Generalized Nash games and equilibrium problems

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Abstract

We reformulate the generalized Nash equilibrium problem as an equilibrium problem so that solving the former problem is reduced to solving the latter problem. We use Ky Fan's Lemma to obtain a new existence result for equilibrium problems, consequently for the generalized Nash equilibrium problems, which does not invoke monotonicity and convexity of the objective function.

Keywords: Equilibrium problem, Generalized Nash equilibrium problem, Ky Fan's Lemma.

1 Introduction

The standard definition of a noncooperative game in normal form, usually requires that each player has a feasible set that is independent of the rival's strategies. In this game, there are N players and for player i it is considered the set $K_i \subset \mathbb{R}^{n_i}$ called the strategy set of player i-th and the function $\theta_i : \prod_i^N K_i \to \mathbb{R}$ called loss function with respect to i-th player. However, it was well understood from the early developments in the field, see e.g. [1], [19] and [20], that in many cases the interaction between the players can also take place at the feasible set level. If one assumes that each player's feasible set can depend on the rival players' strategies we speak of generalized Nash equilibrium problems (GNEP in the sequel). Now we comment on GNEP.

The set of player's is denoted by $I = \{1, 2, ..., N\}$ and for each $i \in I$, $x^i \in \mathbb{R}^{n_i}$ denotes a strategy of the player i-th. It is assumed that $n = \sum_{i \in I} n_i$ and K is the feasible strategy set of the game. Let $\Lambda = \prod_{j \in I, j \neq i} \mathbb{R}^{n_j}$ and $x \in K$, we define $x^i = P_{\mathbb{R}^{n_i}}(x)$ and $x^{-i} = P_\Lambda(x)$ where $P_C(x)$ denotes the orthogonal projection of x on the set C. The set $K(x^{-i})$ denotes the strategy set of the player i-th

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when the other players choose their strategy x^{-i}, therefore, the feasible set for this game is defined as

$$K = \{x \in \mathbb{R}^n : x^i \in K(x^{-i})\}. \tag{1}$$

Now choose $i \in I$ and $\rho \in \mathbb{R}^{n_i}$, we define $x(\rho, i) \in \mathbb{R}^n$ as $(x(\rho, i))^i = \rho$ and $(x(\rho, i))^{-i} = x^{-i}$. Using the above notation, we state the formal definition of GNEP as follows:

GNEP is a Nash game in which each player's strategy depends on the other players' strategies, and it consists of finding an $\bar{x} \in K$ such that, for each $i \in I$, $\bar{x}^i$ solves the minimization problem defined as

$$\min \theta_i(\bar{x}(\rho, i)) \quad \text{subject to} \quad \rho \in K(\bar{x}^{-i}) \subset \Omega_i, \tag{2}$$

where Ω_i is an open convex set, and that $\theta_i : \prod_{j \in I} \Omega_j \to \mathbb{R}$. Throughout this paper we assume that the function $\theta_i(x(\cdot, i)) : \Omega_i \to \mathbb{R}$ is lower semicontinuous and pseudoconvex (see, Definition 2.2 in the following section, or Definition 1 of [14]) satisfying

$$\inf_{\rho \in \Omega_i} \theta_i(x(\rho, i)) < \inf_{\sigma \in K(x^{-i})} \theta_i(x(\sigma, i)). \tag{3}$$

For a survey paper on GNEP, we refer the interested readers to [10].

The outline of this paper is the following. In Section 2 we reformulate GNEP as an equilibrium problem. In Section 3 we prove the existence of solutions for GNEP, using Ky Fan's Lemma, by equilibrium problem approaches. In Section 4 we show that solving GNEP can be reduced to solving equilibrium problem to find solutions of GNEP. We finish this paper with some remarks in Section 5.

2 Reformulation of GNEP

The *Equilibrium Problem* considered in the current paper is defined as the following.

