

# PHYSICAL MEASURES FOR INFINITE-MODAL MAPS

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ABSTRACT. We analyse certain parametrized families of one-dimensional maps with infinitely many critical points from the measure-theoretical point of view. We prove that such families have absolutely continuous invariant probability measures for a positive Lebesgue measure subset of parameters. Moreover we show that both the densities of these measures and their entropy vary continuously with the parameter. In addition we obtain exponential rate of mixing for these measures and also that they satisfy the Central Limit Theorem.

## CONTENTS

1. Introduction	1
2. Idea of the proof	9
3. Refining the partition	10
4. Auxiliary lemmas	14
5. Probability of deep essential returns	22
6. Slow recurrence to the critical set	28
7. Fast expansion for most points	34
8. Exponential bound on derivatives along critical orbits	35
9. Constants depend uniformly on initial parameters	37
References	38

## 1. INTRODUCTION

One of the main goals of Dynamical Systems is to describe the global asymptotic behavior of the iterates of most points under a transformation of a compact manifold, either from a topological or from a probabilistic (or ergodic) point of view. The notion of *uniform hyperbolicity*, introduced by Smale in [Sm], and of *non-uniform hyperbolicity*, introduced by Pesin [P], have been the main tools to rigorously establish general results in the field.

While uniform hyperbolicity is defined using only a finite number of iterates of a given transformation, non-uniform hyperbolicity is a asymptotic notion to begin with, demanding the existence of non-zero Lyapunov exponents almost everywhere with respect to some invariant probability measure.

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On the one hand, the study of consequences of both notions in a general setting has a long history, see [M, S, KH, B, BP, Y, BDV] for details and thorough references.

On the other hand, it is rather hard in general to verify non-uniform hyperbolicity, since we must take into account the behavior of the iterates of the given map when time goes to infinity. This was first achieved in the groundbreaking work of Jakobson [J] on the quadratic family, which was extended for more general one-dimensional families with a unique critical point by many other mathematicians, see e.g. [BC1, R, MS, T, TTY]. One-dimensional families with two critical points were first considered in [Ro] and multimodal maps and maps with critical points and singularities with unbounded derivative were treated in [LT, LV, BLS]. To the best of our knowledge, maps with *infinitely many critical points* were first dealt with in [PRV].

The aim of this paper is prove that the dynamics of the family considered in [PRV], for a positive Lebesgue measure subset of parameters, is non-uniformly hyperbolic and to deduce some consequences from the ergodic point of view. These families naturally appear as one-dimensional models for the dynamical behavior near the unfolding of a double saddle-focus homoclinic connection of a flow in a three-dimensional manifold, see Figure 1 and [Sh]. The main novelty is that we prove *global stochastic behavior* for a family of maps with *infinitely many regions of contraction*.

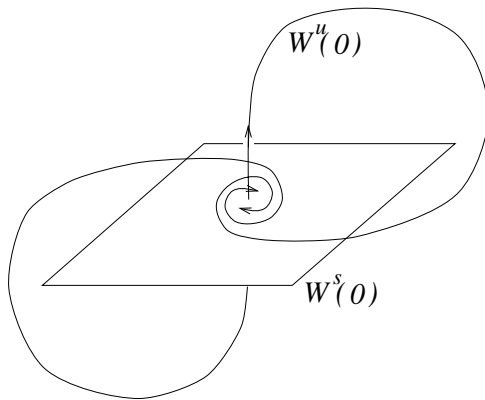


FIGURE 1. Double saddle-focus homoclinic connections

Roughly speaking, the family  $f_\mu$  of one-dimensional circle maps which we consider here is obtained from first-return maps of the three-dimensional flow in Figure 1 to appropriate cross-sections and disregarding one of the variables. This reduction to a one-dimensional model greatly simplifies the study of this kind of unfolding and provides important insight to its behavior. However as we shall see the dynamics of the reduced model is still highly complex.

This family of maps is obtained translating the left-hand side and right-hand side, vertically in opposite directions, of the graph of the map  $f = f_0$  described in Figure 2. This family approximates the behavior of any generic unfolding of  $f_0$ . Such unfolding was first studied in [PRV], where it was shown that for a positive Lebesgue measure subset  $S$  of parameters the map  $f_\mu$ , for  $\mu \in S$ , exhibits a chaotic attractor. This was achieved by proving that the orbits of the critical values of  $f_\mu$  have positive Lyapunov exponent and that  $f_\mu$  has a dense orbit.

Here we complement the topological description of the dynamics of  $f_\mu$  provided by [PRV] for  $\mu \in S$  with a probabilistic description constructing for the same parameters a *physical* probability measure  $\nu_\mu$ . We say that an invariant probability measure  $\nu$  is *physical* or *Sinai-Ruelle-Bowen* (SRB) if there is a positive Lebesgue measure set of points  $x \in \mathbf{S}^1$  such that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi \left( f_\mu^k(x) \right) = \int \varphi d\nu,$$

for any observable (continuous function)  $\varphi : \mathbf{S}^1 \rightarrow \mathbf{R}$ . The set of points  $x \in \mathbf{S}^1$  with this property is called the *basin* of  $\nu$ . SRB measures provide a statistical description of the asymptotic behavior of a large subset of orbits. Combining this with the results from [PRV] we have that  $f_\mu$  has non-zero Lyapunov exponent almost everywhere with respect to  $\nu_\mu$ , i.e.  $f_\mu$  is non-uniformly hyperbolic for  $\mu \in S$ .

The main feature needed for the construction of such measures is to obtain positive Lyapunov exponent for Lebesgue almost every point under the action of  $f_\mu$ ,  $\mu \in S$ . The presence of critical points is a serious obstruction to achieve an asymptotic expansion rate on the derivative of most points. Therefore the control of derivatives along orbits of the critical values is a central subject in the ergodic theory of one-dimensional maps.

The crucial role of the orbits of the critical values on the statistical description of the global dynamics of one-dimensional maps was already present in the pioneer work of Jakobson [J], who considered quadratic maps and obtained SRB measures for a positive Lebesgue measure subset of parameters.

This was later followed by the celebrated papers of Benedicks and Carleson [BC1, BC2], where the parameter exclusion technique was used to show that, for a positive Lebesgue measure subset of parameters, the derivative along the orbit of the unique critical value has exponential growth and satisfies what is nowadays called a *slow recurrence* condition to the critical point. This is enough to construct SRB measures for those parameters.

Recently, in the unimodal setting it was established that indeed the existence of SRB measures, and the exponential growth of the derivative along the orbit of the critical value, are equivalent conditions for Lebesgue almost every parameter for which there are no sinks, see [ALM, AM1, AM2]. See also [BLS] for multimodal maps.

In [PRV] the technique of exclusion of parameters was extended to deal with infinitely many critical orbits. Here we refine this technique to obtain exponential growth of the derivatives and slow recurrence to *the whole critical set for Lebesgue almost every orbit*. By [ABV] this ensures the existence of SRB measures for every parameter  $\mu \in S$ , see Subsection 1.2 and Theorem A.

Moreover we are able to control the measure of the set of points whose orbits are too close to the critical set during the first  $n$  iterates, showing that its Lebesgue measure is exponential in  $n$ , see Theorem B. In addition, the Lebesgue measure of the set of points whose derivative does not grow exponentially fast in the first  $n$  iterates decreases exponentially fast with  $n$ , see Theorem C. By recent general results on the ergodic theory of non-uniformly hyperbolic systems [ALP, G], both estimates above taken together imply exponential decay of correlations for Hölder continuous observables for  $\nu_\mu$  and also that  $\nu_\mu$  satisfies the Central Limit Theorem, for all  $\mu \in S$ , see Subsection 1.3 and Corollary D. We remark that these properties are likewise satisfied by uniformly expanding maps of  $\mathbf{S}^1$ , which are the touchstone of chaotic dynamics, see e.g. [B, V], in spite of the presence of infinitely many points with unbounded contraction (critical points).

Furthermore analyzing our arguments we observe that all the estimates obtained do not depend on the choice of the parameter  $\mu \in S$ . This shows after [A, AOT] that the density  $d\nu_\mu/d\lambda$  of the SRB measure  $\nu_\mu$  with respect to Lebesgue measure and its entropy  $h_{\nu_\mu}(f_\mu)$  vary continuously with  $\mu \in S$ , see Subsection 1.4 and Corollary E. This type of result was recently obtained in [F] for quadratic maps on the set of parameters constructed in [BC1, BC2] using a similar strategy. Hence *statistical properties of the maps  $f_\mu$  for  $\mu \in S$  are stable under small variations of the parameter*, i.e. this family is *statistically stable over  $S$* .

We emphasize that although the general strategy for proving our results follows [BC1, BC2, PRV, F] several new difficulties had to be overcome. Indeed unlike [BC1, BC2, PRV] where the main purpose was to obtain positive Lyapunov exponent along the orbits of critical values, here we need to obtain *positive Lyapunov exponents and slow recurrence to the critical set along almost every orbit*, which forces us to control the distance to the critical set for far more iterates than in [PRV]. Moreover with infinitely many critical points the derivative of the smooth maps we consider here is not globally bounded, unlike any smooth unimodal family [BC1, BC2, F], which demanded a proof of an exponential bound for the derivative along the orbits of critical values. In order to obtain such a bound for a positive Lebesgue measure set of parameters we changed the construction presented in [PRV] adding a new constraint in the exclusion of parameters algorithm.

The paper is organized as follows. We first state precisely our results in Subsections 1.2 to 1.4. We sketch the proof in Section 2. In Section 3 we explain how a sequence  $(\mathcal{P}_n)_{n \geq 0}$  of partitions of  $\mathbf{S}^1$  whose atoms have bounded distortion under action of  $f_\mu^n$  is constructed. Basic lemmas are stated and proved in Section 4. These are used to obtain the main estimates in Section 5. In Sections 6 and 7 we use the main estimates to deduce slow recurrence to the critical set and fast expansion for most points. In Section 8 we explain how an exponential upper bound on the growth of the derivatives along critical orbits can be obtained through an extra condition imposed on the construction performed in [PRV] without loss. Finally in Section 9 we keep track of the estimates obtained during our constructs and show that they do not depend on the parameter  $\mu \in S$ .

**1.1. Statement of the results.** Let  $\hat{f}$  be the interval map  $\hat{f} : [-\varepsilon_1, \varepsilon_1] \rightarrow [-1, 1]$  given by

$$\hat{f}(z) = \begin{cases} az^\alpha \sin(\beta \log(1/z)) & \text{if } z > 0 \\ -a|z|^\alpha \sin(\beta \log(1/|z|)) & \text{if } z < 0, \end{cases} \quad (1.1)$$

where  $a > 0$ ,  $0 < \alpha < 1$ ,  $\beta > 0$  and  $\varepsilon_1 > 0$ , see Figure 2.

Maps  $\hat{f}$  as above have infinitely many critical points, of the form

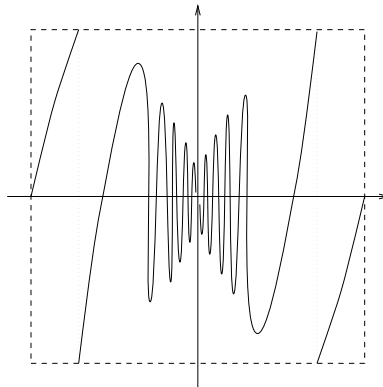
$$x_k = \hat{x} \exp(-k\pi/\beta) \text{ and } x_{-k} = -x_k \text{ for each large } k > 0 \quad (1.2)$$

where  $\hat{x} > 0$  is independent of  $k$ . Let  $k_0 \geq 1$  be the smallest integer such that  $x_k$  is defined for all  $|k| \geq k_0$ , and  $x_{k_0}$  is a local minimum.

We extend this expression to the whole circle  $S^1 = I/\{-1 \sim 1\}$ , where  $I = [-1, 1]$ , in the following way. Let  $\tilde{f}$  be an orientation-preserving expanding map of  $S^1$  such that  $\tilde{f}(0) = 0$  and  $\tilde{f}' > \tilde{\sigma}$  for some constant  $\tilde{\sigma} \gg 1$ . We define  $\varepsilon = 2 \cdot x_{k_0}/(1 + e^{-\pi/\beta})$ , so that  $x_{k_0}$  is the middle point of the interval  $(e^{-\pi/\beta}\varepsilon, \varepsilon)$  and fix two points  $x_{k_0} < \hat{y} < \tilde{y} < \varepsilon$ , with

$$|\hat{f}'(\hat{y})| \gg 1 \quad \text{and also} \quad 2 \frac{1 - \varepsilon^\tau}{1 + e^{-\pi/\beta}} x_{k_0} > \hat{y} > x_{k_0}, \quad (1.3)$$

where  $\tau$  is a small positive constant to be defined in what follows and we take  $k_0 = k_0(\tau)$  sufficiently big (and  $\varepsilon$  small enough) in order that (1.3) holds. Then we take  $f$  to be any

FIGURE 2. Graph of the circle map  $f$ .

smooth map on  $S^1$  coinciding with  $\hat{f}$  on  $[-\hat{y}, \hat{y}]$ , with  $\tilde{f}$  on  $S^1 \setminus [-\tilde{y}, \tilde{y}]$ , and monotone on each interval  $\pm[\hat{y}, \tilde{y}]$ .

Finally let  $f_\mu$  be the following one-parameter family of circle maps unfolding the dynamics of  $f = f_0$

$$f_\mu(z) = \begin{cases} f(z) + \mu & \text{for } z \in (0, \varepsilon] \\ f(z) - \mu & \text{for } z \in [-\varepsilon, 0) \end{cases} \quad (1.4)$$

for  $\mu \in (-\varepsilon, \varepsilon)$ . For  $z \in \mathbf{S}^1 \setminus [-\varepsilon, \varepsilon]$  we assume only that  $|\frac{\partial}{\partial \mu} f_\mu(z)| \geq 2$ . In what follows we write  $z_k^\pm(\mu) = f_\mu(x_k)$  for  $|k| \geq k_0$ .

**Theorem 1.1.** [PRV, Theorem A] *For a given  $\sigma \in (1, \sqrt{\tilde{\sigma}})$  there exists an integer  $N$  such that taking  $k_0 > N$  in the construction of  $f$ , we can find a small positive constant  $\tilde{\rho}$  such that for  $0 < \rho < \tilde{\rho}$  there exists a positive Lebesgue measure subset  $S \subset [-\varepsilon, -\varepsilon^2] \cup [\varepsilon^2, \varepsilon]$  satisfying for every  $\mu \in S$*

- (1) for all  $n \geq 1$  and all  $k_0 \leq |k| \leq \infty$ 
  - (a)  $|(f_\mu^n)'(z_k^\pm(\mu))| \geq \sigma^n$ ;
  - (b) either  $|f_\mu^n(f_\mu(x_l))| > \varepsilon$  or  $|f_\mu^n(f_\mu(x_l)) - x_{m(n)}| \geq e^{-\rho n}$ ;  
where  $x_{m(n)}$  is the critical point nearest  $f_\mu^n(f_\mu(x_l))$ .
- (2)  $\liminf_{n \rightarrow +\infty} n^{-1} \log |(f_\mu^n)'(z)| \geq \log \sigma/3$  for Lebesgue almost every point  $z \in S^1$ ;
- (3) there exists  $z \in S^1$  whose orbit  $\{f_\mu^n(z) : n \geq 0\}$  is dense in  $S^1$ .

The statement of Theorem 1.1 is slightly different from the main statement of [PRV] but the proof is contained therein.

**1.2. Existence of absolutely continuous invariant probability measures.** The purpose of this work is to prove that for parameters  $\mu \in S$  the map  $f_\mu$  admits a unique absolutely continuous invariant probability measure  $\nu_\mu$ , whose basin covers Lebesgue almost every point of  $\mathbf{S}^1$ , and to study some of the main statistical and ergodic properties of these measures.

In what follows we write  $\lambda$  for the normalized Lebesgue measure on  $\mathbf{S}^1$ . Our first result shows the existence of the *SRB* measure.

**Theorem A.** *Let  $\mu \in S$  be given. Then there exists a  $f_\mu$ -invariant probability measure  $\nu_\mu$  which is absolutely continuous with respect to  $\lambda$  and such that for  $\lambda$ -almost every  $x \in \mathbf{S}^1$  and*

every continuous  $\varphi : \mathbf{S}^1 \rightarrow \mathbf{R}$

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f_\mu^j(x)) = \int \varphi d\nu_\mu. \quad (1.5)$$

The proof is based on the technique of parameter exclusion developed in [PRV] to prove Theorem 1.1 and on recent results on hyperbolic times for non-uniformly expanding maps with singularities and criticalities, from [ABV].

In our setting *non-uniform expansion* means the same as item (2) of Theorem 1.1. However due to the presence of (infinitely many) criticalities and the singularity at 0, an extra condition is needed to construct the *SRB* measure: we need to control the average distance to the critical set along most orbits.

We say that  $f_\mu$  has *slow recurrence to the critical set*  $\mathcal{C} = \{x_k : |k| \geq k_0\} \cup \{0\}$  if, for every  $\delta > 0$ , there exists  $\flat > 0$  such that

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} -\log \text{dist}_\flat \left( f_\mu^k(x), \mathcal{C} \right) < \delta \quad \text{for Lebesgue almost every } x \in \mathbf{S}^1, \quad (1.6)$$

where  $\flat$  is a small positive value, and  $\text{dist}_\flat(x, y) = |x - y|$  if  $|x - y| \leq \flat$  and 1 otherwise.

Let  $f_0 : I \setminus \mathcal{C} \rightarrow I$  be a  $C^2$  map. We say that  $\mathcal{C}$  is a *non-flat critical set* if there exist constants  $B > 1$  and  $\beta > 0$  such that

$$\begin{aligned} \mathbf{S1}: \quad & \frac{1}{B} \text{dist}(x, \mathcal{C})^\beta \leq |f_0'(x)| \leq B \text{dist}(x, \mathcal{C})^{-\beta} ; \\ \mathbf{S2}: \quad & |\log |f_0'(x)| - \log |f_0'(y)|| \leq B \frac{|x - y|}{\text{dist}(x, \mathcal{C})^\beta} ; \end{aligned}$$

for every  $x, y \in I \setminus \mathcal{C}$  with  $|x - y| < \text{dist}(x, \mathcal{C})/2$ .

The following result ensuring the existence of finitely many physical probability measures is proved in [ABV].

**Theorem 1.2.** *If  $f_0$  satisfies (S1), (S2), is non-uniformly expanding and has slow recurrence to the critical set  $\mathcal{C}$ , then there are finitely many  $\mu_1, \dots, \mu_l$  ergodic absolutely continuous  $f_0$ -invariant probability measures such that Lebesgue almost every point in  $I$  belongs to the basin of  $\mu_i$  for some  $i \in \{1, \dots, l\}$ .*

The maps  $f_\mu$  satisfy conditions (S1)-(S2) above. Indeed we define  $y_k = 2 \cdot x_k / (1 + e^{-\pi/\beta})$ , for each  $k \geq k_0$ , so that  $x_k$  is the middle point of the interval  $(y_{k+1}, y_k)$ . We note that  $x_k$  is the *closest* critical point to any  $y \in (y_{k+1}, y_k)$ . We also use a similar notation for  $k \leq -k_0$ . We will argue using the following lemmas, which correspond to Lemmas 3.2 and 3.3 proved in [PRV].

**Lemma 1.3.** *There exists  $C > 0$  depending on  $\hat{f}$  only (not depending on  $\varepsilon$  nor  $\mu$ ) such that, for every  $x \in (y_{l+1}, y_l)$  and  $l \geq k_0$ , respectively,  $x \in (y_l, y_{l-1})$  and  $l \leq -k_0$ , we have*

$$\begin{aligned} (1) \quad & C^{-1} |x_l|^{\alpha-2} \cdot |x - x_l|^2 \leq |f(x) - f(x_l)| \leq C |x_l|^{\alpha-2} \cdot |x - x_l|^2; \\ (2) \quad & C^{-1} |x_l|^{\alpha-2} \cdot |x - x_l| \leq |f'(x)| \leq C |x_l|^{\alpha-2} \cdot |x - x_l|. \end{aligned}$$

**Lemma 1.4.** *Let  $s, t \in [y_{l+1}, y_l]$  with  $l \geq k_0$ , respectively,  $s, t \in [y_l, y_{l-1}]$  with  $l \leq -k_0$ . Then*

$$\left| \frac{f'_\mu(s) - f'_\mu(t)}{f'_\mu(t)} \right| \leq K_1 \frac{|s - t|}{|t - x_l|}$$

where  $K_1 > 0$  is independent of  $l, s, t, \varepsilon$  and  $\mu$ .

On the one hand since  $0 < \alpha < 1$ ,  $x \in (y_{l+1}, y_l)$  and  $|x_l| < 1$ , then from item 2 of Lemma 1.3  $C|x_l|^{\alpha-2}|x - x_l| = (C|x_l|^{\alpha-2}|x - x_l|^2)|x - x_l|^{-1} \leq (C|x_l|^{\alpha-2}|x_l|^2)|x - x_l|^{-1} \leq C|x - x_l|^{-1}$ .

On the other hand since  $\alpha - 2 < 0$  and  $|x_l| < 1$  we get  $C^{-1}|x_l|^{\alpha-2}|x - x_l| \geq C^{-1}|x - x_l|$ , showing that (S1) holds for  $f_\mu$  with  $B = C$  and  $\beta = 1$ .

To check that (S2) also holds we write

$$\frac{|f'_\mu(x)|}{|f'_\mu(y)|} = \frac{|f'_\mu(x) - f'_\mu(y) + f'_\mu(y)|}{|f'_\mu(y)|} \leq 1 + \frac{|f'_\mu(x) - f'_\mu(y)|}{|f'_\mu(y)|}$$

and then because  $\log(1 + z) \leq z$  for  $z > -1$  we get

$$|\log |f'_0(x)| - \log |f'_0(y)|| \leq \frac{|f'_\mu(x) - f'_\mu(y)|}{|f'_\mu(y)|} \leq K_1 \frac{|x - y|}{|x - x_l|}.$$

Thus according to Theorem 1.2 and after Theorem 1.1, we only need to show that  $f_\mu$  has slow recurrence to the critical set for  $\mu \in S$  to achieve the result stated in Theorem A. This is done in Sections 4 to 6, where a stronger result is obtained, as explained in what follows.

