mathcal{O}_{\mathbb{P}^n}$ since $H^1(G^*) \neq 0$. Furthermore, the dual E^* is an example of a rank $n+1$ locally-free (thus reflexive) linear sheaf with $c_1(E) > 0$ which is not stable.

4 Special sheaves on smooth quadric hypersurfaces

Now we restrict ourselves to the set-up in [2], and we assume that Q_n is a smooth quadric hypersurface within $\mathbb{P}^{n+1}$, $n \geq 3$; such varieties are cyclic.

Recall that a *special sheaf* E on Q_n is defined [2, Definition 3.4] as either the cohomology of a linear monad

$$\text{(M1)} \quad 0 \to \mathcal{O}_{Q_n}(-1)^{\oplus a} \to \mathcal{O}_{Q_n}^{\oplus b} \to \mathcal{O}_{Q_n}(1)^{\oplus c} \to 0 \ ,$$

or the cohomology of a monad of the following type

$$\text{(M2.1)} \quad 0 \to \Sigma(-1)^{\oplus a} \to \mathcal{O}_{Q_n}^{\oplus b} \to \mathcal{O}_{Q_n}(1)^{\oplus c} \to 0 \ , \text{ if } n \text{ is odd,}$$

$$\text{(M2.2)} \quad 0 \to \Sigma_1(-1)^{\oplus a_1} \oplus \Sigma_2(-1)^{\oplus a_2} \to \mathcal{O}_{Q_n}^{\oplus b} \to \mathcal{O}_{Q_n}(1)^{\oplus c} \to 0 \ , \text{ if } n \text{ is even,}$$

where Σ is the Spinor bundle for n odd, and Σ_1, Σ_2 are the Spinor bundles for n even.

Clearly, instanton sheaves on Q_n are special sheaves of the first kind with zero degree.

Proposition 15. *Let E be a special sheaf on Q_n, $n \geq 3$. Then one of the following conditions holds:*

1. *E is the cohomology of a linear monad, and*
   - $H^0(E(k)) = H^0(E^*(k)) = 0$ *for all* $k \leq -1$,
   - $H^1(E(k)) = 0$ *for all* $k \leq -2$,
   - $H^i(E(k)) = 0$ *for all* k *and* $2 \leq i \leq n-2$,
   - $H^{n-1}(E(k)) = 0$ *for all* $k \geq -n+2$,
   - $H^n(E(k)) = 0$ *for all* $k \geq -n+1$,

     and if E is locally-free:
   - $H^n(E^*(k)) = 0$ *for all* $k \geq -n+1$*; or*
2. *E is the cohomology of a monad of type* (M2.1) *and* (M2.2)*, and*
   - $H^0(E(k)) = H^0(E^*(k)) = 0$ *for all* $k \leq -1$,
   - $H^1(E(k)) = 0$ *for all* $k \leq -2$,
   - $H^i(E(k)) = 0$ *for all* k *and* $2 \leq i \leq n-2$,
   - $H^{n-1}(E(k)) = 0$ *for all* $k \geq -n+1$,
   - $H^n(E(k)) = 0$ *for all* $k \geq -n+1$,

     and if E is locally-free:
   - $H^n(E^*(k)) = 0$ *for all* $k \geq -n+1$

Proof. (1) It is analogous to the proof of Proposition 2.

(2) If n is odd we consider the exact sequences

$$0 \to ker(\delta) \to \mathcal{O}_{Q_n}^{\oplus b} \xrightarrow{\delta} \mathcal{O}_{Q_n}(1)^{\oplus c} \to 0 \quad ,$$

$$0 \to \Sigma(-1)^{\oplus a} \to ker(\delta) \to E \to 0$$

and if n is even we consider the exact sequences

$$0 \to ker(\psi) \to \mathcal{O}_{Q_n}^{\oplus b} \xrightarrow{\psi} \mathcal{O}_{Q_n}(1)^{\oplus c} \to 0 \quad ,$$

$$0 \to \Sigma_1(-1)^{\oplus a_1} \oplus \Sigma_1(-2)^{\oplus a_2} \to ker(\psi) \to E \to 0$$

and we argue as in the proof of Proposition 2 taking into account that

$H^0(\Sigma(k)) = H^0(\Sigma_1(k)) = H^0(\Sigma_2(k)) = 0$ for all $k \leq -1$,

$H^i(\Sigma(k)) = H^i(\Sigma_1(k)) = H^i(\Sigma_2(k)) = 0$ for all k and $1 \leq i \leq n-1$, and

$H^n(\Sigma(k)) = H^n(\Sigma_1(k)) = H^n(\Sigma_2(k)) = 0$ for all $k \geq n$. □

Proposition 16. *Every rank 2 torsion-free special sheaf E on Q_n with $c_1(E) = 0$ is semistable.*

Proof. Since every torsion-free special sheaf E on Q_n satisfies $H^0(E(k)) = H^0(E^*(k)) = 0$, simply use the argument in the proof of Proposition 6. □

Finally, for higher rank locally-free special sheaves on Q_n, we have:

Theorem 17. *Let E be a rank r locally-free special sheaf on Q_n.*

- *If $r \leq 2n-1$ and $c_1(E) = 0$, then E is semistable;*
- *if $r \leq n$ and $c_1(E) \neq 0$, then E is stable.*

Let E be a rank $r \leq n$ reflexive special sheaf on Q_n.

- *If $c_1(E) = 0$, then E is semistable;*
- *if $c_1(E) > 0$, then E is stable.*

It is interesting to note that, by [2, Proposition 4.7], there are no rank $r \leq n-1$ linear sheaves E on Q_n with $c_1(E) < 0$ or rank $r \leq n-2$ linear sheaves E on Q_n with $c_1(E) = 0$.

Proof. For locally-free and reflexive special sheaves which are represented as cohomologies of the monad *(M1)*, the statement follows from Theorem 7 and 10 and for locally-free and reflexive special sheaves which are represented as cohomologies of the monad *(M2.1)* and *(M2.2)* an analogous argument works. □

Note that using the Fløystad type existence theorem for linear sheaves on Q_n established in [2, Proposition 4.7], one can easily produce examples of rank $2n$ locally-free instanton sheaves on Q_n as well as rank $n+1$ locally-free linear sheaves on Q_n which are not semistable, following the ideas in Examples 12 and 14.

However, we do not know whether the bounds in the rank are sharp for locally-free sheaves on Q_n which are the cohomology of monads of type *(M2.1)* and *(M2.2)*. For instance, is there an unstable rank $2n$ locally-free sheaf on Q_n which can be represented as the cohomology of a non-linear special monad?

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