Definition 2.1. *By an equilibrium problem (in short* $\mathrm{EP}(\bar{f}, \bar{K})$*, for the sequel) we understand the problem of finding*

$$x^* \in \bar{K} \quad \text{such that} \quad \bar{f}(x^*, y) \geq 0 \quad \forall y \in \bar{K},$$

where $\bar{K}$ *is a nonempty closed subset of* $\mathbb{R}^m$ *and* $\bar{f} : \bar{K} \times \bar{K} \to \mathbb{R}$ *is a given function. The set of solutions of* $\mathrm{EP}(\bar{f}, \bar{K})$ *will be denoted by* $S(\bar{f}, \bar{K})$*.*

Definition 2.2. *Given* $D \subset \mathbb{R}^m$*, a function* $h : D \to \mathbb{R}$ *is said to be pseudoconvex if for all* $x, y \in D$ *and all* $t \in (0, 1)$ *it holds that*

$$h(z_t) \geq h(x) \quad \Rightarrow \quad h(y) \geq h(z_t),$$

where $z_t = tx + (1-t)y$.

Proposition 2.3. *Consider D and h as in Definition 2.2. Assume that h is lower semicontinuous. Take $\lambda \in \mathbb{R}$ with $\lambda > \inf_{z \in D} h(z)$, in this situation, it holds that $L_h(\lambda) = \overline{L_h^<(\lambda)}$ where $L_h(\lambda) = \{y \in D : h(y) \leq \lambda\}$ and $L_h^<(\lambda) = \{y \in D : h(y) < \lambda\}$.*

Proof. By definition we have that $L_h^<(\lambda) \subset L_h(\lambda)$, consequently $\overline{L_h^<(\lambda)} \subset \overline{L_h(\lambda)} = L_h(\lambda)$ since h is lower semicontinuous. We start proving the converse inclusion. Take $y \in L_h(\lambda)$ so that $h(y) \leq \lambda$. Without loss of generality we can assume that $h(y) = \lambda$ (otherwise $y \in L_h^<(\lambda)$ and therefore, $y \in \overline{L_h^<(\lambda)}$), we can choose $x \in D$ with $\lambda = h(y) > h(x)$, we then put $z_t = tx + (1-t)y$ for each $t \in (0,1)$. Assume that $h(z_{\bar{t}}) \geq h(y)$ for some $\bar{t} \in (0,1)$, then pseudoconvexity of h implies that $h(x) \geq h(z_{\bar{t}})$, as a result of this fact, $h(y) > h(x) \geq h(z_{\bar{t}}) \geq h(y)$ which is a contradiction with our assumption, so we must have $h(z_t) < h(y)$ for all $t \in (0,1)$. The last inequality shows that $z_t \in L_h^<(\lambda) \subset \overline{L_h^<(\lambda)}$ for each $t \in (0,1)$, which in turn implies, $y \in \overline{L_h^<(\lambda)}$ when $t \to 0^+$. Consequently, the demonstration is completed. □

For each $i \in I$ and for each $x \in K$, take $\rho \in K(x^{-i})$, from (3) introduced in the previous section we have that $\theta_i(x(\rho,i)) > \inf_{\sigma \in \Omega_i} \theta_i(x(\sigma,i))$. Now, we denote the sublevel set of the function $\theta_i(x(\cdot,i))$ at the level $\theta_i(x(\rho,i))$ by $L_{\theta_i(x)}(\rho) = \{\sigma \in K(x^{-i}) : \theta_i(x(\sigma,i)) \leq \theta_i(x(\rho,i))\}$, it follows from Proposition 2.3 that $L_{\theta_i(x)}(\rho) = \overline{L^<_{\theta_i(x)}(\rho)}$ where $L^<_{\theta_i}(\rho) = \{\sigma \in K(x^{-i}) : \theta_i(x(\sigma,i)) < \theta_i(x(\rho,i))\}$. It is worthwhile mentioning that for each $\rho \in K(x^{-i})$, the sublevel set $L_{\theta_i(x)}(\rho)$ is a closed convex set and that ρ belongs to its boundary, therefore, the normal cone associated to $L_{\theta_i(x)}(\rho)$ at ρ, denoted by $N_{L_{\theta_i(x)}}(\rho)$, defined as $N_{L_{\theta_i(x)}} : K(x^{-i}) \to \mathcal{P}(R^{n_i})$ and

