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A RIGIDITY THEOREM FOR COMPLETE CMC HYPERSURFACES IN LORENTZ MANIFOLDS

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Abstract. In this paper we use the standard formula for the Laplacian of the squared norm of the second fundamental form and the asymptotic maximum principle of H. Omori and S. T. Yau to classify complete CMC spacelike hypersurfaces of a Lorentz ambient space of nonnegative constant sectional curvature, under appropriate bounds on the scalar curvature.

1. Introduction

In the past decades there has been an increasing interest in studying the structure of spacelike hypersurfaces of Lorentz manifolds of constant sectional curvature. This goes back to 1976, when S. Y. Cheng and S. T. Yau proved ([?]) the Calabi-Bernstein conjecture concerning complete maximal spacelike hypersurfaces of the Lorentz-Minkowsky space, namely, that the only ones are the spacelike hyperplanes.

For the De Sitter space, A. J. Goddard conjectured in [?] that complete CMC spacelike hypersurfaces should be totally umbilical. This turned out to be true only for compact hypersurfaces, due to independent work of S. Montiel ([?]) and J. L. Barbosa and V. Oliker ([?]). For the noncompact case, studying the behavior of the squared norm of the second fundamental form of the immersion A. Brasil, A. Colares and O. Palmas were able to find, in [?], new examples of non-totally umbilical complete CMC hypersurfaces of the De Sitter space.

Our aim in this paper is to study the structure of complete CMC spacelike hypersurfaces M^n of Lorentz ambient spaces of constant sectional curvature $c \geq 0$, once one asks that M has scalar curvature $R \geq c$. In this case, a structure theorem can be proved which gives strong rigidity to M. Indeed, one can classify (and we will do so) all such M as $(n-1)-$cilinders over certain plane curves when $c = 0$, or as totally geodesic hypersurfaces when $c > 0$.

2. Preliminaries

To get into detail, the first tool we need is an asymptopic maximum principle due to H. Omori and S. T. Yau, which we now recall:

Theorem 2.1 ([?]). *Let M^n denote an $n-$dimensional complete Riemannian manifold having Ricci curvature bounded from below. If $f : M \to \mathbb{R}$ is a C^2 function*

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on M such that $\inf_M f > -\infty$, then there exists a sequence $(p_k)_{k\geq 1}$ of points in M such that the following hold:

$$f(p_k) < \inf_M f + \frac{1}{k}, \quad |\nabla f|(p_k) < \frac{1}{k}, \quad \Delta f(p_k) > -\frac{1}{k}.$$

The above theorem has the following useful corollary, which appeared in a weaker form in [**?**].

Corollary 2.2. *Let M^n denote an $n-$dimensional complete Riemannian manifold having Ricci curvature bounded from below. If $f : M \to \mathbb{R}$ is a nonnegative C^2 function on M, such that $\Delta f \geq af^\beta$ for some real numbers $a > 0$ and $\beta > 1$, then $f \equiv 0$.*

Proof. Let $\phi : \mathbb{R}_+^* \to \mathbb{R}_+^*$ be a smooth function to be chosen later, and $g = \phi \circ f$. Then $\nabla g = \phi'(f)\nabla f$ and

$$\begin{aligned}\Delta g &= div(\nabla g) = \phi'(f)\Delta f + \phi''(f)|\nabla f|^2 \\ &= \phi'(f)\Delta f + \frac{\phi''(f)}{\phi'(f)^2}|\nabla g|^2,\end{aligned}$$

so that

$$-\frac{\phi''(f)}{\phi'(f)^2}|\nabla g|^2 + \Delta g = \phi'(f)\Delta f.$$

Letting $\phi(t) = \frac{1}{(1+t)^\alpha}$, $\alpha > 0$, one gets

$$\phi'(t) = -\alpha\phi(t)^{\frac{\alpha+1}{\alpha}}, \quad \frac{\phi''(f)}{\phi'(f)^2} = \left(\frac{\alpha+1}{\alpha}\right)\frac{1}{\phi(t)},$$

and hence

$$\left(\frac{\alpha+1}{\alpha}\right)|\nabla g|^2 - \phi(f)\Delta g = \alpha\phi(f)^{\frac{2\alpha+1}{\alpha}}\Delta f \geq a\alpha\frac{f^\beta}{(1+f)^{2\alpha+1}}.$$

