SECTIONAL-HYPERBOLIC SYSTEMS

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Abstract. We introduce a class of vector fields on n-manifolds containing the singular-hyperbolic systems on 3-manifolds [MPP1], the multidimensional Lorenz attractors [BPV] and the C^1-robustly transitive singular sets in [LGW]. We prove that a system in this class cannot be approximated by ones exhibiting non-hyperbolic closed orbits (this property is false for higher-dimensional singular-hyperbolic systems [TS]). Existence of SRB measures and stochastic stability for attractors in the introduced class is discussed.

1. Introduction

Let M be a compact boundaryless manifold and let $\chi^r(M)$ be the space of C^r vector fields on M with the C^r topology, $r \geq 1$. Denote by X_t the flow generated by $X \in \chi^1(M)$ and by $\Omega(X)$ the nonwandering set of X. If $U \subset M$ and $r \geq 1$ denote by $\mathcal{G}^r(U)$ the set of $X \in \chi^r(M)$ such that every closed orbit in U of every vector field C^r-close to X is hyperbolic. A compact invariant set Λ of X is *isolated* if it is maximal invariant in some compact neighborhood of it called *isolating block*. An isolated set is *attracting* if it has a positively invariant isolating block and an *attractor* is an attracting set which is *transitive*, i.e., the accumulation point set of the positive orbit of one of its points. We prevent the reader that many authors use the name attractor for what we have named attracting set [M].

A *dominated splitting* over Λ is a non-trivial invariant direct sum $T_\Lambda M = E^s_\Lambda \oplus E^c_\Lambda$ such that the angle between $DX_t(x) \cdot v$ and E^c_x approaches exponentially to 0 as $t \to \infty$ for all $v \notin E^s_x$ (this is reason why some authors use the name *projectively hyperbolic* for such sets). A *partially hyperbolic set* is a compact invariant set with a dominated splitting for which the subbundle E^s_Λ is contracting. A partially hyperbolic set with hyperbolic singularities is *singular-hyperbolic* or *hyperbolic* depending on whether the central direction E^c_Λ is volume expanding or splits into an invariant direct sum $E^c_\Lambda = E^u_\Lambda \oplus E^X_\Lambda$ where E^u_Λ is expanding and E^X_Λ is the flow direction. A hyperbolic set Λ is also a singular-hyperbolic set if it is *saddle-type*, i.e., $E^s_x \neq 0$ and $E^u_x \neq 0$ for all $x \in \Lambda$. On the other hand, there are examples of singular-hyperbolic sets which are not hyperbolic as the *geometric* and *multidimensional Lorenz attractors* [ABS], [GW], [BPV].

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A vector field X is called *Axiom A* if $\Omega(X)$ is hyperbolic and the closure of the closed orbits. The class of Axiom A vector fields is important from both deterministic and probabilistic viewpoint. In particular, the Spectral Decomposition Theorem says that the nonwandering set of every Axiom A vector field splits into finitely many disjoint transitive isolated sets. This property motivates [MPP3] to define singular-Axiom A vector field in the following way: A vector field $X \in \chi^1(M)$ is *singular-Axiom A* if $\Omega(X)$ is the closure of the closed orbits and splits into finitely many disjoint transitive isolated sets each one being either an attracting closed orbit or a singular-hyperbolic set (for X or $-X$). It turns out that every Axiom A vector field is singular-Axiom A but not all singular-Axiom A vector fields are Axiom A as, for example, a suitable extension of the geometric Lorenz attractor. Nevertheless, when $dim(M) = 3$ we observe that every singular-Axiom A vector field without cycles belongs to $\mathcal{G}^1(M)$ and, in addition, each element of its nonwandering decomposition is hyperbolic or looks like a geometric Lorenz attractor ([MPP3]). In the converse direction we know that a generic vector field in $\mathcal{G}^1(M)$ is singular-Axiom A without cycles [MPa]. Further properties of singular-hyperbolic sets for vector fields on compact boundaryless 3-manifolds can be found in [BDV].