$$N_{L_{\theta_i(x)}}(\rho) = \{v \in \mathbb{R}^{n_i} : \langle v, \sigma - \rho \rangle \leq 0, \forall \sigma \in L_{\theta_i(x)}\},$$

is a closed, convex, and pointed cone (i.e., $N_{L_{\theta_i(x)}}(\rho) \cap -N_{L_{\theta_i(x)}}(\rho) = \{0\}$) in $\mathcal{P}(R^{n_i})$. It is known that $N_{L_{\theta_i(x)}}$ is a closed operator on $K(x^{-i})$ since $N_{L_{\theta_i(x)}}(\rho) = N_{L^<_{\theta_i(x)}}(\rho)$ for each $\rho \in K(x^{-i})$ (see, Proposition 2.1 in [8]). Now we associate the compact and convex subset $D_i(\rho) = co(N_{L_{\theta_i}}(\rho) \cap S(0,1))$ of $\mathbb{R}^{n_i}$ to $\rho \in K(x^{-i})$ where $S(0,1) = \{u \in \mathbb{R}^{n_i} : \|u\| = 1\}$, it is easy to see that $D_i(\rho)$ is a base for $N_{L_{\theta_i(x)}}(\rho)$ in the sense that $N_{L_{\theta_i(x)}}(\rho) = \cup_{t \geq 0} t D_i(\rho)$ (see, [3]). According to above discussion and notation, we have the following lemma which is an easy consequence of Proposition 2.1 of [8].

Lemma 2.4. *The mapping $T : K \to \mathcal{P}(\mathbb{R}^n)$ defined by $T(x) = \prod_{i=1}^{N} D_i(x^i)$ is a closed operator.*

Now we sate our reformulation. For this purpose we associate the equilibrium problem $\mathrm{EP}(f,K)$ to GNEP where the set K given by (1) and the objective function is defined as

$$f(x,y) = \sup_{u \in T(x)} \langle u, y - x \rangle. \tag{4}$$

It is clear enough that f is well defined because $T(x)$ is compact for each $x \in K$.

3 Existence of equilibria

In this section we consider our existence result for EP(f, K). We will see that our existence result for GNEP is weaker than the one presented in [2] demanding compactness of the set K as well as the continuity of the function $\theta_i(x(\cdot, i))$ for each $i \in I$.

The following Lemma as well as the following three conditions are fundamental in order to guarantee the existence of solutions for EP(f, K).

Lemma 3.1. *(Ky Fan's Lemma) Let Y be a nonempty subset of a real Hausdorff topological vector space X. For each $y \in Y$, consider a closed subset $C(y)$ of X. If the following two conditions hold:*

C1. the convex hull of any finite subset $\{x^1, ..., x^q\}$ of Y, denoted as $co\{x^1, ..., x^q\}$, is contained in $\bigcup_{i=1}^{q} C(x^i)$,

C2. $C(x)$ is compact for at least some $x \in Y$,

then $\bigcap_{y \in Y} C(y) \neq \emptyset$.

A: For any sequence $\{x^j\} \subseteq \bar{K}$ satisfying $\lim_{j\to\infty} \|x^j\| = +\infty$, there exists $j_0 \in \mathbb{N}$ such that for all$j \geq j_0$ it holds that

$$\bar{f}(x^j, y) \geq 0 \quad \forall y \in \bar{K} \setminus B(0, j).$$

B: $C(y) = \{x \in \bar{K} : \bar{f}(x, y) \geq 0\}$ is closed for all $y \in \bar{K}$.

