NON EXISTENCE OF CONVERGENT NORMAL FORM FOR GENERAL GERMS OF UNIPOTENT DIFFEOMORPHISMS

JAVIER RIBÓN

Abstract

A germ of diffeomorphism has convergent normal form if it is formally conjugated to the exponential of a germ of vector field. We prove that there are complex analytic unipotent germs of diffeomorphisms at $(\mathbb{C}^n, 0)$ $(n > 1)$ such that they do not have a convergent normal form. The examples are contained in a family in which the absence of convergence normal form is linked to a geometrical phenomenon. The proof is based on several reductions; it relies on the properties of some linear functional operators that we obtain through the study of polynomial families of diffeomorphisms via potential theory.

1. Introduction

In this paper we prove

Main Theorem. *There exists a unipotent germ of complex analytic diffeomorphism at $(\mathbb{C}^2, 0)$ without convergent normal form.*

Normal forms are very important to study geometrical objects and in particular diffeomorphisms. We denote by $\mathrm{Diff}(\mathbb{C}^n, 0)$ the set of germs of analytic diffeomorphisms at $(\mathbb{C}^n, 0)$ whereas $\widehat{\mathrm{Diff}}(\mathbb{C}^n, 0)$ is the formal completion of $\mathrm{Diff}(\mathbb{C}^n, 0)$. A normal form for $\varphi \in \mathrm{Diff}(\mathbb{C}^n, 0)$ is a diffeomorphism formally conjugated to φ but somehow simpler. Every $\varphi \in \mathrm{Diff}(\mathbb{C}^n, 0)$ admits a unique Jordan decomposition

$$\varphi = \varphi_s \circ \varphi_u = \varphi_u \circ \varphi_s$$

where $\varphi_s \in \widehat{\mathrm{Diff}}(\mathbb{C}^n, 0)$ is semisimple and $\varphi_u \in \widehat{\mathrm{Diff}}(\mathbb{C}^n, 0)$ is unipotent. In other words φ_s is formally linearizable and $j^1\varphi_u - Id$ is nilpotent. Then φ_s has the natural normal form $j^1\varphi_s$. In spite of that φ_u is not formally linearizable unless $\varphi_u \equiv Id$, we need a different approach. Anyway, a unipotent $\varphi \in \mathrm{Diff}(\mathbb{C}^n, 0)$ is always the exponential of a

Date: 12th July 2005.

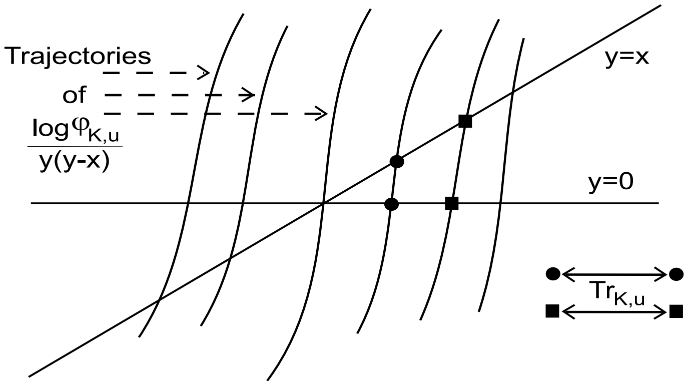
unique formal nilpotent vector field $\log\varphi$. In general $\log\varphi$ is divergent but even if so it is geometrically significant. We are dealing with a well-behaved case when $\log\varphi$ is formally conjugated to X for some germ of vector field X. We say then that φ has *convergent normal form* $\exp(X)$. The convergent normal form provides a continuous model to compare the discrete dynamics of the original diffeomorphism with.

A unipotent $\varphi \in \mathrm{Diff}\,(\mathbb{C},0)$ satisfies $j^1\varphi = Id$; it always has a convergent normal form $\exp(X)$. The study of the regions (Fatou petals) in which φ is analytically conjugated to $\exp(X)$ provides the basis to construct the Ecalle-Voronin system of complete invariants (see [2]). The same strategy can be applied in bigger dimension. For instance Voronin classifies analytically the unipotent diffeomorphisms in $\mathrm{Diff}\,(\mathbb{C}^2,0)$ whose normal form is of the form $x^k\partial/\partial y$ [2]. In spite of this the Main Theorem claims that such a nice model is not always available. In fact we prove that there exist elements of $\mathrm{Diff}\,(\mathbb{C}^2,0)$ of the form

$$\varphi_{K,u}(x,y) = (x + y(y-x)K(x,y), y + y(y-x)u(x,y))$$

with no convergent normal form. Moreover, the obstruction is of geometrical type.

The proof of the Main Theorem is based on the study of the transport mapping . Suppose $\log\varphi_{K,u}$ is a germ of vector field. Then $\log\varphi_{K,u}/[y(y-x)]$ is regular and transversal to both $y=0$ and $y=x$.



Real picture of the transport mapping

We can define a correspondence $Tr_{K,u}$ associating to each point P in $y=0$ the unique point in $y=x$ contained in the trajectory of $\log\varphi_{K,u}/[y(y-x)]$ passing through P. This correspondence is the transport mapping. Even if $\log\varphi_{K,u}$ is divergent we manage to define $Tr_{K,u}$; it is a formal invariant. Moreover we prove that if $\varphi_{K,u}$ has a convergent normal form then its transport mapping is an analytic mapping.

Fix $u \in \mathbb{C}\{x,y\} \setminus (x,y)$ such that $\ln \varphi_{0,u}$ is not a germ of vector field. The rest of the paper is devoted to prove that there exists K in $\mathbb{C}\{x,y\} \cap (x,y)$ such that $Tr_{K,u}$ diverges. We argue by contradiction. Note that $Tr_{K,u}$ can be analytic (for example $Tr_{K,u}(x,0) \equiv (x,x)$) whereas $\log \varphi_{K,u}$ is divergent. The divergence of $Tr_{K,u}$ is even stronger than the non existence of a germ of vector field collinear to $\ln \varphi_{K,u}$. It is obtained through a fine analysis of the nature of the family $(\varphi_{K,u})_K$

We consider polynomial families on $\lambda \in \mathbb{C}$ of the form

$$\varphi_{\Delta,u,\lambda}(x,y) = (x + \lambda y(y-x)\Delta(x,y), y + y(y-x)u(x,y)).$$

The absence of convergent normal form for all $\lambda \in \mathbb{C}$ allows to find by deriving with respect to λ and evaluating at $\lambda = 0$ a linear equation

$$\hat{\epsilon} - \hat{\epsilon} \circ \varphi_{0,u} = y(y-x)\Delta$$

such that $\hat{\epsilon}_\Delta(x,x) - \hat{\epsilon}_\Delta(x,0) \in \mathbb{C}\{x\}$ for every solution $\hat{\epsilon}_\Delta \in \mathbb{C}[[x,y]]$. The proof of the convergence of $\hat{\epsilon}_\Delta(x,x) - \hat{\epsilon}_\Delta(x,0)$ is based on potential theory techniques. The equation $\hat{\epsilon} - \hat{\epsilon} \circ \varphi_{0,u} = y(y-x)\Delta$ has a formal solution $\hat{\epsilon}_\Delta$ for all $\Delta \in \mathbb{C}[[x,y]]$, moreover $\hat{\epsilon}_\Delta(x,x) - \hat{\epsilon}_\Delta(x,0)$ does not depend on the choice of $\hat{\epsilon}_\Delta$. Then the operator $S_u : \mathbb{C}[[x,y]] \to \mathbb{C}[[x]]$ such that

$$S_u(\Delta) = \hat{\epsilon}_\Delta(x,x) - \hat{\epsilon}_\Delta(x,0)$$

is linear, well-defined and $S_u(\mathbb{C}\{x,y\} \cap (x,y))$ is contained in $\mathbb{C}\{x\}$. The situation is much improved. Now it is enough to study a linear operator attached to a diffeomorphism $\varphi_{0,u}$ which is dynamically simple, in particular the property $x \circ \varphi_{0,u} = x$ will be key to prove that $S_u(\mathbb{C}\{x,y\} \cap (x,y))$ contains divergent elements.