If one now takes $\alpha = \frac{\beta-1}{2} > 0$, we arrive at

$$\left(\frac{\alpha+1}{\alpha}\right)|\nabla g|^2 - g\Delta g \geq a\alpha\left(\frac{f}{1+f}\right)^\beta. \tag{2.1}$$

Since g is C^2 and bounded from below, by the previous theorem we get a sequence (p_k) of points in M such that

$$g(p_k) < \inf_M g + \frac{1}{k}, \quad |\nabla g|(p_k) < \frac{1}{k}, \quad \Delta g(p_k) > -\frac{1}{k}.$$

Therefore, $f(p_k) \to \sup_M f$, and substituting into (**??**) we get

$$\frac{\alpha+1}{\alpha k^2} - \frac{1}{k}\left(\inf_M g + \frac{1}{k}\right) \geq a\alpha\left(\frac{f(p_k)}{1+f(p_k)}\right)^\beta.$$

Making $k \to +\infty$, we get $\sup_M f = 0$, and since $f \geq 0$ this gives $f \equiv 0$. □

The second tool is a formula for the Laplacian of the squared norm of the second fundamental form of an isometric immersion. In order to properly state the result, let us recall some facts about isometric immersions:

Any spacelike hypersurface $x : M^n \to \overline{M}^{n+1}$ of an $(n+1)-$dimensional, time-oriented Lorentz manifold $\overline{M}$ is orientable. Moreover, one can choose a globally defined timelike unit normal vector field N on M. If A denotes the second fundamental form of x with respect to N then, at each $p \in M$, A restricts to a self-adjoint linear map $A_p : T_pM \to T_pM$. For $1 \le r \le n$, let $S_r(p)$ denote the $r-$th elementary symmetric function on the eigenvalues of A_p; this way one gets n smooth functions $S_r : M \to \mathbb{R}$, such that

$$\det(tI - A) = \sum_{k=0}^{n} (-1)^k S_k t^{n-k},$$

where $S_0 = 1$ by definition. If $p \in M$ and $\{e_k\}$ is a basis of T_pM formed by eigenvectors of A_p, with corresponding eigenvalues $\{\lambda_k\}$, one immediately sees that

$$S_r = \sigma_r(\lambda_1, \dots, \lambda_n),$$

where $\sigma_r \in \mathbb{R}[X_1, \dots, X_n]$ is the $r-$th elementary symmetric polynomial on the indeterminates $X_1, \dots, X_n$. In particular, if $|A|^2$ stands for $\mathrm{tr}(A^2)$ then it is immediate to check that

$$2S_2 + |A|^2 = S_1^2. \tag{2.2}$$

Also, when the ambient space $\overline{M}$ has constant sectional curvature c, Gauss equation allows one to immediately check that the scalar curvature R of M relates to S_2 in the following manner:

$$2S_2 = n(n-1)(c - R). \tag{2.3}$$

For $1 \le r \le n$, one defines the $r - th$ mean curvature H_r of x by

$$H_r = \frac{(-1)^r}{\binom{n}{r}} S_r = \frac{1}{\binom{n}{r}} \sigma_r(-\lambda_1, \dots, -\lambda_n).$$

In particular, $H_1 = H$ is the mean curvature of x. Such functions satisfy a very useful set of algebraic inequalities, usually reffered to as Newton's inequalities. A proof of them for positive real numbers can be found in [**?**]. Here, we present a more general version of them, together with a sharp condition for equality. For the proof, recall that if a polynomial $f \in \mathbb{R}[X]$ has $k \ge 1$ real roots, then its derivative f' has at least $k - 1$ real roots. In particular, if all roots of f are real, then the same is true of all roots of f'.