**1.3. Stretched exponential decay of correlations and Central Limit Theorem.** Using some recent developments on the statistical behavior of non-uniformly expanding maps [ALP, G] we are able to obtain exponential bounds on the decay of correlations between Hölder continuous observables for  $\nu_\mu$  with  $\mu \in S$ . In addition it follows from standard techniques that  $\nu_\mu$  also satisfies the Central Limit Theorem. In order to achieve this we refined the arguments in [PRV] using strong conditions on the exclusion of parameters extending the estimates obtained therein for critical orbits to get an exponential upper bound on the growth of the derivative along orbits of critical values, as explained in Section 8. Moreover we were able to extend most of the estimates from [PRV] for Lebesgue almost every orbit, yielding an exponential bound on the Lebesgue measure of the set of points whose average distance to the critical set during the first  $n$  iterates is small, as follows.

We first define the average distance to the critical set

$$\mathcal{C}_n^{\flat}(x) = \frac{1}{n} \sum_{j=0}^{n-1} -\log \text{dist}_{\flat}(f_\mu^j(x), \mathcal{C}). \quad (1.7)$$

for a given  $\flat > 0$ . Then we are able to prove the following.

**Theorem B.** *Let  $\mu \in S$  and  $\delta > 0$  be given. Then there are constants  $C_1, \xi_1, \flat > 0$  dependent on  $\hat{f}$ ,  $\sigma$ ,  $k_0$  and  $\delta$  only such that  $\mathcal{R}(x) = \min\{N \geq 1 : \mathcal{C}_n^{\flat}(x) < \delta, \forall n \geq N\}$  satisfies*

$$\lambda\left(\{x \in \mathbf{S}^1 : \mathcal{R}(x) > n\}\right) \leq C_1 \cdot e^{-\xi_1 \cdot n}.$$

We note that in particular this shows that  $f_\mu$  has slow recurrence to the critical set and ensures the existence of the SRB measure  $\nu_\mu$  for  $\mu \in S$  by Theorem 1.2.

We are also able to obtain, using the same techniques, an exponential bound on the set of points whose expansion rate up to time  $n$  is less than the one prescribed by item (2) of Theorem 1.1. This is detailed in Section 7.

**Theorem C.** *Let  $\mu \in S$  be given. Then there exist constants  $C_2, \xi_2 > 0$  dependent on  $\hat{f}$ ,  $\rho$  and  $k_0$  only such that  $\mathcal{E}(x) = \min\{N \geq 1 : |(f_\mu^N)'(x)| > \sigma^{N/3}, \forall n \geq N\}$  satisfies*

$$\lambda\left(\{x \in \mathbf{S}^1 : \mathcal{E}(x) > n\}\right) \leq C_2 \cdot e^{-\xi_2 \cdot n}.$$

In particular we obtain a new proof of item (2) of Theorem 1.1, which *does not follow directly from Theorem A plus the Ergodic Theorem* since it is not obvious whether  $\log |f'|$  is  $\nu_\mu$  integrable.

Theorems B and C together ensure that for  $\mu \in S$  there are constants  $C_3 > 0$  and  $\xi_3 \in (0, 1)$  such that  $\Gamma_n = \{x \in \mathbf{S}^1 : \mathcal{E}(x) > n \text{ or } \mathcal{R}(x) > n\}$  satisfies

$$\lambda(\Gamma_n) \leq C_3 \cdot e^{-\xi_3 \cdot n} \quad (1.8)$$

for all  $n \geq 1$ . This fits nicely into the following statements.

**Theorem 1.5.** *Let  $g : \mathbf{S}^1 \rightarrow \mathbf{S}^1$  be a transitive  $C^2$  local diffeomorphism outside a non-flat critical set  $\mathcal{C}$  such that (1.8) holds. Then*

- (1) [ALP, Theorem 1] *there exists an absolutely continuous invariant probability measure  $\nu$  and some finite power of  $g$  is mixing with respect to  $\nu$ ;*
- (2) [G, Theorem 1.1] *there exist constants  $C, c > 0$  such that the correlation function  $\text{Corr}_n(\varphi, \psi) = \left| \int (\varphi \circ g^n) \cdot \psi d\nu - \int \varphi d\nu \int \psi d\nu \right|$ , for Hölder continuous observables  $\varphi, \psi : \mathbf{S}^1 \rightarrow \mathbf{R}$ , satisfies for all  $n \geq 1$*

$$\text{Corr}_n(\varphi, \psi) \leq C \cdot e^{-c \cdot n}.$$

- (3) [ALP, Theorem 4]  *$\nu$  satisfies the Central Limit Theorem: given a Hölder continuous function  $\phi : \mathbf{S}^1 \rightarrow \mathbf{R}$  which is not a coboundary ( $\phi \neq \psi \circ g - \psi$  for any  $\psi : \mathbf{S}^1 \rightarrow \mathbf{R}$ ) there exists  $\theta > 0$  such that for every interval  $J \subset \mathbf{R}$*

$$\lim_{n \rightarrow \infty} \nu \left( \left\{ x \in \mathbf{S}^1 : \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} \left( \phi(g^j(x)) - \int \phi d\nu \right) \in J \right\} \right) = \frac{1}{\theta \sqrt{2\pi}} \int_J e^{-t^2/2\theta^2} dt.$$

It is then straightforward to deduce the following conclusion.

**Corollary D.** *For every  $\mu \in S$  the map  $f_\mu$  has exponential decay of correlations for Hölder continuous observables and satisfies the Central Limit Theorem with respect to the SRB measure  $\nu_\mu$ .*

**1.4. Continuous variation of densities and of entropy.** We note that during the arguments in Sections 2 to 7 the constants used in every estimation depend uniformly on the values of  $\rho, \sigma$  and  $\varepsilon$  which can be set right from the start of the construction that proves Theorems B and C. This enables us to use recent results of *statistical stability and continuity of the SRB entropy* from [A, AOT], showing that *both the densities of the SRB measures  $\nu_\mu$  and the entropy vary continuously with  $\mu \in S$ .*

Let  $\mathcal{F}$  be a family of  $C^2$  maps of  $\mathbf{S}^1$  outside a fixed non-flat critical set  $\mathcal{C}$  such that for any given  $f \in \mathcal{F}$  and  $\delta_1 > 0$  there exists  $\delta_2 > 0$  satisfying for every measurable subset  $E \subset \mathbf{S}^1$

$$\lambda(E) < \delta_2 \implies \lambda(f^{-1}(E)) < \delta_1,$$

that is  $f_*(\lambda) \ll \lambda$ . We say that a family  $\mathcal{F}$  as above is a *non-degenerate family of maps*.

**Theorem 1.6.** *Let a non-degenerate family  $\mathcal{F}$  of  $C^2$  maps of  $\mathbf{S}^1$  outside a fixed non-flat critical set  $\mathcal{C}$  be given such that for every  $f \in \mathcal{F}$  the corresponding functions  $\mathcal{E}, \mathcal{R} : \mathbf{S}^1 \rightarrow \mathbf{N}$  define a family  $(\Gamma_n)_{n \geq 1}$  satisfying (1.8) with constants  $C_3, \xi_3$  not depending on  $f \in \mathcal{F}$ . Then*

- (1) [A, Theorem A] *the map  $(\mathcal{F}, d_{C^2}) \rightarrow (L^1(\lambda), \|\cdot\|_1), f \mapsto \frac{d\nu_f}{d\lambda} \in L^1(\lambda)$  is continuous, where  $d_{C^2}$  is the  $C^2$  distance and  $\|\cdot\|_1$  the  $L^1$ -norm;*
- (2) [AOT, Corollary C] *the map  $(\mathcal{F}, d_{C^2}) \rightarrow \mathbf{R}, f \mapsto h_{\nu_f}(f)$  is continuous.*



We observe that  $\mathcal{F} = \{f_\mu : \mu \in S\}$  satisfies all the above conditions since

- $\hat{f}$  is a  $C^\infty$  map whose non-zero singularities, albeit infinitely many, are of quadratic type, and near zero  $\hat{f}$  is asymptotic to  $|z|^\alpha$ ;
- $f_\mu$  is obtained from  $\hat{f}$  through a local diffeomorphism extension plus two translations (or rigid rotations when viewed on  $\mathbf{S}^1$ );
- the values of  $\beta, \varepsilon, \sigma, \rho$  can be chosen so that
  - $S$  is given by Theorem 1.1 with positive Lebesgue measure;
  - $f_\mu$  for  $\mu \in S$  satisfies (1.8) with  $C_3, \xi_3 > 0$  depending only on  $\varepsilon, \sigma, \rho$  — this is detailed in Section 9.

Thus we deduce the following corollary which shows that statistical properties of  $f_\mu$  are stable under small variations of the parameter  $\mu$  within the set  $S$ .

**Corollary E.** *The following maps are both continuous:*

$$\begin{array}{ccc} S & \rightarrow & (L^1(\lambda), \|\cdot\|_1) \\ \mu & \mapsto & \frac{d\nu_\mu}{d\lambda} \end{array} \quad \text{and} \quad \begin{array}{ccc} S & \rightarrow & \mathbf{R} \\ \mu & \mapsto & h_{\nu_\mu}(f) \end{array} .$$

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## 2. IDEA OF THE PROOF

From now on we fix a parameter  $\mu \in S$  and write  $\mathcal{C}_\infty = \cup_{n=0}^\infty (f^n)^{-1}(\mathcal{C})$  for the set of pre-orbits of the critical set  $\mathcal{C}$ . Following [PRV] we consider a convenient partition  $\{I(l, s, j)\}$  of the phase space into subintervals, with a bounded distortion property: trajectories with the same itinerary with respect to this partition have derivatives which are comparable, up to a multiplicative constant. This is done as follows. Let  $l \geq k_0$  and  $y_l \in (x_l, x_{l-1})$  be as defined in Subsections 1.1 and 1.2:  $x_l$  is the middle point of  $(y_{l+1}, y_l)$ . We partition  $(x_l, y_l)$  into subintervals

$$I(l, s) = (x_l + e^{-(\pi/\beta)s} \cdot (y_l - x_l), x_l + e^{-(\pi/\beta)(s-1)} \cdot (y_l - x_l)), \quad s \geq 1.$$

We denote by  $I(l, -s)$  the subinterval of  $(y_{l+1}, x_l)$  symmetrical to  $I(l, s)$  with respect to  $x_l$ .

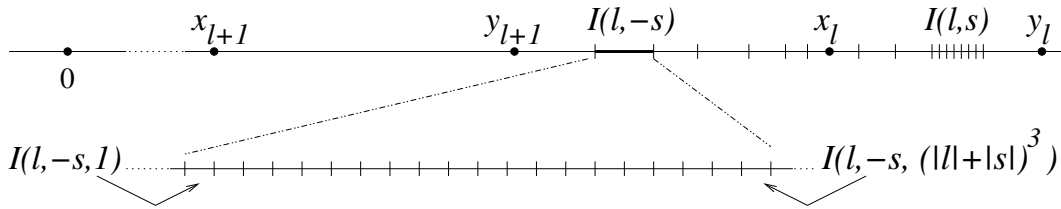


FIGURE 3. The initial partition  $\mathcal{P}_0$ .

We subdivide  $I(l, \pm s)$  into  $(l + |s|)^3$  intervals  $I(l, \pm s, j)$ ,  $1 \leq j \leq (l + |s|)^3$  with equal length and  $j$  increasing as  $I(l, \pm s, j)$  is closer to  $x_l$ , see Figure 3. We also perform entirely symmetric constructions for  $l \leq -k_0$ . Let  $I(\pm k_0, 1, 1)$  be the intervals having  $\pm\varepsilon$  in their boundaries. Clearly we may suppose that  $I(\pm k_0, 1, 1)$  are contained in the region  $\mathbf{S}^1 \setminus [-\tilde{y}, \tilde{y}]$  where  $f$  coincides with  $\tilde{f}$ , and so  $|f'| > \sigma_0 > 1$ . Finally, for completeness, we set  $I(0, 0, 0) = I(0, 0) = \mathbf{S}^1 \setminus [-\varepsilon, \varepsilon]$ .

**Remark 2.1.** *By the definition of  $I(l, s, j)$*

$$|I(l, s, j)| = a_1 \frac{e^{-(\pi/\beta)(|l|+|s|)}}{(|l|+|s|)^3} \quad \text{and} \quad a_2 e^{-(\pi/\beta)(|l|+|s|)} \leq \text{dist}(I(l, s, j), x_l) \leq a_2 e^{-(\pi/\beta)(|l|+|s|-1)}$$

where  $|I|$  denotes the length of the interval  $I$ ,

$$a_1 = \hat{x} \frac{(e^{(\pi/\beta)} - 1)^2}{e^{(\pi/\beta)} + 1} \quad \text{and} \quad a_2 = \hat{x} \frac{e^{(\pi/\beta)} - 1}{e^{(\pi/\beta)} + 1} < 1.$$

Moreover for any  $m \geq 1$  we have  $|x_m - x_{m+1}| = \hat{x} \cdot (1 - e^{-\pi/\beta}) \cdot e^{-\frac{\pi}{\beta}m}$ . In addition we have according to the sign of  $s$

$$\text{dist}(I(l, s, j), 0) = |x_l| \pm \text{dist}(I(l, s, j), x_l)$$

and thus  $|\hat{x} - a_2| \cdot e^{-(\pi/\beta)|l|} \leq \text{dist}(I(l, s, j), 0) \leq (a_2 + \hat{x}) \cdot e^{-(\pi/\beta)|l|}$ .

We will separate the orbit of a point  $x_0 \in I \setminus \mathcal{C}_\infty$  into sequences of consecutive iterates according to whether the point is near  $\mathcal{C}$  or is in the expanding region  $I(0, 0, 0)$ . When  $x_n = f_\mu^n(x_0)$  is near  $\mathcal{C}$ , we say that  $n$  is a *return time* and the expansion may be lost. But since we know that for  $\mu \in S$  the derivatives along the critical orbits grow exponentially fast, we shadow the orbit of  $x_n$  during a *binding period* by the orbit of the nearest critical point and borrow its expansion. At the end of this binding period, the expansion is completely recovered, which will be explained precisely in Section 4.

This picture is complicated by the infinite number of critical points and by the possible returns near another critical point during a binding period. Iterates outside binding periods and return times are *free iterates*, where the derivative is uniformly expanded.

Our main objective is to obtain slow recurrence to  $\mathcal{C}$ , which means that the returns of generic orbits are not too close to  $\mathcal{C}$  on the average. However *even at a free iterate the orbits may be very close to the critical set*, by the geometry of the graph of  $f_0$ , which demands a deeper analysis to achieve slow recurrence to the critical set. Moreover since  $|f'_\mu|$  is not bounded from above in our setting, we do not automatically have an exponential bound on the derivative along orbits of critical values, which is needed to better control the recurrence to  $\mathcal{C}$  and must be proved by a separate argument involving a stronger exclusion of parameters in the algorithm presented in [PRV].

Using the slow recurrence we show that the derivative along the orbit of Lebesgue almost every point grows exponentially fast. Using the estimates from Sections 3 to 5 we are able to obtain more: we deduce the exponential estimates on Theorems B and C in Sections 6 and 7.

Finally the refinement on the parameter exclusion in [PRV] and the dependence of the constants on the choices made during the entire construction are detailed in Sections 8 and 9 respectively, where we show that the estimates are uniform on  $\mu \in S$ .

### 3. REFINING THE PARTITION

We are going to build inductively a sequence of partitions  $\mathcal{P}_0, \mathcal{P}_1, \dots$  of  $I$  (modulus a zero Lebesgue measure set) into intervals. We will define inductively the sets  $R_n(\omega) = \{r_1, \dots, r_{\gamma(n)}\}$  which is the set of the return times of  $\omega \in \mathcal{P}_n$  up to  $n$  and a set  $Q_n(\omega) = \{(l_1, s_1, j_1), \dots, (l_{\gamma(n)}, s_{\gamma(n)}, j_{\gamma(n)})\}$ , which records the indexes of the intervals such that  $f_\mu^{r_i}(\omega) \in I(l_i, s_i, j_i)$ ,  $i = 1, \dots, r_{\gamma(n)}$ .

In the process we will show inductively that for all  $n \in \mathbf{N}_0$

$$\forall \omega \in \mathcal{P}_n \quad f_\mu^{n+1}|_\omega \text{ is a diffeomorphism,} \quad (3.1)$$

which is essential for the construction itself. For  $n = 0$  we define

$$\mathcal{P}_0 = \{I(0, 0, 0)\} \cup \{I(l, s, j) : |l| \geq k_0, s \geq 1, 1 \leq j \leq (|l| + |s|)^3\}.$$

It is obvious that  $\mathcal{P}_0$  satisfies (3.1) for  $n = 0$ . We set  $R_0(I(0, 0, 0)) = \emptyset$  and  $R_0(I(l, s, j)) = \{0\}$ ,  $Q_0(I(l, s, j)) = \{(l, s, j)\}$  for all possible  $(l, s, j) \neq (0, 0, 0)$ .

**Remark 3.1.** *This means that every  $I(l, s, j)$  with  $|l| \geq k_0$ ,  $|s| \geq 1$  and  $j = 1, \dots, (|l| + |s|)^3$  has a return at time 0, by definition. This will be important in Section 6.*

For each  $(l, s)$  with  $|l| \geq k_0$  and  $|s| \geq 1$  such that

$$e^{-(\pi/\beta)|s|} \cdot \frac{1 - e^{-(\pi/\beta)}}{1 + e^{-(\pi/\beta)}} < \tau, \quad \text{i.e.} \quad |s| > s(\tau) = -\frac{\beta}{\pi} \log \left( \tau \cdot \frac{1 + e^{-(\pi/\beta)}}{1 - e^{-(\pi/\beta)}} \right), \quad (3.2)$$

we define the *binding period*  $p(x)$  of  $x \in I(l, s)$  to be the largest integer  $p \geq 0$  such that

$$\begin{aligned} |f_\mu^h(x_l)| \leq \varepsilon \quad \text{and} \quad |f_\mu^h(x) - f_\mu^h(x_l)| &\leq |f_\mu^h(x_l) - x_{m(h-1)}| e^{-\tau h} \\ \text{or} \\ |f_\mu^h(x_l)| > \varepsilon \quad \text{and} \quad |f_\mu^h(x) - f_\mu^h(x_l)| &\leq \varepsilon^{1+\tau} e^{-\tau h} \end{aligned} \quad (3.3)$$

for all  $1 \leq h \leq p$ , where  $x_{m(h)}$  is the critical point nearest  $f_\mu^h(f_\mu(x_l))$  and  $\tau > 0$  is a small constant to be specified during the construction.

Failing condition (3.2) means that  $I(l, s)$  is not close enough to  $\mathcal{C}$  since

$$|x - x_l| \geq e^{-(\pi/\beta)|s|} \cdot (y_l - x_l) \geq e^{-(\pi/\beta)|s|} \cdot \frac{1 - e^{-(\pi/\beta)}}{1 + e^{-(\pi/\beta)}} \cdot |x_l| \geq \tau |x_l|$$

for all  $x \in I(l, s)$ , and in this case there is no expansion loss at the point  $x$ . Indeed by Lemma 1.3 and using the definition of  $x_l$  from (1.2) we get

$$|f'_\mu(x)| \geq C^{-1} \cdot |x_l|^{\alpha-2} \cdot |x - x_l| \geq C^{-1} \cdot |x_l|^{\alpha-2} \cdot \tau |x_l| = \frac{\tau \hat{x}^{\alpha-2}}{C} \cdot e^{(1-\alpha)\frac{\pi}{\beta}|l|} \quad (3.4)$$

Since  $1 - \alpha > 0$  and  $|l| \geq k_0$ , this ensures that  $|f'_\mu(x)| > 1$  if we take  $k_0 = k_0(\tau)$  big enough.

**Remark 3.2.** *As we will explain along the proof, the values of  $k_0$  and  $\tau^{-1}$  will both need to be taken sufficiently big. We note that  $k_0 \rightarrow \infty$  when  $\tau \rightarrow 0^+$ . For more on these dependencies see Section 9.*

We define the *binding period*  $p(l, s)$  of the interval  $I(l, s)$  to be the smallest binding period of all points of this interval, that is  $p(l, s) = \inf\{p(x) : x \in I(l, s)\}$ .

For  $(l, s, j)$  with  $|l| \geq k_0$ ,  $|s| \geq 1$  and  $1 \leq j \leq (|l| + |s|)^3$ , we write

$$I(l, s, j)^+ = I(l, s_1, j_1) \cup I(l, s, j) \cup I(l, s_2, j_2),$$

where  $I(l, s_1, j_1)$  and  $I(l, s_2, j_2)$  are the intervals adjacent to  $I(l, s, j)$  in  $\mathcal{P}_0$ .

Now we assume that  $\mathcal{P}_{n-1}$  is defined, satisfies (3.1) and  $R_{n-1}, Q_{n-1}$  are also defined on each element of  $\mathcal{P}_{n-1}$ . Fixing an interval  $\omega \in \mathcal{P}_{n-1}$  there are three possible situations.

- (1) If  $R_{n-1}(\omega) \neq \emptyset$  and  $n < r_{\gamma(n-1)} + p(l_{\gamma(n-1)}, s_{\gamma(n-1)})$  then we say that  $n$  is a *bound time* for  $\omega$ , put  $\omega \in \mathcal{P}_n$  and set  $R_n(\omega) = R_{n-1}(\omega)$ ,  $Q_n(\omega) = Q_{n-1}(\omega)$ .
- (2) If either  $R_{n-1}(\omega) = \emptyset$ , or  $n \geq r_{\gamma(n-1)} + p(l_{\gamma(n-1)}, s_{\gamma(n-1)})$  and  $f_\mu^n(\omega) \subset I(0, 0, 0) \cup I(\pm k_0, 1, 1)$  and  $f_\mu^n(\omega)$  does not contain any  $I(\pm k_0, 1, 1)$ , then we say that  $n$  is a *free time* for  $\omega$ , put  $\omega \in \mathcal{P}_n$  and set  $R_n(\omega) = R_{n-1}(\omega)$ ,  $Q_n(\omega) = Q_{n-1}(\omega)$ .