The operator S_u was defined in terms of a difference equation. The difference equation can be replaced by a differential equation easier to handle. More precisely $S_u(\Delta) = \hat{\Gamma}(x,x) - \hat{\Gamma}(x,0)$ for every solution $\hat{\Gamma} \in \mathbb{C}[[x,y]]$ of

$$\log \varphi_{0,u}(\hat{\Gamma}) = -y(y-x)\Delta.$$

Since $\log \varphi_{0,u}$ is divergent and $x \circ \varphi_{0,u} = x$ then $\log \varphi_u = \hat{u} y(y-x)\partial/\partial y$ for some divergent $\hat{u} \in \mathbb{C}[[x,y]]$. The collinearity of $\log \varphi_u$ and the germ of vector field $\partial/\partial y$ and some standard functional analysis techniques can be used to prove that $S_u(\mathbb{C}\{x,y\} \cap (x,y)) \subset \mathbb{C}\{x\}$ implies that $\hat{u} \in \mathbb{C}\{x,y\}$. Here we have our contradiction.

We do not say anything about the nature of the transport mapping, besides the fact that it is generically divergent. It would be interesting to know what divergent mappings can be obtained as transport mappings of diffeomorphisms of type $\varphi_{K,u}$.

2. Basic facts

In this section we introduce some well-known facts about diffeomorphisms and vector fields for the sake of completeness.

We say that a formal vector field $\hat{X} = \sum_{j=1}^{n} \hat{a}_j(x_1, \ldots, x_n)\partial/\partial x_j$ where $\hat{a}_j \in \mathbb{C}[[x_1, \ldots, x_n]]$ for all $1 \leq j \leq n$ is nilpotent if $\hat{X}(0) = 0$ and $j^1\hat{X}$ is nilpotent. We denote by $\hat{\mathcal{X}}_N(\mathbb{C}^n, 0)$ and $\mathcal{X}_N(\mathbb{C}^n, 0)$ the sets of formal nilpotent vector fields and germs of nilpotent vector fields respectively. We define $\hat{X}^0(\hat{g}) = \hat{g}$ and $\hat{X}^{j+1} = \hat{X}(\hat{X}^j(\hat{g}))$ for all $\hat{X} \in \hat{\mathcal{X}}_N(\mathbb{C}^n, 0)$ and $\hat{g} \in \mathbb{C}[[x_1, \ldots, x_n]]$. For a formal nilpotent vector field $\hat{X}$ the exponential mapping

$$\exp(t\hat{X}) = \left(\sum_{j=0}^{\infty} \frac{t^j}{j!} \hat{X}^j(x_1), \ldots, \sum_{j=0}^{\infty} \frac{t^j}{j!} \hat{X}^j(x_n) \right)$$

is well-defined and its components belong to $\mathbb{C}[t][[x_1, \ldots, x_n]]$. Moreover if $\hat{X}$ converges then $\exp(t\hat{X})(x, y)$ is the point that we obtain by travelling time t along the trajectory of $\hat{X}$ passing through (x, y). Obviously we have $\exp(0\hat{X}) = Id$.

We say that $\hat{\sigma} = (\hat{\sigma}_1, \ldots, \hat{\sigma}_n) \cap \mathbb{C}[[x_1, \ldots, x_n]]^n$ is a formal diffeomorphism if $j^1\sigma$ is a linear isomorphism. We denote by $\widehat{\mathrm{Diff}}\,(\mathbb{C}^n, 0)$ and $\mathrm{Diff}\,(\mathbb{C}^n, 0)$ the set of formal diffeomorphisms and germs of diffeomorphism respectively. If $j^1\hat{\sigma}$ is unipotent (i.e. if 1 is the only eigenvalue of $j^1\hat{\sigma}$) then we say that $\hat{\sigma}$ is unipotent. We denote by $\widehat{\mathrm{Diff}}_u(\mathbb{C}^n, 0)$ the set of formal unipotent diffeomorphisms. The next result is well-known.

Proposition 2.1. *The exponential mapping $\hat{X} \mapsto \exp(1\hat{X})$ maps bijectively $\hat{\mathcal{X}}_N(\mathbb{C}^n, 0)$ onto $\widehat{\mathrm{Diff}}_u(\mathbb{C}^n, 0)$.*

Let $\hat{\sigma} \in \widehat{\mathrm{Diff}}_u(\mathbb{C}^n, 0)$; we denote by $\log \hat{\sigma}$ the unique formal nilpotent vector field such that $\exp(\log \hat{\sigma}) = \hat{\sigma}$.

Consider an ideal $\hat{I} \subset \mathbb{C}[[x_1, \ldots, x_n]]$. We denote by $Z(\hat{I})$ the set of formal curves $\hat{\gamma} \in (\mathbb{C}[[t]] \cap (t))^n$ such that $\hat{h} \circ \hat{\gamma} = 0$ for all $\hat{h} \in \hat{I}$. Reciprocally, for $\hat{\Delta} \subset (\mathbb{C}[[t]] \cap (t))^n$ we define $I(\hat{\Delta})$ as the set of series $\hat{h} \in \mathbb{C}[[x_1, \ldots, x_n]]$ such that $\hat{h} \circ \hat{\gamma} = 0$ for all $\hat{\gamma} \in \hat{\Delta}$. We have

Proposition 2.2 (Formal theorem of zeros [8], pages 49-50)**.** *Let $\hat{I}$ be an ideal of $\mathbb{C}[[x_1, \ldots, x_n]]$. Then*

$$IZ(\hat{I}) = \sqrt{\hat{I}}.$$

Let $\hat{Y}$ be a formal vector field. We consider the set

$$FI(\hat{Y}) = \{\hat{g} \in \mathbb{C}[[x_1, \ldots, x_n]] \ : \ \hat{Y}(\hat{g}) = 0\}$$

of first integrals of $\hat{Y}$. We say that $\hat{f} \in FI(\hat{Y})$ is primitive if $\sqrt[k]{\hat{f}}$ does not belong to $\mathbb{C}[[x_1, \dots, x_n]]$ for $k > 1$. If $FI(\hat{Y}) \neq \emptyset$ there exists a primitive formal first integral $\hat{f}$; moreover we have [6]

$$FI(\hat{Y}) = \mathbb{C}[[z]] \circ \hat{f}.$$

The primitive first integral can be chosen in $\mathbb{C}\{x_1, \dots, x_n\}$ if $\hat{Y}$ is a germ of vector field [6].

We can give an alternative characterization for the first integrals of the logarithm of a unipotent diffeomorphism.

Lemma 2.1. *Let* $\sigma \in \widehat{\mathrm{Diff}}_u(\mathbb{C}^n, 0)$ *and* $\hat{f} \in \mathbb{C}[[x_1, \dots, x_n]]$. *Then*

$$\log \sigma(\hat{f}) = 0 \Leftrightarrow \hat{f} \circ \sigma = \hat{f}.$$

Proof. We have

$$\hat{f} \circ \exp(t \log \sigma) = \hat{f} + t \log \sigma(\hat{f}) + \frac{t^2}{2!}(\log \sigma)^2(\hat{f}) + \dots$$

Therefore $\log \sigma(\hat{f}) = 0$ implies $\hat{f} \circ \sigma = \hat{f}$.

Suppose $\hat{f} \circ \sigma = \hat{f}$. We obtain $\hat{f} \circ \exp(t \log \sigma) - \hat{f} = 0$ for all $t \in \mathbb{Z}$ and then for all $t \in \mathbb{C}$ since both sides belong to $\mathbb{C}[t][[x, y]]$. That leads us to

$$\log \hat{\sigma}(\hat{f}) = \lim_{t \to 0} \frac{\hat{f} \circ \exp(t \log \sigma) - \hat{f}}{t} = 0.$$

$\square$

3. Formal conjugacy

Throughout this paper we work with germs of diffeomorphism in $(\mathbb{C}^2, 0)$ of the form

$$\varphi_{K,u}(x, y) = (x + y(y - x)K(x, y), y + y(y - x)u(x, y))$$

where $u, K \in \mathbb{C}\{x, y\}$ and $u(0, 0) \neq 0 = K(0, 0)$. In this section we describe a geometrical condition for $\varphi_{K,u}$ not to have a convergent normal form.

