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Necessary optimality conditions for constrained optimization problems under relaxed constraint qualifications

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Abstract We derive first- and second-order necessary optimality conditions for set-constrained optimization problems under the constraint qualification-type conditions significantly weaker than Robinson's constraint qualification. Our development relies on the so-called 2-regularity concept, and unifies and extends the previous studies based on this concept. Specifically, in our setting constraints are given by an inclusion, with an arbitrary closed convex set in the right-hand side. Thus, for the second-order analysis, some curvature characterizations of this set near the reference point must be taken into account.

Keywords Optimization problem · Abstract constraints · Constraint qualification · Optimality condition · Sigma-term

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1 Introduction

Let X and Y be Banach spaces, $f : X \to \mathbf{R}$ be a smooth function, $F : X \to Y$ be a smooth mapping (our smoothness assumptions will be specified below), and Q be a fixed closed convex set in Y. In this paper, we are concerned with the following optimization problem with abstract constraints:

$$\begin{array}{l}\text{minimize} \quad f(x)\\ \text{subject to } x \in D = F^{-1}(Q) = \{x \in X \mid F(x) \in Q\}.\end{array} \tag{1}$$

Classical studies of problem (1) rely on the Lagrange optimality principle and employ the *Lagrangian* of this problem defined by

$$L(x,\, \lambda) = f(x) + \langle \lambda,\, F(x)\rangle \tag{2}$$

for $x \in X$ and $\lambda \in Y^*$. There exists an extensive literature on the first- and second-order necessary optimality conditions for problem (1) (see, e.g., [13, Chapter 3] and references therein). However, most of these works (with some exceptions, to be specified below) assume that the so-called Robinson's constraint qualification (CQ)

$$0 \in \operatorname{int}\,(F(\bar{x}) + \operatorname{im} F'(\bar{x}) - Q) \tag{3}$$

is satisfied at the local solution $\bar{x}$ in question. If Robinson's CQ is violated, necessary optimality conditions in terms of L are in general not valid. Following [12], we refer to problems with this type of behavior as *abnormal* optimization problems.

If the relative interior of $F(\bar{x}) + \operatorname{im} F'(\bar{x}) - Q$ is nonempty (which is always the case fore a finite-dimensional Y), the abnormal case can be covered by the Lagrange principle with the additional multiplier $\lambda_0 \in \mathbf{R}$ corresponding to the objective function (see [13, Proposition 3.18]). However, the value of this generalization is limited by the fact that such first-order optimality condition holds automatically with $\lambda_0 = 0$ provided Robinson's CQ is violated at $\bar{x}$, with the objective function f being irrelevant (see [13, Proposition 3.16]).

In [6,8], the following *generalized Lagrangian* of problem (1) was introduced:

$$L_2(x,\, h,\, \lambda^1,\, \lambda^2) = f(x) + \langle \lambda^1,\, F(x)\rangle + \langle \lambda^2,\, F'(x)h\rangle \tag{4}$$

for $x,\, h \in X$ and $\lambda^1,\, \lambda^2 \in Y^*$, and the corresponding meaningful first- and second-order necessary optimality conditions for abnormal purely equality-constrained optimization problems (i.e., when $Q = \{0\}$) were derived (see also [20,22,2] for the more recent presentations of these results). Element h plays a role of a parameter, and it varies in some set specified by the problem data. This analysis relies on the so-called 2-regularity concept, which will also be the main tool of our development in this paper, being adopted to the general form of constraints in (1).

The results of [6,8] were further extended in [9] to calculus of variations problems; in [7] to optimal control problems; in [19] to the case of milder smoothness requirements, and in [11] to the case when $\operatorname{im} F'(\bar{x})$ is not supposed to be closed. Similar ideas were used in [15,16,18] for deriving optimality conditions for purely inequality-constrained problems (more precisely, for the case when Q is a cone and $\operatorname{int} Q \neq \emptyset$).

Let us also mention a different approach to abnormal optimization problems, developed in [24,3,2]. This approach consists of deriving second-order necessary optimality conditions involving the index of the quadratic form associated with L. However, these results deal with the problems with equality constraints and a finite number of inequality constraints. For the general problem (1) with $\operatorname{int} Q \neq \emptyset$, useful second-order necessary conditions can be found in [13, Theorem 3.50].

This paper can be regarded an extension of the previous work concerned with necessary optimality conditions employing the particular instances of the 2-regularity concept. Specifically, we extend these results to the general setting of problem (1). We deal with a very general setting of an arbitrary closed convex set Q. Note, however, that the results presented below are completely meaningful and new even in the case of a polyhedral Q (i.e., in the context of mathematical programming problems), since up to now, the case of mixed equality and inequality constraints remained uncovered.

The paper is organized as follows. In Section 2, we develop the so-called 2-regularity concept for the constraints defining the set D in (1), which is a weaker regularity concept than the traditional CQs. Section 3 contains two auxiliary lemmas. In Section 4, we prove the principal lemma about the estimate of the distance to the feasible set. This lemma is the main tool for deriving the description of the tangent cones to D at $\bar{x}$, and the necessary optimality conditions in the subsequent sections. Sections 5 and 6 contain our main results, that is, first- and second-order necessary optimality conditions for problem (1) under 2-regularity assumptions. Finally, in Section 7, we present some illustrative examples.

We next briefly discuss our notation. For a given normed linear space X, X^* is its (toplogically) dual space, and $B_\delta(x) = \{\xi \in X \mid \|\xi - x\| \leq \delta\}$ is a ball centered at $x \in X$ and of radius $\delta > 0$. If $K \subset X$ is a cone, $K^\circ = \{l \in X^* \mid \langle l, \xi\rangle \leq 0 \;\forall\, \xi \in K\}$ stands for its polar cone. For a given set $S \subset X$, $\operatorname{int} S$ stands for its interior, $\operatorname{cl} S$ stands for its closure, $\operatorname{cone} S$ stands for its conic hull (the smallest cone containing S), and $S^\perp = \{l \in X^* \mid \langle l, x\rangle = 0 \;\forall\, x \in S\}$ stands for its annihilator. Furthermore, $\sigma(\cdot, S) : X^* \to \mathbf{R}$, $\sigma(l, S) = \sup_{x\in S}\langle l, x\rangle$, is the support function of S, and $\operatorname{dist}(x, S) = \inf_{\xi\in S}\|\xi - x\|$ is the distance from $x \in X$ to S. For a given point $x \in S$, $R_S(x) = \operatorname{cone}(S - x)$ is the so-called radial cone to S at x,

$$T_S(x) = \left\{ h \in X \;\middle|\; \begin{array}{l} \exists\, \{t_k\} \subset \mathbf{R}_+ \setminus \{0\} \text{ such that} \\ \{t_k\} \to 0,\ \operatorname{dist}(x + t_k h, S) = o(t_k) \end{array} \right\} \tag{5}$$

is the contingent cone to S at x, and $N_S(x) = (T_S(x))^\circ$ is the normal cone to S at x (if $x \notin S$ then $N_S(x) = \emptyset$ by definition). Recall that for a convex set S, $T_S(x) = \operatorname{cl} R_S(x)$, and hence, $N_S(x) = (R_S(x))^\circ$.

If Y is another normed linear space, $\mathcal{L}(X, Y)$ ($\mathcal{L}^2(X, Y)$) stands for the space of continuous linear operators (respectively, continuous bilinear mappings) from X (respectively, from $X \times X$) to Y. For a given linear operator $A : X \to Y$, $\operatorname{im} A$ stands for its range (image space), while $\ker A$ stands for its kernel (null space).

2 2-regularity concept

Let $\bar{x} \in D$ be given, and assume that the mapping F is twice Fréchet-differentiable at $\bar{x}$.

Definition 1 The mapping F is said to be *2-regular at the point $\bar{x}$ with respect to the set Q in a direction $h \in X$* if

$$0 \in \operatorname{int}\big(F(\bar{x}) + \operatorname{im} F'(\bar{x}) + F''(\bar{x})[h,\, (F'(\bar{x}))^{-1}(Q - F(\bar{x}))] - Q\big)\,. \tag{6}$$

Note that 2-regularity in the direction $h = 0$ coincides with Robinson's CQ (3) at $\bar{x}$. If the latter condition is satisfied then evidently F is 2-regular at $\bar{x}$ with respect to Q in any direction $h \in X$ (including $h = 0$), but not vice versa. On the other hand, if $Q = \{0\}$ then 2-regularity coincides with the counterpart of this concept for pure equality constraints, as defined in [6] (at least when $\operatorname{im} F'(\bar{x})$ is closed and has a closed complementary subspace in Y; see [4]).

Set

$$D_1(\bar{x}) = (F'(\bar{x}))^{-1}(Q - F(\bar{x})). \tag{7}$$

The set $\bar{x} + D_1(\bar{x})$ can be regarded as the first-order approximation of D near $\bar{x}$. It can be easily checked that

$$R_{D_1(\bar{x})}(0) = (F'(\bar{x}))^{-1}(R_Q(F(\bar{x}))). \tag{8}$$

For a given $h \in X$, define the linear operator $G(\bar{x},\, h) : X \times X \to Y$,

$$G(\bar{x},\, h)(x,\, \xi) = F'(\bar{x})x + F''(\bar{x})[h,\, \xi]. \tag{9}$$

Let M be an arbitrary closed linear subspace in Y such that

$$\operatorname{im} F'(\bar{x}) \subset M \subset \operatorname{im} F'(\bar{x}) - R_Q(F(\bar{x})), \tag{10}$$

Generally, one cannot guarantee the existence of such M, but it exists in some important special cases. For example, if $\operatorname{im} F'(\bar{x})$ is closed (in particular, if $\dim Y < \infty$), then one can take $M = \operatorname{im} F'(\bar{x})$. On the other hand, if Robinson's CQ (3) is satisfied, the right-hand side in (10) coincides with entire Y (see [13, Proposition 2.95]), and one can take $M = Y$.

Define the linear operator $\tilde{G}_M(\bar{x},\, h) : X \times X \to M \times Y$,

$$\tilde{G}_M(\bar{x},\, h)(x,\, \xi) = (F'(\bar{x})\xi,\, G(\bar{x},\, h)(x,\, \xi)) = (F'(\bar{x})\xi,\, F'(\bar{x})x + F''(\bar{x})[h,\, \xi]). \tag{11}$$

Proposition 1 *If $\bar{x} \in D$ then for each $h \in X$ 2-regularity condition* (6) *is equivalent to the equality*

$$\operatorname{im} F'(\bar{x}) + F''(\bar{x})[h,\, (F'(\bar{x}))^{-1}(R_Q(F(\bar{x})))] - R_Q(F(\bar{x})) = Y. \tag{12}$$

Furthermore, if there exists a closed linear subspace M in Y satisfying (10) *then* (6) *and* (12) *are both equivalent to each of the following two conditions:*

$$0 \in \operatorname{int}(\operatorname{im} \tilde{G}_M(\bar{x},\, h) - ((Q - F(\bar{x})) \cap M) \times (Q - F(\bar{x}))), \tag{13}$$

$$\operatorname{im} \tilde{G}_M(\bar{x},\, h) - (R_Q(F(\bar{x})) \cap M) \times R_Q(F(\bar{x})) = M \times Y. \tag{14}$$

Note that int in (13) is taken with respect to $M \times Y$.

Proof Evidently, conditions (6) and (12) can be re-written as

$$0 \in \operatorname{int}\,(F(\bar{x}) + G(\bar{x},\, h)(X \times D_1(\bar{x})) - Q) \tag{15}$$

and

$$G(\bar{x},\, h)(X \times R_{D_1(\bar{x})}(0)) - R_Q(F(\bar{x})) = Y, \tag{16}$$

respectively (in (16), equality (8) is taken into account).

The equivalence of (15) and (16) (and hence, of (6) and (12) as well) can be established by the same argument as the equivalence of (2.194) and (2.195) in [13]. The same applies to the equivalence of (13) and (14), if we recall that M is closed, and hence, $M \times Y$ is a Banach space. It now suffices to show that (12) is equivalent to (14).

Suppose first that (12) holds. By the last inclusion in (10) we obtain that for an arbitrary $\eta \in M$, there exists $\xi^1 \in X$ such that

$$F'(\bar{x})\xi^1 \in \eta + R_Q(F(\bar{x})) \cap M. \tag{17}$$

According to (12), for each $y \in Y$, there exists

$$(x,\, \xi^2) \in X \times (F'(\bar{x}))^{-1}(R_Q(F(\bar{x})))$$

such that

$$F'(\bar{x})x + F''(\bar{x})[h,\, \xi^2] \in y - F''(\bar{x})[h,\, \xi^1] + R_Q(F(\bar{x})). \tag{18}$$

Note that by the first inclusion in (10), $F'(\bar{x})\xi^2 \in R_Q(F(\bar{x})) \cap M$. Set $\xi = \xi^1 + \xi^2$. Then from (17) and (18), we obtain

$$F'(\bar{x})\xi \in \eta + R_Q(F(\bar{x})) \cap M, \quad F'(\bar{x})x + F''(\bar{x})[h,\, \xi] \in y + R_Q(F(\bar{x})).$$

which means that (14) holds too (see (11)).