(3) If the two conditions above fail then we consider two cases:

(a)  $f_\mu^n(\omega)$  does not cover completely any  $I(l, s, j)$ , with  $|l| \geq k_0, |s| \geq 1$  and  $j = 1, \dots, (|l| + |s|)^3$ . Because  $f_\mu^n$  is continuous and  $\omega$  is an interval,  $f_\mu^n(\omega)$  is also an interval and thus it is contained in some  $I(l, s, j)^+$ , for a certain  $|l| \geq k_0, |s| \geq 1$  and  $l = 1, \dots, (|l| + |s|)^3$ , which is called the *host interval*.

If  $|s| > s(\tau)$ , then this  $n$  is an *inessential return time* for  $\omega$  and we set  $R_n(\omega) = R_{n-1}(\omega) \cup \{n\}$ ,  $Q_n(\omega) = Q_{n-1}(\omega) \cup \{(l, s, j)\}$  and put  $\omega \in \mathcal{P}_n$ .

Otherwise,  $|s| \leq s(\tau)$  and there is no expansion loss, thus  $n$  is again a *free time*, put  $\omega \in \mathcal{P}_n$  and set  $R_n(\omega) = R_{n-1}(\omega)$ ,  $Q_n(\omega) = Q_{n-1}(\omega)$ .

(b)  $f_\mu^n(\omega)$  contains at least an interval  $I(l, s, j)$ , with  $|l| \geq k_0, |s| \geq 1$  and  $j = 1, \dots, (|l| + |s|)^3$ , in which case we partition  $\omega$  as follows. We consider the sets

$$\omega'_{l,s,j} = f_\mu^{-n}(I(l, s, j)) \cap \omega \text{ for } \begin{cases} |l| \geq k_0, |s| \geq 1, 1 \leq j \leq (|l| + |s|)^3 \\ \text{and } (|l|, s, j) \neq (k_0, 1, 1) \end{cases};$$

$$\omega'_{0,0,0} = f_\mu^{-n}(I(0, 0, 0) \cup I(\pm k_0, 1, 1)) \cap \omega.$$

Denoting by  $\mathcal{I}$  the set of indexes  $(l, s, j)$  such that  $\omega'_{l,s,j} \neq \emptyset$  we have

$$\omega \setminus f_\mu^{-n}(\mathcal{C}) = \bigcup_{(l,s,j) \in \mathcal{I}} \omega'_{l,s,j}. \quad (3.5)$$

By the induction hypothesis  $f_\mu^n|_\omega$  is a diffeomorphism and then each  $\omega'_{l,s,j}$  is an interval. Moreover  $f_\mu^n(\omega'_{l,s,j})$  covers the whole  $I(l, s, j)$  for  $|l| \geq k_0, |s| \geq 1, j = 1, \dots, (|l| + |s|)^3$ , except eventually for one or two end intervals. When  $f_\mu^n(\omega'_{l,s,j})$  does not cover entirely  $I(l, s, j)$  we enlarge  $\omega'_{l,s,j}$  gluing it with its adjacent intervals in (3.5), getting a new decomposition of  $\omega \setminus f_\mu^{-n}(\mathcal{C})$  into intervals  $\omega_{l,s,j}$  such that

$$I(l, s, j) \subset f_\mu^n(\omega_{l,s,j}) \subset I(l, s, j)^+ \text{ for } |l| \geq k_0, |s| \geq 1, 1 \leq j \leq (|l| + |s|)^3.$$

We put  $\omega_{l,s,j} \in \mathcal{P}_n$  for all  $(l, s, j)$  such that  $\omega_{l,s,j} \neq \emptyset$ , with  $|l| \geq k_0$ . This results in a refinement of  $\mathcal{P}_{n-1}$  at  $\omega$ .

We set  $Q_n(\omega_{l,s,j}) = Q_{n-1}(\omega) \cup \{(l, s, j)\}$  for every non-empty interval  $\omega_{l,s,j}$ . The interval  $I(l, s, j)^+$  is the *host interval* of  $\omega_{l,s,j}$ . We define  $R_n(\omega_{l,s,j}) \cup \{n\}$  and we say that  $n$  is an

- (i) *escape time* for  $\omega_{l,s,j}$  if  $|l| \geq k_0$  and  $s \leq s(\tau)$ .
- (ii) *essential return time* for  $\omega_{l,s,j}$  if  $|l| \geq k_0$  and  $s > s(\tau)$ .

**Remark 3.3.** We note that if  $n$  is a free time or an escape time for  $z$ , then  $x = f_\mu^n(z)$  either is in the region  $\mathbf{S}^1 \setminus [-\varepsilon, \varepsilon]$  and thus  $|f'_\mu(x)| \gg 1$ , or satisfies the inequality (3.4). Hence on free times and escape times we always have expansion of derivatives bounded from below by some uniform constant  $\sigma_0 > 1$ . We stress that we may and will assume that  $\sigma_0 > \max\{e, \sqrt{\bar{\sigma}}\}$  in what follows.

To complete the induction step all we need is to check that (3.1) holds for  $\mathcal{P}_n$ . Since for any interval  $J \subset \mathbf{S}^1$

$$\left. \begin{array}{l} f_\mu^n|_J \text{ is a diffeomorphism} \\ \mathcal{C} \cap f_\mu^n(J) = \emptyset \end{array} \right\} \Rightarrow f_\mu^{n+1}|_J \text{ is a diffeomorphism,}$$

all we are left to prove is that  $\mathcal{C} \cap f_\mu^n(\omega) = \emptyset$  for all  $\omega \in \mathcal{P}_n$ .

Let  $\omega \in \mathcal{P}_n$ . If  $n$  is a free time for  $\omega$  then we are done. If  $n$  is either a return time for  $\omega$ , essential or inessential, or an escape time, then by construction we have that  $f_\mu^n(\omega) \subset I(l, s, j)^+$  for some  $|l| \geq k_0$ ,  $|s| \geq 1$ ,  $j = 1, \dots, (|l| + |s|)^3$  (or for  $(l, s, j) = (0, 0, 0)$ ) and thus  $\mathcal{C} \cap f_\mu^n(\omega) = \emptyset$ . For the binding case we use the following estimate.

**Proposition 3.4.** *Let  $n \geq 1$  and  $\omega \in \mathcal{P}_n$  be such that  $n$  is a binding time for  $\omega$ . Then*

$$\text{either } |f_\mu^n(x)| > x_{k_0} \quad \text{or} \quad \text{dist}(f_\mu^n(x), \mathcal{C}) \geq \rho_0 \cdot e^{-\rho(n-r)}$$

for all  $x \in \omega$ , where  $r = r_{\gamma(n-1)}$  is the last return time for  $\omega$  with  $n < r + p(f_\mu^r(\omega))$  and  $\rho_0 = 1 - e^{-\rho}$ .

This result is enough to conclude that  $\mathcal{C} \cap f_\mu^n(\omega) = \emptyset$ , completing the induction step.

*Proof.* If  $n$  is a binding time for  $\omega$ , then because  $\mu \in S$ , we know from 1.1(1b) that for every  $h \geq 1$  and for all  $|l| \geq k_0$  either

$$|f_\mu^h(f_\mu(x_l))| > \varepsilon \quad \text{or} \quad |f_\mu^h(f_\mu(x_l)) - x_{m(h)}| \geq e^{-\rho h}, \quad (3.6)$$

where  $x_{m(h)}$  is the critical point closest to  $f_\mu^h(f_\mu(x_l))$  as before. In the former case by the definition of binding period we get for all  $x \in \omega$  that

$$\begin{aligned} |f_\mu^n(x)| &\geq |f_\mu^{n-r_{\gamma(n-1)}}(x_{l_{\gamma(n-1)}})| - |f_\mu^n(x) - f_\mu^{n-r_{\gamma(n-1)}}(x_{l_{\gamma(n-1)}})| \\ &\geq \varepsilon - \varepsilon^{1+\tau} e^{-\tau(n-r_{\gamma(n-1)})} \geq \frac{\varepsilon}{x_{k_0}} (1 - \varepsilon^\tau) x_{k_0} = 2 \frac{1 - \varepsilon^\tau}{1 + e^{-\pi/\beta}} x_{k_0} > x_{k_0}, \end{aligned}$$

according to condition (1.3) on the choice of  $k_0$  as a function of  $\tau$ .

In the latter case in (3.6) we get that  $|f_\mu^n(x) - x_{m(n-r_{\gamma(n-1)})}|$  is bounded by

$$\begin{aligned} &|f_\mu^{n-r_{\gamma(n-1)}}(x_{l_{\gamma(n-1)}}) - x_{m(n-r_{\gamma(n-1)})}| - |f_\mu^n(x) - f_\mu^{n-r_{\gamma(n-1)}}(x_{l_{\gamma(n-1)}})| \\ &\geq e^{-\rho(n-r_{\gamma(n-1)})} - |f_\mu^{(n-r_{\gamma(n-1)})}(x_{l_{\gamma(n-1)}}) - x_{m(n-r_{\gamma(n-1)})}| e^{-\tau(n-r_{\gamma(n-1)})} \\ &\geq e^{-\rho(n-r_{\gamma(n-1)})} \cdot (1 - \varepsilon) > 0, \end{aligned} \quad (3.7)$$

by definition of binding (3.3) and because we assume that  $\rho < \tau$ .

To complete the proof we consider the case when  $x_{l_{\gamma(n-1)}}$  is not the closest critical point to  $f_\mu^n(x)$ . We first argue that no  $x' \in \mathcal{C}$  is between  $f_\mu^n(x)$  and  $f_\mu^{n-r_{\gamma(n-1)}}(x_{l_{\gamma(n-1)}})$ . For otherwise using (3.3) and the definition of  $x_{l_{\gamma(n-1)}}$  we would have

$$\begin{aligned} \frac{1}{2} \cdot |x' - x_{m(n-r_{\gamma(n-1)})}| &< |f_\mu^n(x) - f_\mu^{n-r_{\gamma(n-1)}}(x_{l_{\gamma(n-1)}})| \\ &\leq |f_\mu^{n-r_{\gamma(n-1)}}(x_{l_{\gamma(n-1)}}) - x_{m(n-r_{\gamma(n-1)})}| \cdot e^{-\tau(n-r_{\gamma(n-1)})} \\ &\leq \frac{e^{-\tau(n-r_{\gamma(n-1)})}}{2} \cdot |x' - x_{m(n-r_{\gamma(n-1)})}|, \end{aligned}$$

a contradiction because  $e^{-\tau(n-r_{\gamma(n-1)})} < 1$ . Hence there exists  $x' \in \mathcal{C}$  such that  $x'$  and  $x_{m(n-r_{\gamma(n-1)})}$  are consecutive critical points in  $\mathcal{C}$  and both  $f_\mu^n(x)$  and  $f_\mu^{n-r_{\gamma(n-1)}}(x_{l_{\gamma(n-1)}})$  are

between  $x'$  and  $x_{m(n-r_\gamma(n-1))}$ . But then

$$\begin{aligned} |x' - f_\mu^n(x)| &\geq |x' - f_\mu^{n-r_\gamma(n-1)}(x_{l_\gamma(n-1)})| - |f_\mu^n(x) - f_\mu^{n-r_\gamma(n-1)}(x_{l_\gamma(n-1)})| \\ &\geq \frac{1}{2}|x' - x_{m(n-r_\gamma(n-1))}| - |f_\mu^{n-r_\gamma(n-1)}(x_{l_\gamma(n-1)}) - x_{m(n-r_\gamma(n-1))}| e^{-\tau(n-r_\gamma(n-1))} \\ &\geq \frac{1}{2}|x' - x_{m(n-r_\gamma(n-1))}| - \frac{1}{2} \cdot |x' - x_{m(n-r_\gamma(n-1))}| \cdot e^{-\tau(n-r_\gamma(n-1))} \\ &\geq \frac{1}{2}|x' - x_{m(n-r_\gamma(n-1))}| \cdot (1 - e^{-\tau(n-r_\gamma(n-1))}). \end{aligned}$$

Setting  $m = m(n - r_\gamma(n-1))$  for simplicity, we observe that since  $x'$  and  $x_m$  are consecutive critical points we have that  $x'$  is either  $x_{m+1}$  or  $x_{m-1}$ , thus

$$|x' - x_m| \geq 2|f_\mu^{n-r_\gamma(n-1)}(x_{l_\gamma(n-1)}) - x_m| \geq 2e^{-\rho(n-r_\gamma(n-1))}.$$

Combining the two last inequalities and taking into account that  $\rho < \tau$  gives

$$|x' - f_\mu^n(x)| \geq e^{-\rho(n-r_\gamma(n-1))} \cdot (1 - e^{-\tau(n-r_\gamma(n-1))}) \geq e^{-\rho(n-r_\gamma(n-1))} \cdot (1 - e^{-\rho(n-r_\gamma(n-1))}).$$

Choosing  $\rho$  and  $\varepsilon$  close to 0 such that  $e^{-\rho} > \varepsilon$  ( $\rho < \log 2$  and  $\varepsilon < 1/2$  is enough) we get  $1 - \varepsilon > 1 - e^{-\rho}$  and we may then replace  $1 - \varepsilon$  by  $\rho_0 = 1 - e^{-\rho}$  in (3.7), finishing the proof since  $1 - e^{-\rho(n-r_\gamma(n-1))} \geq 1 - e^{-\rho} = \rho_0$ .  $\square$

#### 4. AUXILIARY LEMMAS

Here we collect some intermediate results needed for the proofs of the main estimates. In all that follows we write  $C$  for a constant depending only on the initial map  $\hat{f}$  or  $f_0$ .

**Lemma 4.1.** [PRV, Lemma 3.1] *Given  $\alpha_1, \alpha_2, \beta_1, \beta_2$  with  $\frac{\alpha_1}{\alpha_2} \neq \frac{\beta_1}{\beta_2}$ , there exists  $\delta > 0$  such that, for every  $x$ , at least one of the following assertions hold:*

$$|\alpha_1 \sin x + \beta_1 \cos x| \geq \delta \quad \text{or} \quad |\alpha_2 \sin x + \beta_2 \cos x| \geq \delta.$$

Using this we obtain the following property of bounded distortion for the second derivative near critical points.

**Lemma 4.2.** *There exists a constant  $C > 0$  depending only on  $\hat{f}$  such that for every  $k \geq k_0$  and  $t \in [y_{k+1}, y_k]$  we have*

$$\frac{1}{C} \leq \frac{|f''(t)|}{|f''(x_k)|} \leq C.$$

*Proof.* Indeed we have

$$f'(x) = a|x|^{\alpha-1}[\alpha \sin(\beta \log |x|^{-1}) - \beta \cos(\beta \log |x|^{-1})]$$

$$f''(x) = a|x|^{\alpha-2}[A \sin(\beta \log |x|^{-1}) + B \cos(\beta \log |x|^{-1})]$$

for some  $A$  and  $B$  depending only on  $\alpha$  and  $\beta$ . Applying the previous lemma we get, since  $f'(x_k) = 0$ , that

$$\left| \frac{x_{k-1}}{x_k} \right|^{\alpha-2} \cdot \frac{\min\{|A| \pm |B|\}}{|A| + |B|} \leq \frac{|f''(t)|}{|f''(x_k)|} \leq \left| \frac{x_{k+1}}{x_k} \right|^{\alpha-2} \cdot \frac{|A| + |B|}{\delta}.$$

Thus by (1.2) we obtain

$$e^{\frac{\pi}{\beta}(\alpha-2)} \cdot \frac{\min\{|A| \pm |B|\}}{|A| + |B|} \leq \frac{|f''(t)|}{|f''(x_k)|} \leq e^{-\frac{\pi}{\beta}(\alpha-2)} \cdot \frac{|A| + |B|}{\delta}$$

with  $\min\{|A| \pm |B|\} > 0$ .  $\square$

The next result guarantees that orbits of points in  $[-C\varepsilon^\alpha, -\tilde{y}] \cup [\tilde{y}, C\varepsilon^\alpha]$  remain expanding during a number  $m_0$  of iterates that can be fixed arbitrarily large by reducing  $\varepsilon$  and  $\mu$ . Recall that  $|\tilde{f}'| > \sigma_0 \gg 1$  and that  $f_\mu$  is  $C^1$ -close to  $\tilde{f}$  outside  $[-\tilde{y}, \tilde{y}]$  if  $\mu$  is small.

**Lemma 4.3.** [PRV, Lemma 6.1] *There exist  $c, C > 0$  and  $m_0 \geq c \log(1/\varepsilon)$  such that if  $\tilde{y} < |x| \leq C\varepsilon^\alpha$ , then*

$$f_\mu^i(x) \notin [-\tilde{y}, \tilde{y}] \quad \text{and} \quad |f_\mu'(f_\mu^i(x))| \geq \sigma_0,$$

for all  $1 \leq i \leq m_0$  and all  $\mu \in [-\varepsilon, \varepsilon]$ .

Next we establish some results of bounded distortion and uniformly bounded expansion during binding periods.

**Lemma 4.4** (Bounded distortion on binding periods). *There exists  $A = A(C, \tau) > 1$  such that for all  $x \in I(l, s)$  we have*

$$\frac{1}{A} \leq \left| \frac{(f_\mu^j)'(\xi)}{(f_\mu^j)'(f_\mu(x_l))} \right| \leq A$$

for every  $1 \leq j \leq p(l, s)$  and every  $\xi \in [f_\mu(x_l), f_\mu(x)]$ .

*Proof.* We let  $\eta = f_\mu(x_l)$  and consider  $0 \leq i < j$ . There are two cases to treat, corresponding to the two possibilities in (3.3). If  $|f_\mu^i(\eta)| \leq \varepsilon$  then, by Lemma 1.4,

$$\left| \frac{f'(f_\mu^i(\xi)) - f'(f_\mu^i(\eta))}{f'(f_\mu^i(\eta))} \right| \leq C \frac{|f_\mu^i(\xi) - f_\mu^i(\eta)|}{|f_\mu^i(\eta) - x_{m(i-1)}|} \leq C e^{-\tau i}.$$

If  $|f_\mu^i(\eta)| > \varepsilon$ , then  $|f_\mu^i(\xi) - f_\mu^i(\eta)| \leq \varepsilon^{1+\tau} e^{-\tau i} \ll \varepsilon$  and so the interval bounded by  $f_\mu^i(\xi)$  and  $f_\mu^i(\eta)$  is contained in the region  $S^1 \setminus [-\tilde{y}, \tilde{y}]$ , where  $f = \tilde{f}$ . Thus,

$$\left| \frac{f'(f_\mu^i(\xi)) - f'(f_\mu^i(\eta))}{f'(f_\mu^i(\eta))} \right| \leq C |f_\mu^i(\xi) - f_\mu^i(\eta)| \leq C \varepsilon^{1+\tau} e^{-\tau i} \leq C e^{-\tau i}.$$

Putting together all the above we get  $\sum_{i=0}^{j-1} \left| \frac{f'(f_\mu^i(\xi)) - f'(f_\mu^i(\eta))}{f'(f_\mu^i(\eta))} \right| \leq C \sum_{i=0}^j e^{-\tau i} \leq C$ . Thus

$$\log \left| \frac{(f_\mu^j)'(\xi)}{(f_\mu^j)'(\eta)} \right| \leq \sum_{i=0}^{j-1} \log \left( 1 + \left| \frac{f'(f_\mu^i(\xi))}{f'(f_\mu^i(\eta))} - 1 \right| \right) \leq \sum_{i=0}^{j-1} \left| \frac{f'(f_\mu^i(\xi))}{f'(f_\mu^i(\eta))} - 1 \right| \leq C,$$

and the statement of the lemma follows.  $\square$

Now we prove an exponential bound on the derivative along the orbit of each critical value  $z_k, |k| \leq k_0$ . This is needed to get a lower bound for the binding time  $p$  in terms of the position of the return interval given by  $(l, s)$  in the following Lemma 4.6. This demands a proof since the derivative of  $f_\mu$  is unbounded due to the presence of infinitely many critical points, unlike the quadratic family where we have this property for free.

In our setting we will obtain this by imposing an extra condition, besides item (1b) from Theorem 1.1, in the construction of the set  $S$  of parameters  $\mu$  which we shall consider in the proof of Theorems B and C. This condition is expressed by the inequality

$$\sum_{j=0}^{n-1} -\log \text{dist}(f_\mu^j(z_k), \mathcal{C}) \leq \hat{M} \cdot n, \quad (4.1)$$

for all  $n \geq 1$  which are *not bound times* for the orbit of the critical value  $z_k$ , for every  $|k| \geq k_0$  and for some big fixed constant  $\hat{M} > 0$ . Since for all  $x \in I$  we have  $|x| \geq \text{dist}(x, C)$  this implies

$$\sum_{j=0}^{n-1} -\log |f_\mu^j(z_k)| \leq \hat{M} \cdot n \quad (4.2)$$

and we are able to deduce the following.

**Lemma 4.5.** *Assume that (4.2) holds for some  $\mu \in S$ . Then there exists a constant  $M > 0$  such that for all  $n \geq 1$  and  $|k| \geq k_0$*

$$|(f_\mu^n)'(z_k)| \leq M^n.$$

*Proof.* Note that just by taking the derivative of  $f_\mu$  we get that there exists a constant  $C > 0$  such that  $|f_\mu'(x)| \leq C \cdot |x|^{\alpha-1}$ . On the one hand, for  $n \geq 1$  such that  $n$  is *not a bound time* for  $z_k$  and  $|k| \geq k_0$  we use (4.2) to get

$$\begin{aligned} |(f_\mu^n)'(z_k)| &\leq \prod_{j=0}^{n-1} C \cdot |f_\mu^j(z_k)|^{\alpha-1} \\ &\leq \exp\left(\sum_{j=0}^{n-1} (\log C + (\alpha-1) \cdot \log |f_\mu^j(z_k)|)\right) \\ &\leq \exp\left(n \cdot \log C + (1-\alpha) \cdot \hat{M} \cdot n\right) = \tilde{M}^n, \end{aligned}$$

where  $\tilde{M} = \exp(\log C + (1-\alpha) \cdot \hat{M})$ . We can assume without loss that  $\tilde{M} > A$  where  $A > 1$  is given by Lemma 4.4.