It is natural to ask whether the above results can be extended to compact boundaryless n-manifolds, $n \geq 4$. However, [MPP3] is false in higher dimension as there is a compact boundaryless n-manifold M, $n > 4$, exhibiting a singular-Axiom A vector field which is not in $\mathcal{G}^1(M)$. One more example but now in dimension 4 is the *wild strange attractor* [TS] which is an example of a C^1 vector field X on a compact boundaryless 4-manifold exhibiting an attracting singular-hyperbolic set Λ satisfying $X \notin \mathcal{G}^1(U)$ for every neighborhood U of Λ.

In this paper we try to extend the aforementioned results to compact boundaryless n-manifolds, $n \geq 4$, by introducing the concept of sectional-hyperbolic set. Roughly speaking a compact invariant set is *sectional-hyperbolic* if its singularities are hyperbolic and is partially hyperbolic with *sectionally expanding central direction.* This last property means that the derivative of the flow along the central subbundle E^c *exponentially expands the area of parallelograms.* It is clear from the definition that sectional and singular-hyperbolicity coincide precisely when $dim(M) = 3$. As before we define sectional-Axiom A vector field in the following way: A vector field $X \in \chi^1(M)$ is *sectional-Axiom A* if $\Omega(X)$ is the closure of the closed orbits and splits into finitely many disjoint transitive isolated sets each one being either an attracting closed orbit or a sectional-hyperbolic set (for X or $-X$). We shall prove that a sectional-Axiom A vector field without cycles belongs to $\mathcal{G}^1(M)$ independently of the dimension of M. This provides a somewhat extension of [MPP3] to higher dimensional manifolds. We prove more properties of the sectional-hyperbolicity as it holds not only for all hyperbolic saddle-type sets but also for the C^1 robustly transitive singular sets considered in [LGW], including the multidimensional Lorenz attractor. We also prove that the wild strange attractor is not sectional-hyperbolic.

In the last section we announce the existence of SRB measures and the stochastic stability under diffusion type small random perturbations for C^2 sectional-hyperbolic attractors. As a corollary in dimension 3 we obtain that *a C^2 singular-hyperbolic attractor on a closed 3-manifold has a unique SRB measure and is stochastically stable.* The existence part of this corollary has been announced by Colmenarez [C] assuming that the periodic orbits are dense), and by Pacifico [Pa] in the general case. The stochastic stability part extends the corresponding result for the geometric Lorenz attractor [K] and solves in positive to a question posed in [BDV]. One more corollary of the main result in the last section is that *the multidimensional Lorenz attractor is stochastically stable.* So, we obtain positive answer to a question posed by Viana to the first author.

This paper is divided in three parts. In Section 2 we present the main definitions including the sectional-hyperbolic sets. In Section 3 we discuss dynamical properties of sectional-hyperbolic systems. In Section 4 we state a result concerning the existence of SRB measures and stochastic stability for sectional-hyperbolic attractors leading the detailed proof to a forthcoming work.

2. Main definitions

First we state some basic concepts in topological dynamics. Throughout M denotes a compact manifold and X denotes a C^r vector field in M, $r \geq 0$. The flow of X will be denoted by X_t, $\in \mathbb{R}$. The non-wandering set $\Omega(X)$ of X is the set formed by those points p with the property that for every neighborhood U of p and every $T > 0$ there is $t > T$ such that $X_t(U) \cap U \neq \emptyset$. A compact invariant set Λ of X is *singular* if it contains a singularity of X. We say that Λ is *isolated* if there is a compact neighborhood U of it called *isolating block* such that $\Lambda = \Lambda_X(U)$, where

$$\Lambda_X(U) = \bigcap_{t \in \mathbb{R}} X_t(U)$$

is the maximal invariant set of X in U. We say that Λ is *transitive* if it coincides with the accumulation point set of a positive orbit contained on it. The index $Ind(\sigma)$ of a hyperbolic closed orbit O of X is the dimension of its stable subbundle E^s_O. The first part of the following definition comes from [LGW].