Now suppose that (14) holds, that is, for each pair $(\eta,\, y) \in M \times Y$, there exists $(x,\, \xi) \in X \times X$ such that

$$F'(\bar{x})\xi \in \eta + R_Q(F(\bar{x})) \cap M, \quad F'(\bar{x})x + F''(\bar{x})[h,\, \xi] \in y + R_Q(F(\bar{x})) \tag{19}$$

(see (11)). If we take $\eta = 0$ then from the first relation in (19) it follows that $\xi \in (F'(\bar{x}))^{-1}(R_Q(F(\bar{x})))$. Hence, for each $y \in Y$, there exists $(x,\, \xi) \in X \times (F'(\bar{x}))^{-1}(R_Q(F(\bar{x})))$ such that the second relation in (19) holds, which means that (12) holds too. □

For a given $h \in X$, define the linear operator $\tilde{G}(\bar{x},\, h) : X \times X \to X \times Y$,

$$\tilde{G}(\bar{x},\, h)(x,\, \xi) = (\xi,\, G(\bar{x},\, h)(x,\, \xi)). \tag{20}$$

If 2-regularity condition (15) holds then from [13, Lemma 2.100] we obtain

$$0 \in \operatorname{int}(\operatorname{im} \tilde{G}(\bar{x},\, h) - D_1(\bar{x}) \times (Q - F(\bar{x}))).$$

Thus, Robinson's CQ holds for the constraints $\tilde{G}(\bar{x},\, h)x \in D_1(\bar{x}) \times (Q - F(\bar{x}))$ at $(0,\, 0)$, and hence, $\tilde{G}(\bar{x},\, h)$ is metric regular at $(0,\, 0)$ with respect to

$D_1(\bar{x}) \times (Q - F(\bar{x})))$ at some rate $\tilde{a} = \tilde{a}(\bar{x}, h) > 0$ (see [13, Proposition 2.89]), that is, for all $(\tilde{x}, \tilde{\xi}, \chi, y) \in X \times X \times X \times Y$ close enough to $(0, 0, 0, 0)$, it holds that

$$\begin{aligned} \operatorname{dist}((\tilde{x}, \tilde{\xi}), (\tilde{G}(\bar{x}, h))^{-1}((D_1(\bar{x}) + \chi) \times (Q - F(\bar{x}) + y))) \leq \\ \tilde{a} \operatorname{dist}(\tilde{G}(\bar{x}, h)(\tilde{x}, \tilde{\xi}) - (\chi, y), D_1(\bar{x}) \times (Q - F(\bar{x}))). \end{aligned} \tag{21}$$

Proposition 2 *Let F be 2-regular at $\bar{x}$ with respect to Q in a direction $h \in X$.*

Then there exists $a = a(\bar{x}, h) > 0$ with the following property: for any $\varepsilon > 0$ there exists $\delta = \delta(\bar{x}, h, \varepsilon) > 0$ such that for any mapping $\Phi : X \times X \to Y$ which is Lipschitz-continuous on $(D_1(\bar{x}) \cap B_\varepsilon(0)) \times B_\varepsilon(0)$ with modulus $\ell \in (0, 1/(2\tilde{a}))$ (where $D_1(\bar{x})$ is defined in (7) *and $\tilde{a} > 0$ is taken from* (21)*) and for all $(\tilde{h}, y) \in B_\delta(h) \times B_\delta(0)$, there exists $(x, \xi) \in X \times D_1(\bar{x})$ such that*

$$G(\bar{x}, \tilde{h})(x, \xi) + \Phi(x, \xi) - y \in Q - F(\bar{x}), \tag{22}$$

$$\|x\| + \|\xi\| \leq a \operatorname{dist}(\Phi(0, 0) - y, Q - F(\bar{x})). \tag{23}$$

Proof To begin with, set $\tilde{\delta} = 1/(6\tilde{a}\|F''(\bar{x})\|)$. For an arbitrary $\tilde{h} \in B_{\tilde{\delta}}(h)$, define the mapping $\Psi = \Psi(\bar{x}, \tilde{h}; \cdot) : X \times X \to X \times Y$,

$$\begin{aligned} \Psi(x, \xi) &= (\xi, G(\bar{x}, \tilde{h})(x, \xi) + \Phi(x, \xi)) \\ &= \tilde{G}(\bar{x}, h)(x, \xi) + (0, F''(\bar{x})[\tilde{h} - h, \xi] + \Phi(x, \xi)) \end{aligned} \tag{24}$$

(see (20)). Due to the restrictions on ℓ and the definition of $\tilde{\delta}$, the last term in the right-hand side forms the mapping which is Lipschitz-continuous on $(D_1(\bar{x}) \cap B_\varepsilon(0)) \times B_\varepsilon(0)$ with modulus $\tilde{\ell} < 2/(3\tilde{a})$. Then from (21) and [13, Theorem 2.84 and Remark 2.85]) it follows that Ψ is metric regular at $(0, 0)$ with respect to $D_1(\bar{x}) \times (Q - F(\bar{x}))$ at the rate $a = 3\tilde{a}$, that is, there exists $\delta = \delta(\bar{x}, h, \varepsilon, \tilde{a}(\bar{x}, h)) > 0$ such that for all $(\tilde{x}, \tilde{\xi}, \chi, y) \in B_\delta(0) \times B_\delta(0) \times B_\delta(0) \times B_\delta(0)$ it holds that

$$\begin{aligned} &\operatorname{dist}((\tilde{x}, \tilde{\xi}), \Psi^{-1}((D_1(\bar{x}) + \chi) \times (Q - F(\bar{x}) + y))) \\ &\leq \tilde{a} \operatorname{dist}(\Psi(\tilde{x}, \tilde{\xi}) - (\chi, y), D_1(\bar{x}) \times (Q - F(\bar{x}))). \end{aligned}$$

(It is crucial for our development that δ does not depend on the specific Φ. See [13, Remark 2.85] where, however, it is not pointed out that δ must depend on ε.) By taking $\tilde{x} = \tilde{\xi} = \chi = 0$ and employing the first equality in (24), we obtain (22), (23). To complete the proof, it remains to replace δ by $\tilde{\delta}$ if $\delta > \tilde{\delta}$. □

3 Auxiliary results

Throughout the rest of the paper we assume that F is twice differentiable in a neighborhood of $\bar{x} \in D$, and its second derivative is Lipschitz-continuous in this neighborhood. Then according to the Hadamard lemma [5, Chapter 2], there exists a mapping $\mathcal{R} : X \to \mathcal{L}^2(X, Y)$ such that for each $x \in X$ close enough to 0

$$F(\bar{x} + x) = F(\bar{x}) + F'(\bar{x})x + \frac{1}{2}F''(\bar{x})[x, x] + \mathcal{R}(x)[x, x], \tag{25}$$

and moreover, $\mathcal{R}(0) = 0$ and $\mathcal{R}$ is Lipschitz-continuous near 0.

For arbitrary $y \in Q$ and $\theta \in [0, 1]$, set $Q(y, \theta) = \theta Q + (1-\theta)y$ (the set θQ consists of all elements of the form θq, $q \in Q$, and in particular, $0Q = \{0\}$). Clearly, for all $\theta_1, \theta_2 \in [0, 1]$ such that $\theta_1 \leq \theta_2$, it holds that

$$\{y\} = Q(y, 0) \subset Q(y, \theta_1) \subset Q(y, \theta_2) \subset Q(y, 1) = Q.$$

Lemma 1 *For any bounded set $\Omega \subset X$, there exists $b = b(\Omega) > 0$ such that for any $\xi \in \Omega \setminus \{0\}$, any $t \geq 0$ small enough, and any $\theta \in [0, 1]$ satisfying*

$$(1 - \theta)t^2 \leq \operatorname{dist}(F(\bar{x} + t\xi), Q), \tag{26}$$

it holds that

$$\operatorname{dist}(F(\bar{x} + \theta t\xi), Q(F(\bar{x}), \theta)) \leq b \operatorname{dist}(F(\bar{x} + t\xi), Q).$$

Proof For arbitrary $\xi \in \Omega$, $t \geq 0$ and $\theta \in [0, 1]$ satisfying (26), choose $\tilde{y} = \tilde{y}(\xi, t) \in Q$ such that

$$\|F(\bar{x} + t\xi) - \tilde{y}\| \leq 2 \operatorname{dist}(F(\bar{x} + t\xi), Q), \tag{27}$$

and set $y = \theta\tilde{y} + (1 - \theta)F(\bar{x})$.

Employing (25), (26) and (27), and taking into account that Ω is bounded, we obtain that if t is small enough then

$$\begin{aligned}
\|F(\bar{x} + \theta t\xi) - y\| &= \bigg\| F(\bar{x}) + \theta t F'(\bar{x})\xi + \frac{1}{2}\theta^2 t^2 F''(\bar{x})[\xi, \xi] \\
&\qquad - \theta\tilde{y} - (1 - \theta)F(\bar{x}) + \theta^2 t^2 \mathcal{R}(\theta t\xi)[\xi, \xi] \bigg\| \\
&= \bigg\| \theta \left(F(\bar{x}) + tF'(\bar{x})\xi + \frac{1}{2}t^2 F''(\bar{x})[\xi, \xi] - \tilde{y} \right) \\
&\qquad + \left(-\frac{1}{2}\theta t^2 F''(\bar{x})[\xi, \xi] + \frac{1}{2}\theta^2 t^2 F''(\bar{x})[\xi, \xi] \right) \\
&\qquad + \theta^2 t^2 \mathcal{R}(\theta t\xi)[\xi, \xi] \bigg\| \\
&= \bigg\| \theta(F(\bar{x} + t\xi) - \tilde{y}) - \frac{1}{2}\theta(1 - \theta)t^2 F''(\bar{x})[\xi, \xi] \\
&\qquad - \theta t^2 \mathcal{R}(t\xi)[\xi, \xi] + \theta^2 t^2 \mathcal{R}(\theta t\xi)[\xi, \xi] \bigg\|
\end{aligned}$$

$$
\begin{aligned}
&\le \theta\|F(\bar{x}+t\xi)-\tilde{y}\| + \frac{1}{2}\theta(1-\theta)t^2\|F''(\bar{x})\|\|\xi\|^2 \\
&\quad +\theta(1-\theta)t^2\|\mathcal{R}(t\xi)\|\|\xi\|^2 \\
&\quad +\theta^2t^2\|\mathcal{R}(t\xi)-\mathcal{R}(\theta t\xi)\|\|\xi\|^2 \\
&\le 2\operatorname{dist}(F(\bar{x}+t\xi),\, Q) + \frac{1}{2}\|F''(\bar{x})\|\|\xi\|^2\operatorname{dist}(F(\bar{x}+t\xi),\, Q) \\
&\quad +\|\mathcal{R}(t\xi)\|\|\xi\|^2\operatorname{dist}(F(\bar{x}+t\xi),\, Q) + \ell(1-\theta)t^3\|\xi\|^3 \\
&\le b\operatorname{dist}(F(\bar{x}+t\xi),\, Q),
\end{aligned}
$$

where $\ell > 0$ is a modulus of Lipschitz continuity of $\mathcal{R}$, and

$$
b > 2 + \frac{1}{2}\|F''(\bar{x})\|\sup_{\xi\in\Omega}\|\xi\|^2.
$$

At the same time, $y \in Q(F(\bar{x}),\, \theta)$, and the needed assertion follows. □

We complete this section with the following lemma which will also be needed in subsequent analysis.

Lemma 2 *Let U and V be Banach spaces. For given $l \in U^*$, $a \in \mathbf{R}$, $v \in V$, $A \in \mathcal{L}(U,\, V)$, and for a closed convex set $K \subset V$, $\bar{v} \in K$, and a convex set $T \subset V$, let W be a closed linear subspace in V such that*

$$
\operatorname{im} A \subset W \subset \operatorname{im} A - R_K(\bar{v}), \tag{28}
$$

$$
T + R_K(\bar{v}) \subset T. \tag{29}
$$

Assume further that

$$
v \in \operatorname{im} A + T. \tag{30}
$$

Then condition

$$
\langle l,\, u\rangle + a \ge 0 \quad \forall\, u \in A^{-1}(T - v) \tag{31}
$$

is equivalent to the existence of $\nu \in V^*$ such that

$$
l + A^*\nu = 0, \quad \nu \in (R_K(\bar{v})\cap W)^\circ, \quad a + \langle \nu,\, v\rangle - \sigma(\nu,\, T\cap(v+W)) \ge 0. \tag{32}
$$

Note that in this lemma T is not assumed to be closed.