On the other hand, if  $n$  is a *bound time* for  $z_k$ , let  $t_1 < n$  be the return time for  $z_k$  which originated the binding and  $z_{l_1}$  be the corresponding bound critical value. Then by Lemma 4.4

$$|(f_\mu^n)'(z_k)| \leq A \cdot |(f_\mu^{n-t_1})'(z_{l_1})| \cdot |(f_\mu^{t_1})'(z_k)|.$$

If  $t_1$  is *not a bound time* for  $z_{l_1}$ , then by the bound just proved for all critical values on free times and return situations we bound the last expression by  $A\tilde{M}^{n-t_1}\tilde{M}^{t_1} = A \cdot \tilde{M}^n$ . Otherwise there are  $t_2 < n - t_1$  the return time for  $z_{l_1}$  which originated the binding and  $z_{l_2}$  the corresponding bound critical value and, as above, we get

$$|(f_\mu^n)'(z_k)| \leq A^2 \cdot |(f_\mu^{n-t_1-t_2})'(z_{l_2})| \cdot |(f_\mu^{t_2})'(z_{l_1})| \cdot |(f_\mu^{t_1})'(z_k)|.$$

If  $t_2$  is *not a bound time* for  $z_{l_2}$  we bound this expression by  $A^2\tilde{M}^{n-t_1-t_2}\tilde{M}^{t_2}\tilde{M}^{t_1} \leq A^2\tilde{M}^n$ . Otherwise we repeat the argument. Knowing that [PRV, Section 4] the orbit of every critical value has an initial number  $j_0 \gg 1$  of free iterates, we see that this argument must end in a free time and we arrive at  $|(f_\mu^n)'(z_k)| \leq A^k \tilde{M}^n$  where  $k < n$  is the number of nested binding periods obtained. Since  $\tilde{M} > A$  we conclude the proof of the lemma by setting  $M = \tilde{M}^2$ .  $\square$

Now we obtain the estimates for the binding time assuming that (4.2) holds for  $\mu \in S$ . In Section 8 we explain how to obtain (4.2) for a positive measure subset of parameters  $S$ .

**Lemma 4.6** (Expansion during binding periods). *There are constants  $A_0 = A_0(\varepsilon, \rho, \tau) > 1$ ,  $\iota = \iota(M) > 0$  and  $\theta = \theta(M, \varepsilon, \rho, \tau) \in \mathbf{N}$  such that for  $n \geq 1$  and  $\omega \in \mathcal{P}_n$  with  $R_n(\omega) \neq \emptyset$ , if  $r$  is the last return time for  $\omega$  and  $f_\mu^r(\omega) \subset I(l, s, j)$ , then setting  $p = p(l, s) > 0$  and  $\zeta = (\rho + \tau)/\log \sigma$  we have for  $\tau$  small enough*



- (a)  $\iota(M) \cdot (|l| + |s|) \leq p \leq \frac{2\pi}{\beta \log \sigma} (|l| + |s|)$ ;
- (b)  $|(f_\mu^{p+1})'(f_\mu^r(x))| \geq \frac{1}{C} \cdot \varepsilon^{1+\tau} \cdot \exp\left((1-\zeta)\frac{\pi}{\beta}(|l| + |s|)\right)$  and if  $|l| + |s| \geq \theta$ , then  $|(f_\mu^{p+1})'(f_\mu^r(x))| \geq \exp\left((1-3\zeta)\frac{\pi}{\beta}(|l| + |s|)\right)$ , for every  $x \in \omega$ ;
- (c)  $|(f_\mu^{p+1})'(f_\mu^r(x))| \geq A_0 \cdot \sigma^{(p+1)/3} > 1$  for every  $x \in \omega$ .

*Proof.* To prove item (a), we use the definition of the partition and the construction of the refinement. As  $p > 0$ ,  $(l, s, j) \neq (\pm k_0, 1, 1)$  and so  $|f_\mu^r(x) - x_l| \geq a_2 \cdot e^{-(\pi/\beta)(|l|+|s|)}$ , by Remark 2.1. Using second-order Taylor approximation and Lemma 4.2 we get

$$|f_\mu^{r+1}(x) - f_\mu(x_l)| \geq \frac{1}{C} |f''(x_l)| (a_2 e^{-(\pi/\beta)(|l|+|s|)})^2 \geq \frac{1}{C} \cdot e^{-\frac{\pi}{\beta}|l|(\alpha-2)} \cdot e^{-2(\pi/\beta)(|l|+|s|)},$$

where  $|f''(x_l)| \geq C^{-1}|x_l|^{\alpha-2} = C^{-1} \cdot \hat{x}^{\alpha-2} \cdot e^{-\frac{\pi}{\beta}|l|(\alpha-2)}$  by Lemma 1.3(2). Then for each  $0 \leq j \leq p$ , there is some  $\xi$  between  $f_\mu(x_l)$  and  $f_\mu^{r+1}(x)$  such that

$$\begin{aligned} |f_\mu^{j+r+1}(x) - f_\mu^{j+1}(x_l)| &= |(f_\mu^j)'(\xi)| \cdot |f_\mu^{r+1}(x) - f_\mu(x_l)| \\ &\geq C^{-1} \cdot e^{-\frac{\pi}{\beta}|l|(\alpha-2)} \cdot e^{-2(\pi/\beta)(|l|+|s|)} \cdot |(f_\mu^j)'(\xi)|. \end{aligned} \quad (4.3)$$

Now since  $\alpha - 2 < 0$  and we can take  $|l| \geq k_0$  very big, as a consequence of Lemma 4.4 and of the exponential growth of the derivative at the critical orbits, we get the following bound

$$2 \cdot e^{-2(\pi/\beta)(|l|+|s|)} \sigma^j \leq |f_\mu^{j+r+1}(x) - f_\mu^{j+1}(x_l)| \leq 2$$

Hence  $e^{-2(\pi/\beta)(|l|+|s|)} \sigma^j \leq 1$  for all  $1 \leq j \leq p$ . In particular,

$$-2(\pi/\beta)(|l| + |s|) + p \log(\sigma) \leq 0, \quad \text{implying} \quad p \leq \frac{2(\pi/\beta)(|l| + |s|)}{\log \sigma},$$

thus proving the upper bound in (a).

For the lower bound in (a), we note that by the definition of binding period, we have that

$$\begin{aligned} |f_\mu^{r+p+1}(x) - f_\mu^{p+1}(x_l)| &\geq |f_\mu^{p+1}(x_l) - x_{m(p)}| \cdot e^{-\tau(p+1)} \quad \text{if} \quad |f_\mu^{p+1}(x_l)| \leq \varepsilon \\ &\quad \text{and} \end{aligned} \quad (4.4)$$

$$|f_\mu^{p+r+1}(x) - f_\mu^{p+1}(x_l)| \geq \varepsilon^{1+\tau} \cdot e^{-\tau(p+1)} \quad \text{if} \quad |f_\mu^{p+1}(x_l)| > \varepsilon.$$

So in either case using Lemmas 4.4 and 4.5 we get

$$\begin{aligned} |f_\mu^{p+r+1}(x) - f_\mu^{p+1}(x_l)| &= |(f_\mu^p)'(\xi)| \cdot |f_\mu(x) - f_\mu(x_l)| \\ &\leq A \cdot |(f_\mu^p)'(z_l)| \cdot |f_\mu(x) - f_\mu(x_l)| \\ &\leq A \cdot M^p \cdot e^{-\pi/\beta(|l|+|s|)}. \end{aligned}$$

On the one hand, if  $|f_\mu^{p+1}(x_l)| > \varepsilon$  then we arrive at

$$\varepsilon^{1+\tau} e^{-\tau(p+1)} \leq A \cdot M^p \cdot e^{-\pi/\beta(|l|+|s|)}.$$

On the other hand, if  $|f_\mu^{p+1}(x_l)| \leq \varepsilon$  then by condition (1b) from Theorem 1.1 we arrive at

$$e^{-\rho p} e^{-\tau(p+1)} \leq A \cdot M^p \cdot e^{-\pi/\beta(|l|+|s|)}.$$

In both cases if we take  $M$  big enough, then we get a bound of the form

$$p \geq \iota(M) \cdot (|l| + |s|)$$

concluding the proof of (a).

Now we prove (b). Since  $p + 1$  is not a binding time we must have (4.4) as before. Using Theorem 1.1(1b), setting  $\Delta_{p+1} = \min\{\varepsilon^{1+\tau} \cdot e^{-\tau(p+1)}, e^{-(\rho+\tau)(p+1)}\}$  we get for some  $\xi \in [f_\mu(x_l), f_\mu^{r+1}(x)]$ , by Lemma 4.4, and using second-order Taylor expansion of  $f_\mu$  near  $x_l$  together with Lemma 4.2

$$\begin{aligned} \Delta_{p+1} &\leq |f_\mu^{p+1}(x_l) - f_\mu^{r+p+1}(x)| = |(f_\mu^p)'(\xi)| \cdot |f_\mu(x_l) - f_\mu^{r+1}(x)| \\ &\leq C \cdot |(f_\mu^p)'(f_\mu^{r+1}(x))| \cdot |f_\mu''(x_l)| \cdot |f_\mu^r(x) - x_l|^2 \end{aligned} \quad (4.5)$$

On the other hand, using Lemma 1.3 and again Lemma 4.4 we get

$$|(f_\mu^{p+1})'(f_\mu^r(x))| = |f_\mu'(f_\mu^r(x))| \cdot |(f_\mu^p)'(f_\mu^{r+1}(x))| \geq C^{-1}|x_l|^{\alpha-2} \cdot |f_\mu^r(x) - x_l| \cdot |(f_\mu^p)'(f_\mu^{r+1}(x))|.$$

Hence by the previous expression together with (4.5) we deduce

$$|(f_\mu^{p+1})'(f_\mu^r(x))| \geq \frac{\varepsilon^{1+\tau} \cdot e^{-(\rho+\tau)(p+1)}}{C \cdot |f_\mu^r(x) - x_l|} \geq \frac{1}{C} \cdot \varepsilon^{1+\tau} \cdot e^{-(\rho+\tau)(p+1)} \cdot e^{\frac{\pi}{\beta}(|l|+|s|)}$$

Now we have two possibilities, either  $\varepsilon > e^{-(\rho+\tau)(p+1)}$  or not. In the former case we obtain by the upper bound in item (a)

$$|(f_\mu^{p+1})'(f_\mu^r(x))| \geq e^{-(2+\tau)(\rho+\tau)(p+1)} \cdot e^{\frac{\pi}{\beta}(|l|+|s|)} \geq \exp\left(\left(1 - 3\frac{\rho+\tau}{\log \sigma}\right)\frac{\pi}{\beta}(|l|+|s|)\right).$$

In the latter case we get

$$|(f_\mu^{p+1})'(f_\mu^r(x))| \geq \frac{1}{C} \cdot \varepsilon^{1+\tau} \cdot \exp\left(\left(1 - \frac{\rho+\tau}{\log \sigma}\right)\frac{\pi}{\beta}(|l|+|s|)\right).$$

Observe that by the lower bound in item (a) and since  $k_0 = C \log(1/\varepsilon)$  we have in this case

$$(\rho + \tau) \cdot \iota(M) \cdot (|l| + |s| + 1) \leq (\rho + \tau) \cdot (p + 1) \leq C \cdot k_0.$$

So if  $|l| + |s| \geq \theta$  with  $(\rho + \tau) \cdot \iota(M) \cdot (\theta + 1) > C \cdot k_0$  then only the first alternative can happen. This concludes the proof of item (b).

In order to prove (c) we use Lemma 4.4 once again, the inequality (4.5), Lemma 1.3(2) and Lemma 4.2 to get

$$\begin{aligned} |(f_\mu^{p+1})'(f_\mu^r(x))|^2 &= |(f_\mu^p)'(f_\mu^{r+1}(x))|^2 \cdot |f_\mu'(f_\mu^r(x))|^2 \\ &\geq C^{-1} |(f_\mu^p)'(f_\mu(x_l))| \cdot |(f_\mu^p)'(f_\mu^{r+1}(x))| \cdot C^{-1} |f_\mu''(x_l)|^2 \cdot |f_\mu^r(x) - x_l|^2 \\ &= C^{-1} |(f_\mu^p)'(f_\mu(x_l))| \cdot |f_\mu''(x_l)| \cdot (|(f_\mu^p)'(f_\mu^{r+1}(x))| \cdot |f_\mu''(x_l)| \cdot |f_\mu^r(x) - x_l|^2) \\ &\geq C^{-1} |(f_\mu^p)'(f_\mu(x_l))| \cdot |f_\mu''(x_l)| \cdot (|(f_\mu^p)'(\xi)| \cdot |f_\mu(x_l) - f_\mu^{r+1}(x)|) \\ &= C^{-1} |(f_\mu^p)'(f_\mu(x_l))| \cdot |f_\mu''(x_l)| \cdot |f_\mu^{p+1}(x_l) - f_\mu^{r+p+1}(x)| \\ &\geq C^{-1} \sigma^p \cdot C^{-1} |x_l|^{\alpha-2} \cdot |f_\mu^{p+1}(x_l) - f_\mu^{r+p+1}(x)| \\ &= C^{-2} \sigma^p \cdot \hat{x}^{\alpha-2} \cdot e^{-\frac{\pi}{\beta}|l|(\alpha-2)} \cdot |f_\mu^{p+1}(x_l) - f_\mu^{r+p+1}(x)|. \end{aligned} \quad (4.6)$$

Now we must consider two cases. On the one hand, if  $|f_\mu^{p+1}(x_l)| > \varepsilon$  then, by the definition of  $p$  in (3.3), we must have  $|f_\mu^{p+1}(x_l) - f_\mu^{r+p+1}(x)| > \varepsilon^{1+\tau} e^{-\tau(p+1)}$ . Because  $\alpha - 2 < 0$  and  $|x_l| \leq \varepsilon$ , equation (4.6) implies  $|(f_\mu^{p+1})'(f_\mu^r(x))|^2 \geq C^{-2} \sigma^p \cdot \varepsilon^{\alpha-2} \cdot |f_\mu^{p+1}(x_l) - f_\mu^{r+p+1}(x)|$  and we may write  $|(f_\mu^{p+1})'(f_\mu^r(x))|^2 \geq C^{-1} \sigma^{p+1} \varepsilon^{\alpha-1+\tau} e^{-\tau(p+1)} \geq A_0^2 \cdot \sigma^{2(p+1)/3}$ , as long as we fix  $\tau < \min\{1 - \alpha, \log \sigma/3\}$  and suppose  $\varepsilon$  small enough. On the other hand, if  $|f_\mu^{p+1}(x_l)| \leq \varepsilon$  then, by (3.3)

$$|f_\mu^{p+1}(x_l) - f_\mu^{r+p+1}(x)| > |f_\mu^{p+1}(x_l) - x_{m(p)}| e^{-\tau(p+1)}.$$

We note that now there is only one possibility according to item (1b) of Theorem 1.1:  $|f_\mu^{p+1}(x_l) - x_{m(p)}| \geq e^{-\rho p}$  and thus  $|f_\mu^{p+1}(x_l) - f_\mu^{r+p+1}(x)| > C^{-1}\varepsilon e^{-(\rho+\tau)(p+1)}$ . Hence

$$|(f_\mu^{p+1})'(f_\mu^r(x))|^2 \geq C^{-1} \cdot \sigma^{p+1} \cdot \varepsilon^{\alpha-1} \cdot e^{-(\rho+\tau)(p+1)} \geq A_0^2 \cdot \sigma^{2(p+1)/3}$$

as long as we take  $\varepsilon$ ,  $\rho$  and  $\tau$  small enough. This concludes the proof of the lemma.  $\square$

**Remark 4.7.** Note that  $\iota(M) \rightarrow 0$  when  $M \rightarrow \infty$ .

Now we will obtain estimates of the length of  $|f_\mu^n(\omega)|$ .

**Lemma 4.8** (Lower bounds on the length at return times). *Let  $r$  be an essential or an inessential return time for  $\omega \in \mathcal{P}_{n-1}$ , with host interval  $I(l, s, j)^+$ . Let  $p = p(l, s)$  denote the length of its binding period and set  $Q = Q(l, s, \tau, \rho) = \frac{\varepsilon^{1+\tau}}{(|l|+|s|)^3} \cdot e^{\zeta \frac{\pi}{\beta}(|l|+|s|)}$ . Then the following holds:*

- (1) *Assuming that  $r^* \leq n-1$  is the next return situation for  $\omega$  (either essential, inessential or an escape) we have  $|f_\mu^{r^*}(\omega)| \geq Q \cdot \sigma_0^q \cdot e^{(1-\zeta) \cdot \frac{\pi}{\beta}(|l|+|s|)} \cdot |f_\mu^r(\omega)|$ , where  $q = r^* - (r+p)$ . Moreover we also have  $|f_\mu^{r^*}(\omega)| \geq \sigma_0^q \cdot A_0 \cdot \sigma^{(p+1)/3} \cdot |f_\mu^r(\omega)| \geq A_0 \cdot |f_\mu^r(\omega)| > |f_\mu^r(\omega)|$ .*
- (2) *If  $r$  is the last return time for  $\omega$  up to iterate  $n-1$  and also an essential return, and  $r^*$  is a return time for  $\omega$ , then setting  $q = r^* - (r+p)$  we have*

$$|f_\mu^{r^*}(\omega)| \geq a_1 \cdot Q \cdot \sigma_0^q \cdot e^{-2\zeta \frac{\pi}{\beta}(|l|+|s|)}.$$

Suppose that  $r$  is an escape time for  $\omega \in \mathcal{P}_{n-1}$ .

- (3) *If  $r^* \leq n-1$  is the next return situation for  $\omega$ , then  $|f_\mu^{r^*}(\omega)| \geq \sigma_0^{r^*-r} \cdot |f_\mu^r(\omega)|$ .*

**Remark 4.9.** Note that  $Q = Q(l, s, \tau, \rho) \rightarrow \infty$  when  $|l| + |s| \rightarrow \infty$ .

*Proof.* We start by assuming  $r^* \leq n-1$  as in item (1). By the mean value theorem we have  $|f_\mu^{r^*}(\omega)| \geq |(f_\mu^{r^*-r})'(f_\mu^r(\xi))| \cdot |f_\mu^r(\omega)|$  for some  $\xi \in \omega$ . Using Remark 3.3 and Lemma 4.6 we get by setting  $q = r^* - (r+p)$

$$|f_\mu^{r^*}(\omega)| \geq \left| (f_\mu^q)'(f_\mu^{r+p}(\xi)) \right| \cdot \left| (f_\mu^p)'(f_\mu^r(\xi)) \right| \cdot |f_\mu^r(\omega)| \quad (4.7)$$

$$\begin{aligned} &\geq \sigma_0^q \cdot \frac{1}{C} \cdot \varepsilon^{1+\tau} \cdot e^{(1-\zeta) \frac{\pi}{\beta}(|l|+|s|)} \cdot |f_\mu^r(\omega)| \\ &\geq \sigma_0^q \cdot \frac{\varepsilon^{1+\tau}}{C} \cdot e^{\zeta \frac{\pi}{\beta}(|l|+|s|)} \cdot e^{(1-2\zeta) \frac{\pi}{\beta}(|l|+|s|)} \cdot |f_\mu^r(\omega)|. \end{aligned} \quad (4.8)$$

If  $r$  is an essential return time for  $\omega$ , then  $I(l, s, j) \subset f_\mu^r(\omega)$  and  $|f_\mu^r(\omega)| \geq a_1 \frac{e^{-(\pi/\beta)(|l|+|s|)}}{(|l|+|s|)^3}$ , hence

$$|f_\mu^{r^*}(\omega)| \geq \sigma_0^q \cdot \frac{\varepsilon^{1+\tau} \cdot e^{\zeta \frac{\pi}{\beta}(|l|+|s|)}}{C \cdot (|l|+|s|)^3} \cdot a_1 e^{-2\zeta \frac{\pi}{\beta}(|l|+|s|)}.$$

By the definition of  $Q$  this proves item (2) by taking  $r^* = n$ . Taking into account the definition of  $Q$  in (4.8) we obtain the first part of item (1). For the second part of item (1) just use the inequality from Lemma 4.6(c) in (4.7).

To get item (3) observe that between the iterate  $r$  and  $r^* - 1$  there are only free iterates for  $\omega$ , thus the estimate follows by the uniform expanding rate on free times. Altogether this concludes the proof of the lemma.  $\square$

**Lemma 4.10** (Bounded Distortion). *There is a constant  $D_0 = D_0(\rho, \tau, \sigma) > 0$  such that for  $\omega \in \mathcal{P}_{n-1}$ ,  $n \in \mathbf{N}$ , and for every  $x, y \in \omega$  we have*

$$\left| \frac{(f_\mu^n)'(x)}{(f_\mu^n)'(y)} \right| \leq D_0.$$

*Proof.* Let  $R_{n-1}(\omega) = \{r_1, \dots, r_\gamma\}$  and  $Q_{n-1}(\omega) = \{(l_1, s_1, j_1), \dots, (l_\gamma, s_\gamma, j_\gamma)\}$ , be the sets of return situations (essential returns, inessential returns and escapes) and indexes of host intervals of  $\omega$ , respectively, as defined during the construction of the partition. Let  $\omega_i = f_\mu^{r_i}(\omega)$ ,  $p_i = p(l_i, s_i)$  for  $i = 1, \dots, \gamma$  and, for  $y, z \in \omega$ , let  $y_k = f_\mu^k(y)$  and  $z_k = f_\mu^k(z)$  for  $k = 0, \dots, n-1$ . Observe that  $\omega_i \subset I(l_i, s_i, j_i)^+$  for all  $i$  and

$$\left| \frac{(f_\mu^n)'(z)}{(f_\mu^n)'(y)} \right| = \prod_{k=0}^{n-1} \left| \frac{f'(z_k)}{f'(y_k)} \right| \leq \prod_{k=0}^{n-1} \left( 1 + \left| \frac{f'(z_k) - f'(y_k)}{f'(y_k)} \right| \right). \quad (4.9)$$

On free iterates, if  $y_k \in [-\varepsilon, \varepsilon]$ , then by Lemma 1.4

$$\left| \frac{f'(z_k) - f'(y_k)}{f'(y_k)} \right| \leq K_1 \cdot \left| \frac{z_k - y_k}{y_k - \tilde{x}_k} \right| \leq K_1 \cdot \frac{|f_\mu^k(\omega)|}{\Delta_k(\omega)}, \quad (4.10)$$

where we define  $\Delta_k(\omega) = \text{dist}(f_\mu^k(\omega), \mathcal{C}) = \inf_{x \in \omega} \text{dist}(f_\mu^k(x), \mathcal{C})$  and  $\tilde{x}_k$  is the critical point closest to  $y_k$ . We observe that in this case the interval  $f_\mu^k(\omega)$  is between two consecutive critical points,  $x_{l_k}$  and  $x_{l_{k+1}}$ , and is contained in some  $I(l_k, s_k, j_k)$  with  $s_k \leq s(\tau)$ . Note that by the exponential character of the initial partition, we have

$$|f_\mu^k(\omega)| \leq C \cdot |I(l_k, s_k, 1)^+| \quad \text{and} \quad \Delta_k(\omega) \geq C^{-1} \cdot |I(l_k, s_k)| \quad (4.11)$$

for some constant  $C > 0$  depending only on  $\hat{f}$  (see Remark 2.1).