We denote by $Fix\sigma$ the fixed points set of a germ of diffeomorphism σ. Next, we describe the structure of $\log \varphi_{K,u}$.

Lemma 3.1. *The formal vector field* $\log \varphi_{K,u}$ *is of the form*

$$\log \varphi_{K,u} = y(y - x)\left(u(0, 0)\frac{\partial}{\partial y} + h.o.t.\right).$$

Proof. Since $[y(y-x)=0] = Fix\varphi_{K,u}$ then

$$\exp(t\log\varphi_{K,u})(x,0) \equiv (x,0) \quad \text{and} \quad \exp(t\log\varphi_{K,u})(x,x) \equiv (x,x)$$

for all $t \in \mathbb{Z}$. Indeed, the result is still true for every $t \in \mathbb{C}$ since $\exp(t\log\varphi_{K,u})(x,0)$ and $\exp(t\log\varphi_{K,u})(x,x)$ belong to $(\mathbb{C}[t][[x]])^2$. We have

$$\log\varphi_{K,u}(x,0) \equiv \lim_{t\to 0} \frac{\exp(t\log\varphi_{K,u})(x,0) - \exp(0\log\varphi_{K,u})(x,0)}{t} \equiv 0.$$

We deduce that y divides $\log\varphi_{K,u}$. In an analogous way we obtain that $y-x$ divides $\log\varphi_{K,u}$. Then $\log\varphi_{K,u}/(y(y-x)) = u(0,0)\partial/\partial y + h.o.t.$ as it can be proved by using undetermined coefficients. □

Lemma 3.2. *Fix $\varphi_{K,u}$ and $\hat{g} \in \mathbb{C}[[x]]$. There exists a unique $\hat{f}$ in $\mathbb{C}[[x,y]]$ such that $\log\varphi_{K,u}(\hat{f}) = 0$ and $\hat{f}(x,0) = \hat{g}(x)$.*

Proof. We denote $\hat{Y} = \log\varphi_{K,u}/(y(y-x))$. By lemma 3.1 we have that $\hat{Y}(y)$ is a unit. Then

$$\log\varphi_{K,u}(\hat{f}) = 0 \Leftrightarrow \frac{\partial \hat{f}}{\partial y} = -\frac{\hat{Y}(x)}{\hat{Y}(y)}\frac{\partial \hat{f}}{\partial x}.$$

As a consequence there is a unique formal solution of the previous equation fulfilling the initial condition $\hat{f}(x,0) = \hat{g}(x)$. □

We want to introduce the formal invariants of $\varphi_{K,u}$. The first formal invariant is the fixed points set.

Proposition 3.1. *Let $\tau_1, \tau_2 \in \mathrm{Diff}\,(\mathbb{C}^n, 0)$ and $\hat{\sigma} \in \mathrm{Diff}\,(\mathbb{C}^n, 0)$ such that $\hat{\sigma} \circ \tau_1 = \tau_2 \circ \hat{\sigma}$. Then we have $\hat{\sigma}(Fix\tau_1) = Fix\tau_2$.*

An equivalent statement is the following: Let $\hat{I}_1 = I(Fix\tau_1)$ and $\hat{I}_2 = I(Fix\tau_2)$. Then we have $\hat{I}_2 \circ \hat{\sigma} = \hat{I}_1$.

Proof. Let $\hat{\gamma} \in \mathbb{C}[[t]]^n \cap Z(\hat{I}_1)$. We have $\tau_1 \circ \hat{\gamma}(t) = \hat{\gamma}(t)$; we obtain

$$\hat{\sigma} \circ \tau_1(\hat{\gamma}(t)) = \tau_2 \circ \hat{\sigma}(\hat{\gamma}(t)) \Rightarrow \tau_2 \circ \hat{\sigma}(\hat{\gamma}(t)) = \hat{\sigma}(\hat{\gamma}(t)).$$

We deduce that $\hat{\sigma}\circ\hat{\gamma}(t)$ belongs to $Z(\hat{I}_2)$ and then $\hat{\sigma}(Z(\hat{I}_1)) \subset Z(\hat{I}_2)$. By the analogous argument applied to $\hat{\sigma}^{(-1)}$ we obtain $Z(\hat{I}_2) \subset \hat{\sigma}(Z(\hat{I}_1))$ and then $\hat{\sigma}(Z(\hat{I}_1)) = Z(\hat{I}_2)$. That is equivalent to $IZ(\hat{I}_2) \circ \hat{\sigma} = IZ(\hat{I}_1)$. Since $\hat{I}_1$ and $\hat{I}_2$ are radical ideals then $\hat{I}_2 \circ \hat{\sigma} = \hat{I}_1$ is a consequence of the formal theorem of zeros. □

Remark 3.1. *It is no required the formal theorem of zeros to prove the previous proposition but this proof makes clear that the image by $\hat{\sigma}$ of a parametrization $\hat{\gamma}(t)$ of a formal curve contained in $Fix\tau_1$ is a parametrization $\hat{\sigma} \circ \hat{\gamma}(t)$ of a formal curve contained in $Fix\tau_2$.*

Lemma 3.3. *Let* $\varphi_{K,u}$, $\tau \in \mathrm{Diff}\,(\mathbb{C}^2, 0)$ *and* $\hat{\sigma} \in \widehat{\mathrm{Diff}}\,(\mathbb{C}^2, 0)$ *such that* $\hat{\sigma} \circ \varphi_{K,u} = \tau \circ \hat{\sigma}$. *Then*

(1) $Fix\tau$ *is an analytic set.*
(2) $Fix\tau$ *has two irreducible components* $f_1 = 0$ *and* $f_2 = 0$, *both of them are smooth curves.*
(3) $\hat{\sigma}_*(\log \varphi_{K,u}) = \log \tau$.
(4) $j^0(\log \tau/(f_1 f_2))$ *is transversal to both* $f_1 = 0$ *and* $f_2 = 0$.
(5) *Let* $\hat{f}$ *be a primitive element of* $FI(\log \varphi_{K,u})$. *Then* $\hat{f} \circ \hat{\sigma}^{(-1)}$ *is a primitive element of* $FI(\log \tau)$.
(6) $\hat{f} \circ \hat{\sigma}^{(-1)}_{|f_1=0}$ *and* $\hat{f} \circ \hat{\sigma}^{(-1)}_{|f_2=0}$ *are "injective". In other words, if* $\hat{\gamma}(t)$ *is a minimal parametrization of* $f_j = 0$ *we have* $\nu(\hat{f} \circ \hat{\sigma}^{(-1)} \circ \hat{\gamma}) = 1$.

Proof. Condition (1) is obvious. Conditions (2) and (4) are a consequence of proposition 3.1 and $j^1\hat{\sigma}$ being an isomorphism. Conditions (3) and (5) can be deduced of the uniqueness of the logarithm. Condition (6) is equivalent to prove that $\nu(\hat{f}(x, 0)) = \nu(\hat{f}(x, x)) = 1$ for every primitive $\hat{f}$ in $FI(\log \varphi_{K,u})$. We can suppose that $\hat{f}(x, 0) = x$ since then $\hat{f}$ is primitive and the set of primitive elements of $FI(\log \varphi_{K,u})$ is $\widehat{\mathrm{Diff}}\,(\mathbb{C}, 0) \circ \hat{f}$. The relation $\log \varphi_{K,u}/(y(y - x)) = u(0, 0)\partial/\partial y + h.o.t$ implies $j^1\hat{f} = x$. Therefore $\nu(\hat{f}(x, 0)) = \nu(\hat{f}(x, x)) = 1$. □

Consider a couple (S, g) where S is a germ of analytic set and g is a function on S. We typically consider a couple $(\gamma, |Jac\varphi_{K,u}|_{|\gamma})$ where γ is a germ of curve contained in $Fix\varphi_{K,u}$ and $|Jac\varphi_{K,u}|$ is the determinant of the jacobien matrix.

Proposition 3.2. *The couples*

$$(y = 0, |Jac\varphi_{K,u}|_{|y=0}) \quad \text{and} \quad (y = x, |Jac\varphi_{K,u}|_{|y=x})$$

are formal invariants of $\varphi_{K,u}$.