Proposition 2.3. *Let $n > 1$ be an integer, and $\lambda_1, \dots, \lambda_n$ be real numbers. Define, for $0 \le r \le n$, $S_r = S_r(\lambda_i)$ as above, and $H_r = H_r(\lambda_i) = \binom{n}{r}^{-1} S_r(\lambda_i)$.*

(a) *For $1 \le r < n$, one has $H_r^2 \ge H_{r-1}H_{r+1}$. Moreover, if equality happens for $r = 1$ or for some $1 < r < n$, with $H_{r+1} \ne 0$ in this case, then $\lambda_1 = \dots = \lambda_n$.*
(b) *If $H_1, H_2, \dots, H_r > 0$ for some $1 < r \le n$, then $H_1 \ge \sqrt{H_2} \ge \sqrt[3]{H_3} \ge \dots \ge \sqrt[r]{H_r}$. Moreover, if equality happens for some $1 \le j < r$, then $\lambda_1 = \dots = \lambda_n$.*
(c) *If, for some $1 \le r < n$, one has $H_r = H_{r+1} = 0$, then $H_j = 0$ for all $r \le j \le n$. In particular, at most $r - 1$ of the λ_i are different from zero.*

Proof. In order to prove (a) we use induction on the number $n > 1$ of real numbers. For $n = 2$, $H_1^2 \ge H_0H_2$ is equivalent to $(\lambda_1 - \lambda_2)^2 \ge 0$, with equality if and only if $\lambda_1 = \lambda_2$. Suppose the inequalities true for $n - 1$ real numbers, with equality when $H_{r+1} \ne 0$ if and only if all of them are equal. Given $n \ge 3$ real numbers $\lambda_1, \dots, \lambda_n$, let

$$f(x) = (x + \lambda_1) \dots (x + \lambda_n) = \sum_{r=0}^{n} \binom{n}{r} H_r(\lambda_i) x^{n-r}.$$

Then

$$f'(x) = \sum_{r=0}^{n-1}(n-r)\binom{n}{r}H_r(\lambda_i)x^{n-r-1}.$$

On the other hand, there exist real numbers $\gamma_1, \ldots, \gamma_{n-1}$ such that

$$\begin{aligned} f'(x) &= n(x+\gamma_1)\cdots(x+\gamma_{n-1}) = n\sum_{r=0}^{n-1}S_r(\gamma_i)x^{n-1-r} \\ &= \sum_{r=0}^{n-1} n\binom{n-1}{r}H_r(\gamma_i)x^{n-1-r}. \end{aligned}$$

Since $(n-r)\binom{n}{r} = n\binom{n-1}{r}$, comparing coefficients gives us $H_r(\lambda_i) = H_r(\gamma_i)$ for $0 \leq r \leq n-1$. Hence, it follows from the induction hypothesis that, for $1 \leq r \leq n-2$,

$$H_r^2(\lambda_i) = H_r^2(\gamma_i) \geq H_{r-1}(\gamma_i)H_{r+1}(\gamma_i) = H_{r-1}(\lambda_i)H_{r+1}(\lambda_i).$$

Moreover, if equality happens for the λ_i, with $H_{r+1}(\lambda_i) \neq 0$, then it will also happen for the γ_i, with $H_{r+1}(\gamma_i) \neq 0$. Again from the induction hypothesis, it follows that $\gamma_1 = \cdots = \gamma_{n-1}$, and thus $\lambda_1 = \cdots = \lambda_n$.

To finish, it suffices to prove that $H_{n-1}^2(\lambda_i) \geq H_{n-2}(\lambda_i)H_n(\lambda_i)$, with equality for $H_n \neq 0$ if and only if all of the λ_i are equal. If $\lambda_i = 0$ for some $1 \leq i \leq n$, equality is obvious. If not, $H_n \neq 0$ and

$$\begin{aligned} H_{n-1}^2 \geq H_{n-2}H_n \quad &\Leftrightarrow \quad \left[\binom{n}{n-1}^{-1}\sum_i \frac{H_n}{\lambda_i}\right]^2 \geq \left[\binom{n}{n-2}^{-1}\sum_{i<j}\frac{H_n}{\lambda_i\lambda_j}\right]H_n \\ &\Leftrightarrow \quad (n-1)\left(\sum_i \frac{1}{\lambda_i}\right)^2 \geq 2n\sum_{i<j}\frac{1}{\lambda_i\lambda_j}. \end{aligned}$$