Otherwise for free iterates  $y_k \in S^1 \setminus [-\varepsilon, \varepsilon]$  we get

$$\begin{aligned} \sum_{\substack{r_i + p_i < k < r_{i+1} \\ |y_k| > \varepsilon}} \left| \frac{f'(z_k) - f'(y_k)}{f'(y_k)} \right| &\leq \frac{L}{\tilde{\sigma}} \sum_{\substack{r_i + p_i < k < r_{i+1} \\ |y_k| > \varepsilon}} |z_k - y_k| \leq \frac{L}{\tilde{\sigma}} \sum_{\substack{r_i + p_i < k < r_{i+1} \\ |y_k| > \varepsilon}} |f_\mu^k(\omega)| \\ &\leq \frac{L}{\tilde{\sigma}} \sum_{r_i + p_i < k < r_{i+1}} \sigma_0^{k-r_{i+1}} \cdot |f^{r_{i+1}}(\omega)| \leq K_2 \cdot \frac{|\omega_{i+1}|}{\Delta_{r_{i+1}}(\omega)} \end{aligned} \quad (4.12)$$

by definition of  $f_\mu$  on  $S^1 \setminus [-\varepsilon, \varepsilon]$ , since  $|f'_\mu| | S^1 \setminus [-\varepsilon, \varepsilon] | > \tilde{\sigma}$  and  $|f''_\mu| | S^1 \setminus [-\varepsilon, \varepsilon] | \leq L$  for some constant  $L$ . We recall also that  $\Delta_{r_{i+1}}(\omega) < 1$  by definition.

For an escape time  $k = r_i$  with  $i \in \{1, \dots, \gamma\}$  we have either  $|y_k| \leq \varepsilon$  and then we have the inequality (4.10), or  $|y_k| > \varepsilon$  and we get as in (4.12)

$$\left| \frac{f'(z_k) - f'(y_k)}{f'(y_k)} \right| \leq \frac{L}{\tilde{\sigma}} \cdot |z_k - y_k| \leq \frac{L}{\tilde{\sigma}} \cdot |f_\mu^k(\omega)| \leq K_3 \cdot \frac{|\omega_i|}{\Delta_{r_i}(\omega)}. \quad (4.13)$$

Hence up to now all cases are bounded by the same type of expression.

Next we find a bound for iterates during binding times. Let us fix  $i = 1, \dots, \gamma$  such that  $r_i$  is *not an escape time* for  $\omega$ , i.e. it is either an essential or inessential return time. Then for  $k = r_i$  we have the same bound (4.10). For  $r_i < k \leq r_i + p_i$  we get for some  $\xi \in \omega$

$$\begin{aligned} |z_k - y_k| &= |(f^{k-r_i})'(f_\mu^{r_i}(\xi))| \cdot |z_{r_i} - y_{r_i}| \leq |(f^{k-r_i})'(f_\mu^{r_i}(\xi))| \cdot |f_\mu^{r_i}(\omega)| \\ &\leq C \cdot |(f^{k-r_i-1})'(f_\mu^{r_i+1}(\xi))| \cdot |f''(x_{l_i})| \cdot |f_\mu^{r_i}(\xi) - x_{l_i}| \cdot |\omega_i|, \end{aligned}$$

where we have used the Taylor expansion of  $f'$  near the critical point  $x_{l_i}$  together with Lemma 4.2. By definition of  $p_i$  we have two possibilities. On the one hand, for the first case in (3.3) there exists  $w \in [f_\mu(x_{l_i}), f_\mu^{r_i+1}(\xi)]$  such that, using second order Taylor expansion and Lemma 4.2 again

$$\begin{aligned} |f_\mu^{k-r_i}(x_{l_i}) - x_{m(k-r_i-1)}| e^{-\tau(k-r_i)} &\geq |f_\mu^k(\xi) - f_\mu^{k-r_i}(x_{l_i})| \\ &= |(f^{k-r_i-1})'(w)| \cdot |f_\mu^{r_i+1}(\xi) - f_\mu(x_{l_i})| \\ &\geq C^{-1} |(f^{k-r_i-1})'(w)| \cdot |f''(x_{l_i})| \cdot |f_\mu^{r_i}(\xi) - x_{l_i}|^2 \\ &\geq (AC)^{-1} |(f^{k-r_i-1})'(f_\mu^{r_i+1}(\xi))| \cdot |f''(x_{l_i})| |f_\mu^{r_i}(\xi) - x_{l_i}|^2, \end{aligned} \quad (4.14)$$

where we have used Lemma 4.4 in the last inequality. The last two expression together show that

$$|z_k - y_k| \cdot |f_\mu^{r_i}(\xi) - x_{l_i}| \leq (AC^2) \cdot |f_\mu^{k-r_i}(x_{l_i}) - x_{m(k-r_i-1)}| e^{-\tau(k-r_i)} \cdot |\omega_i|.$$

This and Lemma 1.4 provide

$$\begin{aligned} \left| \frac{f'(z_k) - f'(y_k)}{f'(y_k)} \right| &\leq K_1 \left| \frac{z_k - y_k}{y_k - \tilde{x}_k} \right| \leq AC^2 K_1 e^{-\tau(k-r_i)} \cdot \frac{|\omega_i| \cdot |f_\mu^{k-r_i}(x_{l_i}) - x_{m(k-r_i-1)}|}{|f_\mu^{r_i}(\xi) - x_{l_i}| \cdot |y_k - \tilde{x}_k|} \\ &\leq D \cdot e^{-\tau(k-r_i)} \cdot \frac{|\omega_i|}{\Delta_i(\omega)} \cdot \frac{|f_\mu^{k-r_i}(x_{l_i}) - x_{m(k-r_i-1)}|}{|f_\mu^{k-r_i}(x_{l_i}) - x_{m(k-r_i-1)}| - |y_k - f_\mu^{k-r_i}(x_{l_i})|} \\ &\leq D \cdot \frac{e^{-\tau(k-r_i)}}{1 - e^{-\tau(k-r_i)}} \cdot \frac{|\omega_i|}{\Delta_i(\omega)} \leq D_1 \cdot e^{-\tau(k-r_i)} \cdot \frac{|\omega_i|}{\Delta_{r_i}(\omega)}. \end{aligned}$$

On the other hand, for the second case in (3.3) we get a similar inequality in (4.14) providing

$$|z_k - y_k| \cdot |f_\mu^{r_i}(\xi) - x_{l_i}| \leq (AC^2) \cdot \varepsilon^{1+\tau} e^{-\tau(k-r_i)} \cdot |\omega_i|$$

and thus by definition of  $\tilde{f}$  we get

$$\begin{aligned} \left| \frac{f'(z_k) - f'(y_k)}{f'(y_k)} \right| &\leq \frac{L \cdot |z_k - y_k|}{\tilde{\sigma}} \leq \frac{ACL}{\tilde{\sigma}} e^{-\tau(k-r_i)} \frac{|\omega_i| \cdot \varepsilon^{1+\tau}}{|f_\mu^{r_i}(\xi) - x_{l_i}|} \\ &\leq D_2 \cdot e^{-\tau(k-r_i)} \cdot \frac{|\omega_i|}{\Delta_{r_i}(\omega)}. \end{aligned}$$

This shows that for every  $i = 1, \dots, \gamma$  which is *not* an escape time, we have

$$\sum_{k=r_i}^{r_i+\ell} \left| \frac{f'(z_k) - f'(y_k)}{f'(y_k)} \right| \leq D_3 \cdot \frac{|\omega_i|}{\Delta_{r_i}(\omega)} \leq \frac{1}{C} \cdot \frac{|I(l_i, s_i, j_i)^+|}{|I(l_i, s_i)|}, \quad (4.15)$$

for all  $\ell = 1, \dots, p_i$ , where we have used the definition of  $\omega_i$  and of host interval, together with the same estimate as in (4.11).

Taking into account (4.10), (4.12), (4.13) and (4.15) and summing over all iterates we arrive at

$$\sum_{k=0}^{n-1} \left| \frac{f'(z_k) - f'(y_k)}{f'(y_k)} \right| \leq D_4 \sum_{k \in F_1} \frac{|f_\mu^k(\omega)|}{\Delta_k(\omega)} + \left( \frac{L}{\tilde{\sigma}} + K_2 \right) \sum_{k \in F_2} |f_\mu^k(\omega)|. \quad (4.16)$$

Here the left hand side sum is over the set  $F_1$  of free iterates together with return situations (essential and inessential returns and escapes) from  $k = 0$  to  $k = n - 1$ . The right hand side sum is over the set  $F_2$  of free iterates which are not followed by any return, from  $r_\gamma + p_\gamma$  to  $n$ .

Moreover  $D_4$  is a constant depending only on  $\varepsilon, \tau$  and  $\tilde{\sigma}$ . So if we can bound (4.16) uniformly we then find a uniform bound to (4.9) also and complete the proof of the lemma.

The right hand side sum in (4.16) is easily bounded as follows

$$\sum_{k \in F_2} |f_\mu^k(\omega)| \leq \sum_{k=0}^{n-1} \tilde{\sigma}^{k-n} |f_\mu^n(\omega)| \leq C,$$

since  $|f_\mu^n(\omega)|$  is always less than 1.

Now we bound the left hand side sum

$$\begin{aligned} \sum_{k \in F_1} \frac{|f_\mu^k(\omega)|}{\Delta_k(\omega)} &\leq \sum_{\substack{k \in F_1 \\ (l_k, s_k) = (0,0)}} \frac{|f_\mu^k(\omega)|}{\Delta_k(\omega)} + \sum_{|l| \geq k_0} \sum_{|s| \geq 1} \sum_{\substack{k \in F_1 \\ (l_k, s_k) = (l,s)}} \frac{|f_\mu^k(\omega)|}{\Delta_k(\omega)} \\ &\leq C \cdot \frac{\sigma_1}{\sigma_1 - 1} + \sum_{|l| \geq k_0} \sum_{|s| \geq 1} \frac{\sigma_1}{\sigma_1 - 1} \cdot \frac{1}{C} \cdot \frac{|f_\mu^{q(l,s)}(\omega)|}{|I(l,s)|}, \end{aligned}$$

by (4.11), where  $q(l,s) = \max\{0 \leq k \leq n-1 : (\hat{l}_k, \hat{s}_k) = (l,s)\}$  and we convention that whenever  $\{0 \leq k \leq n-1 : (\hat{l}_k, \hat{s}_k) = (l,s)\} = \emptyset$  we have  $\frac{|f_\mu^{q(l,s)}(\omega)|}{|I(l,s)|} = 0$ . We have used the following estimate for any given fixed value of  $(l,s)$

$$\sum_{\{k: \hat{s}_k = s\}} |f_\mu^k(\omega)| \leq |f_\mu^{q(l,s)}(\omega)| \sum_{\{k: (\hat{l}_k, \hat{s}_k) = (l,s)\}} \sigma_1^{k-q(l,s)} \leq \frac{\sigma_1}{\sigma_1 - 1} \cdot |f_\mu^{q(l,s)}(\omega)| \leq C \cdot |I(l,s,j)^+|,$$

because writing  $\{k : \hat{s}_k = s\} = \{k_1 < k_2 < \dots < k_h\}$  we have  $|f_\mu^{k_i}(\omega)| \leq \sigma_1^{-1} \cdot |f_\mu^{k_{i+1}}(\omega)|$  for  $i = 1, \dots, h$ , where  $1 < \sigma_1 = \min\{\sigma_0, e^{(1-2\zeta) \cdot \frac{\pi}{\beta}(k_0+1)}\} \leq \min\{\sigma_0, e^{(1-2\zeta) \cdot \frac{\pi}{\beta}(|l|+|s|)}\}$ , after Lemma 4.6(b) together with Remark 3.3.

We observe that by construction we must have  $|I(l,s,j)^+|/|I(l,s)| \leq 9(|l| + |s|)^{-3}$  and so we arrive at

$$\sum_{k \in F_1} \frac{|f_\mu^k(\omega)|}{\Delta_k(\omega)} \leq C \sum_{|l| \geq k_0} \sum_{|s| \geq 1} \frac{9}{(|l| + |s|)^3} < \infty,$$

finishing the proof of the lemma. □

## 5. PROBABILITY OF DEEP ESSENTIAL RETURNS

Here we use the results from Section 4 to estimate the probability of having an orbit with a given sequence of host intervals at essential return situations.

For each  $x \in I \setminus \mathcal{C}_\infty$ , let  $u_n(x)$  denote the number of essential returns or escapes of the orbit of  $x$  between 1 and  $n$ , let  $0 \leq t_1 \leq \dots \leq t_{u_n} \leq n$  be the instants of occurrence of the essential returns or escapes and let  $(l_1, s_1, j_1), \dots, (l_{u_n}, s_{u_n}, j_{u_n})$  be the corresponding critical points and depths. Note that by construction  $t_1(x) = 0$  for all  $x \in I \setminus (\mathcal{C} \cup I(0,0,0))$  and  $t_1(x) = 1$  for any  $x \in I(0,0,0)$ , see Remark 3.1.

**Lemma 5.1** (No return probability). *For every  $n \geq 0$  there exists no non-degenerate interval  $\omega \in \mathcal{P}_n$  such that  $\omega \in \mathcal{P}_{n+k}$  for all  $k \geq 1$ . Moreover there exist constants  $0 < \xi_0 < 1$  and  $K_0 > 0$  (depending only on  $\sigma, \sigma_0$  and on  $\zeta$  from Lemma 4.6), and  $n_0 \geq 1$  such that for every  $n > n_0$*

$$\lambda\left(\bigcup\{\omega \in \mathcal{P}_n : u_n | \omega = 1\}\right) \leq K_0 \cdot e^{-\xi_0 n}.$$

*Proof.* If  $\omega \in \mathcal{P}_{n+k}$  for all  $k \geq 0$ , then  $\omega$  is not refined in all future iterates. This means that  $f_\mu^{n+k}(\omega)$  has no essential returns nor escapes for  $k \geq 1$ . Hence every iterate is either free or a binding time associated to a inessential return. Let  $p_0, p_1, p_2, \dots$  be the length of the binding period associated to every (if any) inessential return time  $t = r_0 < r_1 < r_2 < \dots$  for  $\omega$  after  $t$ , where  $0 \leq t \leq n$  is the last essential return time or escape before  $n$ . Let  $k \geq 0$  and  $r_i + p_i \leq n+k < r_{i+1}$  for some  $i \geq 0$ , where we set  $r_{i+1} = +\infty$  if  $r_i$  is the last inessential return time after  $t$  (it may happen that  $r_1 = +\infty$  in which case  $i = 0$  and there are no inessential returns after  $t$ ). Lemma 4.8 ensures that

$$2 \geq |f_\mu^{n+k}(\omega)| \geq 2^i \cdot \sigma_0^{n+k-t-\sum_{k=0}^i p_i} |f_\mu^t(\omega)|, \quad (5.1)$$

for arbitrarily big values of  $k \geq 0$ . Note that if there are no inessential returns, then  $i = 0$  and so

$$|f_\mu^t(\omega)| \leq 2 \cdot \sigma_0^{t+p_0-n-k}. \quad (5.2)$$

Otherwise there is at least a minimum expansion rate of 2 at the end of each binding period by Lemma 4.8, hence

$$|f_\mu^t(\omega)| \leq 2^{1-i} \cdot \sigma_0^{-(n+k-r_i-p_i)}.$$

We conclude that  $|f_\mu^t(\omega)| = 0$  which is not possible for a non-degenerate interval. This proves the first part of the Lemma.

Now let  $\omega \in \mathcal{P}_n$  be such that the restriction  $u_n | \omega$  of  $u_n$  to  $\omega$  is constant and equal to 1. Then either  $\omega = I(0, 0, 0)$  with  $t_1 = 1$  the unique essential return up to iterate  $n$  and  $f_\mu(\omega) = I(l, s, j)$  with  $(l, s, j) \neq (0, 0, 0)$ ; or  $\omega = I(l, s, j)$  with  $|l| \geq k_0, |s| \geq 1$  and  $j = 1, \dots, (|l| + |s|)^3$ , having a single essential return  $t_1 = 0$  up to iterate  $n$ . We concentrate on the latter case and write  $p_0, p_1, p_2, \dots$  and  $0 = t_1 = r_0 < r_1 < r_2 < \dots$  the binding periods associated to their respective inessential return times of the orbit of  $\omega$  as before. Then by (5.1) and (5.2) either  $n \leq p_0$  or

$$\begin{aligned} |\omega| &\leq 2 \cdot \sigma_0^{p_0-n} \quad \text{if } p_0(\omega) < n \leq r_1(\omega) \leq +\infty, \quad \text{or} \\ |\omega| &\leq 2^{1-i} \cdot \sigma_0^{-(n-\sum_{k=0}^{i-1} p_k(\omega))} \quad \text{if there is } i \geq 1 \text{ such that } r_i(\omega) \leq n < r_{i+1}(\omega). \end{aligned}$$

We note that in the case of  $f_\mu(\omega) = I(l, s, j)$ , that is, when  $t_1(\omega) = 1$ , we can repeat the arguments for  $\omega_1 = f_\mu(\omega)$ , arriving at the same bounds for  $|\omega|$  except for an extra factor of  $\tilde{\sigma}$  since  $|f_\mu(\omega)| \geq \tilde{\sigma}|\omega|$ . Hence we may write according to three cases above

$$\begin{aligned} \lambda(\cup \{\omega \in \mathcal{P}_n : u_n | \omega = 1\}) &\leq \sum_{n \leq p_0(\omega)} |\omega| + \sum_{\substack{|\omega| \leq 2\sigma_0^{p_0(\omega)-n} \\ p_0(\omega) \leq n \leq r_1(\omega) \leq +\infty}} |\omega| + \sum_{\substack{|\omega| \leq 2^{1-i} \cdot \sigma_0^{-n+\sum_{k=0}^{i-1} p_k(\omega)} \\ r_i(\omega) \leq n < r_{i+1}(\omega)}} |\omega| + S_3 \\ &= S_0 + S_1 + S_2 + S_3, \end{aligned}$$

where, by the above comment, we may assume that every sum ranges over  $\omega \in \mathcal{P}_0 \setminus I(0, 0, 0)$  and  $S_3$  corresponds to the sum over the partition elements in  $I(0, 0, 0) \cap \mathcal{P}_1$ , which is bounded by  $(S_0 + S_1 + S_2)/\tilde{\sigma}$ .

For  $S_0$  we use Lemma 4.6(a) to deduce that the summands in  $S_0$  are the elements of  $\mathcal{P}_0$  such that

$$n \leq p_0(\omega) \leq \frac{2\pi}{\beta \log \sigma} (|l| + |s|), \quad \text{i.e.} \quad |l| + |s| \geq \frac{\beta \log \sigma}{2\pi} \cdot n,$$

thus by Remark 2.1, setting  $C_0 = \frac{\beta \log \sigma}{2\pi}$

$$\begin{aligned} S_0 &\leq \sum_{|l|+|s| \geq C_0 \cdot n} a_1 \frac{e^{-(\pi/\beta)(|l|+|s|)}}{(|l|+|s|)^3} \leq a_1 \sum_{k \geq C_0 \cdot n} k \cdot \frac{e^{-(\pi/\beta)k}}{k^3} \\ &\leq \frac{a_1}{(C_0 \cdot n)^2} \cdot \frac{e^{-(\pi/\beta)C_0 \cdot n}}{(1 - e^{-C_0 \cdot n})} \leq K'_0 \cdot \sigma^{-n/2}. \end{aligned} \quad (5.3)$$

We write  $S_1 = S_{11} + S_{12}$  where

$$S_{11} = \sum_{\substack{p_0(\omega) \leq n/2 \\ |\omega| \leq 2\sigma_0^{-n/2}}} |\omega| \quad \text{and} \quad S_{12} = \sum_{n/2 < p_0(\omega)} |\omega| \leq K'_0 \cdot \sigma^{-n/4}$$

and we have used the bound (5.3). We also split  $S_2 = S_{21} + S_{22}$  according to whether  $n - \sum_{k=0}^{i-1} p_k(\omega) \geq n/2$  or not, obtaining

$$S_{21} = \sum_{|\omega| \leq 2^{1-i} \cdot \sigma_0^{-n/2}} |\omega| \quad \text{and} \quad S_{22} = \sum_{\substack{|\omega| \leq 2^{1-i} \cdot \sigma_0^{-n/2} \\ n - \sum_{k=0}^{i-1} p_k(\omega) < n/2}} |\omega|.$$

Since  $2^{1-i} \leq 2$  we get  $S_{11} + S_{21} \leq 2 \cdot S_{11}$  and the summands  $\omega \in \mathcal{P}_0$  satisfy  $|\omega| \leq 2 \cdot \sigma_0^{-n/2}$ , thus by Remark 2.1 we get

$$|l| + |s| \geq \frac{n}{2} \cdot \frac{\log \sigma_0}{3 + \pi/\beta} + \frac{3 - \log(2/a_1)}{3 + \pi/\beta} \geq C_1 \cdot \frac{n}{4},$$

where  $C_1 = \log \sigma_0 / (3 + \pi/\beta)$  for every big enough  $n$ . Then  $S_{11} + S_{21} \leq 2K''_0 \cdot \sigma_0^{-n/5}$  by the same calculations as in (5.3) with slightly different constants.