Remark 1 It can be easily seen that under the assumptions (28), (29), condition (30) is equivalent to

$$
T \cap (v + W) \ne \emptyset. \tag{33}
$$

Indeed, if (30) holds then there exists $w \in T$ such that $v \in w + \operatorname{im} A$, and hence, according to the first inclusion in (28), $w \in v + \operatorname{im} A \subset v + W$. Thus, $w \in T \cap (v + W)$, which proves (33).

On the other hand, if (33) holds then there exists $w \in W$ such that $v + w \in T$, and hence, according to the first second inclusion in (29), and according to (29), $v \in T - W \subset T + \operatorname{im} A + R_K(\bar{v}) \subset \operatorname{im} A + T$, that is, (30) holds.

Proof Suppose that (31) holds. Set $K_0 = (K - \bar{v}) \cap W$, $T_0 = T \cap (v + W)$. Evidently, $R_{K_0}(0) = R_K(\bar{v}) \cap W$ and hence, by (29),

$$\begin{aligned} T_0 + R_{K_0}(0) &= T \cap (v + W) + R_K(\bar{v}) \cap W \\ &\subset (T + R_K(\bar{v})) \cap (v + W + W) \\ &\subset T \cap (v + W) \\ &= T_0. \end{aligned} \tag{34}$$

In the Banach space $\mathbf{R} \times W$ (recall that W is closed), consider the set

$$S = \{(\alpha,\, w) \in \mathbf{R} \times W \mid \alpha > \langle l,\, u\rangle + a,\ w \in Au + v - T_0,\ u \in U\}$$

(by the first inclusion in (28), this set indeed belongs to $\mathbf{R} \times W$). Evidently, S is convex. We next show that

$$(0,\, 0) \notin S, \quad \operatorname{int} S \neq \emptyset, \tag{35}$$

where int is taken with respect to $\mathbf{R} \times W$.

The first relation in (35) readily follows from (31). According to the second inclusion in (28),

$$\operatorname{im} A - R_{K_0}(0) = W, \tag{36}$$

and hence, according to [13, Proposition 2.95],

$$0 \in \operatorname{int}(\operatorname{im} A - K_0),$$

where int is taken with respect to W. By the generalized open mapping theorem [13, Theorem 2.70] we now obtain that

$$0 \in \operatorname{int}(AB_1(0) - K_0). \tag{37}$$

By Remark 1, $T_0 \neq \emptyset$. Fix an arbitrary $w^0 \in T_0$, and set

$$S_0 = \{\alpha \in \mathbf{R} \mid \alpha > \|l\| + a\} \times (v - w^0 + A(B_1(0)) - K_0).$$

Because of (37), S_0 evidently has a nonempty interior with respect to $\mathbf{R} \times W$, and it remains to show that $S_0 \subset S$.

Fix and arbitrary pair $(\alpha,\, w) \in S_0$. Then $\alpha > \|l\| + a$, and there exist $u \in B_1(0)$ and $v^0 \in K_0$ such that $w = v - w^0 + Au - v^0$. Hence,

$$\alpha > \|l\| + a \geq \langle l,\, u\rangle + a.$$

On the other hand, note that $K_0 \subset R_{K_0}(0)$, and hence, $v^0 \in R_{K_0}(0)$. From (34) it now follows that

$$w = Au + v - (w^0 + v^0) \in Au + v - T_0,$$

that is, $(\alpha,\, w) \in S$, which completes the proof of (35).

From convexity of S and (35), by the separation theorem (e.g., [13, Theorem 2.13]) we obtain the existence of $(\nu_0,\, \mu) \in (\mathbf{R} \times W^*) \setminus \{(0,\, 0)\}$ such that

$$\nu_0(\langle l,\, u\rangle + a + \alpha) + \langle \mu,\, Au + v - w\rangle \geq 0 \quad \forall\, u \in U,\ \forall\, \alpha > 0,\ \forall\, w \in T_0.$$

It easily follows that

$$\nu_0 \geq 0, \quad \nu_0 l + A^*\mu = 0, \quad \nu_0 a + \langle \mu,\, v - w\rangle \geq 0 \quad \forall\, w \in T_0. \tag{38}$$

Moreover, by (34), and by the last inequality in (38), for any fixed $w \in T_0$ we obtain that

$$\nu_0 a + \langle \mu,\, v\rangle \geq \langle \mu,\, w + \eta\rangle \quad \forall\, \eta \in R_{K_0}(0),$$

and hence,

$$\langle \mu,\, \eta\rangle \leq 0 \quad \forall\, \eta \in R_{K_0}(0). \tag{39}$$

If we suppose that $\nu_0 = 0$ then the second relation in (38) implies the equality $A^*\mu = 0$. Combined with (39), this contradicts (36). Thus, $\nu_0 > 0$, and hence, in (38) we can put $\nu_0 = 1$. According to the Hahn-Banach theorem [13, Theorem 2.10], we can take $\nu \in V^*$ as a continuous extension of the functional $\mu \in W^*$ to the entire V, and with this choice of ν, (38), (39) imply (32) (recall that $v - w \in W\ \forall\, w \in T_0$).

On the other hand, if (32) holds then

$$\begin{aligned}\langle l,\, u\rangle + a + \langle \nu,\, Au + v - w\rangle &= \langle l + A^*\nu,\, u\rangle + a + \langle \nu,\, v - w\rangle \\ &\geq 0 \quad \forall\, u \in U, \forall\, w \in T \cap (W + v).\end{aligned}$$

Take here $u \in A^{-1}(T - v)$ and $w = v + Au \in T \cap (v + W)$ (recall the first inclusion in (28)), then $\langle l,\, u\rangle + a \geq 0$, i.e., (31) holds. □

With K being a closed convex cone and $T = K$, and with $a = 0$, $v = \bar{v} = 0$, this result implies the following characterization of the polar cone to $A^{-1}(K)$: for each closed linear subspace W in V satisfying $\operatorname{im} A \subset W \subset \operatorname{im} A - K$, it holds that

$$(A^{-1}(K))^\circ = A^*(K \cap W)^\circ.$$

According to [14, Lemma 5.8], $(K \cap W)^\circ$ equals weak* closure of $K^\circ + W^\perp$. Hence, under any additional assumption implying that $K^\circ + W^\perp$ is weakly* closed (which is of course not automatic), the last relation results in the more customary formula:

$$(A^{-1}(K))^\circ = A^*(K^\circ + W^\perp) = A^*K^\circ, \tag{40}$$

since $W^\perp \subset (\operatorname{im} A)^\perp = \ker A^*$. Among the additional assumptions guaranteeing the existence of W with the needed properties, let us mention the following:

– V is finite-dimensional and K is polyhedral (in which case, one can take $W = \operatorname{im} A$).
– Robinson's CQ holds for the constraints $Au \in K$ at 0, that is, $\operatorname{im} A - K = V$ (in which case, one can take $W = V$).

In the former case, relation (40) is the well-known Farkas lemma [13, Proposition 2.201]. In the latter case, relation (40) is a particular case of [13, Lemma 3.27].

4 Characterization of the contingent cone

Lemma 3 *Let F be 2-regular at $\bar{x}$ with respect to Q in a direction $h \in X$.*

Then there exists $c = c(\bar{x}, h) > 0$ such that for all $\tilde{h} \in X$ close enough to h and all $t > 0$ small enough, it holds that

$$\operatorname{dist}(\bar{x} + t\tilde{h}, D) \leq c \operatorname{dist}(F(\bar{x} + t\tilde{h}), Q)/t. \tag{41}$$

Proof If $F''(\bar{x}) = 0$ or $h = 0$ then 2-regularity condition (6) reduced to Robinson's CQ (3), and hence, the estimate stronger than (41) follows from Robinson's stability theorem [13, Theorem 2.87]. Thus we may suppose that $F''(\bar{x}) \neq 0$ and $h \neq 0$.

For an arbitrary fixed $\varepsilon > 0$, let $a > 0$ and $\delta \in (0, 2\|h\|]$ be defined according to Proposition 2. Set $\Omega = B_\delta(h)$, and define $b > 0$ according to Lemma 1. Set

$$\tilde{\delta} = \max\{\varepsilon, a\delta\}, \quad \tilde{c} = 2b/\delta, \quad \gamma = 1/(32a\tilde{c}\tilde{\delta}\|F''(\bar{x})\|\|h\|). \tag{42}$$

Throughout the rest of the proof, let $\tilde{h} \in B_{\delta/2}(h)$ (note that since $\delta \leq 2\|h\|$, this implies the inequality $\|\tilde{h}\| \leq 2\|h\|$), and let $t \in [0, 1]$ be small enough.

If $t\|\tilde{h}\| < \operatorname{dist}(F(\bar{x} + t\tilde{h}), Q)/(\gamma t)$, then (41) holds with $c = 1/\gamma$. That is why we further suppose that

$$\operatorname{dist}(F(\bar{x} + t\tilde{h}), Q)/t^2 \leq \gamma\|\tilde{h}\|. \tag{43}$$

Set $\tau = \tilde{c} \operatorname{dist}(F(\bar{x} + t\tilde{h}), Q)/t$, $\theta = 1 - \tau - \tau t$. From (43) it follows that $\tau \to 0$, $\theta \to 1$ as $t \to 0$. In particular, if t is small enough then $\tau, \theta \in [0, 1]$ and $\theta\tilde{h} \in B_\delta(h)$.

Choose $\tilde{y} \in Q(F(\bar{x}), \theta)$ such that $\|F(\bar{x} + \theta t\tilde{h}) - \tilde{y}\| \leq 2 \operatorname{dist}(F(\bar{x} + \theta t\tilde{h}), Q(F(\bar{x}), \theta))$, and set $y = (F(\bar{x} + \theta t\tilde{h}) - \tilde{y})/(\tau t)$. Then by (9), (25), for $(\xi, x) \in X \times X$ we have

$$\begin{aligned} F(\bar{x} + \theta t\tilde{h} + \tau\xi + \tau tx) &= (F(\bar{x} + \theta t\tilde{h} + \tau\xi + \tau tx) - F(\bar{x} + \theta t\tilde{h})) \\ &\quad + (F(\bar{x} + \theta t\tilde{h}) - \tilde{y}) + \tilde{y} \\ &= \tau F'(\bar{x})\xi \\ &\quad + \tau t(F'(\bar{x})x + F''(\bar{x})[\theta\tilde{h}, \xi]) + \tau t^2 F''(\bar{x})[\theta\tilde{h}, x] \\ &\quad + \frac{1}{2}\tau^2 F''(\bar{x})[\xi + tx, \xi + tx] \\ &\quad + \mathcal{R}(\theta t\tilde{h} + \tau\xi + \tau tx)[\theta t\tilde{h} + \tau\xi + \tau tx, \theta t\tilde{h} + \tau\xi + \tau tx] \\ &\quad - \mathcal{R}(\theta t\tilde{h})[\theta t\tilde{h}, \theta t\tilde{h}] + \tau ty + \tilde{y} \\ &= \tau F'(\bar{x})\xi + \tau t(G(\bar{x}, \theta\tilde{h})(x, \xi) + \Phi_t(x, \xi) + y) + \tilde{y}, \end{aligned} \tag{44}$$

where

$$\begin{aligned}\Phi_t(x,\,\xi) &= \Phi_t(\bar{x},\,\tilde{h};\,x,\,\xi)\\ &= tF''(\bar{x})[\theta\tilde{h},\,x] + \frac{\tau}{2t}F''(\bar{x})[\xi+tx,\,\xi+tx]\\ &\quad + \frac{1}{\tau t}(\mathcal{R}(\theta t\tilde{h}+\tau\xi+\tau tx)[\theta t\tilde{h}+\tau\xi+\tau tx,\,\theta t\tilde{h}+\tau\xi+\tau tx]\\ &\quad -\mathcal{R}(\theta t\tilde{h})[\theta t\tilde{h},\,\theta t\tilde{h}]). \end{aligned} \tag{45}$$

The mapping $(x,\,\xi)\to tF''(\bar{x})[\theta\tilde{h},\,x] : X\times X\to Y$ (see the first term in the right-hand side of (45)) is Lipschitz-continuous on the entire $X\times X$ with modulus less than $1/(8a)$ (recall that $t>0$ is taken small enough).