Denoting $\alpha_i = 1/\lambda_i$, the last inequality above is equivalent to

$$(n-1)\left(\sum_{i=1}^n \alpha_i\right)^2 \geq 2n\sum_{i<j}\alpha_i\alpha_j.$$

Letting $T(\alpha_i) = (n-1)\left(\sum_{i=1}^n \alpha_i\right)^2 - 2n\sum_{i<j}\alpha_i\alpha_j$, we get

$$\begin{aligned} T(\alpha_i) &= n\left(\sum_{i=1}^n \alpha_i\right)^2 - \left(\sum_{i=1}^n \alpha_i\right)^2 - 2n\sum_{i<j}\alpha_i\alpha_j \\ &= n\left[\left(\sum_{i=1}^n \alpha_i\right)^2 - 2\sum_{i<j}\alpha_i\alpha_j\right] - \left(\sum_{i=1}^n \alpha_i\right)^2 \\ &= n\sum_{i=1}^n \alpha_i^2 - \left(\sum_{i=1}^n \alpha_i\right)^2 \geq 0, \end{aligned}$$

by Cauchy-Schwarz inequality. Also, in this case equality happens if and only if all of the α_i (and then all of the λ_i) are equal. Note that the above reasoning also proves that $H_1^2 = H_2$ if and only if all of the λ_i are equal, for $T(\lambda_i) = n^2(n-1)[H_1^2(\lambda_i) - H_2(\lambda_i)]$.

Regarding (b), observe that $H_1 \geq H_2^{1/2}$ follows from (a). On the other hand, if $H_1 \geq H_2^{1/2} \geq \cdots \geq H_k^{1/k}$ for some $2 \leq k < r$, then

$$H_k^2 \geq H_{k-1}H_{k+1} \geq H_k^{\frac{k-1}{k}} H_{k+1},$$

or still $H_k^{1/k} \geq H_{k+1}^{1/(k+1)}$. It now follows immediately from the above inequalities that, if $H_k^{1/k} = H_{k+1}^{1/(k+1)}$ for some $1 \leq k < r$, then $H_k^2 = H_{k-1}H_{k+1}$. Therefore, item (a) gives $\lambda_1 = \cdots = \lambda_n$.

To prove (c) suppose, without loss of generality, $r < n-1$. Since $H_r = H_{r+1} = 0$, one has equality in Newton's inequality

$$H_{r+1}^2 \geq H_r H_{r+2}.$$

If $H_{r+2} \neq 0$, it follows from (a) that $\lambda_1 = \cdots = \lambda_n = \lambda$. Hence, $H_r = 0 \Rightarrow \lambda = 0$, from where $H_{r+2} = 0$, a contradiction. Therefore $H_{r+2} = 0$, and analogously $H_j = 0$ for all $r \leq j \leq n$. To finish, it suffices to note that the polynomial $f(x)$ of item (a) is, in this case, just

$$f(x) = \sum_{j=0}^{n} S_j x^{n-j} = \sum_{j=0}^{r-1} S_j x^{n-j}.$$

$\square$

For $0 \leq r \leq n$ one defines the $r-$th Newton transformation P_r on M by setting $P_0 = I$ (the identity operator) and, for $1 \leq r \leq n$, via the recurrence relation

$$P_r = (-1)^r S_r I + A P_{r-1}. \tag{2.4}$$

A trivial induction shows that

$$P_r = (-1)^r (S_r I - S_{r-1}A + S_{r-2}A^2 - \cdots + (-1)^r A^r),$$

so that Cayley-Hamilton theorem gives $P_n = 0$. Moreover, since P_r is a polynomial in A for every r, it is also self-adjoint and commutes with A. Therefore, all bases of T_pM diagonalizing A at $p \in M$ also diagonalize all of the P_r at p. Let $\{e_k\}$ be such a basis. Denoting by A_i the restriction of A to $\langle e_i \rangle^{\perp} \subset T_pM$, it is easy to see that

$$\det(tI - A_i) = \sum_{k=0}^{n-1} (-1)^k S_k(A_i) t^{n-1-k},$$

where

$$S_k(A_i) = \sum_{\substack{1 \leq j_1 < \ldots < j_k \leq n \\ j_1, \ldots, j_k \neq i}} \lambda_{j_1} \cdots \lambda_{j_k}.$$

With the above notations, it is also immediate to check that $P_r e_i = (-1)^r S_r(A_i) e_i$, so that (lemma 2.1 of [**?**])

(a) $S_r(A_i) = S_r - \lambda_i S_{r-1}(A_i)$.
(b) $\mathrm{tr}(P_r) = (-1)^r \sum_{i=1}^n S_r(A_i) = (-1)^r (n-r) S_r$.
(c) $\mathrm{tr}(AP_r) = (-1)^r \sum_{i=1}^n \lambda_i S_r(A_i) = (-1)^r (r+1) S_{r+1}$.
(d) $\mathrm{tr}(A^2 P_r) = (-1)^r \sum_{i=1}^n \lambda_i^2 S_r(A_i) = (-1)^r (S_1 S_{r+1} - (r+2) S_{r+2})$.