Proof. Suppose $\hat{\sigma} \circ \varphi_{K,u} = \tau \circ \hat{\sigma}$ for $\tau \in \mathrm{Diff}\,(\mathbb{C}^2, 0)$ and $\hat{\sigma} \in \widehat{\mathrm{Diff}}\,(\mathbb{C}^2, 0)$. We have

$$(|Jac\hat{\sigma}| \circ \varphi_{K,u})|Jac\varphi_{K,u}| = (|Jac\tau| \circ \hat{\sigma})|Jac\hat{\sigma}|$$

by the chain rule. Let $\gamma(t) \in (\mathbb{C}\{t\} \cap (t))^2$ be a parametrization of either $y = 0$ or $y = x$. We have $\varphi_{K,u} \circ \gamma(t) = \gamma(t)$; that implies

$$|Jac\varphi_{K,u}| \circ \gamma(t) = |Jac\tau| \circ (\hat{\sigma} \circ \gamma(t))$$

as we wanted to prove. □

Consider the formal mapping $Tr_{K,u} : (y = 0) \to (y = x)$ such that $\hat{f}_{K,u} \circ Tr_{K,u} = \hat{f}_{K,u}$ where $\hat{f}_{K,u}$ is a primitive formal first integral of

$\log \varphi_{K,u}$. By condition (6) in lemma 3.3 we have that $\hat{f}_{K,u}(x,0)$ and $\hat{f}_{K,u}(x,x)$ belong to $\widehat{\text{Diff}}\,(\mathbb{C},0)$. As a consequence we obtain

$$Tr_{K,u}(x) = (\hat{f}_{K,u}(x,x))^{(-1)} \circ \hat{f}_{K,u}(x,0)$$

The mapping $Tr_{K,u}$ does not depend on the choice of $\hat{f}_{K,u}$. We call $Tr_{K,u}$ the transport mapping. If $\log \varphi_{K,u}$ is a germ of vector field then $Tr_{K,u}(x)$ is the only point in $y = x$ contained in the same trajectory of $\log \varphi_{K,u}/[y(y-x)]$ than $(x,0)$.

Proposition 3.3. *The transport mapping $Tr_{K,u}$ associated to a diffeomorphism $\varphi_{K,u}$ is a formal invariant.*

Suppose $\hat{\sigma} \circ \varphi_{K,u} = \tau \circ \hat{\sigma}$ for $\tau \in \text{Diff}\,(\mathbb{C}^2,0)$ and $\hat{\sigma} \in \widehat{\text{Diff}}\,(\mathbb{C}^2,0)$. By proposition 3.1 the formal curves $\gamma_1 = \hat{\sigma}(y=0)$ and $\gamma_2 = \hat{\sigma}(y=x)$ are in fact analytic. We define $Tr_\tau : \gamma_1 \to \gamma_2$ such that $\hat{g} \circ Tr_\tau = \hat{g}$ for every primitive $\hat{g}$ in $FI(\log \tau)$.

Proof. We keep the notations in the previous paragraph. We choose $\hat{\sigma}(x,0)$ and $\hat{\sigma}(x,x)$ as formal parameterizations of γ_1 and γ_2 respectively. We choose a primitive $\hat{f} \in FI(\log \varphi_{K,u})$; the series $\hat{g} = \hat{f} \circ \hat{\sigma}^{(-1)}$ is a primitive element of $FI(\log \tau)$ (lemma 3.3). We obtain

$$Tr_\tau(x) = (\hat{f} \circ \hat{\sigma}^{(-1)}(\hat{\sigma}(x,x)))^{(-1)} \circ (\hat{f} \circ \hat{\sigma}^{(-1)}(\hat{\sigma}(x,0))) = Tr_{K,u}(x).$$

□

Now we introduce an obstruction to have a convergent normal form. A unipotent diffeomorphism τ has convergent normal form if $\log \tau$ is formally conjugated to the exponential of a germ of vector field.

Proposition 3.4. *Suppose that there exist $X \in \mathcal{X}_N(\mathbb{C}^2,0)$ and $\hat{\sigma}$ in $\widehat{\text{Diff}}\,(\mathbb{C}^2,0)$ such that $\hat{\sigma} \circ \varphi_{K,u} = \exp(X) \circ \hat{\sigma}$. Then $Tr_{K,u}$ is a convergent mapping.*

Proof. The expression of $Tr_{K,u}$ in coordinates $x \mapsto (x,0)$, $x \mapsto (x,x)$ and of $Tr_{\exp(X)}$ in coordinates $x \mapsto \hat{\sigma}(x,0)$, $x \mapsto \hat{\sigma}(x,x)$ are the same. Since it is clear that the expression of $Tr_{\exp(\mathrm{X})}$ in convergent coordinates is convergent then it is enough to prove that $\hat{\sigma}(x,0)$ and $\hat{\sigma}(x,x)$ belong to $\mathbb{C}\{x\}^2$.

We have $|Jac\varphi_{K,u}| = 1+(2y-x)u(0,0)+h.o.t.$ Therefore $|Jac\varphi_{K,u}|_{|y=0}$ is injective. Consider a convergent minimal parametrization $\eta(x)$ of $\hat{\sigma}(y=0)$; there exists $\hat{h} \in \widehat{\text{Diff}}\,(\mathbb{C},0)$ such that $\hat{\sigma}(x,0) = \eta \circ \hat{h}(x)$. Since $|Jac(\exp(X))| \circ \hat{\sigma}(x,0) = |Jac\varphi_{K,u}|(x,0)$ then

$$\frac{\partial}{\partial x}(|Jac(\exp(X))| \circ \eta(x))(0) = -\frac{u(0,0)}{\partial \hat{h}/\partial x(0)} \neq 0.$$

As a consequence

$$\hat{h} = (|Jac(\exp(X))| \circ \eta(x) - 1)^{(-1)} \circ (|Jac\varphi_{K,u}|(x,0) - 1)$$

belongs to Diff $(\mathbb{C}, 0)$. That implies $\hat{\sigma}(x,0) = \eta \circ \hat{h} \in \mathbb{C}\{x\}^2$. The proof for $\hat{\sigma}(x,x)$ is analogous. $\square$

Remark 3.2. *In order to find a unipotent diffeomorphism without convergent normal form it is enough to exhibit a $\varphi_{K,u}$ such that $Tr_{K,u}$ is divergent.*

Remark 3.3. *We do not prove it in this paper but a diffeomorphism $\varphi_{K,u}$ such that $Tr_{K,u}$ is an analytic mapping has convergent normal form. In particular a diffeomorphism $\varphi_{0,u} = (x, y + y(y-x)u(x,y))$ has convergent normal form.*

4. Polynomial families

We define

$$\varphi_{K,u,\lambda} = (x + \lambda y(y-x)K(x,y), y + y(y-x)u(x,y)$$

where $u(0,0) \neq 0 = K(0,0)$ and $\lambda \in \mathbb{C}$. We denote by $\hat{f}_\lambda$ the only element of $FI(\log \varphi_{K,u,\lambda})$ such that $\hat{f}_\lambda(x,0) = x$. The transport mapping $Tr_{\lambda K,u}$ satisfies

$$Tr_{\lambda K,u}(x) = \left(\hat{f}_\lambda(x,x)\right)^{(-1)} \circ \hat{f}_\lambda(x,0) = \left(\hat{f}_\lambda(x,x)\right)^{(-1)}.$$

As a consequence $Tr_{\lambda K,u}$ is convergent if and only if $\hat{f}_\lambda(x,x) \in \mathbb{C}\{x\}$.

Lemma 4.1. *We have*

$$\frac{\log \varphi_{K,u,\lambda}}{y(y-x)} = \left(\sum_{0 \leq k,l} a^1_{k,l}(\lambda) x^k y^l \right) \frac{\partial}{\partial x} + \left(\sum_{0 \leq k,l} a^2_{k,l}(\lambda) x^k y^l \right) \frac{\partial}{\partial y}$$

where $a^j_{k,l} \in \mathcal{O}(\mathbb{C})$ for all $k,l \geq 0$ and $j \in \{1,2\}$.

The lemma can be proved by using undetermined coefficients.