Furthermore, for each $(\xi^1,\,x^1)$, $(\xi^2,\,x^2)\in B_{\tilde{\delta}}(0)\times B_{\tilde{\delta}}(0)$, by the definition of τ, the last relation in (42), and (43), we obtain

$$\begin{aligned}\Big\|\frac{\tau}{2t}F''(\bar{x})[\xi^1+tx^1,\,\xi^1+tx^1]\\ -\frac{\tau}{2t}F''(\bar{x})[\xi^2+tx^2,\,\xi^2+tx^2]\Big\| &= \frac{\tau}{2t}\|F''(\bar{x})[\xi^1+\xi^2+t(x^1+x^2),\,\xi^1-\xi^2\\ &\quad +t(x^1-x^2)]\|\\ &\le \frac{2\tau\tilde{\delta}}{t}\|F''(\bar{x})\|(\|\xi^1-\xi^2\|+\|x^1-x^2\|)\\ &= \frac{2\tilde{c}\tilde{\delta}}{t^2}\operatorname{dist}(F(\bar{x}+t\tilde{h}),\,Q)\|F''(\bar{x})\|(\|\xi^1-\xi^2\|\\ &\quad +\|x^1-x^2\|)\\ &\le 2\tilde{c}\gamma\tilde{\delta}\|F''(\bar{x})\|\|\tilde{h}\|(\|\xi^1-\xi^2\|+\|x^1-x^2\|)\\ &= \|\tilde{h}\|/(16a\|h\|)\\ &\le 1/(8a). \end{aligned} \tag{46}$$

Thus, the mapping $(x,\,\xi)\to\frac{\tau}{2t}F''(\bar{x})[\xi+tx,\,\xi+tx] : X\times X\to Y$ (see the second term in the right-hand side of (45)) is Lipschitz-continuous on $B_{\tilde{\delta}}(0)\times B_{\tilde{\delta}}(0)$ with modulus less than $1/(8a)$.

Finally, employing the properties of R, it can be shown that the mapping $(x,\,\xi)\to\frac{1}{\tau t}(\mathcal{R}(\theta t\tilde{h}+\tau\xi+\tau tx)[\theta t\tilde{h}+\tau\xi+\tau tx,\,\theta t\tilde{h}+\tau\xi+\tau tx]-\mathcal{R}(\theta t\tilde{h})[\theta t\tilde{h},\,\theta t\tilde{h}]) : X\times X\to Y$ (see the third term in the right-hand side of (45)) is Lipschitz-continuous on $B_{\tilde{\delta}}(0)\times B_{\tilde{\delta}}(0)$ with modulus less than $1/(4a)$.

Summarizing, we conclude that $\Phi_t(\cdot,\,\cdot)$ is Lipschitz-continuous on $B_{\tilde{\delta}}(0)\times B_{\tilde{\delta}}(0)$ with modulus less than $1/(2a)$, and $\Phi_t(0,\,0)=0$. Furthermore, from the definition of τ and θ it follows that (26) holds with $\xi=\tilde{h}$ (recall that $t>0$ is taken small enough). Hence, according to the definition of y, to the choice of $\tilde{y}$, and to the second relation in (42), by Lemma 1 it holds that

$$\begin{aligned}\|y\| &= \|(F(\bar{x}+\theta t\tilde{h})-\tilde{y})/(\tau t)\|\\ &\le \frac{2\operatorname{dist}(F(\bar{x}+\theta t\tilde{h}),\,Q(F(\bar{x}),\,\theta))}{\tilde{c}\operatorname{dist}(F(\bar{x}+t\tilde{h}),\,Q)}\end{aligned}$$

$$
\begin{aligned}
&= \frac{\delta \operatorname{dist}(F(\bar{x}+\theta t\tilde{h}),\, Q(F(\bar{x}),\, \theta))}{b \operatorname{dist}(F(\bar{x}+t\tilde{h}),\, Q)} \\
&\leq \delta.
\end{aligned}
$$

Therefore, since a and δ were chosen according to Proposition 2, and $\tilde{\delta} \geq \varepsilon$, there exist $(x,\, \xi) \in X \times X$ and $(\eta,\, \tilde{\eta}) \in Q$ such that

$$
F'(\bar{x})\xi = \tilde{\eta} - F(\bar{x}), \quad G(\bar{x},\, \theta\tilde{h})(x,\, \xi) + \Phi_t(x,\, \xi) + y = \eta - F(\bar{x}), \tag{47}
$$

$$
\|x\| + \|\xi\| \leq a \operatorname{dist}(\Phi(0,\, 0) + y,\, Q - F(\bar{x})) \leq a\|y\| \leq a\delta \leq \tilde{\delta} \tag{48}
$$

(see the first relation in (42)).

Since $\tilde{y} \in Q(F(\bar{x}),\, \theta)$, there exists $\hat{y} \in Q$ such that $\tilde{y} = \theta\hat{y} + (1-\theta)F(\bar{x})$. By (44) and (47) we then obtain

$$
\begin{aligned}
F(\bar{x} + t\tilde{h} - (1-\theta)t\tilde{h} + \tau\xi + \tau t x) &= F(\bar{x} + \theta t\tilde{h} + \tau\xi + \tau t x) \\
&= \tau(\tilde{\eta} - F(\bar{x})) + \tau t(\eta - F(\bar{x})) + \theta\hat{y} \\
&\quad + (1-\theta)F(\bar{x}) \\
&= (1 - \tau - \tau t - \theta)F(\bar{x}) + \tau\tilde{\eta} + \tau t\eta + \theta\hat{y} \\
&= \tau\tilde{\eta} + \tau t\eta + \theta\hat{y} \\
&\in Q,
\end{aligned} \tag{49}
$$

where the definition of θ and convexity of Q are taken into account. Moreover, by (48), employing the definitions of θ and τ, we obtain

$$
\begin{aligned}
\|-(1-\theta)t\tilde{h} + \tau\xi + \tau t x\| &\leq 2(1-\theta)t\|h\| + \tau\tilde{\delta} + \tau t\tilde{\delta} \\
&= (\tilde{\delta} + t\tilde{\delta} + 2(1+t)t\|h\|)\tau \\
&\leq 2\tilde{c}\tilde{\delta} \operatorname{dist}(F(\bar{x}+t\tilde{h}),\, Q)/t
\end{aligned} \tag{50}
$$

(recall that $t > 0$ is taken small enough). Relations (49) and (50) imply (41). This completes the proof. □

Lemma 3 will be used below in order to derive necessary optimality conditions for problem (1). This development is based on the description of first and second-order tangent sets to the feasible set D. According to Robinson's stability theorem, under Robinson's CQ (3), estimate (41) can be replaced by a stronger one:

$$
\operatorname{dist}(\bar{x} + t\tilde{h},\, D) \leq c \operatorname{dist}(F(\bar{x} + t\tilde{h}),\, Q).
$$

However, without Robinson's CQ, this is in general not the case. In the rest of this section, we demonstrate how Lemma 3 may help to characterize the contingent cone to D at $\bar{x}$ under the assumptions weaker than Robinson's CQ (3).

For a given linear operator $A \in \mathcal{L}(X,\, Y)$, define the set

$$
T_Q^2(y,\, d;\, A) = \left\{ w \in Y \;\middle|\; \begin{array}{l} \exists\, \{t_k\} \subset \mathbf{R}_+ \setminus \{0\},\ \{x^k\} \subset X \text{ such that} \\ \{t_k\} \to 0,\ \{x^k\} \to 0, \\ \operatorname{dist}\left(y + t_k d + t_k A x^k + \frac{1}{2}t_k^2 w,\, Q\right) = o(t_k^2) \end{array} \right\}. \tag{51}
$$

This set is somewhat related to the so-called upper second-order approximations for Q at y in the direction d and with respect to A, defined in [13, Definition 3.82]. In particular,

$$T_Q^2(y,\, d) = T_Q^2(y,\, d;\, 0) \tag{52}$$

is the usual (outer) second-order tangent set to Q at y in the direction d, as defined in [13, Definition 3.28]. On the other hand, $T_Q^2(F(\bar{x}),\, F'(\bar{x})h;\, F'(\bar{x}))$, $h \in X$, was introduced in [25, Definition 2.1].

Note that if for given $y \in Q$ and $d \in Y$ and some $A \in \mathcal{L}(X,\, Y)$ it holds that $T_Q^2(y,\, d;\, A) \neq \emptyset$ then $d \in T_Q(y)$. It can be easily seen that

$$T_Q^2(y,\, d;\, A) \subset T_{T_Q(y)-\operatorname{im} A}(d) = \operatorname{cl}(T_Q(y) - \operatorname{im} A - \operatorname{cone}\{d\}), \tag{53}$$

and if $\dim Y < \infty$ and Q is a polyhedral set then

$$T_Q^2(y,\, d;\, A) = T_{T_Q(y)-\operatorname{im} A}(d) = T_Q(y) - \operatorname{im} A - \operatorname{cone}\{d\} \tag{54}$$

for any $d \in T_Q(y) = R_Q(y)$ (the closely related results for usual second-order tangent sets can be found in [13, p. 168]).

For the sake of brevity, for each $h \in X$ put

$$\mathcal{T}^2(h) = T_Q^2(F(\bar{x}),\, F'(\bar{x})h;\, F'(\bar{x})),$$

i.e., according to (51),

$$\mathcal{T}^2(h) = \left\{ w \in Y \;\middle|\; \begin{array}{l} \exists\, \{t_k\} \subset \mathbf{R}_+ \setminus \{0\},\ \{x^k\} \subset X \text{ such that} \\ \{t_k\} \to 0,\ \{x^k\} \to 0, \\ \operatorname{dist}\big(F(\bar{x}) + t_k F'(\bar{x})h + t_k F'(\bar{x})x^k + \tfrac{1}{2}t_k^2 w,\, Q\big) = o(t_k^2) \end{array} \right\}. \tag{55}$$

Define the sets

$$H_2(\bar{x}) = \{h \in X \mid F''(\bar{x})[h,\, h] \in \mathcal{T}^2(h)\}, \tag{56}$$

$$\bar{H}_2(\bar{x}) = \{h \in H_2(\bar{x}) \mid (6) \text{ holds}\}. \tag{57}$$

It can be easily checked that both these sets are cones.

Theorem 1 *The following inclusions are valid:*

$$\bar{H}_2(\bar{x}) \subset T_D(\bar{x}) \subset H_2(\bar{x}). \tag{58}$$

Proof If $h \in T_D(\bar{x})$ then, according to (5), there exist $\{t_k\} \subset \mathbf{R}_+ \setminus \{0\}$ and $\{r^k\} \subset X$ such that $\{t_k\} \to 0$, $r^k = o(t_k)$, and $F(\bar{x} + t_k h + r^k) \in Q$ for all k large enough. For such k, by twice differentiability of F at $\bar{x}$ we obtain

$$Q \ni F(\bar{x}+t_k h+r^k) = F(\bar{x})+t_k F'(\bar{x})h+t_k F'(\bar{x})r^k/t_k+\frac{1}{2}t_k^2 F''(\bar{x})[h,\, h]+o(t_k^2).$$

Hence, $F''(\bar{x})[h,\, h] \in \mathcal{T}^2(h)$ (in (55) one must take $x^k = r^k/t_k$), and inclusion $h \in H_2(\bar{x})$ follows by (56). The second inclusion in (58) is thus proved.

In order to prove the first inclusion in (58), consider an arbitrary $h \in \bar{H}_2(\bar{x})$, and fix the sequences $\{t_k\}$ and $\{x^k\}$ related to this h by (56), (57), and (55). For each k, set $h^k = h + x^k$. By (25), taking into account that $\mathcal{R}(0) = 0$ and $\mathcal{R}$ is continuous at 0, we obtain

$$\begin{aligned}\operatorname{dist}(F(\bar{x}+t_kh^k),\,Q) &\leq \operatorname{dist}\left(F(\bar{x})+t_kF'(\bar{x})h^k+\frac{1}{2}t_k^2F''(\bar{x})[h^k,\,h^k],\,Q\right)\\ &\quad+t_k^2\|\mathcal{R}(t_kh^k)[h^k,\,h^k]\|\\ &\leq \operatorname{dist}\Bigg(F(\bar{x})+t_kF'(\bar{x})h+t_kF'(\bar{x})x^k\\ &\quad+\frac{1}{2}t_k^2F''(\bar{x})[h,\,h],\,Q\Bigg)+o(t_k^2)\\ &= o(t_k^2). \end{aligned} \tag{59}$$

Applying Lemma 3 with $\tilde{h} = h^k$ and $t = t_k$ for k large enough, we further conclude that

$$\operatorname{dist}(\bar{x}+t_kh,\,D) \leq \operatorname{dist}(\bar{x}+t_kh^k,\,D)+t_k\|x^k\| = o(t_k)$$

where (59) is taken into account (recall also that $\{x^k\} \to 0$). Hence, $h \in T_D(\bar{x})$ (see (5)). $\square$

The statement of Theorem 1 can be easily modified so that it will be giving a characterization of the so-called inner tangent cone

$$T_D^i(\bar{x}) = \{h \in X \mid \operatorname{dist}(\bar{x}+th,\,D) = o(t),\ t \geq 0\}$$

to D at $\bar{x}$ instead of the contingent cone $T_D(\bar{x})$. In order to do this, one should replace $\mathcal{T}^2(h)$ in (56) by its inner counterpart

$$T_Q^{i,\,2}(F(\bar{x}),\,F'(\bar{x})h,\,F'(\bar{x})) = \left\{ w \in Y \;\middle|\; \begin{array}{l} \forall\, t \geq 0\ \exists\, x(t) \in X \text{ such that} \\ x(t) \to 0 \text{ as } t \to 0, \\ \operatorname{dist}\big(F(\bar{x})+tF'(\bar{x})h \\ \ +tF'(\bar{x})x(t)+\frac{1}{2}t^2w,\,Q\big) = o(t^2) \end{array} \right\}.$$

With these modifications, both inclusions in (58) remain valid.