C: For any finite subset $\{x^1, ..., x^q\}$ of $\bar{K}$, it holds that

$$\max_{i=1,\ldots,q} \bar{f}(x, x^i) \geq 0 \ \ \forall x \in co\{x^1, ..., x^q\}.$$

Theorem 3.2. *Consider* EP(f, K) *satisfying the condition A. In this situation, we have that $S(f, K) \neq \emptyset$.*

Proof. We first prove that the condition B is met for EP(f, K) with K given by (1) and f given by (4), i.e., we show that $C(y)$ is closed for an arbitrary $y \in K$. Assume that $\{z^j\}_{j=1}^{\infty} \subset C(y)$ so that $z^j \to z$, there exists $v^j \in T(z^j)$ satisfying $f(z^j, y) = \sup_{v \in T(z^j)} \langle v, y - z^j \rangle = \langle v^j, y - z^j \rangle \geq 0$ since $T(z^j)$ is compact for each j. On the other hand $\| v^j \| \leq 1$ for each j, therefore, without loss of generality we can assume that $v^j \to \bar{v}$, hence

$$f(z, y) = \sup_{v \in T(z)} \langle v, y - z \rangle \geq \langle \bar{v}, y - z \rangle = \lim_{j \to +\infty} \langle v^j, y - z^j \rangle \geq 0,$$

where the leftmost inequality follows from the fact that $\bar{v} \in T(z)$, using Lemma 2.4, consequently the condition B is satisfied.

Choose an arbitrary $\{x^1, ..., x^q\} \subset Y = K$, we show that the condition C1 in Lemma 3.1 is satisfied for the family of the closed sets $\{C(y)\}_{y \in K}$ where $C(y) = \{x \in K : f(x, y) \geq 0\}$. In other words, we demonstrate that $x \in \bigcup_{i=1}^{q} C(x^i)$ for each $x \in co\{x^1, ..., x^q\}$. Write $x = \sum_{i=1}^{q} \lambda_i x^i$ where $\lambda_i \geq 0$ and $\lambda = \sum_{i=1}^{q} \lambda_i$, without loss of generality we assume that $\lambda_i > 0$ for $i = 1, ..., q$. We have that $x \notin \bigcup_{i=1}^{q} C(x^i)$ if and only if $f(x, x^i) < 0$ for $i = 1, ..., q$, consequently

$$\sum_{i=1}^{q} \lambda_i f(x, x^i) = \sum_{i=1}^{q} \lambda_i \sup_{u^i \in T(x)} \langle u^i, x^i - x \rangle = \sum_{i=1}^{q} \sup_{u^i \in T(x)} \langle u^i, \lambda_i x^i - \lambda_i x \rangle < 0,$$

which is a contradiction with the fact that

$$0 = \sup_{u \in T(x)} \langle u, \sum_{i=1}^{q} \lambda_i x^i - x \rangle = \sup_{u \in T(x)} \sum_{i=1}^{q} \langle u, \lambda_i x^i - \lambda_i x \rangle \leq \sum_{i=1}^{q} \sup_{u^i \in T(x)} \langle u^i, \lambda_i x^i - \lambda_i x \rangle,$$

hence, the condition C1 in Lemma 3.1 holds for the family of the closed sets $\{C(y)\}_{y \in K}$.

Now for each $j \in \mathbb{N}$ we define the following two compact sets as

$$K_j = \{x \in K : \| x \| \leq j\},$$

and

$$C^j(y) = K_j \cap C(y).$$

Without loss of generality we can assume that $K_j \neq \emptyset$ for all $j \in \mathbb{N}$. Using Lemma 3.1 we have that

$$S(f, K_j) = \bigcap_{y \in K_j} C^j(y) \neq \emptyset \quad \forall j \in \mathbb{N}.$$

Take $x^j \in S(f, K_j)$ for each $j \in \mathbb{N}$. If $\{x^j\}_{j=1}^{\infty}$ has a bounded subsequence, then $\{x^j\}_{j=1}^{\infty}$ has cluster points which solve $\mathrm{EP}(f, K)$. Otherwise, we have that $\lim_{j\to\infty} \|x^j\| = +\infty$, in which case, we can choose a sequence $\{y^j\}_{j=1}^{\infty} \subset K$ such that

$$f(x^j, y^j) < 0 \quad \text{with} \quad y^j \in K \setminus B(0, j) \quad \forall j \in \mathbb{N},$$

this contradicts with condition A, and therefore, we conclude that $S(f, K) \neq \emptyset$. $\square$