Definition 1. *A compact invariant set Λ of a C^r vector field X on a compact manifold M is:*

- `Strongly homogeneous` *of index $Ind(\Lambda)$ if there is a neighborhood $U \subset M$ of Λ such that $Ind(O) = Ind(\Lambda)$, for every hyperbolic periodic orbit $O \subset U$ of every vector field that is C^r close to X.*
- C^r `robustly transitive` *if Λ is isolated and there is an isolating block U of Λ such that $\Lambda_Y(U)$ is a non-trivial transitive set of Y, for every vector field Y that is C^r close to X.*

Next we introduce some concepts in differentiable dynamics. Let X be a C^1 vector field in M. If $\Lambda \subset M$ a *subbundle* F_Λ over Λ is a continuous map $x \in \Lambda \mapsto F_x$ where F_x is a linear subspace of T_xM. If Λ is an invariant set of X we say that F_Λ is invariant if $DX_t(x)(F_x) = F_{X_t(x)}$ for every $x \in \Lambda$ and every $t \in \mathbb{R}$. In such a case we say that F_Λ is *contracting* if there are positive constants K, λ such that

$$\| DX_t/_{F_x} \| \leq Ke^{-\lambda t}, \quad \forall x \in \Lambda, \ \forall t > 0.$$

We say that F_Λ is *expanding* if it is contracting for the reversed vector field $-X$. A compact invariant set Λ of X is *hyperbolic* if there is a continuous invariant tangent bundle decomposition $T_\Lambda M = E^s_\Lambda \oplus E^X_\Lambda \oplus E^u_\Lambda$ such that:

1. E^s_Λ is a contracting subbundle.
2. E^u_Λ is an expanding subbundle.
3. $E^X_x = < X(x) >$ for every $x \in \Lambda$.

A closed orbit of X is hyperbolic if its full orbit is a hyperbolic set of X.

Afterward we introduce the notion of *sectionally expanding subbundle.* The Jacobian of a linear map T will be denoted by $Det(T)$.

Definition 2. *Let F_Λ be an invariant subbundle over a compact invariant subset Λ of X. We say that F_Λ is* `sectionally expanding` *if each restriction $DX_t/_L$ to a two-dimensional subspace $L \subset F_\Lambda$ is area expanding in the following sense: There are positive constants K, λ such that for every $x \in \Lambda$ and every two-dimensional subspace $L_x \subset F_x$ one has*

$$| Det(DX_t(x)/_{L_x}) | \geq Ke^{\lambda t}, \qquad \forall t > 0.$$

One can generalize this definition to k-sectionally expanding subbundle, $1 \leq k \leq dim(F_\Lambda)$, just requering k-dimensional subspaces instead of two-dimensional ones. Sectionally expanding expresses nothing but that the derivative along F_Λ *expands the area of parallelograms* instead of *the length of vectors.* Actually the difference between sectionally expanding and expanding subbundles is that the first one allows at most one negative Lyapunov exponent whereas the second one does not. In particular, an expanding subbundle is sectionally expanding but not conversely.

Recall the definition of dominated splitting and partially hyperbolicity. Let Λ be a compact invariant set of X. A continuous invariant splitting $T_\Lambda M = E^s_\Lambda \oplus E^c_\Lambda$ is *dominated* if for every $x \in \Lambda$ one has $E^s_x \neq 0$, $E^c_x \neq 0$ and there are positive constants K, λ such that

$$\| DX_t/_{E^s_x} \| \cdot \| DX_{-t}/_{E^c_{X_t(x)}} \| \leq Ke^{-\lambda t}, \quad \forall x \in \Lambda, \ \forall t > 0.$$

If in addition the subbundle E^s_Λ is contracting then we say that Λ is *partially hyperbolic.* The subbundle E^c_Λ is called *central subbundle* of Λ.