The result of Theorem 1 for pure equality constraints was derived in [6]. Some earlier version of the result from [6] under stronger smoothness assumptions can be found in [23]. For other related material see also [26, 8, 10, 20, 21, 2, 17, 19].

Definition 2 The mapping F is said to be *2-regular at the point $\bar{x}$ with respect to the set Q* if it is 2-regular at this point with respect to Q in any direction $h \in H_2(\bar{x}) \setminus \{0\}$, that is $H_2(\bar{x}) \setminus \{0\} = \bar{H}_2(\bar{x}) \setminus \{0\}$.

From Theorem 1, we immediately obtain

Corollary 1 *Let F be 2-regular at $\bar{x}$ with respect to Q.*
Then $T_D(\bar{x}) = H_2(\bar{x})$.

Recall that Robinson's CQ (3) implies 2-regularity in any direction, and hence in this case $H_2(\bar{x}) = \bar{H}_2(\bar{x})$. Furthermore, according to the discussion following the definition of $T_Q^2(y, d; A)$, from (56) it follows that

$$H_2(\bar{x}) \subset (F'(\bar{x}))^{-1}(T_Q(F(\bar{x}))). \tag{60}$$

Moreover, if Robinson's CQ (3) is satisfied then this inclusion holds as an equality. More precisely, in this case, the equality $\mathcal{T}^2(h) = Y$ holds for each $h \in (F'(\bar{x}))^{-1}(T_Q(F(\bar{x})))$. Indeed, according to (5), there exists a sequence $\{t_k\} \subset \mathbf{R}_+ \setminus \{0\}$ such that $\{t_k\} \to 0$ and $\operatorname{dist}(F(\bar{x}) + t_k F'(\bar{x})h,\, Q) = o(t_k)$. Furthermore, from (3) and the Robinson–Ursescu stability theorem [13, Theorem 2.83] it follows that the linearized mapping $\xi \to F(\bar{x}) + F'(\bar{x})\xi : X \to Y$ is metric regular at 0 with respect to Q. In particular, for any $w \in Y$ and each k large enough, there exists $\xi^k \in X$ such that

$$F(\bar{x}) + F'(\bar{x})(t_k h + \xi^k) \in Q - \frac{1}{2} t_k^2 w,$$

$$\|\xi^k\| = O\left(\operatorname{dist}\left(F(\bar{x}) + t_k F'(\bar{x})h + \frac{1}{2} t_k^2 w,\, Q\right)\right) = o(t_k).$$

It remains to put $x^k = \xi^k / t_k$. With this choice,

$$F(\bar{x}) + t_k F'(\bar{x})h + t_k F'(\bar{x})x^k + \frac{1}{2} t_k^2 w \in Q$$

and $\{x^k\} \to 0$, and hence, according to (55), $w \in \mathcal{T}^2(h)$.

Summarizing, in the case of Robinson's CQ (3), Corollary 1 reduces to the classical description of the contingent cone (see, e.g., [13, Corollary 2.91]). However, Theorem 1 and Corollary 1 are applicable far beyond the case of Robinson's CQ.

For pure equality constraints, Corollary 1 was proved in [27] for the special case when $F'(\bar{x}) = 0$.

For pure inequality-type constraints, the counterparts of Theorem 1 and Corollary 1 can be found in [15,16,18]. Finally, for the case when Q is a cone and $F(\bar{x}) = 0$, Theorem 1 and Corollary 1 were derived in [1].

5 "First-order" necessary conditions

In this section, we derive the "first-order" necessary optimality conditions. These conditions are "first-order" in the following sense: they employ only the first derivative of the objective function, and as will be demonstrated below, they are the extension of the customary first-order necessary optimality conditions. Thus, let f be Fréchet-differentiable at $\bar{x}$.

From Theorem 1, we immediately obtain the following *primal* "first-order" necessary condition.

Theorem 2 *Let $\bar{x}$ be a local solution of problem* (1).
Then

$$\langle f'(\bar{x}),\, h\rangle \geq 0 \quad \forall\, h \in \bar{H}_2(\bar{x}).$$

In the remainder of this section, we derive the *primal-dual* "first-order" necessary condition, of which Theorem 2 is a particular case.

Define the so-called *second-order tightened critical cone* of problem (1) at $\bar{x}$:

$$C_2(\bar{x}) = \{h \in H_2(\bar{x}) \mid \langle f'(\bar{x}),\, h\rangle \leq 0\}. \tag{61}$$

Define also the cone

$$\bar{C}_2(\bar{x}) = C_2(\bar{x}) \cap \bar{H}_2(\bar{x}) = \{h \in C_2(\bar{x}) \mid (6) \text{ holds}\}. \tag{62}$$

Theorem 3 *Let $\bar{x}$ be a local solution of problem* (1).
Then for any $h \in \bar{C}_2(\bar{x})$, there exists $\lambda^2 = \lambda^2(h) \in Y^*$ such that

$$-f'(\bar{x}) - (F''(\bar{x})[h])^*\lambda^2 \in ((F'(\bar{x}))^{-1}(R_Q(F(\bar{x}))))^\circ, \quad (F'(\bar{x}))^*\lambda^2 = 0, \tag{63}$$

$$\lambda^2 \in N_Q(F(\bar{x})). \tag{64}$$

Proof For an arbitrary $h \in \bar{C}_2(\bar{x})$, fix the sequences $\{t_k\}$ and $\{x^k\}$ related to this h by (61), $C_2(\bar{x})$ and $H_2(\bar{x})$, and by (55). Then there exists a sequence $\{\tilde{y}^k\} \subset Q$ such that $\omega^k = o(t_k^2)$, where

$$\omega_k = \left\| F(\bar{x}) + t_kF'(\bar{x})h + t_kF'(\bar{x})x^k + \frac{1}{2}t_k^2F''(\bar{x})[h,\, h] - \tilde{y}^k \right\|.$$

Fix an arbitrary $(x,\, \xi) \in X \times D_1(\bar{x})$ such that $F'(\bar{x})x + F''(\bar{x})[h,\, \xi] \in Q - F(\bar{x})$, and for each k set

$$\tau_k = (\max\{\|x^k\|t_k^2,\, \omega_k,\, t_k^3\})^{1/2}, \quad \theta_k = 1 - \tau_k - \tau_kt_k,$$

$$h^k = \theta_k(h + x^k) + \tau_k\xi/t_k + \tau_kx.$$

Evidently, $\tau_k \to 0$ (and moreover, $\tau_k = o(t_k)$), $\theta_k \to 1$, and $\{h^k\} \to h$ as $k \to \infty$. In particular, $\tau_k,\, \theta_k \in [0,\, 1]$ for all k large enough.

By Lemma 3 we obtain the existence of a sequence $\{r^k\} \subset X$ such that for all k large enough it holds that

$$F(\bar{x} + t_kh^k + r^k) \in Q, \quad \|r^k\| = O(\mathrm{dist}(F(\bar{x} + t_kh^k),\, Q)/t_k). \tag{65}$$

According to the choice of $(x,\, \xi)$, there exists and $(\eta,\, \tilde{\eta}) \in Q$ such that

$$F'(\bar{x})\xi = \tilde{\eta} - F(\bar{x}), \quad F'(\bar{x})x + F''(\bar{x})[h,\, \xi] = \eta - F(\bar{x}). \tag{66}$$

For each k set $y^k = \theta_k\tilde{y}^k + \tau_k\tilde{\eta} + t_k\tau_k\eta$. Then $y^k \in Q$, where the definition of θ_k and convexity of Q are taken into account. Moreover, according to (25), the properties of $\mathcal{R}(\cdot)$, the definitions of h^k, τ_k, θ_k, y^k and ω_k, and (66), we obtain

$$\begin{aligned}
\|F(\bar{x}+t_kh^k)-y^k\| &= \left\|F(\bar{x})+t_kF'(\bar{x})h^k+\frac{1}{2}t_k^2F''(\bar{x})[h^k,\, h^k]-y^k\right\|+O(t_k^3)\\
&= \Bigg\|F(\bar{x})+\theta_kt_kF'(\bar{x})h+\theta_kt_kF'(\bar{x})x^k\\
&\quad +\frac{1}{2}\theta_k^2t_k^2F''(\bar{x})[h,\, h]+\tau_kF'(\bar{x})\xi\\
&\quad +\tau_kt_k(F'(\bar{x})x+F''(\bar{x})[h,\, \xi])-y^k\Bigg\|\\
&\quad +O(\tau_k^2)+O(\tau_kt_k^2)+O(t_k^3)\\
&\le \theta_k\Bigg\|F(\bar{x})+t_kF'(\bar{x})h+t_kF'(\bar{x})x^k\\
&\quad +\frac{1}{2}t_k^2F''(\bar{x})[h,\, h]-\tilde{y}^k\Bigg\|\\
&\quad +\left\|\frac{1}{2}\theta_k^2t_k^2F''(\bar{x})[h,\, h]-\frac{1}{2}\theta_kt_k^2F''(\bar{x})[h,\, h]\right\|\\
&\quad +\|\tau_k(F'(\bar{x})\xi-\tilde{\eta})+\tau_kt_k(F'(\bar{x})x+F''(\bar{x})[h,\, \xi]-\eta)\\
&\quad -(1-\theta_k)F(\bar{x})\|+O(\tau_k^2)\\
&= \theta_k\omega_k+\frac{1}{2}\theta_k(1-\theta_k)t_k^2\|F''(\bar{x})\|\|h\|^2+O(\tau_k^2)\\
&= O(\tau_k^2).
\end{aligned}$$

Thus, by the second relation in (65),

$$\|r^k\| = O(\tau_k^2/t_k) = o(\tau_k). \tag{67}$$

Since $\bar{x}$ is a local solution of problem (1), by the first relation in (65), by (67), by the definitions of h^k and τ_k, and by (62) and (61), we obtain that for k large enough

$$\begin{aligned}
0 &\le f(\bar{x}+t_kh^k+r^k)+F(\bar{x})\\
&= \theta_kt_k\langle f'(\bar{x}),\, h\rangle+\theta_kt_k\langle f'(\bar{x}),\, x^k\rangle+\tau_k\langle f'(\bar{x}),\, \xi\rangle+o(\tau_k)+O(t_k^2)\\
&\le \tau_k\langle f'(\bar{x}),\, \xi\rangle+o(\tau_k).
\end{aligned}$$

Dividing the left- and right-hand side by τ_k and passing onto the limit as $k\to\infty$, we conclude that $\langle f'(\bar{x}),\, \xi\rangle\ge 0$.

We thus proved that (0, 0) is a (global) solution of the problem

$$\begin{aligned}
&\text{minimize} \quad \langle f'(\bar{x}),\, \xi\rangle\\
&\text{subject to } (x,\, \xi)\in D_2(\bar{x},\, h),
\end{aligned}$$

where

$$D_2(\bar{x},\, h) = \{(x,\, \xi)\in X\times D_1(\bar{x}) \mid F'(\bar{x})x+F''(\bar{x})[h,\, \xi]\in Q-F(\bar{x})\},$$

and moreover, 2-regularity condition (6) is precisely Robinson's CQ for the constraints of this problem at (0, 0). Hence, by [13, Theorem 3.9], there exist $\lambda^2 = \lambda^2(h) \in Y^*$ such that

$$(F'(\bar{x}))^* \lambda^2 = 0,$$

$$-f'(\bar{x}) - (F''(\bar{x})[h])^* \lambda^2 \in N_{D_1(\bar{x})}(0) = ((F'(\bar{x}))^{-1}(R_Q(F(\bar{x}))))^\circ,$$

$$\lambda^2 \in N_{Q-F(\bar{x})}(0) = N_Q(F(\bar{x})),$$

where (8) is taken into account. The needed assertion is thus proved. $\square$

Define the generalized Lagrangian of problem (1) according to (4). According to the comments following Lemma 2, if $R_Q(F(\bar{x}))$ is closed (i.e., $R_Q(F(\bar{x})) = T_Q(F(\bar{x}))$), and if one can choose a closed linear subspace M in Y satisfying (10) and such that the cone $(R_Q(F(\bar{x})))^\circ + M^\perp$ is weakly* closed (which is always the case for mathematical programming problems), then combination of (63) and (64) is equivalent to the existence of $\lambda^1 = \lambda^1(h) \in Y^*$ such that

$$\frac{\partial L_2}{\partial x}(\bar{x}, h, \lambda^1, \lambda^2) = 0, \quad (F'(\bar{x}))^* \lambda^2 = 0, \tag{68}$$

$$\lambda^1 \in N_Q(F(\bar{x})), \quad \lambda^2 \in N_Q(F(\bar{x})). \tag{69}$$

(Note that M does not appear in this set of conditions!) However, the existence of such M is not automatic, and generally, we cannot guarantee the existence of λ^1 satisfying (68), (69). At the same time, the somewhat weaker assertion is valid.