For  $S_{22}$  we note that  $n - \sum_{k=0}^{i-1} p_k(\omega) < n/2$  implies  $\sum_{k=0}^{i-1} p_k(\omega) > n/2$  and so, by Lemma 4.6(b) we get

$$2 \geq |f^{r_i}(\omega)| > e^{(1-2\zeta)\frac{\pi}{\beta} \sum_{k=0}^{i-1} p_k(\omega)} \cdot |\omega| > e^{n\pi(1-2\zeta)/(2\beta)} \cdot |\omega|$$

and hence again by Remark 2.1, for every big enough  $n$ , we must have  $|l| + |s| > C_2 \cdot n/4$  where  $C_2 = \pi(1 - 2\zeta)/(3\beta + \pi)$ . We deduce that  $S_{22} \leq K'''_0 \cdot e^{-(\pi/\beta)C_2 \cdot n/4}$  following the same calculations in (5.3).

Putting all together we see that there are constants  $0 < \xi_0 < 1$ ,  $K_0 > 0$  and  $n_0 \geq 1$  such that  $S_0 + S_1 + S_2 + S_3 \leq K_0 \cdot e^{-\xi_0 n}$  for all  $n \geq n_0$ , with  $\xi_0$  and  $K_0$  dependent on  $\sigma, \sigma_0$  and  $\zeta$ , as stated.  $\square$

Let  $v \leq u \leq n$  and  $v$  pairs of positive integers  $(\eta_1, v_1), \dots, (\eta_v, v_v)$  be given,  $\eta_i \geq k_0$ ,  $v_i \geq 1$  and  $\eta_i + v_i \geq \Theta$ , where

$$\Theta = \Theta(b) = \frac{\beta}{\pi} \cdot \log \frac{a_2}{b}. \quad (5.4)$$

This value of  $\Theta$  is chosen in such a way that  $\text{dist}(I(\eta, v), \mathcal{C}) < b$  if, and only if,  $\eta + v \geq \Theta$ . We assume that  $b$  is so small that  $\Theta \geq \theta$  (recall that  $\theta$  was defined in Lemma 4.6) and, following Remark 4.9, that  $\Theta$  is big enough in order that  $Q(l, s, \tau, \rho) > 1$  for all  $|l| + |s| \geq \Theta$ . For  $x \in I \setminus \mathcal{C}_\infty$  with  $u_n(x) = u$ , let  $1 = t_1 < \dots < t_u \leq n$  be the return situations (essential returns or escapes) of  $x$ ,  $(l_i, s_i)$  the corresponding indexes of the host intervals and  $d_n(x) = v$



the number of pairs  $(l_i, s_i)$  such that  $|l_i| + |s_i| \geq \Theta$ . Denote by  $1 \leq r_1 < \dots < r_v \leq u$  the indexes corresponding to the deep return situations. We then define the event

$$A_{(\eta_1, v_1), \dots, (\eta_v, v_v)}^{u, v}(n) = \{x \in I : u_n(x) = u, d_n(x) = v \text{ and } |f_\mu^{t_{r_i}}(x)| \in I(\eta_i, v_i), i = 1, \dots, v\}.$$

**Proposition 5.2** (Probability of essential returns with specified depths). *If  $\Theta$  is large enough (depending on  $D_0$  from Lemma 4.10), then for every big  $n \geq u \geq v \geq 1$  and  $\vartheta = \max\{\alpha, 3\zeta\}$*

$$\lambda \left( A_{(\eta_1, v_1), \dots, (\eta_v, v_v)}^{u, v}(n) \right) \leq \binom{u}{v} \exp \left[ (2\vartheta - 1) \frac{\pi}{\beta} \sum_{i=1}^v (\eta_i + v_i) \right].$$

*Proof.* We start by fixing  $n \in \mathbf{N}$ ,  $u \in \{1, \dots, n\}$ ,  $v \in \{1, \dots, u\}$  and taking  $\omega_0 \in \mathcal{P}_0$ . Let  $\omega \in \mathcal{P}_n$  be such that  $\omega \subset \omega_0$  and  $u_n | \omega = u$  and  $1 = t_1 < \dots < t_u \leq n$  be the return situations (essential returns or escapes) of  $\omega$ .

For  $m = 1, \dots, u$  we write  $\omega_m = \omega((l_1, s_1, j_1), \dots, (l_m, s_m, j_m)) \in \mathcal{P}_{t_i}$  the subset of  $\omega_0$  satisfying

$$f_\mu^{t_i}(\omega_m) \subset I(l_i, s_i, j_i)^+, i \in \{1, \dots, m-1\} \text{ and } I(l_m, s_m, j_m) \subset f_\mu^{t_m}(\omega_m) \subset I(l_m, s_m, j_m)^+,$$

by the definition of the sequence of partitions  $\mathcal{P}_n$ . Note that we get a nested sequence of sets

$$\omega_0 \supsetneq \omega_1 \supsetneq \dots \supsetneq \omega_u = \omega.$$

We set  $\mathcal{T} = \{\omega \in \mathcal{P}_n : \omega \subset \omega_0, u_n | \omega = u\}$  and consider the sequence of deep return situations (essential or escapes) of each  $\omega \in \mathcal{T}$ :  $1 \leq r_1 < \dots < r_v \leq u$ , that is, the indexes of the return situations such that for  $i = 1, \dots, v$

$$t_{r_i} \text{ satisfies } |f_\mu^{t_{r_i}}(x)| \in I(\eta_i, v_i) \text{ for all } x \in \omega \text{ and } \eta_i + v_i \geq \Theta.$$

Now we define by induction a sequence of partitions of  $\mathcal{T}$  which will enable us to determine the estimates we need.

Start by putting  $\mathcal{V}_0 = \cup\{\omega \in \mathcal{T}\}$ . Assume that  $\mathcal{V}_{r_i+h}^{j_1, \dots, j_i}$  is already defined with  $1 \leq j_\ell \leq (\eta_\ell + v_\ell)^3$  corresponding to the host intervals of the return situations,  $\ell = 1, \dots, i$ . If  $r_i + h + 1 < r_{i+1}$ , we set

$$\mathcal{V}_{r_i+h+1}^{j_1, \dots, j_i} = \bigcup \left\{ \omega_{r_i+h+1} \cap f_\mu^{-t_{r_i+h+1}(\omega)}(I \setminus B(\mathcal{C}, b)) \mid \omega \in \mathcal{T} \right\} \cap \mathcal{V}_{r_i+h}^{j_1, \dots, j_i},$$

where  $B(\mathcal{C}, b) = \cup_{x \in \mathcal{C}} B(x, b)$  is the  $b$ -neighborhood of the critical set  $\mathcal{C}$ . If  $r_i + h + 1 = r_{i+1}$ , then we set

$$\mathcal{V}_{r_{i+1}}^{j_1, \dots, j_{i+1}} = \bigcup \left\{ \omega_{r_{i+1}} \cap f_\mu^{-t_{r_{i+1}}(\omega)}(I(\eta_{i+1}, v_{i+1}, j_{i+1})) \mid \omega \in \mathcal{T} \right\} \cap \mathcal{V}_{r_i+h}^{j_1, \dots, j_i}.$$

Now we will compare  $\lambda(\mathcal{V}_{r_{i+1}}^{j_1, \dots, j_{i+1}})$  with  $\lambda(\mathcal{V}_{r_i}^{j_1, \dots, j_i})$ . For this we proceed as follows.

Given  $\omega \in \mathcal{T}$  and  $\omega_{r_{i-1}} \in \mathcal{V}_{r_{i-1}}^{j_1, \dots, j_{i-1}}$  we write  $\omega_{r_i}^{j_i} = \omega_{r_{i-1}} \cap \mathcal{V}_{r_i}^{j_1, \dots, j_i}$  for the set of points in  $\omega_{r_{i-1}}$  which remain in the next level of the partition. There are several cases to be considered.

- (1)  $t_{r_{i-1}}$  is an essential return with depth  $(l, s)$ .

In this case since  $\omega_{r_i}^{j_i} \subset \omega_{r_{i-1}} \in \mathcal{P}_{t_{r_{i-1}}}$  we can use Lemma 4.10 to write

$$\begin{aligned} \frac{|\omega_{r_i}^{j_i}|}{|\omega_{r_{i-1}}|} &\leq D_0 \cdot \frac{|f_\mu^{t_{r_i}}(\omega_{r_i}^{j_i})|}{|f_\mu^{t_{r_i}}(\omega_{r_{i-1}})|} \leq D_0 \cdot \frac{|I(\eta_i, v_i, j_i)^+|}{a_1 Q e^{-2\zeta \frac{\pi}{\beta}(|l|+|s|)}} \leq D_0 \cdot \frac{9a_1 e^{-\frac{\pi}{\beta}(\eta_i+v_i)}}{a_1 Q \cdot (\eta_i + v_i)^3 e^{-2\zeta \frac{\pi}{\beta}(|l|+|s|)}} \\ &= \frac{C}{(\eta_i + v_i)^3} \cdot \frac{e^{-\frac{\pi}{\beta}(\eta_i+v_i)}}{e^{-2\zeta \frac{\pi}{\beta}(|l|+|s|)}} \leq \frac{C}{(\eta_i + v_i)^3} \cdot \frac{e^{-\frac{\pi}{\beta}(\eta_i+v_i)}}{e^{-2\zeta \frac{\pi}{\beta}(\eta_{i-1}+v_{i-1})}}, \end{aligned}$$

where  $C = C(\rho, \tau, \varepsilon, \sigma)$  and in the second and third inequalities we used Remark 2.1 and Lemma 4.8(2). In the last inequality we argue as follows.

If  $r_i = r_{i-1} + 1$ , then  $|l| + |s| = \eta_{i-1} + v_{i-1}$  by definition. If  $r_i > r_{i-1} + 1$ , then by definition of deep returns  $|l| + |s| < \Theta \leq \eta_{i-1} + v_{i-1}$ .

(2)  $t_{r_{i-1}}$  is an escape time having host interval  $I(l, s, j)^+$  with  $(l, s, j) \neq (\pm k_0, 1, 1)$ .

We only need to get a lower bound for  $|f_\mu^{t_{r_i}}(\omega_{r_{i-1}})|$ . By the Mean Value Theorem

$$\begin{aligned} |f_\mu^{t_{r_i}}(\omega_{r_{i-1}})| &= |(f_\mu^{t_{r_i}-t_{r_{i-1}}})'(\xi)| \cdot |\omega_{r_{i-1}}| = |(f_\mu^{t_{r_i}-t_{r_{i-1}}-1})'(f_\mu(\xi))| \cdot |f'_\mu(\xi)| \cdot |\omega_{r_{i-1}}| \\ &\geq (\sigma_0^{t_{r_i}-t_{r_{i-1}}-1}) \cdot \frac{1}{C} |f''_\mu(x_l)| \cdot |\omega_{r_{i-1}}| \geq \frac{1}{C} \cdot |x_l|^{\alpha-2} \cdot |\omega_{r_{i-1}}| \\ &\geq \frac{1}{C} \cdot e^{(2-\alpha)\frac{\pi}{\beta}|l|} \cdot a_1 \cdot \frac{e^{-\frac{\pi}{\beta}(|l|+|s|)}}{(|l| + |s|)^3} \\ &= \frac{a_1}{C} \cdot e^{(2-\alpha)\frac{\pi}{\beta}(|l|+|s|)} \cdot \frac{e^{-\frac{\pi}{\beta}(|l|+|s|)}}{(|l| + |s|)^3} \cdot e^{(\alpha-2)\frac{\pi}{\beta}|s|} \\ &\geq C(\tau) \cdot \frac{e^{(1-\alpha)\frac{\pi}{\beta}(|l|+|s|)}}{(|l| + |s|)^3} \geq C(\tau) \cdot e^{-\alpha\frac{\pi}{\beta}(|l|+|s|)}, \end{aligned}$$

where we have used that  $|s| \leq s(\tau)$  in the last inequalities and in the first inequality that

$$|f'_\mu(\xi)| = |f'_\mu(\xi) - f'_\mu(x_l)| = |f''_\mu(\hat{\xi})| \geq \frac{1}{C} \cdot |f''_\mu(x_l)| \geq \frac{1}{C} \cdot |x_l|^{\alpha-2}$$

by Lemma 4.2. Thus we arrive at a similar bound

$$\frac{|\omega_{r_i}^{j_i}|}{|\omega_{r_{i-1}}|} \leq \frac{C}{(\eta_i + v_i)^3} \cdot \frac{e^{-\frac{\pi}{\beta}(\eta_i+v_i)}}{e^{-\alpha\frac{\pi}{\beta}(\eta_{i-1}+v_{i-1})}}. \quad (5.5)$$

(3)  $t_{r_{i-1}}$  is an escape time having as host interval  $I(k_0, 1, 1)$  or  $I(-k_0, 1, 1)$ .

From Lemma 4.3 we have that points in  $f_\mu^{t_{r_i}-1}(\omega)$  remain outside  $[-\varepsilon, \varepsilon]$  during a minimum number  $m_0 \geq \log(1/\varepsilon)$  of iterates with derivative bigger than  $\sigma_0$  which can be taken larger than  $e$ . Hence

$$\begin{aligned} |f_\mu^{t_{r_i}}(\omega_{r_{i-1}})| &= |(f_\mu^{t_{r_i}-t_{r_{i-1}}})'(\xi)| \cdot |\omega_{r_{i-1}}| \geq \sigma_0^m \cdot |\omega_{r_{i-1}}| \geq e^{\log(1/\varepsilon)} \cdot \frac{e^{-\frac{\pi}{\beta}(k_0+1)}}{(k_0 + 1)^3} \\ &\geq C \frac{e^{\log(1/\varepsilon)}}{(\log(1/\varepsilon))^3} \cdot e^{-\frac{\pi}{\beta}(k_0+1)} \geq C \cdot e^{-\alpha\frac{\pi}{\beta}(k_0+1)}, \end{aligned}$$

where we have used that  $k_0 \leq C \cdot \log(1/\varepsilon)$  and assumed that  $\varepsilon$  is small enough so that  $e^{\log(1/\varepsilon)} \geq (\log(1/\varepsilon))^3$ . Thus we arrive again at a bound of the form (5.5).

Now we are ready to compare the measures of the sets  $\mathcal{V}_{r_i}^{j_1, \dots, j_i}$  as follows for  $1 \leq i \leq v$

$$\lambda(\mathcal{V}_{r_i}^{j_1, \dots, j_i}) = \sum_{\omega_{r_{i-1}} \in \mathcal{V}_{r_{i-1}}^{j_1, \dots, j_{i-1}}} \frac{|\omega_{r_i}^{j_i}|}{|\omega_{r_i-1}|} \cdot |\omega_{r_i-1}| \leq \frac{C}{(\eta_i + v_i)^3} \cdot \frac{e^{-\frac{\pi}{\beta}(\eta_i + v_i)}}{e^{-\vartheta \frac{\pi}{\beta}(\eta_{i-1} + v_{i-1})}} \cdot \lambda(\mathcal{V}_{r_{i-1}}^{j_1, \dots, j_{i-1}}),$$

where  $\vartheta = \max\{2\zeta, \alpha\} \in (0, 1)$ . Using the above we get

$$\begin{aligned} \lambda(\mathcal{V}_u^{j_1, \dots, j_v}) &\leq \dots \leq \lambda(\mathcal{V}_v^{j_1, \dots, j_v}) \leq \frac{C}{(\eta_v + v_v)^3} \cdot \frac{e^{-\frac{\pi}{\beta}(\eta_v + v_v)}}{e^{-\vartheta \frac{\pi}{\beta}(\eta_{v-1} + v_{v-1})}} \cdot \lambda(\mathcal{V}_{r_{v-1}}^{j_1, \dots, j_{v-1}}) \\ &\leq \left( \prod_{i=1}^s \frac{C}{(\eta_i + v_i)^3} \right) \exp \left( -\frac{\pi}{\beta} \sum_{i=1}^v (\eta_i + v_i) + \vartheta \frac{\pi}{\beta} \sum_{i=1}^v (\eta_{i-1} + v_{i-1}) \right) \cdot \lambda(\mathcal{V}_0). \end{aligned}$$

Finally we need to consider all possible combinations of the events  $\mathcal{V}_u^{j_1, \dots, j_v}$  which are included in  $A_{(\eta_1, v_1), \dots, (\eta_v, v_v)}^{u, v}(n)$ . Note that for any given  $v \leq u$  there are  $\binom{u}{v}$  ways of having  $v$  deep returns among  $u$  return situations, and by symmetry for any sequence of deep returns with given depth  $(\eta_i, v_i)$  there are  $4 \cdot (\eta_i + v_i)^3$  different possibilities of falling in an element of the partition  $\mathcal{P}_0$ . Thus since  $\lambda(\mathcal{V}_0) \leq \lambda(\omega_0)$  we arrive at

$$\begin{aligned} \lambda \left( A_{(\eta_1, v_1), \dots, (\eta_v, v_v)}^{u, v}(n) \right) &\leq \binom{u}{v} \cdot \prod_{i=1}^v 4(\eta_i + v_i)^3 \cdot \left( \prod_{i=1}^v \frac{C}{(\eta_i + v_i)^3} \right) \\ &\quad \cdot \exp \left( (\vartheta - 1) \frac{\pi}{\beta} \sum_{i=1}^v (\eta_i + v_i) \right) \cdot \sum_{\omega_0 \in \mathcal{P}_0} e^{\vartheta \frac{\pi}{\beta}(\eta_0 + v_0)} \lambda(\omega_0) \\ &\leq \binom{u}{v} \cdot \exp \left( (2\vartheta - 1) \frac{\pi}{\beta} \sum_{i=1}^v (\eta_i + v_i) \right), \end{aligned}$$

where we have used that  $\sum_{\omega_0 \in \mathcal{P}_0} e^{\vartheta \frac{\pi}{\beta}(\eta_0 + v_0)} \lambda(\omega_0) < \infty$  and also that  $\sum_{i=1}^v (\eta_i + v_i) \geq v\Theta$  and that  $\Theta$  can be taken as large as needed. The proof is complete.  $\square$

Now we set using the same notations as before with  $\eta \geq k_0$ ,  $v \geq 1$  and  $\eta + v \geq \Theta$

$$\begin{aligned} A_{(\eta, v), j}^{v, u}(n) &= \{x \in I : u_n(x) = u, d_n(x) = v \text{ and } |f_\mu^{tr_j}(x)| \in I(\eta, v)\}, \\ A_{(\eta, v)}^{v, u}(n) &= \{x \in I : u_n(x) = u, d_n(x) = v, \text{ and } \exists 1 \leq j \leq v \text{ s.t. } |f_\mu^{tr_j}(x)| \in I(\eta, v)\}, \\ A_{(\eta, v)}(n) &= \{x \in I : \exists t \leq n \text{ s.t. } t \text{ is an essential return and } |f_\mu^t(x)| \in I(\eta, v)\}, \end{aligned}$$

and derive the following corollary which will be used during the final arguments.

**Corollary 5.3.** *We have for  $1 \leq j \leq v \leq u \leq n$  that*

- (1)  $\lambda(A_{(\eta, v), j}^{v, u}(n)) \leq \binom{u}{v} e^{(2\vartheta-1)\frac{\pi}{\beta}(\eta+v)}$ ;
- (2)  $\lambda(A_{(\eta, v)}^{v, u}(n)) \leq v \binom{u}{v} e^{(2\vartheta-1)\frac{\pi}{\beta}(\eta+v)}$ ;
- (3)  $\lambda(A_{(\eta, v)}(n)) \leq n^3 e^{o(\Theta)n} e^{(4\vartheta-1)\frac{\pi}{\beta}(\eta+v)}$ ,

if  $\Theta$  is sufficiently big and  $\zeta > \vartheta/3$  is small enough, where  $o(\Theta) \rightarrow 0$  when  $\Theta \rightarrow \infty$ .

*Proof.* We note that since

$$A_{(\eta, v), j}^{v, u}(n) \subseteq \bigcup_{\eta_i + v_i \geq \Theta, \eta_i \geq k_0, v_i \geq 1, i \neq j} A_{(\eta_1, v_1), \dots, (\eta_{i-1}, v_{i-1}), (\eta, v), (\eta_{i+1}, v_{i+1}), \dots, (\eta_u, v_u)}^{v, u}(n)$$

then  $\lambda(A_{(\eta,v),j}^{v,u}(n)) \leq \binom{u}{v} \left( \sum_{l+s \geq \Theta} e^{(2\vartheta-1)\frac{\pi}{\beta}(l+s)} \right)^{v-1} \cdot e^{(2\vartheta-1)\frac{\pi}{\beta}(\eta+v)} \leq e^{(2\vartheta-1)\frac{\pi}{\beta}(\eta+v)}$ , as long as  $\Theta$  is big enough in order that  $\sum_{l+s \geq \Theta} e^{(2\vartheta-1)\frac{\pi}{\beta}(l+s)} \leq 1$ . From this we get item (2) since

$$A_{(\eta,v)}^{v,u}(n) \subset \bigcup_{j=1}^v A_{(\eta,v),j}^{v,u}(n).$$

For item (3) we note that

$$A_{(\eta,v)}(n) \subset \bigcup_{v=1}^u \bigcup_{u=v}^n A_{(\eta,v)}^{v,u}(n),$$

but since there is a deep essential return at iterate  $t$  (which is *not* an escape situation) before  $n$ , we know that the binding period  $p$  is larger than  $\iota(M)(\eta+v) > \iota(M) \cdot \Theta$ .