Now we state the main definition of this work.

Definition 3. *A compact invariant set is* `sectional-hyperbolic` *if its singularities are hyperbolic and is partially hyperbolic with sectionally expanding central subbundle.*

The above definition will be used to define a class of vector fields on n-manifolds. The motivation is the definition of singular-Axiom A vector field in [MPP1] p. 3396. A *sink* of a vector field is a hyperbolic closed orbit with zero unstable subbundle E^u and a *source* is a sink for the time reversed vector field.

Definition 4. *A C^r vector field on a closed manifold is* `sectional-Axiom A` *if $\Omega(X)$ is the closure of the closed orbits of X and writes as a disjoint union of isolated transitive sets*

$$\Omega(X) = \Lambda_1 \cup \cdots \cup \Lambda_k,$$

where each Λ_i is either a sink or a sectional-hyperbolic set for X or $-X$. We say that X `has no cycles` *if there are no regular orbits linking the Λ_i's in a cyclic way.*

It turns out that the class of sectional-Axiom A vector fields includes the Axiom A ones and, as we shall see later, the multidimensional Lorenz attractor as well.

3. Dynamical properties of sectional-hyperbolic systems

In this section we discuss the dynamical properties of the sectional-hyperbolic sets. To start with we relate sectional-hyperbolicity with robustly transitivity. Indeed, the following can be seen as complement of the main result in [LGW].

Theorem A. *Let Λ be a strongly homogeneous C^1 robustly transitive singular set of a C^1 vector field X on a compact manifold M. If every singularity $\sigma \in \Lambda$ of X is hyperbolic with $Ind(\sigma) > Ind(\Lambda)$ then Λ is a sectional-hyperbolic set of X.*

Proof. We already know from [LGW] that Λ has a partially hyperbolic splitting

$$T_\Lambda M = E^s_\Lambda \oplus E^c_\Lambda$$

defined in the following way: A point $\in \Lambda$ is called periodic if it belongs to a periodic orbit. By an argument involving the Pugh Closing Lemma we can assume that every periodic orbit of X in Λ is hyperbolic and that the periodic points are dense in Λ. Then,

$$E^s_x = \lim_{n\to\infty} \tilde{E}^s_{p_n} \qquad E^c_x = \lim_{n\to\infty} (\tilde{E}^X_{p_n} \oplus \tilde{E}^u_{p_n}),$$

where p_n is a sequence of periodic points converging to x and the splitting $T_{p_n}M = \tilde{E}^s_{p_n} \oplus \tilde{E}^X_{p_n} \oplus \tilde{E}^u_{p_n}$ is the hyperbolic splitting of the orbit of p_n.

By using the strongly homogeneous hypothesis, the Connecting Lemma [H] and standard facts in the theory of homoclinic loops (e.g. Chapter 6 in [AH] or [FS]) one can prove that every singularity $\sigma \in \Lambda$ is *generalized Lorenz-like*, i.e., σ has at least one real eigenvalue $\lambda_0 < 0$ such that if we define

- $\lambda_-(\sigma) = \max\{Re(\lambda) : \lambda \neq \lambda_0$ is an eigenvalue of σ with negative real part$\}$;
- $\lambda_+(\sigma) = \min\{Re(\lambda) : \lambda$ is an eigenvalue of σ with positive real part$\}$,

then $\#\lambda_-(\sigma) \geq 1$ and

$$\lambda_-(\sigma) < \lambda_0 < 0 < -\lambda_0 < \lambda_+(\sigma).$$

(In the present case one has $\#\lambda_-(\sigma) = Ind(\Lambda)$ and so $Ind(\sigma) = Ind(\Lambda) + 1$ for every singularity $\sigma \in \Lambda$. Note also that a generalized Lorenz-like singularity is in fact a generalization of the concept of Lorenz-like singularity in [MPP1].)