Theorem 4 *Let $\bar{x}$ be a local solution of problem* (1).

Then for any $h \in \bar{C}_2(\bar{x})$, there exists $\lambda^2 = \lambda^2(h) \in Y^*$ such that for any closed linear subspace M in Y satisfying (10)*, there exists $\lambda^1 = \lambda^1(h;\, M) \in Y^*$ such that*

$$\frac{\partial L_2}{\partial x}(\bar{x}, h, \lambda^1, \lambda^2) = 0, \quad (F'(\bar{x}))^* \lambda^2 = 0, \tag{70}$$

$$\lambda^1 \in N_{Q \cap (F(\bar{x}) + M)}(F(\bar{x})), \quad \lambda^2 \in N_Q(F(\bar{x})). \tag{71}$$

Proof It suffices to apply Theorem 3 and Lemma 2 with $U = X$, $V = Y$, $l = f'(\bar{x}) + (F''(\bar{x})[h])^* \lambda^2$, $a = 0$, $v = 0$, $A = F'(\bar{x})$, $K = Q$, $T = R_Q(F(\bar{x}))$, $\bar{v} = F(\bar{x})$, $W = M$. $\square$

If Robinson's CQ (3) is satisfied then the result just derived reduces to the customary primal-dual first-order necessary optimality condition. Indeed, according to the discussion above, (60) holds as an equality, and moreover,

$$\bar{H}_2(\bar{x}) = H_2(\bar{x}) = (F'(\bar{x}))^{-1}(T_Q(F(\bar{x}))).$$

Then by (61) and (62), $\bar{C}_2(\bar{x})$ reduces to the usual critical cone

$$C(\bar{x}) = \{h \in (F'(\bar{x}))^{-1}(T_Q(F(\bar{x}))) \mid \langle f'(\bar{x}), h\rangle \le 0\}.$$

In particular, this set contains $h = 0$, and with this h, Theorem 4 reduces to the customary first-order necessary optimality condition (see, e.g., [13, Theorem 3.9]). Indeed, for $x \in X$, $\lambda = \lambda^1 \in Y^*$ and any $\lambda^2 \in Y^*$

$$L_2(x,\, 0,\, \lambda^1,\, \lambda^2) = L(x,\, \lambda),$$

where L is the standard Lagrangian of problem (1), given in (2). Moreover, according to [13, Proposition 2.95], the right-hand side of (10) coincides with entire Y, and hence, condition (10) is satisfied with $M = Y$. Thus, (70), (71) reduce to

$$\frac{\partial L}{\partial x}(\bar{x},\, \lambda) = 0, \quad \lambda \in N_Q(F(\bar{x})). \tag{72}$$

Furthermore, Robinson's CQ (3) implies (see [13, Proposition 2.97]) that the second relations in (70) and (71) can hold only with $\lambda^2 = 0$, and hence, (70), (71) reduce to (72) for any $h \in C(\bar{x})$ (not only for $h = 0$). At the same time, (70), (71) (perhaps with $\lambda^2 \neq 0$) may provide meaningful information about the point $\bar{x}$ under consideration even when Robinson's CQ is violated but the weaker 2-regularity condition holds.

6 "Second-order" necessary conditions

This section is devoted to the "second-order" necessary optimality conditions. These conditions are "second-order" in the following sense: they employ the first two derivatives of the objective function, and as will be demonstrated below, they are the extension of the customary second-order necessary optimality conditions. Thus, let f be twice Fréchet-differentiable at $\bar{x}$. At the same time, we need to assume that F is three times Fréchet-differentiable at $\bar{x}$.

For given linear operator $A \in \mathcal{L}(X,\, Y)$, linear subspace M in Y, and $\eta \in Y$, define the set

$$T_Q^3(y,\, d;\, A;\, M,\, \eta) = \left\{ \begin{array}{l|l} (w^1,\, w^2) & \exists\, \{t_k\} \subset \mathbf{R}_+ \setminus \{0\},\ \{x^k\} \subset X \\ \in (\eta + M) \times Y & \text{such that } \{t_k\} \to 0,\ \{x^k\} \to 0, \\ & \operatorname{dist}\big(y + t_k d + \frac{1}{2} t_k^2 w^1 + \frac{1}{2} t_k^2 A x^k \\ & + \frac{1}{3!} t_k^3 w^2,\, Q\big) = o(t_k^3) \end{array} \right\} \tag{73}$$

and the set

$$T_Q^3(y,\, d;\, A) = T_Q^3(y,\, d;\, A;\, Y,\, \eta), \tag{74}$$

which does not depend on the specific choice of η. In particular,

$$T_Q^3(y,\, d) = T_Q^3(y,\, d;\, 0)$$

can be regarded as the usual (outer) third-order tangent set to the set Q at the point y in the direction d. Clearly, for any linear operator $A \in \mathcal{L}(X,\, Y)$

$$T_Q^3(y,\, d;\, A) \subset T_Q^2(y,\, d) \times Y, \tag{75}$$

and

$$T_Q^3(y,\, d;\, A;\ M,\, \eta) \subset T_Q^3(y,\, d;\, A) \tag{76}$$

for each linear subspace M in Y, and each $\eta \in Y$.

The following lemma will be used in the proof of Theorem 6 below.

Lemma 4 *For any $y \in Q$, $d \in Y$, any linear operator $A \in \mathcal{L}(X, Y)$, and any $\eta \in Y$, it holds that*

$$T_Q^3(y,\, d;\, A) + R_Q(y) \times R_Q(y) \subset T_Q^3(y,\, d;\, A).$$

Proof Take an arbitrary $(w^1,\, w^2) \in T_Q^3(y,\, d;\, A)$. According to (73), (74), there exist sequences $\{t_k\} \subset \mathbf{R}_+ \setminus \{0\}$, $\{x^k\} \subset X$, and $\{r^k\} \subset Y$, such that $\{t_k\} \to 0$, $\{x^k\} \to 0$, $r^k = o(t_k^3)$, and $\forall\, k$

$$y + t_k d + \frac{1}{2}t_k^2 w^1 + \frac{1}{2}t_k^2 Ax^k + \frac{1}{3!}t_k^3 w^2 + r^k \in Q. \tag{77}$$

Fix arbitrary $\theta_1 \geq 0$, $\theta_2 \geq 0$. From the classical inverse function theorem it follows that there exists a sequence $\{\tau_k\} \subset \mathbf{R}_+$ such that $\{\tau_k\} \to 0$, and for all k large enough

$$\frac{\tau_k}{1 - \frac{\theta_1}{2}\tau_k^2 - \frac{\theta_2}{3!}\tau_k^3} = t_k.$$

Clearly $\tau_k = t_k + o(t_k)$, and $t_k = \tau_k + o(\tau_k)$. For each k set $\alpha_k = 1 - \frac{\theta_1}{2}\tau_k^2 - \frac{\theta_2}{3!}\tau_k^3$, $\rho_k = r^k/\alpha_k$ (note that $\alpha_k = 1 + O(\tau_k^2)$ and $\rho_k = o(\tau_k^3)$). Then for arbitrary y^1, $y^2 \in Q$ we have

$$\begin{aligned}
&y + \tau_k d + \frac{1}{2}\tau_k^2(w^1 + \theta_1(y^1 - y)) \\
&\qquad\qquad + \frac{1}{2}\tau_k^2 Ax^k \\
&\quad + \frac{1}{3!}\tau_k^3(w^2 + \theta_2(y^2 - y)) + \rho^k = \left(1 - \frac{\theta_1}{2}\tau_k^2 - \frac{\theta_2}{3!}\tau_k^3\right) y + \tau_k d + \frac{1}{2}\tau_k^2 w^1 \\
&\qquad + \frac{1}{2}\tau_k^2 Ax^k + \frac{1}{3!}\tau_k^3 w^2 + \rho^k + \frac{\theta_1}{2}\tau_k^2 y^1 \\
&\qquad + \frac{\theta_2}{3!}\tau_k^3 y^2 \\
&= \left(1 - \frac{\theta_1}{2}\tau_k^2 - \frac{\theta_2}{3!}\tau_k^3\right)\left(y + \frac{\tau_k}{\alpha_k} d\right. \\
&\qquad + \frac{1}{2}\frac{\tau_k^2}{\alpha_k^2}\alpha_k w^1 + \frac{1}{2}\frac{\tau_k^2}{\alpha_k^2}\alpha_k Ax^k \\
&\qquad \left. + \frac{1}{3!}\frac{\tau_k^3}{\alpha_k^3}\alpha_k^2 w^2 + \frac{1}{\alpha_k}\rho^k\right) \\
&\qquad + \frac{\theta_1}{2}\tau_k^2 y^1 + \frac{\theta_2}{3!}\tau_k^3 y^2 \\
&= \left(1 - \frac{\theta_1}{2}\tau_k^2 - \frac{\theta_2}{3!}\tau_k^3\right)\left(y + t_k d + \frac{1}{2}t_k^2 w^1\right. \\
&\qquad \left. + \frac{1}{2}t_k^2 Ax^k + \frac{1}{3!}t_k^3 w^2 + r^k\right) \\
&\qquad + \frac{\theta_1}{2}\tau_k^2 y^1 + \frac{\theta_2}{3!}\tau_k^3 y^2 + O(\tau_k^4),
\end{aligned}$$

and because of (77) and convexity of Q, the sum of the first three terms in the right-hand side of the last relation belongs to Q for all k large enough. We thus proved that

$$\operatorname{dist}\left(y + \tau_k d + \frac{1}{2}\tau_k^2(w^1 + \theta_1(y^1 - y)) + \frac{1}{2}\tau_k^2 A x^k \right.$$
$$\left. + \frac{1}{3!}\tau_k^3(w^2 + \theta_2(y^2 - y)),\, Q\right) = o(\tau_k^3).$$

Therefore, according to (73), (74), $(w^1 + \theta_1(y^1 - y),\, w^2 + \theta_2(y^2 - y)) \in T_Q^3(y,\, d;\, A)$. □

For the sake of brevity, for each $h \in X$ put

$$\mathcal{T}^3(h) = T_Q^3(F(\bar{x}),\, F'(\bar{x})h;\, F'(\bar{x})), \tag{78}$$

i.e., according to (73), (74),

$$\mathcal{T}^3(h) = \left\{ \begin{array}{c|l} (w^1,\, w^2) & \exists\, \{t_k\} \subset \mathbf{R}_+ \setminus \{0\},\ \{x^k\} \subset X \text{ such that} \\ \in Y \times Y & \{t_k\} \to 0,\ \{x^k\} \to 0, \\ & \operatorname{dist}\big(F(\bar{x}) + t_k F'(\bar{x})h + \frac{1}{2}t_k^2 w^1 + \frac{1}{2}t_k^2 F'(\bar{x})x^k \\ & + \frac{1}{3!}t_k^3 w^2,\, Q\big) = o(t_k^3) \end{array} \right\}. \tag{79}$$

Define the set

$$\Xi_3(\bar{x},\, h) = \left\{ \xi \in X \;\middle|\; \begin{array}{l} \exists\, x \in X \text{ such that} \\ (F'(\bar{x})\xi + F''(\bar{x})[h,\, h],\, F'(\bar{x})x + 3F''(\bar{x})[h,\, \xi] \\ + F'''(\bar{x})[h,\, h,\, h]) \in \mathcal{T}^3(h) \end{array} \right\}. \tag{80}$$

In the next theorem we present the *primal* "second-order" necessary condition.

Theorem 5 *Let $\bar{x}$ be a local solution of problem* (1).