We proved Theorem 3.2 for a particular class of equilibrium problems $\mathrm{EP}(\bar{f}, \bar{K})$, i.e, with $\bar{K} = K$ as defined in (1) and $\bar{f} = f$ as defined in (4). We observed that two conditions B and C hold for $\mathrm{EP}(f, K)$ by the nature of this problem. Furthermore, the condition A is satisfied for any $\mathrm{EP}(f, K)$ provided that K is bounded, for instance. In general, one can argue the same as the one presented in Theorem 3.2 to verify that $\mathrm{EP}(\bar{f}, \bar{K})$ attains solutions whenever $\mathrm{EP}(\bar{f}, \bar{K})$ satisfies the conditions A, B and C.

4 Link between GNEP and EP(f, K)

The main idea of this section is to show that instead of solving GNEP one can solve $\mathrm{EP}(f, K)$ with K defined as (1) and f defined as (4).

Theorem 4.1. *Assume that K is convex, then each solution for* $\mathrm{EP}(f, K)$ *solves GNEP.*

Proof. Take $\bar{x} \in S(f, K)$, by definition we have that $f(\bar{x}, y) = \sup_{u \in T(\bar{x})} \langle u, y - \bar{x} \rangle \geq 0$ for each $y \in K$. From Lemma 1 of [7], we know that there exists $\bar{u} \in T(\bar{x})$ such that $f(\bar{x}, y) = \langle \bar{u}, y - \bar{x} \rangle \geq 0$ $\forall y \in K$. Fix $i \in I$ and then choose an arbitrary $\rho \in K(\bar{x}^{-i})$. We define $y = \bar{x}(\rho, i) \in K$ and consequently we get $\langle \bar{u}, y - \bar{x} \rangle = \langle \bar{u}^i, \rho - \bar{x}^i \rangle \geq 0$. Since ρ was arbitrary in $K(x^{-i})$, $\bar{u}^i \neq 0$ and $\bar{u}^i$ belongs to the normal cone of the level set $L_{\theta_i}(\bar{x}^i)$, we conclude from [3] (or Proposition 3.2 in [4]) that $\bar{x}^i$ minimizes $\theta_i(\bar{x}(\cdot, i))$ over $K(\bar{x}^{-i})$. Since i was arbitrary in I, $\bar{x}$ is a solution of GNEP. $\square$

It is notable that in Theorem 4.1 the link between $S(f, K)$ and the solution set of GNEP is valid even if the constraint set K is not compact. In addition, Theorem 4.1 remains true whenever θ_i is lower semicontinuous while Theorem 2.1 in [11] requests compactness of K and continuously differentiability of θ_i. Another theorem similar to Theorem 4.1 has been recently contributed to literature (see Theorem 3.1 in [2]) demanding continuity of θ_i where GNEP is reformulated as a variational inequality problem.

5 Final remarks

In view of equilibrium problems, Theorem 3.2 is a new existence result for equilibrium problems which does not invoke monotonicity condition and convexity condition on objective function (see, [14] and [18]) while it requests convexity of the feasible set.

Instead of variational inequality techniques investigated in [2] and [11] for GNEP, in this work we used equilibrium problem techniques for the same problem which contains variational inequality problems as its particular case [17]. Theorem 3.2 is a new theorem that guarantees the existence of solutions for GNEP. Nevertheless, on can utilize existence results which have already been contributed to literature (see, e.g., [5], [6], [9], [12], [13], [14] and [18]) to guarantee the existence of solutions for GNEP, this issue has been studied for NEP (Nash equilibrium problem in noncooperative games) where it is shown that $S(f, K)$ coincides with the solution set of NEP (see, e.g, [17]), in other words, the necessary condition in Theorem 4.1 is a sufficient condition in this case.

In view of computational methods for finding the solutions of GNEP in practice, taking into account Theorem 4.1, one can utilize the algorithms proposed in [15], [16] and [17] to find solutions of GNEP.

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