Next we obtain the following property:

(P) There is a neighborhood $\mathcal{U}$ of X in the space of all C^1 vector fields such that if $Y \in \mathcal{U}$, then there are $0 < \lambda < 1$ and a neighborhood $\mathcal{V} \subset \mathcal{U}$ of Y such that if $Z \in \mathcal{V}$ and p is a periodic point of Z with period t_p and orbit contained in U, then

$$|| DZ_{-t_p}(p)/_{\tilde{E}^u_p} || \leq \lambda^{t_p}.$$

To prove (P) we proceed as in the proof of Theorem 3.6(a) in [MPP1] p. 412 except that, in the present case, we use the strongly homogeneous hypothesis instead of the absence of sinks or sources used there.

Afterward we apply the argument in [MPP1] p. 404. Indeed, since the Grassmann manifold of two-dimensional planes is compact (e.g. [MS] p. 55), to prove that E^c_Λ is sectionally expanding we only need to prove that

$$\liminf_{t\to\infty} | detDX_{-t}(x)/_{L_x} | = 0$$

for every $x \in \Lambda_X(U)$ and every two-dimensional subspace $L_x \subset E^c_x$. By contradiction assume that this is not so. By applying the Ergodic Closing Lemma for flows as in [MPP1] p. 405, using the fact that every singularity of Λ is generalized Lorenz-like, it is possible to find, for every $\gamma < 0$ close to 0, a vector field $Y^n \in \mathcal{U}$ as in (P), a periodic point p_n with period t_{p_n} of Y^n and a two-dimensional subspace $L_{p_n} \subset E^c_{p_n}$ such that

$$| detDY^n_{t_{p_n}}(p_n)/L_{p_n} | \geq (e^\gamma)^{t_{p_n}}.$$

But

$$|| DY^n_{t_{p_n}}(p_n) || \geq \sqrt{| detDY^n_{t_{p_n}}(p_n)/_{L_{p_n}} |},$$

so

$$|| DY^n_{t_{p_n}}(p_n) || \geq (e^{\gamma/2})^{t_{p_n}}.$$

Using (P) we obtain

$$\lambda^{t_{p_n}} \geq (e^{\gamma/2})^{t_{p_n}}$$

which is absurd since λ is fixed and γ is close to 0. This proves the result. □

Corollary 5. *The multidimensional Lorenz attractor is sectional-hyperbolic.*

Proof. For completeness we give a brief description of the multidimensional Lorenz attractor [BPV]. Consider an expanding map $f : T^k \to T^k$ of the k-dimensional torus $T^k = S^1 \times \overset{k}{\cdots} \times S^1$, $k \geq 2$. Denote by D^2 the closed unit ball in $I\!R^2$ and denote by $N = T^k \times D^2$. Note that N carries a foliation by 2-disks $\mathcal{F}^s = \{* \times D^2 : * \in T^k\}$ with leaf space $N/\mathcal{F}^s = T^k$. Denote by $Int(N)$ the interior of N. Let $F : V \to Int(N)$ be a C^∞ diffemorsphism which preserves and contracts $\mathcal{F}^s$ such that the map induced by F in the leaf space $N/\mathcal{F}^s = T^k$ is f. It follows that the maximal invariant set $A_F = \cap_{n\geq 0} F^n(N)$ is a hyperbolic attractor with stable foliation $\mathcal{F}^s$ of F. Consider the suspension X^F of F which is a C^∞ vector field defined in the suspended manifold M^F with a global cross section $\Sigma = N \times 0$. To prove transitivity of the attractor [BPV] impose that the modulus of the derivative of f is uniformly bigger than $\max\{2, 2\frac{\Delta}{R}\}$ where R and Δ are respectively the injectivity radius of the exponential map and the diameter of T^k.