Proposition 4.1. *Let $\hat{f}_\lambda$ be the unique formal first integral of* $\log \varphi_{K,u,\lambda}$ *such that $\hat{f}_\lambda(x,0) = x$. Then $\hat{f}_\lambda$ can be expressed in the form*

$$\hat{f}_\lambda = x + y \sum_{j+k \geq 1} f_{j,k}(\lambda) x^j y^k$$

where $f_{j,k} \in \mathbb{C}[\lambda]$ and $\deg f_{j,k} \leq j + k$ *for all $j + k \geq 1$.*

Proof. We can use undetermined coefficients, the lemma 4.1 and the equation

$$\frac{\log \varphi_{K,u,\lambda}}{y(y-x)}(\hat{f}_\lambda) = 0$$

to prove that $f_{j,k} \in \mathcal{O}(\mathbb{C})$ for all $j+k \geq 1$. Now consider

$$\tau_{K,u,\lambda}(x,y) = \left(\frac{x}{\lambda}, \frac{y}{\lambda}\right) \circ \varphi_{K,u,1/\lambda} \circ (\lambda x, \lambda y).$$

We have

$$\tau_{K,u,\lambda}(x,y) = (x + y(y-x)K(\lambda x, \lambda y), y + \lambda y(y-x)u(\lambda x, \lambda y)).$$

We can proceed like in lemma 3.1 to prove

$$\log \tau_{K,u,\lambda}(x,y) = \lambda y(y-x)\left(u(0,0)\frac{\partial}{\partial y} + h.o.t.\right).$$

There is an analogue of lemma 4.1 for $\log \tau_{K,u,\lambda}/(\lambda y(y-x))$. Again such an expression can be used to prove that the unique first integral

$$\hat{g}_\lambda = x + y \sum_{j+k \geq 1} g_{j,k}(\lambda) x^j y^k$$

of $\log \tau_{K,u,\lambda}$ such that $\hat{g}_\lambda(x,0) = x$ satisfies $g_{j,k} \in \mathcal{O}(\mathbb{C})$ for all $j+k \geq 1$. The relation between $\varphi_{K,u\lambda}$ and $\tau_{K,u,\lambda}$ implies

$$\hat{f}_{1/\lambda}(\lambda x, \lambda y) = \lambda \hat{g}_\lambda(x,y).$$

We obtain $f_{j,k}(1/\lambda)\lambda^{j+k} = g_{j,k}(\lambda)$ for all $j+k \geq 1$. Since $f_{j,k}$ and $g_{j,k}$ are integer functions we deduce that $f_{j,k}$ is a polynomial of degree at most $j+k$ for all $j+k \geq 1$. □

We have $\hat{f}_\lambda(x,x) = x + \sum_{j+k \geq 1} f_{j,k}(\lambda) x^{j+k+1}$. Next result is crucial.

Proposition 4.2 ([5, 4]). *Let $\hat{P} = \sum_{j \geq 0} P_j(\lambda) x^j$ where $P_j \in \mathbb{C}[\lambda]$ and $\deg P_j \leq Aj + B$ for some $A, B \in \mathbb{R}$ and all $j \in \mathbb{N}$. Then either $\hat{P}(\lambda, x)$ is convergent in a neighborhood of $x = 0$ or $\hat{P}(\lambda) \in \mathbb{C}[[x]] \setminus \mathbb{C}\{x\}$ for all $\lambda \in \mathbb{C}$ outside a polar set.*

A polar set is pretty small. Its measure is zero as well as its Haussdorff dimension. Moreover, it is totally disconnected.

Corollary 4.1. *Fix $K \in \mathbb{C}\{x,y\} \cap (x,y)$ and $u \in \mathbb{C}\{x,y\} \setminus (x,y)$. Either $(\lambda, x) \mapsto Tr_{\lambda K,u}(x)$ is convergent in a neighborhood of $x = 0$ or $x \mapsto Tr_{\lambda K,u}(x)$ is divergent for all $\lambda \in \mathbb{C}$ outside a polar set.*

Proposition 4.3. *Fix $u \in \mathbb{C}\{x,y\} \setminus (x,y)$. Suppose $\varphi_{K,u}$ has a convergent normal form for all $K \in \mathbb{C}\{x,y\} \cap (x,y)$. Then the equation*

$$\hat{\epsilon} - \hat{\epsilon} \circ \varphi_{0,u} = y(y-x)\Delta(x,y)$$

has a solution $\hat{\epsilon}_\Delta \in \mathbb{C}[[x,y]]$ such that $\hat{\epsilon}_\Delta(x,x) - \hat{\epsilon}_\Delta(x,0) \in \mathbb{C}\{x\}$ for all $\Delta \in \mathbb{C}\{x,y\} \cap (x,y)$.

Proof. Fix $\Delta \in \mathbb{C}\{x,y\} \cap (x,y)$. Let $\hat{f}_\lambda$ be the unique first integral of $\log \varphi_{\Delta,u,\lambda}$ such that $\hat{f}_\lambda(x,0) = x$. We have $\hat{f}_\lambda \circ \varphi_{\Delta,u,\lambda} = \hat{f}_\lambda$ for all $\lambda \in \mathbb{C}$ by lemma 2.1. That implies

$$\frac{\partial(\hat{f}_\lambda \circ \varphi_{\Delta,u,\lambda})}{\partial \lambda}_{|\lambda=0} = \frac{\partial \hat{f}_\lambda}{\partial \lambda}_{|\lambda=0}.$$

Since $\hat{f}_0 = x$ then

$$y(y-x)\Delta(x,y) + \left(\frac{\partial \hat{f}_\lambda}{\partial \lambda}_{|\lambda=0} \right) \circ \varphi_{0,u} = \frac{\partial \hat{f}_\lambda}{\partial \lambda}_{|\lambda=0}.$$

We define $\hat{\epsilon}_\Delta(x,y) = (\partial \hat{f}_\lambda / \partial \lambda)(x,y,0)$. We have $\hat{\epsilon}_\Delta(x,0) = 0$ by definition of $\hat{f}_\lambda$. By the hypothesis and the proposition 4.2 we obtain that $(\lambda, x) \mapsto \hat{f}_\lambda(x,x)$ is convergent in a neighborhood of $x=0$. As a consequence $\hat{\epsilon}_\Delta(x,x) \in \mathbb{C}\{x\}$ and then $\hat{\epsilon}_\Delta(x,x) - \hat{\epsilon}_\Delta(x,0) \in \mathbb{C}\{x\}$. □

The diffeomorphism $\varphi_{0,u}$ satisfies $x \circ \varphi_{0,u} = x$, moreover it has a convergent normal form. That is a significant progress since in order to find a $\varphi_{K,u}$ without a convergent normal form we only have to deal with simpler diffeomorphisms.

5. The equation $\hat{\epsilon} - \hat{\epsilon} \circ \varphi_{0,u} = y(y-x)\Delta(x,y)$

The main in result in this section is proving that the difference equation can be replaced by a differential equation. This is a key tool in proving the main theorem.

We denote $\varphi_{0,u} = (x, y + y(y-x)u(x,y))$ by φ_u.

Lemma 5.1. $\log \varphi_u$ *is of the form* $y(y-x)(u(0,0) + h.o.t.)\partial/\partial y$.

Proof. By lemma 3.1 it is enough to prove that $\log \varphi_u(x) = 0$. We have that $x \circ \exp(t \log \varphi_u) - x = 0$ for $t \in \mathbb{Z}$; moreover since both sides belong to $\mathbb{C}[t][[x,y]]$ we obtain $x \circ \exp(t \log \varphi_u) = x$ for $t \in \mathbb{C}$. Hence

$$\log \varphi_u(x) = \lim_{t \to 0} \frac{x \circ \exp(t \log \varphi_u) - x}{t} = 0.$$

□

Lemma 5.2. *The equation* $\hat{\epsilon} - \hat{\epsilon} \circ \varphi_u = y(y-x)\Delta(x,y)$ *has a solution* $\hat{\epsilon} = \hat{\epsilon}_\Delta \in \mathbb{C}[[x,y]]$ *for all* $\Delta \in \mathbb{C}[[x,y]]$.