Associated to each Newton transformation P_r one has the second order linear differential operator $L_r : \mathcal{D}(M) \to \mathcal{D}(M)$, given by

$$L_r(f) = \mathrm{tr}(P_r Hess f).$$

When $\overline{M}^{n+1}$ is a constant sectional curvature Riemannian space, it was proved by H. Rosenberg in [?] that

$$L_r(f) = div(P_r\nabla f), \tag{2.5}$$

where div stands for the divergence of a vector field on M, and his proof works as well for Lorentz ambient spaces $\overline{M}^{n+1}$. Therefore, for $f, g \in \mathcal{D}(M)$, it follows from the properties of the divergence of vector fields that

$$L_r(fg) = fL_r(g) + gL_r(f) + 2\langle P_r\nabla f, \nabla g\rangle. \tag{2.6}$$

With the above notations, the following result is by now classical. For a proof, we refer the reader to [?] or [?].

$$\begin{aligned} L_1(S_1) &= -\Delta S_2 - |\nabla A|^2 + |\nabla S_1|^2 \\ &\quad -\mathrm{tr}(AP_1)(|A|^2 + cn) + S_1[\mathrm{tr}(A^2P_1) + c\mathrm{tr}(P_1)], \end{aligned} \tag{2.7}$$

Substituting (??), (??) and (??) into (??), and taking into account the properties of the divergence of vector fields, after some manipulations one easily arrives at

$$\begin{aligned} \frac{1}{2}\Delta|A|^2 &= div(A\nabla S_1) + |\nabla A|^2 - |\nabla S_1|^2 \\ &\quad +\mathrm{tr}(AP_1)(|A|^2 + cn) - S_1[\mathrm{tr}(A^2P_1) + c\mathrm{tr}(P_1)]. \end{aligned} \tag{2.8}$$

3. CMC hypersurfaces having bounds on the scalar curvature

In what follows, $\overline{M}_c^{n+1}$ denotes complete a Lorentz manifold of constant sectional curvature c.

Theorem 3.1. *Let $x : M^n \to \overline{M}_c^{n+1}$, $c \geq 0$, be a complete spacelike hypersurface of constant mean curvature H. If M has scalar curvature $R \geq c$, then:*

(a) *$R = c$ on M.*
(b) *If $c = 0$ and $H \neq 0$, then $S_j = 0$ for all $2 \leq j \leq n$.*
(c) *If $c > 0$, then M is totally geodesic and closed.*

Proof. It follows from equation (??) that

$$\frac{1}{2}\Delta|A|^2 = |\nabla A|^2 + \mathrm{tr}(AP_1)|A|^2 - S_1\mathrm{tr}(A^2P_1) + c[n\mathrm{tr}(AP_1) - S_1\mathrm{tr}(P_1)], \tag{3.1}$$

and an easy computation shows that

$$\mathrm{tr}(AP_1)|A|^2 - S_1\mathrm{tr}(A^2P_1) = -2S_2|A|^2 + S_1^2S_2 - 3S_1S_3 \tag{3.2}$$

and

$$n\mathrm{tr}(AP_1) - S_1\mathrm{tr}(P_1) = -2nS_2 + (n-1)S_1^2. \tag{3.3}$$

Now, applying the second one of Newton's inequalities, one gets

$$\frac{S_1S_3}{n\binom{n}{3}} = H_1H_3 \leq H_2^2 = \frac{S_2^2}{\binom{n}{2}^2}$$

so that

$$3S_1S_3 \leq \frac{2(n-2)}{n-1}S_2^2. \tag{3.4}$$

On the other hand, the first of Newton's inequalities gives

$$(n-1)S_1^2 - 2nS_2 = (n-1)n^2(H_1^2 - H_2) \geq 0. \tag{3.5}$$

Let's now interpret the condition on R in terms of $|A|^2$: from (**??**) we get $S_2 \leq 0$. Therefore,