Pick a point $\xi \in A_F \subset \Sigma$ and consider the piece of orbit $\overline{\xi F(\xi)}$ of X^F from ξ to its first return point $F(\xi)$ to Σ. Consider a small neighborhood of the form $U \times 0$ of ξ in C. The set $V = X_{[0,1]}(U \times 0)$ is a neighborhood of $\overline{xF(x)}$ in M^F. By making a surgery on V we can replace the flow of the vector field restriction $X^F/_V$ by a Cherry-like one. More precisely we create a singularity O inside V with stable manifold $W^s(O)$ of dimension $k+1$. Observe that O has a strong stable manifold $W^{ss}(O)$ of dimension k. In this way we obtain a new vector field Y having $\Sigma = T^k \times D^2$ as a cross-section. Moreover, there is a return map Φ induced by Y in Σ such that:

(P1) Φ admits $\mathcal{F}^s$ as a contracting C^∞ foliation.
(P2) The quotient space of $\mathcal{F}^s$ is diffeomorphic to T^k and the map $\phi : T^k \setminus \{\xi\} \to T^k$ induced by Φ in the leaf space of $\mathcal{F}^s$ has derivative bigger that $\max\{2, 2\frac{\Delta}{R}\}$.

These properties are preserved by small C^1 perturbations Z of Y. The *multidimensional Lorenz attractor* is the attracting set of Y defined by

$$A(Y) = \bigcup_{T>0} closure \left(\bigcup_{t>T} Z_t(\Sigma) \right).$$

To prove that $A(Y)$ is singular-hyperbolic we need to prove that its singularity is hyperbolic and that it is partially hyperbolic with sectionally expanding central direction (see Definition 3). We can prove this directly by verifying the required properties in the above construction. Here we proceed in an indirect way using Theorem 5. More precisely, we shall prove that $A(Y)$ is a strongly homogeneous C^1 robustly transitive set with a unique singularity O which is hyperbolic and satisfies $Ind(O) > Ind(A(Y))$. That $A(Y)$ has a unique singularity O which is hyperbolic is obvious from the construction. To prove that $A(Y)$ is C^1 robustly transitive we can appeal to part (1) of the Main Theorem in [BPV] p.885. Next we observe that $A(Y)$ is strongly homogeneous as the dimension Ind of the stable direction of every periodic orbit P close to $A(Y)$ of a vector field Z that is C^1 close to Y is precisely 2. Hence $Ind(A(Y)) = 2$. Observe also that the stable dimension $Ind(O)$ of the singularity O is precisely 3. It then follows that

$$Ind(O) = 3 > 2 = Ind(A(Y)).$$

So we are done by Theorem A. □

The result below extends Theorem B in [MPP1] p. 3396 to higher dimensions.

Theorem B. *A C^r sectional-Axiom A vector field without cycles of a closed n-manifold M is in $\mathcal{G}^r(M)$ for any $r \geq 1$.*

Proof. The proof follows the argument in the proof of Theorem B in [MPP1] p. 3399 and the lemma below. Recall that a subset U and $r \geq 1$ we denote by $\mathcal{G}^r(U)$ the set of C^r vector fields X such that every closed orbit in U of every vector field that is C^r close to X is hyperbolic. Observe that $\mathcal{G}^1(U)$ is the local version of the space $\mathcal{G}^1(M)$ considered in [H].

Lemma 6. *If Λ is a sectional-hyperbolic set of a C^r vector field X on a compact manifold M, then there is an open subset $U \subset M$ containing Λ such that every compact invariant* non-singular *subset in U of every vector field that is C^r close to X is hyperbolic. In particular $X \in \mathcal{G}^r(U)$ and if Λ is transitive, then Λ is strongly homogeneous as well.*

Indeed, by the compactness of the Grassmann space of two-dimensional subspaces we have that the sectionally expansiveness of a subbundle is preserved by C^r perturbations. In particular, there is a neighborhood U of Λ such that every compact invariant non-singular set $H \subset U$ of every Y that is C^r close to X has a partially hyperbolic splitting with sectionally expanding central direction. One can prove as in [MPP3] that H is hyperbolic. From this it follows that $X \in \mathcal{G}^1(U)$. On the other hand, if Λ is transitive, then it is connected and so the dimension of its stable subbundle E^s_Λ is constant. In such a case Λ is strongly homogeneous with index such a constant. This proves Lemma 6. □

As a complement to Theorem B let us mention that there are vector fields in $\mathcal{G}^r(M)$ which are not sectional-Axiom A for all $r \geq 1$. Besides, Theorem B motivates the question how large the class of sectional-Axiom A vector fields is. In particular, are the sectional-Axiom A vector fields dense in $\mathcal{G}^r(M)$?