Proof. We define $\Delta_0 = \Delta$. Consider the equation

$$\log \varphi_u(\epsilon(\Delta_0)) = -y(y-x)\Delta_0.$$

Since $\log \varphi_u/(y(y-x))$ is regular there exists a solution $\epsilon(\Delta_0) \in \mathbb{C}[[x,y]]$ such that $\nu(\epsilon(\Delta_0)) \geq \nu(\Delta_0) + 1$. We consider

$$(\epsilon(\Delta_0) + \epsilon_1) - (\epsilon(\Delta_0) + \epsilon_1) \circ \varphi_u = y(y-x)\Delta_0.$$

This equation is equivalent to

$$\epsilon_1 - \epsilon_1 \circ \varphi_u = \sum_{k \geq 2} \frac{1}{k!} (\log \varphi_u)^{k-1} (-y(y-x)\Delta_0(x,y)).$$

We denote the term in the right-hand side by $y(y-x)\Delta_1$; we have $\nu(\Delta_1) \geq \nu(\Delta_0)+1$. We can proceed by induction. Given Δ_j there exists $\epsilon(\Delta_j)$ such that $\log \varphi_u(\epsilon(\Delta_j)) = -y(y-x)\Delta_j$ and $\nu(\epsilon_j) \geq \nu(\Delta_j) + 1$. As previously we define

$$\Delta_{j+1} = \sum_{k \geq 2} \frac{1}{k!} (\log \varphi_u)^{k-1} (-y(y-x)\Delta_j(x,y)).$$

We obtain $\nu(\Delta_{j+1}) \geq \nu(\Delta_j) + 1$ for all $j \geq 0$. Therefore $\sum_{j \geq 0} \epsilon(\Delta_j)$ converges in the Krull topology to a solution $\hat{\epsilon}_\Delta \in \mathbb{C}[[x,y]]$ of the equation $\hat{\epsilon} - \hat{\epsilon} \circ \varphi_u = y(y-x)\Delta(x,y)$. □

Lemma 5.3. *Fix* $\Delta \in \mathbb{C}[[x,y]]$. *The series* $\hat{\epsilon}_\Delta(x,x) - \hat{\epsilon}_\Delta(x,0)$ *does not depend on the solution* $\hat{\epsilon}_\Delta \in \mathbb{C}[[x,y]]$ *of* $\hat{\epsilon} - \hat{\epsilon} \circ \varphi_u = y(y-x)\Delta(x,y)$.

Proof. It is enough to prove that $\hat{\epsilon}(x,x) - \hat{\epsilon}(x,0) = 0$ for a solution $\hat{\epsilon} \in \mathbb{C}[[x,y]]$ of $\hat{\epsilon} - \hat{\epsilon} \circ \varphi_u = 0$. The series $\hat{\epsilon}$ belongs to $FI(\log \varphi_u)$ by lemma 2.1. Moreover, lemma 5.1 implies $\partial \hat{\epsilon}/\partial y = 0$. Hence $\hat{\epsilon}$ belongs to $\mathbb{C}[[x]]$ and clearly $\hat{\epsilon}(x,x) - \hat{\epsilon}(x,0) = 0$. □

Given $\Delta \in \mathbb{C}[[x,y]]$ and a solution $\hat{\epsilon}_\Delta \in \mathbb{C}[[x,y]]$ of the equation $\hat{\epsilon} - \hat{\epsilon} \circ \varphi_u = y(y-x)\Delta(x,y)$ we define $S_u(\Delta) = \hat{\epsilon}_\Delta(x,x) - \hat{\epsilon}_\Delta(x,0)$. The lemmas 5.2 and 5.3 imply that $S_u : \mathbb{C}[[x,y]] \to \mathbb{C}[[x]]$ is a well-defined linear functional. Proposition 4.3 implies that if there is no $\varphi_{K,u}$ without convergent normal form then $S_u(\mathbb{C}\{x,y\} \cap (x,y)) \subset \mathbb{C}\{x\}$.

Lemma 5.4. *Let* $u \in \mathbb{C}\{x,y\} \setminus (x,y)$. *Then we have*

$$S_u \left(\frac{\log \varphi_u}{y(y-x)} [y(y-x)\Delta(x,y)] \right) = 0$$

for all $\Delta \in \mathbb{C}[[x,y]]$.

Proof. Let $\hat{\epsilon}_0 \in \mathbb{C}[[x,y]]$ be a solution of $\hat{\epsilon} - \hat{\epsilon} \circ \varphi_u = y(y-x)\Delta(x,y)$. Now $\hat{\epsilon}_t = \hat{\epsilon} \circ \exp(t \ln \varphi_u)$ satisfies the equation

$$\hat{\epsilon}_t - \hat{\epsilon}_t \circ \varphi_u = [y(y-x)\Delta(x,y)] \circ \exp(t \ln \varphi_u)$$

for all $t \in \mathbb{C}$. Moreover, we have

$$\hat{\epsilon} \circ \exp(t \ln \varphi_u)(x,x) - \hat{\epsilon} \circ \exp(t \ln \varphi_u)(x,0) = \hat{\epsilon}(x,x) - \hat{\epsilon}(x,0).$$

That implies

$$S_u \left(\frac{[y(y-x)\Delta(x,y)] \circ \exp(t \ln \varphi_u) - [y(y-x)\Delta(x,y)]}{ty(y-x)} \right) = 0$$

for all $t \in \mathbb{C}^*$. By deriving with respect to t and evaluating at $t=0$ we obtain the thesis of the lemma. □

Next we replace our difference equation with a differential equation.

Proposition 5.1. *Let $u \in \mathbb{C}\{x,y\} \setminus (x,y)$ and $\Delta \in \mathbb{C}[[x,y]]$. Consider a solution $\hat{\Gamma}_\Delta$ of $\log \varphi_u(\hat{\Gamma}) = -y(y-x)\Delta$. Then we have*

$$S_u(\Delta) = \hat{\Gamma}_\Delta(x,x) - \hat{\Gamma}_\Delta(x,0).$$

Proof. We have

$$\hat{\Gamma}_\Delta - \hat{\Gamma}_\Delta \circ \varphi_u = y(y-x)\Delta + \log \varphi_u \left(\sum_{k \geq 2} \frac{(\log \varphi)^{k-2}(y(y-x)\Delta(x,y))}{k!} \right).$$

Consider a solution $\hat{\epsilon}_\Delta \in \mathbb{C}[[x,y]]$ of $\hat{\epsilon} - \hat{\epsilon} \circ \varphi_u = y(y-x)\Delta(x,y)$. We have

$$(\hat{\Gamma}_\Delta - \hat{\epsilon}_\Delta)(x,x) - (\hat{\Gamma}_\Delta - \hat{\epsilon}_\Delta)(x,0) = 0$$

by lemma 5.4. □

Corollary 5.1. *Suppose $\log \varphi_u \in \mathcal{X}_N(\mathbb{C}^2,0)$. Then $S_u(\mathbb{C}\{x,y\})$ is contained in $\mathbb{C}\{x\}$.*

Remark 5.1. *Even if $\log \varphi_u \in \mathcal{X}_N(\mathbb{C}^2,0)$ implies $S_u(\Delta) \in \mathbb{C}\{x\}$ (for $\Delta \in \mathbb{C}\{x,y\}$) in general there is no convergent solution $\hat{\epsilon}_\Delta$ of the equation $\hat{\epsilon} - \hat{\epsilon} \circ \varphi_{0,u} = y(y-x)\Delta(x,y)$. The divergence of $S_u(\Delta)$ is subtler than the divergence of every $\hat{\epsilon}_\Delta$.*

6. The induced differential equation

Let $v \in \mathbb{C}[[x,y]]$. We can define the operator $D_v : \mathbb{C}\{x,y\} \to \mathbb{C}\{x\}$ such that $D_v(H) = \hat{\epsilon}_H(x,x) - \hat{\epsilon}_H(x,0)$ where $\hat{\epsilon}_H \in \mathbb{C}[[x,y]]$ is a solution of $\partial \hat{\epsilon}/\partial y = vH$. The definition of $D_v(H)$ does not depend on the choice of $\hat{\epsilon}_H$. This section is devoted to prove