$$|A|^2 - S_1^2 = -2S_2 \geq 0. \tag{3.6}$$

Substituting (**??**), (**??**), (**??**) and (**??**) into (**??**), and taking into account that $c \geq 0$, one successively gets

$$\begin{aligned}
\frac{1}{2}\Delta(|A|^2 - S_1^2) &= \frac{1}{2}\Delta|A|^2 \geq -2S_2|A|^2 + S_1^2 S_2 - \frac{2(n-2)}{n-1}S_2^2 \\
&= -2S_2\left(|A|^2 - \frac{S_1^2}{2} + \left(\frac{n-2}{n-1}\right)\left(\frac{S_1^2 - |A|^2}{2}\right)\right) \\
&= \frac{n}{2(n-1)}(|A|^2 - S_1^2)\left(|A|^2 - \frac{S_1^2}{n}\right) \\
&\geq \frac{n}{2(n-1)}(|A|^2 - S_1^2)^2. \qquad (3.7)
\end{aligned}$$

Our purpose is to apply corollary **??** to $|A|^2$ in (**??**). To this end, we also need to know that our manifold M has Ricci curvature bounded from below. Take $p \in M$, a unit vector $v \in T_pM$ and an orthonormal basis $\{e_1, \ldots, e_{n-1}, e_n = v\}$ of T_pM. Gauss equation gives

$$\begin{aligned}
(n-1)Ric_p(v) &= \sum_{i<n} R(e_i, e_n, e_i, e_n) \\
&= (n-1)c + \sum_{i<n}[-\langle Ae_i, e_i\rangle\langle Ae_n, e_n\rangle + \langle Ae_i, e_n\rangle^2] \\
&= (n-1)c - (S_1 - \langle Ae_n, e_n\rangle)\langle Ae_n, e_n\rangle + \sum_{i<n}\langle Ae_i, e_n\rangle^2 \\
&\geq (n-1)c + \langle Ae_n, e_n\rangle^2 - S_1\langle Ae_n, e_n\rangle \\
&\geq (n-1)c - \frac{S_1^2}{4}. \qquad (3.8)
\end{aligned}$$

Therefore, applying corollary **??** with $\beta = 2$ we get $|A|^2 = S_1^2$ on M, which is the same as $S_2 = 0$, or $R = c$ on M. To prove items (b) and (c), it now follows immediately from (**??**), (**??**), (**??**) and (**??**) that $\nabla A = 0$, $H_1H_3 = H_2^2 = 0$ and $cH_1^2 = cH_2 = 0$ on M.

If $c > 0$ then $H_1^2 = H_2 = 0$ on M, so that M totally geodesic. Moreover, (**??**) shows that the Ricci curvature of M satisfies $Ric \geq 1$, so that M is closed, by Myers' theorem. If $c = 0$ and $H_1 = H \neq 0$, then $H_3 = 0$ on M, and item (c) of proposition **??** gives $S_j = 0$ on M, for all $2 \leq j \leq n$. $\square$

Hereafter, by $\mathbb{L}^{n+1}$ we mean the standard $(n+1)-$dimensional Lorentz-Minkowsky space of zero sectional curvature. More precisely, $\mathbb{L}^{n+1}$ is the $(n+1)-$dimensional Euclidean space $\mathbb{E}^{n+1}$ with coordinates $(x_1, \ldots, x_{n+1})$, furnished with the Lorentz metric corresponding to the quadratic form

$$q(x) = x_1^2 + \cdots + x_n^2 - x_{n+1}^2.$$

Also, $\mathbb{S}_1^{n+1}$ denotes the $(n+1)-$dimensional De Sitter space, that is,

$$\mathbb{S}_1^{n+1} = \{p \in \mathbb{L}^{n+2};\ \langle p, p\rangle = 1\}.$$

It is a standard result that $\mathbb{S}_1^{n+1}$ is a Lorentz manifold of constant sectional curvature identically 1.