We finish this section with the following corollary.

Corollary 7. *The wild strange attractor in* [TS] *is singular-hyperbolic but not sectional-hyperbolic.*

Proof. The attractor is singular-hyperbolic by construction ([TS] p. 138) but not sectional-hyperbolic by Lemma 6 and Theorem 4 in [TS]. □

4. Ergodic properties of sectional-hyperbolic systems

In this section we discuss some ergodic properties of sectional-hyperbolic sets leading the details for a forthcoming work.

Consider the family of transition probability measures $P^\varepsilon(t, x, \cdot)$ on a manifold M given for every $x \in M$ and $t \in \mathbb{R}$ (or $t \in \mathbb{Z}_+$) and $\varepsilon > 0$ small enough and define Markov chains x_t^ε, $t \in \mathbb{R}$ in the following way: if $x_t^\varepsilon = x$ then $x_{t+\tau}^\varepsilon$ has probability $P^\varepsilon(\tau, x, A)$ of being in A. The Markov chain x_t^ε for $t \in \mathbb{R}$ is called a *small random perturbation* of a flow X_t if for every continuous function h on M, we have

$$\lim_{\varepsilon \to 0} \left| \int_M P^\varepsilon(t, x, dy) h(y) - h(X_t(x)) \right| = 0\,.$$

Similarly, the Markov chain x_n^ε for $n \in \mathbb{Z}_+$ is called a small random perturbation of a map f if for every continuous function h on M, we have

$$\lim_{\varepsilon \to 0} \left| \int_M P^\varepsilon(n, x, dy) h(y) - h(f^n(x)) \right| = 0.$$

We say that ν^ε on M is a stationary measure for the Markov chain x_t^ε if for all Borel set A and any $\tau > 0$, we have

$$\int_M \nu^\varepsilon(dx) P^\varepsilon(\tau, x, A) = \nu^\varepsilon(A).$$

Under general hypothesis [K] weak limits of stationary measures ν^ε when $\varepsilon \longrightarrow 0$ are invariant under the flow (or the map, depending on the case).

Definition 8. *Let X be a C^2 vector field and μ be an invariant probability measure of X. We say that μ is a* `SRB measure` *if μ has at least one positive Lyapunov exponent a.e. and μ has absolutely continuous invariant measure on unstable manifolds.*

Denote by $\mathcal{B}(M)$ the set of borelians of a manifold M.

Definition 9. *Let X a vector field on a manifold M and Λ be an attractor of X having a unique SRB measure μ. Let $P^\varepsilon \colon \mathbb{R}^+ \times M \times \mathcal{B}(M) \to [0,1]$ be the transition probability measures associated to a fixed small random perturbation x_t^ε of X and $\{\mu^\varepsilon\}_{\varepsilon>0}$ be a family of stationary measures of P^ε. We say that Λ is* `stochastically stable` *if for every real number sequence $\varepsilon_i \to 0^+$ such that $\mu^{\varepsilon_i} \to \nu$ in the weak sense one has $\nu = \mu$.*

By stochastic stability for diffusion type perturbations it is meant that we are going to use transition probabilities of the form

$$P^\varepsilon(\tau, x, A) = \int_A p^\varepsilon(\tau, x, y) dy,$$

where dy means integration with respect to the natural Lebesgue measure of the manifold and $p^\varepsilon(\tau,x,y)$ is a solution of the diffusion equation

$$\frac{\partial p^\varepsilon}{\partial t}(t,x,y) = (\varepsilon L + X)p^\varepsilon(t,x,y)$$

with L being an elliptic operator and X a vector field. Note that the elliptic operator introduces the possibility of collision with particle in a media (or heat equation), that gives the random part of the Markov chains. Typical solution of this equation comes with a factor that has Gaussian behavior. That is, $p^\varepsilon(t,x,y) \sim \exp(\frac{-|X_t(x)-y|}{\varepsilon})$.