Let  $m = t + p \geq n$ . By definition  $A_{(\eta,v)}(n) \subseteq A_{(\eta,v)}(m)$  and the maximum number  $u$  of essential return situations in the first  $m$  iterates of  $x$  is bounded by  $m/(\iota(M) \cdot \Theta \cdot \tilde{v})$ , where  $\tilde{v}$  is the number of deep essential returns among the  $u$  essential return situations. Since we know that  $\tilde{v} \geq 1$  we get  $u \leq (\iota(M)\Theta)^{-1} \cdot m$  and this implies that

$$\begin{aligned} \lambda(A_{(\eta,v)}(n)) &\leq \lambda(A_{(\eta,v)}(m)) \leq \sum_{v=1}^u \sum_{u=v}^m \lambda(A_{(\eta,v)}^{v,u}(m)) \\ &\leq \sum_{v=1}^{m/(\iota(M)\Theta)} \sum_{u=v}^m v \binom{u}{v} e^{(2\vartheta-1)\frac{\pi}{\beta}(\eta+v)} = e^{(2\vartheta-1)\frac{\pi}{\beta}(\eta+v)} \sum_{v=1}^{(\iota(M)\Theta)^{-1} \cdot m} v \cdot \sum_{u=v}^m \binom{u}{v} \\ &< e^{(2\vartheta-1)\frac{\pi}{\beta}(\eta+v)} \cdot m \cdot \sum_{v=1}^{(\iota(M)\Theta)^{-1} \cdot m} v \binom{m}{v} \\ &< m \cdot e^{(2\vartheta-1)\frac{\pi}{\beta}(\eta+v)} \cdot \binom{m}{(\iota(M)\Theta)^{-1} \cdot m} \cdot \sum_{v=1}^{(\iota(M)\Theta)^{-1} \cdot m} v \\ &< \frac{m^3}{(\iota(M)\Theta)^2} \cdot e^{(2\vartheta-1)\frac{\pi}{\beta}(\eta+v)} \cdot e^{o(\Theta)m} \end{aligned}$$

where we used the bound  $\binom{m}{(\iota(M)\Theta)^{-1} \cdot m} \leq e^{o(\Theta)m}$  which can be obtained by a straightforward application of Stirling's Formula, as long as  $\Theta$  is big enough. Finally since  $m - n < p \leq \frac{2\pi}{\beta \log \sigma}(\eta+v)$  by Lemma 4.6(a) we have for  $\vartheta$  small enough

$$\begin{aligned} \lambda(A_{(\eta,v)}(n)) &\leq n^3 e^{o(\Theta)n} e^{(2\vartheta-1)\frac{\pi}{\beta}(\eta+v)} \cdot e^{o(\Theta)(m-n)} \cdot \left(\frac{m}{n}\right)^3 \\ &\leq n^3 e^{o(\Theta)n} \cdot e^{(\frac{2\pi}{\log \sigma} + 2\vartheta-1)\frac{\pi}{\beta}(\eta+v)} \cdot \left(1 + \frac{2\pi(\eta+v)}{n\beta \log \sigma}\right)^3 \leq n^3 e^{o(\Theta)n} e^{(4\vartheta-1)\frac{\pi}{\beta}(\eta+v)}, \end{aligned}$$

which completes the proof of the lemma.  $\square$

## 6. SLOW RECURRENCE TO THE CRITICAL SET

Now we make use of the lemmas from Section 5 to prove Theorem B and consequently also Theorem A. We start by recalling the definition of  $\mathcal{C}_n^b(x)$  from (1.7) and that  $u_n(x)$  is the number of essential return situations of the  $f_\mu$ -orbit of  $x$  between 0 and  $n$ .

We let  $0 \leq t_1 < \dots < t_{u_n} \leq n$  be the essential return times or escape times of the orbit of  $x$  and write  $(l_1, s_1, j_1), \dots, (l_{u_n}, s_{u_n}, j_{u_n})$  for the corresponding critical points and depths at each essential return situation, as in Section 5. We recall also that  $d_n(x)$  is the number of pairs  $(l_i, s_i)$  such that  $l_i + s_i \geq \Theta$ , where  $\Theta = \Theta(b)$  is defined in (5.4).

We consider the sequence of deep return situations of  $x$ :  $1 \leq r_1 < \dots < r_{d_n} \leq u_n$  among the sequence of return situations and then define

$$\mathcal{D}_n^b(x) = \sum_{k=1}^{d_n(x)} (|l_{r_k}| + |s_{r_k}|)$$

which is constant on the elements of  $\mathcal{P}_n$  and get the following bound.

**Proposition 6.1.** *There exists  $B_0 = B_0(\sigma, \rho, \tau) > 0$  such that for every  $\omega \in \mathcal{P}_n$  such that  $d_n | \omega \geq 2$  we have  $\mathcal{C}_n^b(x) \leq \frac{B_0}{n} \cdot \mathcal{D}_n^b(x)$  for all  $x \in \omega$ .*

We start by proving the following.

**Lemma 6.2.** *Let  $1 \leq t_i < n$  be a essential return, inessential return or escape situation for  $\omega \in \mathcal{P}_n$ , with binding time  $p_i = p(l_i, s_i)$  (which we set to zero in the case of an escape situation). Then there exists a constant  $B_1 > 0$  such that for all  $x \in \omega$*

$$\mathcal{C}(t_i, t_i + p_i) = \sum_{k=t_i}^{t_i+p_i} -\log \text{dist}_b \left( f_\mu^k(x), \mathcal{C} \right) \leq B_1 \cdot (|l_i| + |s_i|).$$

Moreover if  $p_i > 0$ , then  $\text{dist} \left( f_\mu^{t_i+k}(x), \mathcal{C} \right) > \text{dist} \left( f_\mu^{t_i}(x), \mathcal{C} \right)$  for  $k = 1, \dots, p$  as long as  $\rho > 0$  is small enough. In particular  $\mathcal{C}(t_i, t_i + p_i) = 0$  for escape situations or return situations which are not deep, i.e. the host interval  $I(l, s, j)$  is such that  $|l| + |s| \leq \Theta$ .

*Proof.* Let us fix  $x \in \omega$  in what follows. We consider first the case of  $t_i$  being an essential return or escape situation with  $|l_i| + |s_i| \leq \Theta$ . Thus

$$\log \text{dist}_b \left( f_\mu^{t_i}(x), \mathcal{C} \right) = 0.$$

If  $t_i$  is an escape situation there is nothing else to prove because  $p_i = 0$ . Otherwise  $p_i > 0$  and we have two possibilities during the binding times  $t_i + k$  with  $k = 1, \dots, p_i$ : either  $|f_\mu^{t_i+k}(x)| > \varepsilon$ , in which case we have again  $\log \text{dist}_b \left( f_\mu^{t_i+k}(x), \mathcal{C} \right) = 0$ ; or  $|f_\mu^{t_i+k}(x)| \leq \varepsilon$ . In this case by Proposition 3.4 and the definition of binding time, if

$$\text{dist} \left( f_\mu^{t_i+k}(x), \mathcal{C} \right) \leq \text{dist} \left( f_\mu^{t_i}(x), \mathcal{C} \right),$$

then

$$\rho_0 \cdot e^{-\rho k} \leq \text{dist} \left( f_\mu^{t_i+k}(x), \mathcal{C} \right) \leq \text{dist} \left( f_\mu^{t_i}(x), \mathcal{C} \right) \leq a_2 \cdot e^{-\frac{\pi}{\beta}(|l_i|+|s_i|)}$$

and thus

$$\frac{2\pi}{\beta \log \sigma} (|l_i| + |s_i|) \geq p_i \geq k \geq \frac{1}{\rho} \log \frac{\rho_0}{2} + \frac{\pi}{\beta \rho} (|l_i| + |s_i|)$$

which is impossible as long as  $\pi/(\beta \rho) > 2\pi/(\beta \log \sigma)$ . Therefore choosing  $k_0$  big enough and  $\rho$  sufficiently small we get  $\text{dist} \left( f_\mu^{t_i+k}(x), \mathcal{C} \right) > \text{dist} \left( f_\mu^{t_i}(x), \mathcal{C} \right)$  for all  $k = 1, \dots, p_i$  and so  $\mathcal{C}(t_i, t_i + p_i) = 0$  for escapes and returns which are not deep. This proves the last part of the statement of the lemma.

On the other hand, if  $t_i = t_{r_k}$  we get

$$\text{dist}_b \left( f_\mu^{r_i}(x), \mathcal{C} \right) \geq a_2 \cdot e^{-\frac{\pi}{\beta}(|l_{r_k}|+|s_{r_k}|)} \quad (6.1)$$

and so if  $t_{r_k}$  is an escape or a return we have the contribution

$$-\log \text{dist}_b(f_\mu^{r_i}(x), \mathcal{C}) \leq -\log a_2 + \frac{\pi}{\beta}(|l_{r_k}| + |s_{r_k}|)$$

for the sum  $\mathcal{C}_n^b(x)$ . For escape situations the proof ends here. For return times we must consider the next binding period. Recall that we are assuming (4.1) holds.

Now we write for  $h = 1, \dots, p_{r_k}$ ,  $l = l_{r_k}$  and  $i = r_k$

$$\text{dist}(f_\mu^{t_i+h}(x), \mathcal{C}) \geq \begin{cases} \varepsilon - x_{k_0} & \text{if } |f_\mu^{t_i+h}(x)| > \varepsilon \\ \text{dist}(f_\mu^h(x_l), \mathcal{C}) - \text{dist}(f_\mu^h(x_l), f_\mu^{t_i+h}(x)) & \text{if } |f_\mu^{t_i+h}(x)| \leq \varepsilon \end{cases} \quad (6.2)$$

and in the second alternative above  $-\log \text{dist}(f_\mu^{t_i+h}(x), \mathcal{C})$  is bounded by

$$-\log((1 - e^{-\tau h}) \cdot \text{dist}(f_\mu^h(x_l), \mathcal{C})) \leq \frac{e^{-\tau h}}{1 - e^{-\tau h}} - \log \text{dist}(f_\mu^h(x_l), \mathcal{C}).$$

Summing over all  $h = 1, \dots, p_i$  we get in the worst case

$$\sum_{h=1}^{p_i} -\log \text{dist}(f_\mu^{t_i+h}(x), \mathcal{C}) \leq C(\tau) + \sum_{h=1}^{p_i} -\log \text{dist}(f_\mu^h(x_l), \mathcal{C}). \quad (6.3)$$

To bound the last sum we use the assumption (4.1) on free times of the orbit of  $x_l$ . We need to sum up to the first *free time* of the orbit of  $x_l$  after  $p_i$ . But if  $p_i$  is a bound time for the orbit of  $x_l$ , then by [PRV, Lemma 5.3(a)] its binding period must be smaller than  $\frac{2\rho}{\log \sigma} p_i$ . Thus there exists a free time  $n$  for the orbit of  $x_l$  with  $p_i \leq n \leq (1 + \frac{2\rho}{\log \sigma}) p_i$ . Hence

$$\sum_{h=1}^{p_i} -\log \text{dist}(f_\mu^h(x_l), \mathcal{C}) \leq \hat{M} \cdot (1 + \frac{2\rho}{\log \sigma}) \cdot p_i$$

and so (6.3) is bounded by  $C(\tau) + \hat{M} \cdot (1 + \frac{2\rho}{\log \sigma}) \cdot p_i$ .

For  $\varepsilon$  small enough we have that  $k_0$  and  $p_i$  are very big and in both cases (6.2) we get  $\mathcal{C}(t_{r_k}, t_{r_k} + p_{r_k}) \leq \tilde{M} \cdot p_{r_k}$  for a constant  $\tilde{M}$ . By Lemma 4.6(a) we obtain

$$\mathcal{C}(t_{r_k}, t_{r_k} + p_{r_k}) \leq B_1 \cdot (|l_{r_k}| + |s_{r_k}|)$$

for a constant  $B_1 > 0$ , concluding the proof of the lemma.  $\square$

Next we show that the depth of an inessential return or free time is not greater than the depth of the essential return situation that precedes it.

**Lemma 6.3.** *Let  $t_i$  be an essential return or an escape for  $\omega \in \mathcal{P}_n$  with  $I(l_i, s_i, j_i) \subset f_\mu^{t_i}(\omega) \subset I(l_i, s_i, j_i)^+$ . Then for each consecutive inessential return  $t_i < t_i(1) < \dots < t_i(v) < n$  before the next essential return or escape and for each free time, i.e. for all iterates  $t_i(k) + p_i(k) < j \leq t_i(k+1)$  for  $k = 0, \dots, v-1$  with  $p_i(k)$  the binding period of the  $k$ th return, the host interval  $I(l, s, j) \supset f_\mu^j(\omega)$  is such that  $|l| + |s| < |l_i| + |s_i|$ .*

*Proof.* By items (1) and (3) of Lemma 4.8 we have  $|f_\mu^j(\omega)| > |f_\mu^{t_i}(\omega)| > |I(l_i, s_i, j_i)|$ . Thus because each  $j$  is an inessential return or a free time we get

$$a_1 \frac{e^{-(\pi/\beta)(|l|+|s|)}}{(|l|+|s|)^3} > |f_\mu^{t_i(k)}(\omega)| > a_1 \frac{e^{-(\pi/\beta)(|l_i|+|s_i|)}}{(|l_i|+|s_i|)^3}.$$

As  $z^{-3} \cdot e^{-(\pi/\beta)z}$  is decreasing for  $z > 0$ , we conclude that  $|l| + |s| < |l_i| + |s_i|$ .  $\square$

If we have deep returns we can use sharper bounds to obtain a relation between the logarithmic distance to the critical set along the orbit between consecutive essential returns and the depth of the first return. Note that by (3.4) on “deep” free iterates between critical points we may write

$$|f'_\mu(x)| \geq \left( \frac{\tau \hat{x}^{\alpha-2}}{C} \cdot e^{-(1-\alpha)\frac{\pi}{\beta}|s(\tau)|} \right) \cdot e^{(1-\alpha)\frac{\pi}{\beta}(|l|+|s|)} > e^{(1-\hat{\alpha})\frac{\pi}{\beta}(|l|+|s|)} \quad (6.4)$$

for some  $\alpha < \hat{\alpha} < 1$ , since on these times we have  $|s| \leq s(\tau)$  and we assume that  $|l| + |s| \geq \Theta$  is so big that the expression in parenthesis in (6.4) is larger than  $e^{-(\hat{\alpha}-\alpha)\Theta}$ .

**Lemma 6.4.** *Let  $t_i$  be a deep essential return or an escape for  $\omega \in \mathcal{P}_n$  with  $I(l_i, s_i, j_i) \subset f_\mu^{t_i}(\omega) \subset I(l_i, s_i, j_i)^+$ . Let  $t_i = t_i(0) < t_i(1) < \dots < t_i(v) < t_i(v+1) = t_{i+1}$  be the consecutive inessential returns before the next essential return or escape and let  $p_i(k)$  be the corresponding binding times for  $k = 0, \dots, v+1$ . Define  $J = \cup_{k=0}^v \{t_i(k) + p_i(k) + 1, \dots, t_i(k+1)\}$  the set of free iterates and of inessential return times. Assume that every such iterate  $h \in J$  is contained in a “deep” interval  $I(l_h, s_h, j_h)$ , i.e.  $l_h + s_h \geq \Theta$ . Then there exists a constant  $B_2 > 0$  such that*

$$\sum_{h \in J} -\log \text{dist}_b(f_\mu^h(x), \mathcal{C}) \leq B_2(|l_i| + |s_i|).$$

*Proof.* By the choice of  $\Theta$  we have  $Q > 1$  in Lemma 4.8 and (6.4) also holds. Then we deduce

$$|f_\mu^{t_{i+1}}(\omega)| \geq \prod_{k=0}^v e^{(1-3\zeta)\frac{\pi}{\beta}(|l_{t_i(k)}|+|s_{t_i(k)}|)} \prod_{h=t_i(k)+p_i(k)}^{t_i(k+1)-1} e^{(1-\hat{\alpha})\frac{\pi}{\beta}(|l_h|+|s_h|)} \cdot |f_\mu^{t_i+p_i}(\omega)|$$

and then since we may assume that  $3\zeta < \hat{\alpha}$ , by the definition of essential returns and using Lemma 4.6(b) we get

$$\frac{e^{-\frac{\pi}{\beta}(|l_{i+1}|+|s_{i+1}|)}}{(|l_{i+1}| + |s_{i+1}|)^3} \geq \exp\left(\frac{\pi}{\beta}\left((1-\hat{\alpha})\sum_{h \in J}(|l_h| + |s_h|)\right)\right) \cdot \frac{e^{-3\zeta\frac{\pi}{\beta}(|l_i|+|s_i|)}}{(|l_i| + |s_i|)^3}$$

or equivalently

$$3 \log\left(\frac{|l_i| + |s_i|}{|l_{i+1}| + |s_{i+1}|}\right) + 3\zeta\frac{\pi}{\beta}(|l_i| + |s_i|) \geq \frac{\pi}{\beta}(|l_{i+1}| + |s_{i+1}|) + \frac{\pi}{\beta}\left((1-\hat{\alpha})\sum_{h \in J}(|l_h| + |s_h|)\right). \quad (6.5)$$

Since  $|l_{i+1}| + |s_{i+1}| \gg 1$  we also have that the left hand side of (6.5) is smaller than

$$\left(3\zeta + \frac{3\beta \log(|l_i| + |s_i|)}{\pi(|l_i| + |s_i|)}\right) \frac{\pi}{\beta}(|l_i| + |s_i|) \leq \left(3\zeta + \frac{3\beta \log \Theta}{\pi \cdot \Theta}\right) \frac{\pi}{\beta}(|l_i| + |s_i|) \leq 5\zeta\frac{\pi}{\beta}(|l_i| + |s_i|)$$

for  $\Theta$  sufficiently big, because  $|l_i| + |s_i| \geq \Theta$  and  $\log(z)/z$  is decreasing for  $z > e$ . Thus we get

$$5\zeta(|l_i| + |s_i|) \geq |l_{i+1}| + |s_{i+1}| + (1-\hat{\alpha})\sum_{h \in J}(|l_h| + |s_h|). \quad (6.6)$$

Now since every iterate is “deep” for all  $x \in \omega$  we have the bound

$$\sum_{h \in J} -\log \text{dist}_b(f_\mu^h(x), \mathcal{C}) \leq -\#J \cdot \log a_2 + \frac{\pi}{\beta}\sum_{h \in J}(|l_h| + |s_h|) \leq B_2 \cdot (|l_i| + |s_i|)$$

for a constant  $B_2 > 0$  depending on  $\zeta$  and  $\hat{\alpha}$  from (6.6), as long as  $|l_h| + |s_h| \geq \Theta$  is sufficiently big. This completes the proof of the lemma.  $\square$

*Proof of Proposition 6.1:* Let us fix  $x \in \omega \in \mathcal{P}_n$  with  $d_n \mid \omega \geq 2$  and  $i \in \{1, \dots, u_n(x) - 1\}$ . According to Remark 2.1 and the definition of deep essential return situation we have

$$\text{dist}_b(f_\mu^{r_i}(x), \mathcal{C}) \geq a_2 \cdot e^{-\frac{\pi}{\beta}(|l_i| + |s_i|)}. \quad (6.7)$$

Note that the above truncated distance is 1 on the return situations  $t_i$  which are not deep, by the choice of  $\Theta$ . Moreover this distance is also 1 for all iterates between such  $t_i$  (not deep) and the next return situation  $t_{i+1}$  by Lemma 6.3 for the inessential return and free iterates, and by Lemma 6.2 for the bound iterates.

Hence *we only have to take care of the deep essential return or escape times* plus the next iterates before the following essential return situation. The sum of the logarithms of the truncated distance on binding periods, given by Lemma 6.2, is bounded by a constant times the depth of the return which originated the binding. In addition, the same sum over the free and the inessential return iterates is likewise bounded by the depth of the essential return or escape  $t_i$ , by Lemma 6.4. If we keep the notations introduced in the statements of Lemmas 6.3 and 6.4, then we may write

$$\begin{aligned} \mathcal{C}(t_i, t_{i+1}) &\leq \sum_{k=0}^v \left[ \mathcal{C}(t_i(k), t_i(k) + p_i(k)) + \mathcal{C}(t_i(k) + p_i(k) + 1, t_i(k+1)) \right] \\ &\leq B_1 \sum_{k=0}^v \left( |l_{t_i(k)}| + |s_{t_i(k)}| \right) + B_2(|l_i| + |s_i|) \\ &\leq B_1 \sum_{k=0}^v \mathcal{C}(t_i(k) + p_i(k) + 1, t_i(k+1)) + B_2(|l_i| + |s_i|) \\ &\leq B_2 \cdot (1 + B_1) \cdot (|l_i| + |s_i|) \end{aligned}$$

Setting  $B_0 = B_2(1 + B_1)$  this finishes the proof of Proposition 6.1.  $\square$

**6.1. The expected value of the distance at return times.** The statement of Proposition 6.1 together with Lemma 5.1 and Proposition 5.2 ensure that, to obtain slow recurrence to the critical set, we need to bound  $\mathcal{D}_n^b(x)/n$  for Lebesgue almost every  $x \in I$ . Indeed we have for every big enough  $n$

$$\{x \in I : \mathcal{C}_n^b(x) > \delta\} \subseteq \bigcup \{ \omega \in \mathcal{P}_n : u_n \mid \omega \equiv 1 \} \cup \left\{ x \in I : u_n(x) \geq 2 \text{ and } \mathcal{D}_n^b(x) > \frac{n}{B_0} \cdot \delta \right\}$$

and Lemma 5.1 shows that the left hand side subset of the above union has exponentially small measure. We now show that Proposition 5.2 implies a similar bound for the right hand subset.

**Lemma 6.5.** *For every  $z \in (0, \frac{(1-2\theta)\pi}{2\beta})$  there exists  $\Theta_1 = \Theta_1(z, \tau, \rho, \sigma)$  such that for  $\Theta > \Theta_1$*

$$\int e^{z \cdot \mathcal{D}_n^b(x)} d\lambda(x) \leq e^{o(\Theta) \cdot n},$$

where  $o(\Theta) \rightarrow 0$  when  $\Theta \rightarrow \infty$ .