Corollary 3.2. *Let $x : M^n \to \mathbb{S}_1^{n+1}$ be a complete spacelike hypersurface of constant mean curvature H in the De Sitter space. If M has scalar curvature $R \geq 1$, then*

$$x(M) = \{p \in \mathbb{S}_1^{n+1};\ \langle p, w\rangle = 0\},$$

for some $w \in \mathbb{L}^{n+2}$ such that $\langle w, w\rangle = -1$.

Proof. In [?] and [?], the authors characterized closed CMC spacelike hypersurfaces of $\mathbb{S}_1^{n+1}$ as being totally umbilical. More precisely, any such hypersurface equals

$$M_{\tau,w} = \{x \in \mathbb{S}_1^{n+1};\ \langle x, w\rangle = \tau\},$$

for some $\tau \in \mathbb{R}$ and $w \in \mathbb{L}^{n+2}$ such that $\langle w, w\rangle = -1$. The corollary now follows from the fact that $M_{\tau,w}$ has mean curvature $\frac{\tau}{\tau^2+1}$. □

Corollary 3.3. *Let $x : M^n \to \mathbb{L}^{n+1}$ be a complete spacelike hypersurface of constant mean curvature $H > 0$. If the scalar curvature R of M is nonnegative, then $x(M)$ is an $(n-1)-$cilinder over a plane curve γ. Moreover, up to isometries of $\mathbb{L}^{n+1}$, one has*

$$\gamma(x_1, \ldots, x_n) = (0, \ldots, 0, g(x_1)),$$

where

$$g(x_1) = \frac{\sqrt{|c|}}{nH} - \sqrt{\left(x_1 - \frac{\sqrt{c^2-1}}{nH}\right)^2 + \frac{1}{n^2H^2}}$$

and $c \in \mathbb{R}$ is arbitrary such that $|c| \geq 1$.

Proof. It follows from the preceding theorem that the relative nullity ν of M is identically $n-1$. Therefore, from a theorem of Ferus ([?], chap. 5) the distribution of relative nullity is smooth and integrable, and the leaves are complete and totally geodesic in M^n and in $\mathbb{L}^{n+1}$. Hence, the argument of the proof of Hartman-Nirenberg's theorem (see also chap. 5 of [?]) works ipsis literis, and any such M is a cilinder over a plane curve.

On the other hand, it is well known ([?]) that every complete spacelike hypersurface M^n of $\mathbb{L}^{n+1}$ is the graph

$$(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, f(x_1, \ldots, x_n))$$

of some function $f : \mathbb{R}^n \to \mathbb{R}$ having gradient $|\nabla f| < 1$. Moreover, in this case (see also [?]) the mean curvature H of M satisfies

$$nH = -div\left(\frac{\nabla f}{\sqrt{1 - |\nabla f|^2}}\right).$$

Hence, up to an isometry of $\mathbb{L}^{n+1}$, we can assume that M is the graph of $f(x_1, \ldots, x_n) = g(x_1)$, and the equation above for the mean curvature of M becomes, after an easy manipulation,

$$nH = -\frac{g''}{[1 - (g')^2]^{3/2}}.$$

This ODE gives

$$nHg' + \frac{g'g''}{[1 - (g')^2]^{3/2}} = 0,$$

and integrating from 0 to x_1 (assuming $g(0) = 0$ without loss of generality) we get

$$nHg + \frac{1}{\sqrt{1-(g')^2}} = c,$$

where c is some real constant such that $|c| \geq 1$. Now, it is not difficult to check that

$$g(x_1) = \sqrt{\alpha^2+\beta^2} - \sqrt{(x_1-\alpha)^2+\beta^2}$$

is the solution for the above ODE, where

$$\alpha = \frac{\sqrt{c^2-1}}{nH}, \quad \beta = \frac{1}{nH}.$$

□

Remark 3.4. Some nonnegativity on the scalar curvature R is necessary for theorem **??** to be true. Indeed, the hyperbolic space $\mathbb{H}^n(r) \subset \mathbb{L}^{n+1}$, of constant sectional curvature $-1/r^2$, is CMC with $H = \pm 1/r$, but has scalar curvature $R = -1/r$. For the De Sitter space, the examples of A. Brasil, A. Colares and O. Palmas in [**?**] are of non-totally umbilical, complete CMC hypersurfaces, thus not having nonnegative scalar curvature.

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