The proof of the theorem below follows from the arguments used to prove the stochastic stability of the geometric Lorenz attractor [K].

Theorem C. *A C^2 sectional-hyperbolic attractor on a compact boundaryless manifold has a unique SRB measure and is stochastically stable for diffusion type small random perturbations.*

This theorem has the following interesting corollaries:

Corollary 10. *A singular-hyperbolic attractor of a C^2 vector field on a closed 3-manifold has a unique SRB measure and is stochastically stable for diffusion type small perturbations.*

Corollary 11. *A C^2 multidimensional Lorenz attractor* [BPV] *is stochastically stable under diffusion type small random perturbations.*

As already mentioned in the Introduction, the existence part of Corollary 10 has been recently announced by Colmenarez [C] (assuming the denseness of periodic orbits) and by M. Pacifico [Pa] (in general case). Corollary 11 answers positively to a question posed by M. Viana to the first author.

Let us present an outline of the proof of Theorem C leading the details to a forthcoming work. Instead of $P^\varepsilon(t,x,A)$ for all $t \in \mathbb{R}^+$ we will consider only discrete Markov chains $P^\varepsilon(\tau,x,A)$ for τ fixed and greater than zero. In other words we will calculate the stationary measure for the diffusion type small random perturbation for the flow by calculating the stationary measure for the diffeomorfism X_τ defining $P^\varepsilon(1,x,A) = P^\varepsilon(\tau,x,A)$. This step is fully justified since stationary measure for diffusion type perturbations are unique for each ε. And, since $\int P^\varepsilon(n,x,A)d\mu^\varepsilon(x) = \mu^\varepsilon(A)$ we can use arbitrary big iterates of the dynamics to approximate the stationary measure. Let μ^{ε_i} be a subsequence of stationary measures converging weakly to some measure μ^*, i.e.,

$$\lim_{i\to\infty} \mu^{\varepsilon_i} = \mu^*.$$

We approximate $P^\varepsilon(n,x,A)$ using the following steps:

Step 1.- We can reduce the problem of calculating the probability of arriving to a measurable set A beginning at x in n steps through any Markov chain by calculating the probability using only δ-pseudo-orbits (cf. Lemma 1.1 of [K]).

Step 2.- If we are far from the singularities we can apply Lemma 6 to shadow δ-pseudo-orbits by true orbits.

Step 3.- The orbits in step 2 are the significant ones since the probability of arriving to a $\varepsilon^{1-\gamma}$ neighborhood of the stable manifolds of the singularities is of the order of ε^{γ}.

Step 4.- For the orbits in step 2 we have expansion and bounded distortion properties, and following similar methods developed in [K] we have

$$P^{\varepsilon}(n,x,A) \leq D\,Leb^{u}(A)|2s| + \mathcal{O}(\varepsilon)$$

for all measurable set A, where $\mathcal{O}(\varepsilon)$ goes to zero with ε.

From here,

$$\mu^{\varepsilon}(A) \leq D\,Leb^{u}(A)|2s| + \mathcal{O}(\varepsilon).$$

Replacing $\varepsilon = \varepsilon_i$ and passing to the limit we obtain

$$\mu^{*}(A) \leq D\,Leb^{u}(A)|2s|.$$

Therefore μ^* is absolutely continuous with respect to Leb^u and satisfies the entropy formula. This is known to imply that μ^* is an SRB measure. As there is only one measure satisfying that entropy formula for transitive attractors we obtain the desired stochastic stability. □

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