*Proof.* The integral in the statement equals the following series

$$\sum_{\substack{1 \leq v \leq u \leq n \\ (\eta_1, v_1), \dots, (\eta_v, v_v)}} \exp \left( z \sum_{k=1}^v (\eta_k + v_k) \right) \cdot \lambda \left( A_{(\eta_1, v_1), \dots, (\eta_v, v_v)}^{u, v}(n) \right),$$



where  $\eta_k + v_k \geq \Theta$ ,  $v_k \geq 1$  and  $\eta_k \geq k_0$  for  $k = 1, \dots, v$ . Proposition 5.2 provides the bound

$$\sum_{\substack{1 \leq v \leq u \leq n \\ (\eta_1, v_1), \dots, (\eta_v, v_v)}} \binom{u}{v} e^{z \sum_{k=1}^v (\eta_k + v_k) + (2\vartheta - 1) \frac{\pi}{\beta} \sum_{k=1}^v (\eta_k + v_k)} = \sum_{\substack{1 \leq v \leq u \leq n \\ (\eta_1, v_1), \dots, (\eta_v, v_v)}} \binom{u}{v} e^{(z + (2\vartheta - 1) \frac{\pi}{\beta}) \cdot \sum_{k=1}^v (\eta_k + v_k)}.$$

Now setting  $\Delta = \sum_{k=1}^v (\eta_k + v_k)$  and

$$K(v, \Delta) = \#\left\{((l_1, s_1), \dots, (l_v, s_v)) : \sum_{k=1}^v (l_k + s_k) = \Delta, l_k \geq k_0, s_k \geq 1, l_k + s_k \geq \Theta, 1 \leq k \leq v\right\}$$

we may rewrite the last series as  $\sum_{1 \leq v \leq u \leq n} \sum_{\Delta \geq v\Theta} \binom{u}{v} K(v, \Delta) \cdot e^{(z + (2\vartheta - 1) \frac{\pi}{\beta}) \Delta}$ . To estimate  $K(v, \Delta)$  we observe that

$$K(v, \Delta) \leq \#\left\{(n_1, \dots, n_{2v}) : \sum_{k=1}^{2v} n_k = \Delta \text{ and } n_k \geq 0, k = 1, \dots, 2v\right\} = \binom{\Delta + 2v - 1}{2v - 1},$$

where  $\binom{n}{k} = \frac{n!}{k!(n-k)!}$  is a binomial coefficient. By a standard application of Stirling's Formula we get

$$K(v, \Delta) \leq \left(C^{1/\Delta} \cdot \left(1 + \frac{2v - 1}{\Delta}\right) \cdot \left(1 + \frac{\Delta}{2v - 1}\right)^{(2v-1)/\Delta}\right)^\Delta \leq e^{z \cdot \Delta},$$

since  $\Delta \geq v\Theta$  ensures that the expression in parenthesis can be made arbitrarily close to 1 if  $\Theta$  is taken bigger than some constant  $\Theta_0 = \Theta_0(z)$ , where  $0 < C < 1$  is a constant independent of  $\Theta$  and we assume that  $z > 0$  is small. Hence we arrive at

$$\begin{aligned} \int e^{z\mathcal{D}_n^b(x)} d\lambda(x) &\leq \sum_{1 \leq v \leq u \leq n} \sum_{\Delta \geq v\Theta} \binom{u}{v} e^{(2z + (2\vartheta - 1) \frac{\pi}{\beta}) \Delta} \leq \sum_{v=0}^n \binom{n}{v} \cdot C \cdot e^{(2z + (2\vartheta - 1) \frac{\pi}{\beta}) \Theta v} \\ &\leq \left(1 + C \cdot e^{(2z + (2\vartheta - 1) \frac{\pi}{\beta}) \Theta}\right)^n = e^{o(\Theta) \cdot n}, \end{aligned}$$

as long as  $0 < z < (1 - 2\vartheta)\pi/(2\beta)$  and  $\Theta > \Theta_1 > \max\{\theta, \Theta_0\}$  is big enough so that  $Q > 1$  in Lemma 4.8 and (6.4) holds, as stated.  $\square$

As a consequence of this bound we can use Tchebychev's inequality with  $z$  and  $\Theta$  as in the statement of Lemma 6.5 to obtain

$$\lambda\left(\left\{\mathcal{D}_n^b \geq \frac{n}{B_0} \cdot \delta\right\}\right) = \lambda\left(\left\{e^{z\mathcal{D}_n^b} > e^{z \cdot \delta \cdot n / B_0}\right\}\right) \leq e^{-z \cdot \delta \cdot n / B_0} \int e^{z \cdot \mathcal{D}_n^b} d\lambda \leq e^{-z \cdot \delta \cdot n / B_0} \cdot e^{o(\Theta) \cdot n}.$$

Now observe that we may take  $b = b(\delta) > 0$  so small that  $\Theta(b)$  becomes big enough to satisfy all constraints and moreover  $o(\Theta) < \delta \cdot \frac{z}{n}$ .

As already explained, this together with Lemma 5.1 implies that there are  $C, \xi > 0$ , where  $\xi = \xi(\delta)$ , such that for all  $n \geq 1$

$$\lambda(\{x \in I : \mathcal{C}_n(x) > \delta\}) \leq C e^{-\xi n}.$$

Then since

$$\{x \in I : \mathcal{R}(x) > n\} \subseteq \bigcup_{k > n} \{x \in I : \mathcal{C}_k(x) > \delta\}$$

we conclude that there are constants  $C_1, \xi_1 > 0$  such that Theorem B holds.

## 7. FAST EXPANSION FOR MOST POINTS

Here we use the results from the previous sections to prove Theorem C and as a consequence obtain Corollary D. We start by setting

$$E_n = \left\{ x \in I : \exists 1 \leq k \leq n \text{ s.t. } \text{dist}(f_\mu^k(x), \mathcal{C}) < e^{-\rho \cdot n}, f_\mu^k(x) \in I(l, s) \text{ and } |l| \geq k_0, |s| > s(\tau) \right\}$$

and proving the following bound.

**Lemma 7.1.** *There are constants  $C, \xi > 0$  dependent on  $\hat{f}$ ,  $k_0$ ,  $\zeta$ ,  $\rho$ , and  $\tau$  only such that for all  $n \geq 1$*

$$\lambda(E_n) \leq C \cdot e^{-\xi \cdot n}.$$

*Proof.* Let us take  $x \in E_n$  and let  $k \in \{1, \dots, n\}$  be the iterate which is very close to the critical set. Observe that by Remark 2.1 the constraint on the distance implies

$$-\rho n > \log \text{dist}(f_\mu^k(x), \mathcal{C}) \geq \log a_2 - \frac{\pi}{\beta}(|l| + |s|) \quad \text{and so} \quad |l| + |s| \geq \frac{\beta \rho}{20\pi} \cdot n.$$

Since this iterate is in the binding region, there must be an essential return  $t < k$  whose depth is at least as larger as  $|l| + |s|$ , by the results of Lemmas 6.2 and 6.3.

Hence according to the definition of  $A_{(\eta, v)}(n)$  from Section 5 if  $n \geq \Theta$

$$\begin{aligned} E_n &\subset \bigcup \left\{ A_{(\eta, v)}(n) : (\eta, v) \text{ is such that } \eta + v \geq \frac{\beta \rho}{20\pi} \cdot n \right\}, \quad \text{thus by Corollary 5.3} \\ \lambda(E_n) &\leq n^3 e^{o(\Theta)n} \cdot \sum_{\substack{\eta+v \geq \beta \rho n / (20\pi) \\ \eta \geq k_0, v \geq s(\tau)}} e^{(4\vartheta-1)\frac{\pi}{\beta}(\eta+v)} \leq n^3 e^{o(\Theta)n} \cdot \sum_{\Delta \geq \beta \rho n / (20\pi)} \Delta e^{(4\vartheta-1)\frac{\pi}{\beta}\Delta} \\ &\leq C \cdot n^3 e^{o(\Theta)n} \cdot e^{(4\vartheta-1)\frac{\rho}{20}n} \leq C \cdot e^{(4\vartheta-1)\frac{\rho}{100}n} \end{aligned}$$

for some constant  $C > 0$  with  $\xi = (4\vartheta - 1)\frac{\rho}{100}$  for a big enough  $\Theta$ . This finishes the proof of the lemma.  $\square$

**Lemma 7.2.** *If  $n$  is big enough,  $\rho$  small enough (depending only on  $\sigma$  and  $A$  from Lemma 4.4) and  $x \in I \setminus E_n$ , then  $|(f_\mu^n)'(x)| \geq \sigma^{n/3}$ .*

*Proof.* Let us take  $x \in I \setminus E_n$  and let  $0 < r_1 < \dots < r_k < n$  be the consecutive returns (either essential or inessential) of the first  $n$  iterates of the orbit of  $x$ , and  $p_1, p_2, \dots, p_k$  the respective binding periods. We also set  $q_i = r_{i+1} - (r_i + p_i)$  the free periods and possibly escape times between consecutive returns, for  $i = 1, \dots, k-1$ ,  $q_0 = r_1$  and  $q_k = n - (r_k + p_k)$  if  $n > r_k + p_k$  or  $q_{k+1} = 0$  otherwise.

We split the argument in the following two cases. If  $n \geq r_k + p_k$  then

$$|(f_\mu^n)'(x)| = \prod_{i=0}^k \left( |(f_\mu^{q_i})'(f_\mu^{r_i+p_i}(x))| \cdot |(f_\mu^{p_i})'(f_\mu^{r_i}(x))| \right) \geq \sigma_0^{\sum_{i=0}^{k+1} q_i} \cdot A_0^k \cdot \sigma^{\sum_{i=1}^k p_i} \geq \sigma^n,$$

since  $\sigma_0 > \tilde{\sigma} > \sigma$  and  $A_0 > 1$  by Lemma 4.6(c), and also by (3.4) we may assume that at escape times the expansion rate is at least  $\sigma$ .

On the other hand, if  $n < r_k + p_k$  then using Lemma 4.4 and that  $x \in I \setminus E_n$

$$\begin{aligned} |(f_\mu^n)'(x)| &= |(f_\mu^{r_k})'(x)| \cdot |(f_\mu)'(f_\mu^{r_k}(x))| \cdot |(f_\mu^{n-r_k-1})'(f_\mu^{r_k+1}(x))| \\ &\geq |(f_\mu^{r_k})'(x)| \cdot e^{-\rho r_k} \cdot \frac{1}{A} \cdot |(f_\mu^{n-r_k-1})'(x_l)| \\ &\geq A^{-1} \cdot \sigma^{r_k} \cdot e^{-\rho r_k} \cdot \sigma^{n-r_k-1} \\ &\geq \exp\left((n-1)\left(\log \sigma - \rho \frac{r_k}{n} - \frac{\log A}{n}\right)\right) \geq \sigma^{n/3}, \end{aligned}$$

for  $\rho > 0$  small enough and  $n$  big enough since  $r_k \leq n$ , where  $x_l$  is the critical point associated to  $f_\mu^{r_k}(x)$  and we have used also the calculation for the previous case to estimate  $|(f_\mu^{r_k})'(x)|$ .  $\square$

Finally since

$$\{x \in I : \mathcal{E}(x) > n\} \subseteq \bigcup_{k>n} \left\{x \in I : |(f_\mu^k)'(x)| < \sigma^{k/3}\right\}$$

we conclude by Lemmas 7.1 and 7.2 that there are  $C_2, \xi_2 > 0$  satisfying

$$\lambda\left(\{x \in I : \mathcal{E}(x) > n\}\right) \leq \sum_{k>n} \lambda(E_n) \leq C_2 \cdot e^{-\xi_2 \cdot n},$$

concluding the proof of Theorem C.

## 8. EXPONENTIAL BOUND ON DERIVATIVES ALONG CRITICAL ORBITS

Here we explain how to obtain the bound (4.1) for the parameters in the set  $S$ . First we claim that it is enough to obtain the following bound for a sufficiently small value of  $b > 0$

$$\mathcal{C}_n^b(z_k) = \sum_{j=0}^{n-1} -\log \text{dist}_b(f_\mu^j(z_k), \mathcal{C}) \leq \hat{M} \cdot n, \quad (8.1)$$

where  $\hat{M} > 0$  is a big constant and this bound holds for all  $n \geq 1$  which is *not a bound time* for  $z_k$ , for every critical value  $z_k$ ,  $|k| \geq k_0$ . Indeed, fixing  $n, k$  and  $b > 0$ , if (8.1) holds then we can write

$$\begin{aligned} \sum_{j=0}^{n-1} -\log \text{dist}(f_\mu^j(z_k), \mathcal{C}) &\leq \sum_{\substack{\text{dist}(f_\mu^j(z_k), \mathcal{C}) < b \\ 0 \leq j < n}} -\log \text{dist}_b(f_\mu^j(z_k), \mathcal{C}) + \sum_{\substack{\text{dist}(f_\mu^j(z_k), \mathcal{C}) \geq b \\ 0 \leq j < n}} -\log b \\ &\leq \hat{M} \cdot n - \log b \cdot \#\{0 \leq j < n : \text{dist}(f_\mu^j(z_k), \mathcal{C}) \geq b\} \\ &\leq (\hat{M} - \log b) \cdot n, \end{aligned}$$

proving the claim.

For any given  $\mu \in S$  and  $|k| \geq k_0$ , let  $n$  satisfy  $1 = r_0 < r_1 < \dots < r_u \leq n$ , where  $r_1, \dots, r_u$  are the essential return situations of the orbit of  $z_k$ , in the sense of the construction performed in [PRV]. We denote by  $(l_i, s_i)$  the depth corresponding to  $r_i$  for  $i = 1, \dots, u$ , and set  $(l_0, s_0)$  to be such that  $z_k(\mu) = f_\mu(x_k) \in I(l_0, s_0)$ . To prove (8.1) it suffices to obtain the following pair of relations

$$\mathcal{C}_n^b(z_k) \leq B \sum_{i=0}^{u_n(k)} (|l_i| + |s_i|) \leq B \cdot \frac{n}{2}, \quad (8.2)$$

where  $u_n(k)$  denotes the number of return situations of the orbit of  $z_k$  up to the  $n$ th iterate, for every time  $n \geq 1$  which is *not a bound time* for the orbit of  $z_k$ , for all critical values

$z_k$ ,  $|k| \geq k_0$ , and  $B$  is a positive constant. Obviously both inequalities in (8.2) together imply (8.1) with  $\hat{M} = B/2$ .

To obtain (8.2) we first assume that the right hand side inequality has been proved for all  $n \geq 1$ ,  $|k| \geq k_0$  and  $\mu \in S$  and deduce the left hand side inequality in the same setting by induction on the number  $n$  of iterates, as follows.

First observe that since  $\mu \in S$  satisfies  $|\mu| \geq \varepsilon^2$ , then  $z_k(\mu) = f(x_k) \pm \mu$  is outside  $(-\varepsilon^2/2, \varepsilon^2/2)$  if  $|k|$  is big enough, which implies that the depth  $l_0 + s_0$  of  $z_k(\mu)$  is bounded from above by some constant  $K = K(\varepsilon)$  for all  $|k| \geq k_0$ . Moreover by [PRV, Section 4] we know that for a fixed  $\alpha < \gamma < 1$

$$\varepsilon > |f_\mu^i(z_k(\mu))| \geq |z_k(\mu)|^{\gamma^i} \quad \text{and} \quad f_\mu^i(z_k(\mu)) \in I(l_i, s_i) \quad \text{with} \quad |s_i| \leq s(\tau)$$

for every  $0 \leq i < j_0 = j_0(k, \mu)$  where  $j_0$  is uniformly bounded from above for all  $|k| \geq k_0$  and  $\mu \in S$ . Hence by Remark 2.1

$$\text{dist} \left( f_\mu^i(z_k(\mu)), \mathcal{C} \right) \geq a_2 e^{-\frac{\pi}{\beta}(|l_i| + |s_i|)} \geq a_2 e^{-\frac{\pi}{\beta}s(\tau)} e^{-\frac{\pi}{\beta}|l_i|} \geq a_2 C \tau \frac{|f_\mu^i(z_k(\mu))|}{a_2 + \hat{x}} \geq K(\tau) |z_k(\mu)|^{\gamma^i}$$

and thus

$$\sum_{j=0}^{j_0} -\log \text{dist}_b \left( f_\mu^j(z_k(\mu)), \mathcal{C} \right) \leq -j_0 \cdot \log K(\tau) - \log |z_k(\mu)| \cdot \sum_{j=0}^{j_0} \gamma^j \leq \hat{B} \cdot (|l_0| + |s_0|)$$

for a constant  $\hat{B} > 0$  which depends on  $\tau$ . Since these initial iterates are all free, we have shown that the left hand side of (8.2) holds for the  $j_0 > 1$  initial iterates of the orbit of every critical value for any  $B \geq \hat{B}$ .

Now assume that the left hand side of (8.2) is true for a free time  $n - 1$  of the orbit of a critical value  $z_k$  for a fixed  $\mu \in S$ .

If  $n$  is a free time for  $z_k$  and the last essential return situation was not deep, i.e. the depth was smaller than  $\Theta$ , then the depth of  $f_\mu^n(z_k)$  is also smaller than  $\Theta$  by arguments akin to Lemma 6.3. Hence this situation does not contribute to the sum  $\mathcal{C}_n^b(z_k)$ .

If  $n$  is a free time for  $z_k$  and the last essential return situation was deep, then let  $r \leq n - 1$  be the last essential return time with depth  $(l_r, s_r)$  and  $t > n$  be the next return situation (either essential, inessential or escape). We claim that

$$\sum_{j=r}^{t-1} -\log \text{dist}_b \left( f_\mu^j(z_k(\mu)), \mathcal{C} \right) \leq B \cdot (|l_r| + |s_r|), \quad (8.3)$$

which shows that the induction can be carried out up to the iterate  $t - 1 \geq n$ .

To prove the claim, note that similar arguments to those proving Lemma 6.4 show that the part of the sum (8.3) corresponding to free times after the binding period is bounded by  $B_2 \cdot (|l_r| + |s_r|)$ . Hence we get (8.3) as long as  $B \geq B_2$ .

Finally, if  $n$  is a return situation for  $z_k$ , either essential, inessential or an escape, we let  $p$  be the binding period corresponding to the return and consider the next free iterate  $n + p$  of the orbit of  $z_k$ .

If  $n$  is an escape, then  $p = 0$  and we are done by Remark 2.1. Otherwise  $0 < p \leq \frac{2\rho}{\log \sigma} \cdot n < n$  by [PRV, Lemma 5.3] and so we can use the induction hypothesis to get (6.2) with  $t_i = n$ ,  $h = 1, \dots, p$ ,  $x = z_k$  and  $x_l = x_{l_n}$ , the critical point which will shadow the orbit of  $z_k$  during the binding period. Hence we obtain as before the bound (6.3).

Note that if  $|l_n| + |s_n| \leq \Theta$ , then the truncated distance is always 1 and we are done, by the same arguments in the proof of Lemma 6.2. If we have a deep return then, analogously, we need to consider the next free time  $t \geq p$  of the orbit of the bound critical value  $z_{l_n}$  in order to properly use the induction hypothesis. We have  $t \leq (1 + 2\rho/\log \sigma) \cdot p$  again by [PRV, Lemma 5.3] from which we get

$$\sum_{j=n+1}^{n+p} -\log \text{dist}_b \left( f_\mu^j(z_k(\mu)), \mathcal{C} \right) \leq C(\tau) + \left( 1 + \frac{2\rho}{\log \sigma} \right) \cdot p \leq B \cdot (|l_n| + |s_n|)$$

by induction and assuming that the right hand side of (8.2) holds. We also used the upper bound in Lemma 4.6(a) and assumed that

$$B > \frac{C(\tau)}{\Theta} + \frac{2\pi}{\beta \log \sigma} \cdot \left( 1 + \frac{2\rho}{\log \sigma} \right).$$

Thus if  $B$  is sufficiently big, then the inductive step can be performed in every situation. This shows that the left hand side inequality in (8.2) is true *if* the right hand side inequality holds.

Now we explain why we can assume that parameters  $\mu \in S$  satisfy the right hand side of (8.2) for all critical values. The condition (30) used in [PRV, p. 463]

$$B(n, \omega, k) = \sum_{j=1}^{n-1} p(j, \omega, k) < \frac{n}{2} \tag{8.4}$$

to test whether a given interval  $\omega$  of parameters should be excluded or not, can be replaced by the following

$$\sum_{i=0}^{u_n(k)} (|l_i| + |s_i|) \leq \frac{n}{2} \tag{8.5}$$

without loss since  $B(n, \omega, k) \leq C \sum_{i=0}^{u_n(k)} (|l_i| + |s_i|)$  by Claim (3) in [PRV, p. 478]. Indeed it is this last inequality which is used in the arguments proving [PRV, Lemma 5.7] establishing the exponential bound on the set of excluded parameters. Therefore repeating the algorithm presented there step by step with the new condition (8.5) instead of (8.4) leads to the construction of a positive Lebesgue measure set  $S$  satisfying Theorem 1.1 and (8.2) for all  $n \geq 1$ , every  $|k| \geq k_0$  and for every  $\mu \in S$ . This concludes the proof of (8.1).

## 9. CONSTANTS DEPEND UNIFORMLY ON INITIAL PARAMETERS

We finally complete the proof of Corollary E by explicitly showing the dependence of the constants used in the estimates on Sections 2 to 7.

In the statements of the lemmas and propositions in the aforementioned sections we stated explicitly the direct dependence of the constants appearing in each claim from earlier statements. For constants which depend only on  $\hat{f}$  we used the plain letter  $C$ .

It is straightforward to see that every constant depends on values that ultimately rest on the choice of initial values for  $\sigma, \sigma_0$  and  $k_0$  and on the the choice of  $\rho$  and  $\tau$ , which are taken to be small enough and where  $0 < \rho < \tau$  is the unique restriction, used solely in the proof of Proposition 3.4. Note that by definition  $\varepsilon = \varepsilon(k_0)$  and that  $k_0 = k_0(\tau)$  according to (1.3). Thus  $\tau$  can be made as small as needed.

Hence by choosing  $1 < \sigma < \sqrt{\sigma} < \sigma_0$  and a small  $\delta$ , we may then take  $0 < \rho < \tau$  as small as we need to obtain a small  $\varepsilon > 0$  (and  $k_0$  big enough, as a consequence, see Remark 3.2), and then find  $b > 0$  in order that  $\Theta = \Theta(b)$  be big enough so that the constants  $C_1, C_2, \xi_1, \xi_2$ ,

and consequently  $C_3, \xi_3$  on the statements of Section 1, are defined depending only on  $\alpha, \beta$ , which depend only on  $\hat{f}$ . So  $C_1, C_2, C_3, \xi_1, \xi_2, \xi_3$  depend on  $\sigma, \sigma_0, \rho$  and  $\tau$ , but *do not depend* on  $\mu \in S$ .

This concludes the proof of Corollary E.

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