Then for any $h \in \bar{C}_2(\bar{x})$ it holds that

$$\langle f'(\bar{x}),\, \xi\rangle + f''(\bar{x})[h,\, h] \geq 0 \quad \forall\, \xi \in \Xi_3(\bar{x},\, h).$$

Proof For an arbitrary $h \in \bar{C}_2(\bar{x})$, fix $\xi \in \Xi_3(\bar{x},\, h)$, the element x, and the sequences $\{t_k\}$ and $\{x^k\}$ related to these h and ξ by (79) and by (80). For each k, set $h^k = h + \frac{1}{2}t_k\xi + \frac{1}{2}t_k x^k + \frac{1}{3!}t_k^2 x$. According to (62), Lemma 3 is applicable with $\tilde{h} = h^k$ and $t = t_k$ for k large enough, and we obtain the existence of a sequence $\{r^k\} \subset X$ such that

$$F\left(\bar{x} + t_k h + \frac{1}{2}t_k^2\xi + \frac{1}{2}t_k^2 x^k + \frac{1}{3!}t_k^3 x + r^k\right) = F(\bar{x} + t_k h^k + r^k) \in Q$$

for all k large enough, and

$$\begin{aligned}
\|r^k\| &= O(\mathrm{dist}(F(\bar{x}+t_kh^k),\, Q)/t_k) \\
&= O\left(\mathrm{dist}\left(F\left(\bar{x}+t_kh+\frac{1}{2}t_k^2\xi+\frac{1}{2}t_k^2x^k+\frac{1}{3!}t_k^3x\right),\, Q\right)/t_k\right) \\
&= O\left(\mathrm{dist}\left(F(\bar{x})+t_kF'(\bar{x})h+\frac{1}{2}t_k^2\left(F'(\bar{x})\xi+F''(\bar{x})[h,\,h]\right)\right.\right. \\
&\quad +\frac{1}{2}t_k^2F'(\bar{x})x^k \\
&\quad \left.\left.+\frac{1}{3!}t_k^3\left(F'(\bar{x})x+3F''(\bar{x})[h,\,\xi]+F'''(\bar{x})[h,\,h,\,h]\right),\, Q\right)/t_k\right)+o(t_k^2) \\
&= o(t_k^2),
\end{aligned}$$

where (79) and (80) were taken into account. Since $\bar{x}$ is a local solution of problem (1), we then obtain that for all k large enough

$$\begin{aligned}
0 &\leq f\left(\bar{x}+t_kh+\frac{1}{2}t_k^2\xi+\frac{1}{2}t_k^2x^k+\frac{1}{3!}t_k^3x+r^k\right)-f(\bar{x}) \\
&= f\left(\bar{x}+t_kh+\frac{1}{2}t_k^2\xi+o(t_k^2)\right)-f(\bar{x}) \\
&= \langle f'(\bar{x}),\, h\rangle t_k+\frac{1}{2}\left(\langle f'(\bar{x}),\, \xi\rangle+f''(\bar{x})[h,\,h]\right)t_k^2+o(t_k^2) \\
&\leq \frac{1}{2}\left(\langle f'(\bar{x}),\, \xi\rangle+f''(\bar{x})[h,\,h]\right)t_k^2+o(t_k^2),
\end{aligned}$$

where (61) and (62) are taken into account. It remains to divide the right- and the left-hand side by t_k^2, and to pass onto the limit as $k\to\infty$. $\square$

According to (75), (78), from (80) it follows that

$$\varXi_3(\bar{x},\, h)\subset\{\xi\in X\mid F'(\bar{x})\xi+F''(\bar{x})[h,\,h]\in T_Q^2(F(\bar{x}),\, F'(\bar{x})h)\}.$$

Moreover, if Robinson's CQ (3) is satisfied then this inclusion holds as an equality. More precisely, in this case, the equality

$$\mathcal{T}^3(h)=T_Q^2(F(\bar{x}),\, F'(\bar{x})h)\times Y$$

holds for each $h\in X$. Indeed, take an arbitrary $w^1\in T_Q^2(F(\bar{x}),\, F'(\bar{x})h)$. Then according to (55), (52), there exists a sequence $\{t_k\}\subset\mathbf{R}_+\setminus\{0\}$ such that $\{t_k\}\to 0$ and $\mathrm{dist}(F(\bar{x})+t_kF'(\bar{x})h+\frac{1}{2}t_k^2w^1,\, Q)=o(t_k^2)$. Since the linearized mapping $\xi\to F(\bar{x})+F'(\bar{x})\xi: X\to Y$ is metric regular at 0 with respect to Q, we obtain that for any $w^2\in Y$ and each k large enough, there exists $\xi^k\in X$ such that

$$F(\bar{x})+F'(\bar{x})(t_kh+\xi^k)\in Q-\frac{1}{2}t_k^2w^1-\frac{1}{3!}t_k^3w^2,$$

$$\|\xi^k\|=O\left(\mathrm{dist}\left(F(\bar{x})+t_kF'(\bar{x})h+\frac{1}{2}t_k^2w^1+\frac{1}{3!}t_k^3w^2,\, Q\right)\right)=o(t_k^2).$$

It remains to put $x^k = \xi^k/t_k^2$. With this choice,

$$F(\bar{x}) + t_k F'(\bar{x})h + \frac{1}{2}t_k^2 w^1 + \frac{1}{2}t_k^2 F'(\bar{x})x^k + \frac{1}{3!}t_k^3 w^2 \in Q$$

and $\{x^k\} \to 0$, and hence, according to (79), $(w^1,\, w^2) \in \mathcal{T}^3(h)$.

Taking into account the comments following Theorem 4, we now conclude that in the case of Robinson's CQ (3), Theorem 5 reduces to the well-known result (see, e.g., [13, Lemma 3.44]).

For each $h \in X$ put

$$\mathcal{T}^3(h;\, M) = T_Q^3(F(\bar{x}),\, F'(\bar{x})h;\, F'(\bar{x});\, M,\, F''(\bar{x})[h,\, h]), \tag{81}$$

i.e., according to (73),

$$\mathcal{T}^3(h;\, M) = \left\{ \begin{array}{l|l} (w^1,\, w^2) & \exists\, \{t_k\} \subset \mathbf{R}_+ \setminus \{0\},\ \{x^k\} \subset X \\ \in (F''(\bar{x})[h,\, h] + M) \times Y & \text{such that } \{t_k\} \to 0,\ \{x^k\} \to 0, \\ & \operatorname{dist}\left(F(\bar{x}) + t_k F'(\bar{x})h + \frac{1}{2}t_k^2 w^1 \right. \\ & \left. +\frac{1}{2}t_k^2 F'(\bar{x})x^k + \frac{1}{3!}t_k^3 w^2,\, Q\right) \\ & = o(t_k^3) \end{array} \right\} \tag{82}$$

Note that according to (76), (78) and (81), it holds that

$$\mathcal{T}^3(h;\, M) \subset \mathcal{T}^3(h). \tag{83}$$

We proceed with the *primal-dual* form of the "second-order" necessary condition.

Theorem 6 *Let $\bar{x}$ be a local solution of problem* (1).

Then for any closed linear subspace M in Y satisfying (10)*, any $h \in \bar{C}_2(\bar{x})$, and any convex set $\mathcal{T} \subset \mathcal{T}^3(h;\, M)$, there exist $\lambda^1 = \lambda^1(h;\, M) \in Y^*$ and $\lambda^2 = \lambda^2(h;\, M) \in Y^*$ such that* (70), (71) *hold, and*

$$\frac{\partial^2 L_2}{\partial x^2}\left(\bar{x},\, h,\, \lambda^1,\, \frac{1}{3}\lambda^2\right)[h,\, h] - \sigma((\lambda^1,\, \lambda^2),\, \mathcal{T}) \geq 0. \tag{84}$$

Proof If $\mathcal{T} = \emptyset$ then $\sigma((\lambda^1,\, \lambda^2),\, \mathcal{T}) = -\infty$ for each $\lambda^1,\, \lambda^2 \in Y^*$, and the assertion of this theorem follows trivially from Theorem 4. Throughout the rest of the proof we assume that $\mathcal{T} \neq \emptyset$.

Set $U = X \times X$, $V = Y \times Y$, and let $l \in U^*$ and the linear operator $A \in \mathcal{L}(U,\, V)$ be defined by $\langle l,\, u\rangle = \langle f'(\bar{x}),\, \xi\rangle$ and

$$Au = (F'(\bar{x})\xi,\, F'(\bar{x})x + 3F''(\bar{x})[h,\, \xi]) \tag{85}$$

respectively, $u = (x,\, \xi) \in U$ (compare (85) with (11)). Furthermore, set $a = f''(\bar{x})[h,\, h]$, $v = (F''(\bar{x})[h,\, h],\, F'''(\bar{x})[h,\, h,\, h])$. Finally, let $K = Q \times Q$, $\bar{v} = (F(\bar{x}),\, F(\bar{x}))$ and $T = \mathcal{T} + R_K(\bar{v})$, and let $W = M \times Y$.

With these definitions, (28) follows from (10) and Proposition 1 (see (14)), while (29) is automatic. Moreover, by the inclusion $\mathcal{T} \subset \mathcal{T}^3(h;\, M)$ and by (82),

$$\begin{aligned} T \cap (v + W) &= (\mathcal{T} + R_Q(F(\bar{x})) \times R_Q(F(\bar{x}))) \cap ((F''(\bar{x})[h,\, h] + M) \times Y) \\ &= \mathcal{T} + (R_Q(F(\bar{x})) \cap M) \times R_Q(F(\bar{x})). \end{aligned} \tag{86}$$

Since $\mathcal{T} \neq \emptyset$, we conclude that (33) holds, which, according to Remark 1, is equivalent to (30). Finally, by Lemma 4, by (80), (83), by the inclusion $\mathcal{T} \subset \mathcal{T}^3(h;\, M)$, and by Theorem 5, we obtain (31).

The needed result now readily follows from Lemma 2 and from (86). $\square$

Proposition 3 *Let $\bar{x}$ be a local solution of problem* (1)*, and assume that $R_Q(F(\bar{x}))$ is closed, and there exists a closed linear subspace M in Y satisfying* (10) *and such that the cone $(R_Q(F(\bar{x})))^\circ + M^\perp$ is weakly* closed.*

Then the assertion of Theorem 6 holds with this M, and with (68), (69) *instead of* (70), (71).

Proof Since $R_Q(F(\bar{x}))$ is closed and $(R_Q(F(\bar{x})))^\circ + M^\perp$ is weakly* closed, it holds that

$$\begin{aligned} N_{Q\cap(F(\bar{x})+M)}(F(\bar{x})) &= (R_Q(F(\bar{x})) \cap M)^\circ \\ &= (R_Q(F(\bar{x}))^\circ + M^\perp \\ &= N_Q(F(\bar{x})) + M^\perp. \end{aligned}$$

For any $h \in \bar{C}_2(\bar{x})$ and any closed convex set $\mathcal{T} \subset \mathcal{T}^3(h;\, M)$, choose λ^1 and λ^2 according to Theorem 6. Then according to (70), (71) and (87), there exist $\mu^1 \in N_Q(F(\bar{x}))$ and $\mu^2 \in M^\perp$ such that $\lambda^1 = \mu^1 + \mu^2$.

Take an arbitrary $(w^1,\, w^2) \in \mathcal{T}$, then according to (82), it holds that $w^1 \in F''(\bar{x})[h,\, h] + M$. Hence

$$\langle \lambda^1,\, F''(\bar{x})[h,\, h] - w^1 \rangle = \langle \mu^1,\, F''(\bar{x})[h,\, h] - w^1 \rangle,$$

and taking into account (10), it can be easily seen now that (68), (69) and (84) do hold with λ^1 replaced by μ^1. $\square$

Taking into account the discussion following Theorems 4 and 5, it can be easily seen that in the case of Robinson's CQ (3), Theorem 6 with $M = Y$ reduces to the standard second-order necessary optimality condition (see, e.g., [13, Theorem 3.45]). Specifically, in this case, the assertion of Theorem 6 takes the following form: for any $h \in \bar{C}(\bar{x})$, and any convex set $T \subset T_Q^2(F(\bar{x}),\, F'(\bar{x})h)$, there exists $\lambda = \lambda^1 = \lambda^1(h) \in Y^*$ such that (72) holds, and

$$\frac{\partial^2 L}{\partial x^2}(\bar{x},\, \lambda)[h,\, h] - \sigma(\lambda,\, T) \geq 0. \tag{87}$$

Furthermore, it is well-known that the so-called σ-term in (87) is always nonpositive (see [13, (3.109)]). The same can be proved for the σ-term in (84), at least under some additional assumptions.