Proposition 6.1. *Let $v \in \mathbb{C}[[x,y]]$. If $D_v(\mathbb{C}\{x,y\}) \subset \mathbb{C}\{x\}$ then v belongs to $\mathbb{C}\{x,y\}$.*

Fix $\epsilon, \delta > 0$. We define the Banach space $B_{\epsilon,\delta}$ whose elements are the series $H = \sum_{0 \leq j,k} H_{j,l} x^j y^k$ such that

$$||H||_{\epsilon,\delta} = \sum_{0 \leq j,k} |H_{j,l}| \epsilon^j \delta^k < +\infty.$$

We have $B_{\epsilon,\delta} \subset \mathbb{C}\{x,y\}$. Moreover, a function $H \in B_{\epsilon,\delta}$ is holomorphic in $B(0,\epsilon) \times B(0,\delta)$ and continuous in $\overline{B}(0,\epsilon) \times \overline{B}(0,\delta)$. Given v in $\mathbb{C}[[x,y]]$ we can define for $j \geq 1$ the linear functionals $D_v^j : B_{\epsilon,\delta} \to \mathbb{C}$ such that

$$D_v(H) = \sum_{j \geq 1} D_v^j(H) x^j$$

for all $H \in B_{\epsilon,\delta}$.

Lemma 6.1. *Let $v \in \mathbb{C}[[x,y]]$. Then D_v^j is a linear continuous functional for all $j \in \mathbb{N}$.*

Proof. We denote $H = \sum_{0 \leq k,l} H_{k,l}(H) x^k y^l$. We have that

$$D_v^j = \sum_{k+l<j} c_{k,l}^j H_{k,l}$$

where $c_{k,l}^j \in \mathbb{C}$ for all $j \geq 1$ and $k+l < j$. As a consequence it is enough to prove that $H_{k,l} : B_{\epsilon,\delta} \to \mathbb{C}$ is a continuous functional for all $0 \leq k, l$. We apply Cauchy's integration formula to obtain

$$H_{k,l}(H) = \frac{1}{(2\pi i)^2} \int_{\partial B(0,\epsilon) \times \partial B(0,\delta)} \frac{H(x,y)}{x^{k+1} y^{l+1}} dx dy.$$

That implies

$$|H_{k,l}(H)| \leq \frac{\sup_{\partial B(0,\epsilon) \times \partial B(0,\delta)} |H|}{\epsilon^k \delta^l} \leq \frac{||H||_{\epsilon,\delta}}{\epsilon^k \delta^l}.$$

As a consequence $||H_{k,l}|| \leq \epsilon^{-k} \delta^{-l}$. □

Lemma 6.2. *Let $v \in \mathbb{C}[[x,y]]$. Either $\limsup_{j \to \infty} \sqrt[j]{||D_v^j||} < +\infty$ or $D_v(H) \notin \mathbb{C}\{x\}$ for all H in a dense subset of $B_{\epsilon,\delta}$.*

Proof. Suppose $\limsup_{j \to \infty} \sqrt[j]{||D_v^j||} = +\infty$. We choose a sequence (a_j) of positive numbers such that $a_j \to \infty$ and

$$\limsup_{j \to \infty} \frac{\sqrt[j]{||D_v^j||}}{a_j} = +\infty.$$

Hence $\limsup_{j \to \infty} ||D_v^j / a_j^j|| = +\infty$. We deduce that

$$\limsup_{j \to \infty} |D_v^j(H)| / a_j^j = +\infty$$

for all H in a dense subset E of $B_{\epsilon,\delta}$ by the uniform boundedness principle. Moreover, since

$$\limsup_{j\to\infty} \sqrt[j]{|D_v^j(H)|} \geq \liminf_{j\to\infty} a_j = +\infty$$

then $D_v(H) \notin \mathbb{C}\{x\}$ for all $H \in E$. $\square$

Proposition 6.2. *Let $v \in \mathbb{C}[[x,y]]$. Suppose $D_v(B_{\epsilon,\delta}) \subset \mathbb{C}\{x\}$. Then there exists $\eta_{\epsilon,\delta} > 0$ such that $D_v(H) \in \mathcal{O}(B(0,\eta_{\epsilon,\delta}))$ for all $H \in B_{\epsilon,\delta}$.*

Proof. There exists $\eta_{\epsilon,\delta} > 0$ such that $\limsup_{j\to\infty} \sqrt[j]{||D_v^j||} \leq 1/\eta_{\epsilon,\delta}$ by lemma 6.2. As a consequence

$$\limsup_{j\to\infty} \sqrt[j]{|D_v^j(H)|} \leq \limsup_{j\to\infty} \left(\sqrt[j]{||D_v^j||} \sqrt[j]{||H||_{\epsilon,\delta}} \right) \leq 1/\eta_{\epsilon,\delta}.$$

That implies that $D_v(H) \in \mathcal{O}(B(0,\eta_{\epsilon,\delta}))$ for all $H \in B_{\epsilon,\delta}$. $\square$

Proof of proposition 6.1. Since $D_v(H) \in \mathbb{C}\{x\}$ for all $H \in B_{1,1}$ there exists $C \geq 1$ such that $||D_v^j|| \leq C^j$ for all $j \geq 1$ by lemma 6.2. We denote $v = \sum_{0\leq k,l} v_{k,l}x^k y^l$. We have

$$D_v^1(1) = v_{0,0} \Rightarrow |v_{0,0}| \leq ||D_v^1||||1||_{1,1} \leq C.$$

We want to estimate $v_{k,0}$, …, $v_{0,k}$ for all $k \geq 0$. We obtain

$$Hil^k \begin{pmatrix} v_{k,0} \\ v_{k-1,1} \\ \vdots \\ v_{0,k} \end{pmatrix} = \begin{pmatrix} D_v^{k+1}(1) \\ D_v^{k+2}(y) \\ \vdots \\ D_v^{2k+1}(y^k) \end{pmatrix}$$

where Hil^k is the $(k+1)\times(k+1)$ Hilbert matrix; this is a real symmetric matrix such that $Hil^k_{a,b} = 1/(a+b-1)$ for all $1 \leq a,b \leq k+1$. The Hilbert matrix is the matrix associated to the bilinear form

$$< P, Q >= \int_0^1 P(r)Q(r)dr$$

in the basis 1, …, x^k of the space of real polynomials in one variable of degree at most k. Therefore Hil^k is not singular and all its eigenvalues are positive numbers. In order to estimate $||v_{k,0},\dots,v_{0,k}||_2$ we want to estimate the spectral norm of the inverse of Hil^k, i.e. $||(Hil^k)^{-1}||_2$. Since $(Hil^k)^{-1}$ is hermitian then

$$||(Hil^k)^{-1}||_2 = \max eigenvalues((Hil^k)^{-1}) = \frac{1}{\min eigenvalues(Hil^k)}.$$

Let $\rho = 1 + \sqrt{2}$ and $K = (8\pi 2^{3/4})/(1+\sqrt{2})^4$; we obtain

$$||(Hil^k)^{-1}||_2 = \frac{\rho^{4k}}{K\sqrt{k}}(1+o(1))$$

as $k \to \infty$ [3]. We have $|D_v^{k+l+1}(y^l)| \leq ||D_v^{k+l+1}|| \, ||y^l||_{1,1} \leq C^{k+l+1}$. As a consequence we obtain

$$||v_{k,0}, \ldots, v_{0,k}||_2 \leq \frac{\rho^{4k}}{K\sqrt{k}}\sqrt{k+1}C^{2k+1}(1+o(1)).$$

Then

$$|v_{l,m}| \leq \frac{\rho^{4(l+m)}}{K\sqrt{l+m}}\sqrt{l+m+1}C^{2(l+m)+1}(1+o(1))$$

for $0 \leq l, m$ where $\lim_{l+m\to\infty} o(1) = 0$. We deduce that v belongs to $\mathcal{O}(B(0, 1/(\rho^4 C^2)) \times B(0, 1/(\rho^4 C^2)))$. □

7. End of the proof of the Main Theorem

The following proposition basically implies the Main Theorem.