Proposition 4 *For any closed linear subspace M in Y satisfying* (10)*, any $h \in \bar{C}_2(\bar{x})$, any convex set $\mathcal{T} \subset \mathcal{T}^3(h;\, M)$, and any $\lambda^1 \in Y^*$ and $\lambda^2 \in Y^*$ satisfying* (68), (69)*, it holds that*

$$\sigma((\lambda^1,\, \lambda^2),\, \mathcal{T}) \leq 0. \tag{88}$$

Proof Fix an arbitrary pair $(w^1, w^2) \in \mathcal{T}$. According to (10) and (82),

$$w^1 \in F''(\bar{x})[h, h] + M \subset F''(\bar{x})[h, h] + \operatorname{im} F'(\bar{x}) - R_Q(F(\bar{x})), \tag{89}$$

and there exist $\{t_k\} \subset \mathbf{R}_+ \setminus \{0\}$, $\{x^k\} \subset X$ and $\{\rho^k\} \subset Y$ such that $\{t_k\} \to 0$, $\{x^k\} \to 0$, $\rho^k = o(t_k^3)$, and $\forall\, k$

$$F(\bar{x}) + t_k F'(\bar{x})h + \frac{1}{2}t_k^2 w^1 + \frac{1}{2}t_k^2 F'(\bar{x})x^k + \frac{1}{3!}t_k^3 w^2 + \rho^k \in Q. \tag{90}$$

By the inclusion $h \in \bar{C}_2(\bar{x})$, from (53), (56), (61), (62), and from (68) and Theorem 2, it follows that

$$\langle \lambda^1,\, F'(\bar{x})h\rangle + \langle \lambda^2,\, F''(\bar{x})[h, h]\rangle = 0, \tag{91}$$

$$F'(\bar{x})h \in T_Q(F(\bar{x})), \quad F''(\bar{x})[h, h] \in \operatorname{cl}(T_Q(F(\bar{x})) - \operatorname{im} F'(\bar{x})). \tag{92}$$

From (68), (69), (91) and (92) we immediately obtain that

$$\langle \lambda^1,\, F'(\bar{x})h\rangle = \langle \lambda^2,\, F''(\bar{x})[h, h]\rangle = 0. \tag{93}$$

By the first inclusion in (69), and by (90) and (93), $\forall\, k$

$$\begin{aligned} 0 &\geq \left\langle \lambda^1,\, t_k F'(\bar{x})h + \frac{1}{2}t_k^2 w^1 + \frac{1}{2}t_k^2 F'(\bar{x})x^k + \frac{1}{3!}t_k^3 w^2 + \rho^k \right\rangle \\ &= \frac{1}{2}t_k^2 \langle \lambda^1,\, w^1\rangle + o(t_k^2), \end{aligned}$$

which implies the inequality

$$\langle \lambda^1,\, w^1\rangle \leq 0. \tag{94}$$

Similarly, by (68), (69), and by (89), (90) and (93), $\forall\, k$

$$\begin{aligned} 0 &\geq \left\langle \lambda^2,\, t_k F'(\bar{x})h + \frac{1}{2}t_k^2 w^1 + \frac{1}{2}t_k^2 F'(\bar{x})x^k + \frac{1}{3!}t_k^3 w^2 + \rho^k \right\rangle \\ &\geq \frac{1}{3!}t_k^3 \langle \lambda^2,\, w^2\rangle + o(t_k^3), \end{aligned}$$

which implies the inequality

$$\langle \lambda^2,\, w^2\rangle \leq 0. \tag{95}$$

Combining (94) and (95), we obtain the needed inequality (88). □

It is important to note, however, that the σ-term in (84) can be dropped in the case of polyhedral Q. Indeed, in this case, Q possesses the so-called *conicity* property at $F(\bar{x})$, that is, $T_Q(F(\bar{x})) = R_Q(F(\bar{x}))$, and by the first inclusion in (92) we obtain that $F'(\bar{x})h \in R_Q(F(\bar{x}))$. It can now be easily seen from (82) that the set $\mathcal{T}^3(h;\, M)$ contains $(0, 0)$, and one can apply Theorem 6 with, e.g., $\mathcal{T} = \{(0, 0)\}$.

We complete this section with the following observations. For $Q = \{0\}$ (the case of a purely equality-constrained problem) and $M = \operatorname{im} F'(\bar{x})$ (which subsumes that $\operatorname{im} F'(\bar{x})$ is closed), Theorems 1, 4 and 6 reduce to the results obtained in [6]. On the other hand, if Q is a cone and $\operatorname{int} Q \neq \emptyset$, Theorems 1 and 4 reduces to the results obtained in [16,18].

7 Illustrative examples

The first example illustrates the role of the additional multiplier λ^2 in Theorems 3 and 4, and demonstrates the use of Theorem 4 in order to classify a given feasible point with violated Robinson's CQ as a non-optimal one.

Example 1 Let $X = \mathbf{R}^3$, $Y = \mathbf{R}^2$, $f(x) = \langle l,\, x\rangle$, $l \in \mathbf{R}^3$, $F(x) = (x_1x_3,\, x_1^2 + x_2^2 - x_3^2)$, $Q = \{y \in \mathbf{R}^2 \mid y_1 = 0,\, y_2 \le 0\}$. The point $\bar{x} = 0$ is feasible in problem (1), and $F(\bar{x}) = 0$, $F'(\bar{x}) = 0$. Thus, Robinson's CQ (3) is violated, and the related standard necessary optimality conditions cannot be used in order to check if $\bar{x}$ is a local solution of problem (1).

Since Q is a polyhedral set in a finite-dimensional space, (54) is valid, and hence $T_Q^2(0,\, 0;\, 0) = T_Q(0) = Q$. With this equality at hand, one can easily obtain from (56) and (61) that

$$C_2(\bar{x}) = \{h \in \mathbf{R}^3 \mid h_1 = 0,\ h_2^2 \le h_3^2,\ l_2h_2 + l_3h_3 \le 0\}.$$

Furthermore, 2-regularity condition (6) takes the form

$$0 \in \operatorname{int}\left(F''(\bar{x})[h,\, X] - Q\right),$$

and for each $h \in C_2(\bar{x}) \setminus \{0\}$, the right-hand side of the latter relation equals the entire Y. Thus, by (62), $\bar{C}_2(\bar{x}) = C_2(\bar{x}) \setminus \{0\}$.

For any $h \in C_2(\bar{x})$, (68) and the first relation in (69) hold with all $\lambda^1 \in \mathbf{R}^2$ and $\lambda^2 \in \mathbf{R}^2$ satisfying the relations

$$\lambda_2^1 \ge 0, \quad l_1 + \lambda_1^2 h_3 = 0, \quad l_2 + 2\lambda_2^2 h_2 = 0, \quad l_3 - 2\lambda_2^2 h_3 = 0, \quad \lambda_2^2 \ge 0. \tag{96}$$

It can be easily seen that if $l_2 \ne 0$ or $l_3 \ne 0$ then one can choose $h \in C_2(\bar{x}) \setminus \{0\}$ in such a way that λ_2^2 satisfying the last two equalities in (96) does not exist. By Theorem 4 we conclude that in this case, $\bar{x}$ cannot be a local solution of problem (1).

At the same time, if $l_2 = l_3 = 0$ then (96) holds for each $h \in C_2(\bar{x}) \setminus \{0\}$ with $\lambda_1^2 = -l_1/h_3$ and $\lambda_2^2 = 0$, and the resulting λ^2 satisfies the second inclusion in (69). It can be seen that in this case, $\bar{x}$ is indeed a solution of problem (1). Thus, Theorem 4 completely characterizes optimality in this example, but of course, this will not necessarily remain true if higher-order terms will be added to f and/or F (see Example 2 below).

Our next example demonstrates the situation when Theorem 4 is not sharp enough to classify a feasible point as a non-optimal one, while Theorem 6 does the job.

Example 2 Let $X = \mathbf{R}^4$, $Y = \mathbf{R}^3$, $f(x) = x_1$, $F(x) = (x_1x_3 + x_3^3,\, x_1^2 + x_2^2 - x_3^2,\, x_1^2 - x_4^2)$, $Q = \{y \in \mathbf{R}^2 \mid y_1 = y_2 = 0,\, y_3 \le 0\}$. The point $\bar{x} = 0$ is feasible in problem (1), $F(\bar{x}) = 0$, $F'(\bar{x}) = 0$, and Robinson's CQ (3) is violated.

By the same argument as in Example 1, we obtain

$$C_2(\bar{x}) = \{h \in \mathbf{R}^4 \mid h_1 = 0,\ h_2^2 = h_3^2\},$$

and that $\bar{C}_2(\bar{x}) = \{h \in C_2(\bar{x}) \mid h_2 \ne 0,\ h_4 \ne 0\}$.

For any $h \in \bar{C}_2(\bar{x})$, (68), (69) hold with all $\lambda^1 \in \mathbf{R}^3$ and $\lambda^2 \in \mathbf{R}^3$ satisfying the relations

$$\lambda_3^1 \geq 0, \quad 1 + \lambda_1^2 h_3 = 0, \quad \lambda_2^2 h_2 = 0, \quad \lambda_2^2 h_3 = 0, \quad \lambda_3^2 h_4 = 0, \quad \lambda_3^2 \geq 0. \tag{97}$$

Thus the "first-order" necessary conditions of Theorem 4 are satisfied at $\bar{x}$. At the same time, take $h = (0,\, 1,\, 1,\, 1) \in \bar{C}_2(\bar{x})$. For this h, (97) implies that $\lambda^2 = (-1,\, 0,\, 0)$, and hence

$$\frac{\partial^2 L_2}{\partial x^2}\left(\bar{x},\, h,\, \lambda^1,\, \frac{1}{3}\lambda^2\right)[h,\, h] = -2\lambda_3^1 + 2\lambda_1^2 < 0$$

for all $\lambda^1 \in \mathbf{R}^3$ and $\lambda^2 \in \mathbf{R}^3$ satisfying (97). It remains to take into account that in the case of a polyhedral Q, the σ-term in (84) can be dropped. Thus, by Theorem 6 we conclude that $\bar{x}$ cannot be a local solution of problem (1).

Note that if we replace the inequality constraint in this example by equality constraint (that is, replace Q above by $Q = \{0\}$) then

$$C_2(\bar{x}) = \{h \in \mathbf{R}^4 \mid h_1 = h_4 = 0,\ h_2^2 = h_3^2\},$$

and $\bar{C}_2(\bar{x}) = \emptyset$. Thus, the results developed earlier for problems with 2-regular equality constraints cannot be applied.

Our last example illustrates the role of the σ-term in (84).

Example 3 Let $X = \mathbf{R}^5$, $Y = \mathbf{R}^4$, $f(x) = x_2 + x_3 - x_1^2$,

$$F(x) = (x_1,\, x_2,\, x_3 x_5,\, x_3^2 + x_4^2 - x_5^2),$$

$Q = \{y \in \mathbf{R}^4 \mid y_2 \geq a y_1^2,\, y_3 = 0,\, y_4 \leq 0\}$, with $a \geq 1$ playing the role of a parameter. It can be easily seen that the point $\bar{x} = 0$ is a solution of problem (1). Furthermore, $F(\bar{x}) = 0$, $\operatorname{im} F'(\bar{x}) = \{y \in \mathbf{R}^4 \mid y_3 = y_4 = 0\}$, $\operatorname{im} F'(\bar{x}) - Q = \{y \in \mathbf{R}^4 \mid y_3 = 0,\, y_4 \geq 0\}$, and Robinson's CQ (3) is violated.

From (55), (56), (61), and (62), it can be easily derived that

$$C_2(\bar{x}) = \{h \in \mathbf{R}^5 \mid h_2 = h_3 = 0,\ h_4^2 - h_5^2 \leq 0\} \tag{98}$$

and furthermore, $\bar{C}_2(\bar{x}) = \{h \in C_2(\bar{x}) \mid h_5 \neq 0\}$. (Note that a nontrivial sequence $\{x^k\}$ must be taken in (55) in order to show that the left-hand side of (98) is contained in $C_2(\bar{x})$.)

Take, e.g., $h = (1,\, 0,\, 0,\, 1,\, 1) \in \bar{C}_2(\bar{x})$. Note that

$$(R_Q(F(\bar{x})))^\circ + (\operatorname{im} F'(\bar{x}))^\perp = \{y \in \mathbf{R}^4 \mid y_1 = 0,\, y_2 \leq 0\}$$

is a closed set, and hence, as discussed above, (68), (69) can be used instead of (70), (71), and for the given h, (68), (69) hold with $\lambda^1 = (0,\, -1,\, \lambda_3^1,\, \lambda_4^1)$ and $\lambda^2 = (0,\, 0,\, -1,\, 0)$ for any λ_3^1 and λ_4^1. Furthermore, for any choice of λ_3^1 and λ_4^1, it holds that

$$\frac{\partial^2 L_2}{\partial x^2}\left(\bar{x},\, h,\, \lambda^1,\, \frac{1}{3}\lambda^2\right)[h,\, h] = -2, \tag{99}$$

and hence, (84) would not hold with the σ-term being dropped. At the same time, from (82) it easily follows that

$$\mathcal{T}^3(h;\, M) = \left\{ \begin{array}{l} w = (w^1,\, w^2) \\ \in \mathbf{R}^4 \times \mathbf{R}^4 \end{array} \,\middle|\, \begin{array}{l} w_2^1 \geq 2a, \\ w_3^1 = w_4^1 = 0, \\ w_3^2 = 0,\ w_4^2 \leq 0 \end{array} \right\},$$

which is a convex set, and for $\mathcal{T} = \mathcal{T}^3(h;\, M)$ it holds that

$$\sigma((\lambda^1,\, \lambda^2),\, \mathcal{T}) = \sup\{-w_2^1 \mid w_2^1 \geq 2a\} = -2a \leq -2.$$

From (99) it now follows that (84) is satisfied.

We complete this section with the following observation: in each of the examples above, one can add to F any terms of order greater than 3. This would change none of our conclusions but would make these conclusions even less evident and more difficult to reach by different known tools.

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