Proposition 7.1. *Let $u \in \mathbb{C}\{x,y\} \setminus (x,y)$. Suppose that $\log \varphi_{0,u}$ is not convergent. Then there exists $K \in \mathbb{C}\{x,y\} \cap (x,y)$ such that $\varphi_{K,u}$ does not have a convergent normal form.*

Proof. Suppose the result is false. Hence $S_u(\mathbb{C}\{x,y\} \cap (x,y)) \subset \mathbb{C}\{x\}$ by proposition 4.3. Let $\hat{u} \in \mathbb{C}[[x,y]]$ be the formal unit such that $\log \varphi_{0,u} = y(y-x)\hat{u}(x,y)\partial/\partial y$ (see lemma 5.1). By hypothesis $\hat{u}$ is a divergent power series. By proposition 5.1 the series $\hat{\Gamma}_\Delta(x,x) - \hat{\Gamma}_\Delta(x,0)$ belongs to $\mathbb{C}\{x\}$ for all solution $\hat{\Gamma}_\Delta \in \mathbb{C}[[x,y]]$ of

$$\frac{\partial \hat{\Gamma}}{\partial y} = -\frac{\Delta(x,y)}{\hat{u}(x,y)}$$

and all $\Delta \in \mathbb{C}\{x,y\} \cap (x,y)$. Since $D_{-x/\hat{u}}(\mathbb{C}\{x,y\}) \subset \mathbb{C}\{x\}$ then $-x/\hat{u} \in \mathbb{C}\{x,y\}$ by proposition 6.1. We deduce that $\hat{u} \in \mathbb{C}\{x,y\}$; that is a contradiction. □

To end the proof of the Main Theorem it is enough to exhibit an example of a diffeomorphism $\varphi_{0,u}$ such that $\log \varphi_{0,u}$ is divergent by proposition 7.1.

If $u(0,y) \in \mathcal{O}(\mathbb{C})$ then $(y \circ \varphi_{0,u})(0,y)$ is an integer function different than y. Then $\log(\varphi_{0,u})_{|x=0}$ is divergent [1]. That implies $\hat{u}(0,y) \notin \mathbb{C}\{y\}$ and then $\hat{u}(x,y) \notin \mathbb{C}\{x,y\}$. In particular we can choose $u = 1$.

8. Remarks and generalizations

In our approach a unipotent $\tau \in \text{Diff}\,(\mathbb{C}^n, 0)$ has convergent normal form if $\log \tau$ does. We can say then that the normal form is strong. There is an alternative definition: We say that $\tau \in \text{Diff}\,(\mathbb{C}^n, 0)$ has a weak convergent normal form if there exists a germ of vector field Y vanishing at 0 whose exponential is formally conjugated to τ. This definition is suppler but not so geometrically significant. For instance $Id = \exp(0) \in \text{Diff}\,(\mathbb{C}, 0)$ is the exponential of every germ of vector field whose first jet is $2\pi i z \partial/\partial z$. It is natural to restrict our study to the strong case. Anyway, in the family $(\varphi_{K,u})$ the strong and weak concepts of convergent normal form coincide. Hence a general $\varphi_{K,u}$ does not have a weak convergent normal form.

There exists φ_{K_0,u_0} without convergent normal form. That is the generic situation. Consider the set

$$E = \{\varphi_{K,u} : K \in (x, y) \text{ and } u(0, 0) = 1\} \subset \text{Diff}_u(\mathbb{C}^2, 0).$$

For every $\varphi_{K,u}$ there exists $\mu \in \mathbb{C}^*$ such that $(x/\mu, y/\mu) \circ \varphi_{K,u} \circ (\mu x, \mu y)$ is in E. Then to study the existence of convergent normal forms in the family $(\varphi_{K,u})$ we can restrict ourselves to E. In particular we can suppose $\varphi_{K_0,u_0} \in E$. The set E is an affine space whose underlying vector space is $(x, y) \times (x, y)$. Then E is the union of the complex lines

$$L_{A,B} : \lambda \mapsto \varphi_{K_0+\lambda A, u_0+\lambda B}.$$

where $A, B \in \mathbb{C}\{x, y\} \cap (x, y)$. By arguing like in section 4 and applying proposition 4.2 we can prove that for all line $L_{A,B}$ through φ_{K_0,u_0} the transport mapping is divergent outside of a polar set. The absence of convergent normal form is clearly the generic situation in E.

Consider K, u such that $\varphi_{K,u}$ does not have a convergent normal form. We define

$$\varphi^n_{K,u} = (z_1 + z_2(z_2 - z_1)K(z_1, z_2), z_2 + z_2(z_2 - z_1)u(z_1, z_2), z_3, \ldots, z_n).$$

Then $\varphi_{K,u} \in \widehat{\text{Diff}}_u(\mathbb{C}^n, 0) \cap \text{Diff}\,(\mathbb{C}^n, 0)$ does not have convergent normal form for all $n \geq 2$. Hence there are unipotent germs of diffeomorphism without convergent normal form for any dimension greater than 2.

Let $f \in \mathbb{C}\{x, y\} \cap (x, y)$. Consider the family

$$\varphi^f_{K,u} = (x + f(x, y)K(x, y), y + f(x, y)u(x, y))$$

where $K(0, 0) = 0 \neq u(0, 0)$. The choice $f = y(y - x)$ is by no means special. We can choose f such that its decomposition $x^m f_1^{n_1} \ldots f_p^{n_p}$ in irreducible factors satisfies $\nu((f_1 \ldots f_p)(0, y)) > 1$. This condition means that in a suitable domain $\sharp((f = 0) \cap (x = c)) > 1$ for all $c \neq 0$ in a neighborhood of 0. It is the condition we need to define an analogue of the transport mapping. Fix u such that $\ln \varphi_{0,u}^{f}$ is divergent; we can adapt the results in this paper to prove that there exists $\varphi_{K,u}^{f}$ with no convergent normal form for some $K \in \mathbb{C}\{x,y\} \cap (x,y)$. The two main difficulties in the proof are:

- The formal invariants are slightly more complicated [7]. This phenomenon is isolated in the example $f = y^{a_0}(y-x)^{a_1}$ for $a_j \in \mathbb{N}$. If $a_j > 1$ the function $|Jac\varphi_{K,u}|_{|y=jx}$ is identically equal to 1, it is a trivial formal invariant. Anyway, there are always non-constant functions on $y=0$ and $y=x$ which are formal invariants. This is crucial to prove proposition 3.4 since otherwise we can not claim that the action of a formal conjugation on a fixed points curve is convergent. The rest of the proof is basically the same.
- The technical details in the proofs are in general trickier. That is the situation if the curve $f=0$ is complicated, for instance if its components are singular. Anyway, the proof basically follows the same lines. The additions are intended to make the strategy in this paper work. We chose the case $f = y(y-x)$ because the presentation is clearer but it contains all the main ideas.

References

[1] P. Ahern and J.-P. Rosay. Entire functions in the classification of differentiable germs tangent to the identity, in one or two variables. *Trans. of the American Math. Soc.*, 347(2):543–572, 1995.

[2] Yu. S. Il'yashenko, S.M. Voronin, et al. *Nonlinear Stokes phenomena*, volume 14 of *Advances in soviet mathematics*. American mathematical society, 1992.

[3] G. A. Kalyabin. Asymptotics of the smallest eigenvalues of Hilbert-type matrices. *Funct. Anal. Appl.*, 35(1):67–70, 2001.

[4] R. Pérez Marco. A note on holomorphic extensions. *Preprint. UCLA. http://xxx.lanl.gov/abs/math.DS/0009031*, 2000.

[5] R. Pérez Marco. Total convergence or general divergence in small divisors. *Communications in Mathematical Physics*, 3(223):451–464, 2001.

[6] J.-F. Mattei and R. Moussu. Holonomie et intégrales premières. *Ann. Sci. Ecole Norm. Sup.*, (13):469–523, 1980.

[7] J. Ribon. Difféomorphismes de $(\mathbb{C}^2, 0)$ tangents à l'identité qui préservent la fibration de Hopf. *Comptes Rendues de l'Académie des Sciences de Paris*, (Note soumise).

[8] J.C. Tougeron. *Ideaux de fonctions différentiables.* Springer, 1972.