STABILITY AND DECAY PROPERTIES OF SOLITARY WAVE SOLUTIONS FOR THE GENERALIZED BO-ZK EQUATION

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ABSTRACT. In this paper we study the generalized BO-ZK equation in two dimensions. We classify the existence and non-existence of solitary waves depending on the sign of the dispersions and on the nonlinearity. By using the approach introduced by Cazenave and Lions we study the nonlinear stability of solitary waves. We also prove some decay and regularity properties of such waves.

1. INTRODUCTION

This paper is concerned with (non)existence, stability and some decay properties of solitary wave solutions for the two-dimensional generalized Benjamin-Ono-Zakharov-Kuznetsov equation (BO-ZK henceforth),

$$u_t + u^p u_x + \alpha \mathscr{H} u_{xx} + \varepsilon u_{xyy} = 0, \quad (x, y) \in \mathbb{R}^2, \ t \in \mathbb{R}^+. \tag{1.1}$$

Here $p > 0$ is a real constant, the constant ε measures the transverse dispersion effects and is normalized to ± 1, the constant α is a real parameter and $\mathscr{H}$ is the Hilbert transform defined by

$$\mathscr{H} u(x, y, t) = \text{p.v.} \frac{1}{\pi} \int_{\mathbb{R}} \frac{u(z, y, t)}{x - z} \, dz,$$

where p.v. denotes the Cauchy principal value. When $p = 1$, the equation (1.1) appears in electromigration and the interaction of the nanoconductor with the surrounding medium [21, 26], by considering Benjamin-Ono dispersive term with the anisotropic effects included via weak dispersion of ZK-type. In fact, the equation (1.1) is a generalization of the one-dimensional Benjamin-Ono equation (see also [15]).

Now a days, several physical situations in two dimensions are described by generalizations of well-known one-dimensional equations. The most known and studied are the KP and ZK equations, which are generalizations of the KdV equation. As far as we know, equation (1.1) was recently derived in [26], where from the physical viewpoint existence of solitary waves was studied.

The generalized Benjamin-Ono equation,

$$u_t + u^p u_x + \alpha \mathscr{H} u_{xx} = 0, \quad x \in \mathbb{R}, \ t \in \mathbb{R}^+$$

has been studied by several authors considering both the initial value problem and the nonlinear stability. The initial value problem has been studied, recently, for instance in [8, 23, 24, 31, 35, 37], whereas the issue of existence and stability of solitary waves has been studied in [1]–[5]. On the other hand, the Zakharov-Kuznetsov equation

$$u_t + u^p u_x + \alpha u_{xxx} + \varepsilon u_{xyy} = 0, \quad (x,y) \in \mathbb{R}^2, \ t \in \mathbb{R}^+,$$

has been less studied. Indeed, as far as we know the only results concerning the existence and nonlinear stability of solitary waves was given in [12] and well-posedness (for $p = 1$) was studied in [19].

It can be seen that the flow associated to (1.1) satisfies the conservation quantities $\mathscr{F}$ and E, where

$$\mathscr{F}(u) = \frac{1}{2} \int_{\mathbb{R}^2} u^2 \, dxdy$$

and

$$E(u) = \frac{1}{2} \int_{\mathbb{R}^2} \left(\varepsilon u_y^2 - \alpha u \mathscr{H} u_x - \frac{2}{(p+1)(p+2)} \, u^{p+2} \right) \, dxdy.$$

In the present paper we will investigate the existence of solitary wave solutions of (1.1) and some of their properties. As it was pointed out in [26] there are no exact solitary waves to (1.1).

In order to describe our results, the space $\mathscr{Z}$ shall denote the closure of $C_0^\infty(\mathbb{R}^2)$ for the norm

$$\|\varphi\|_{\mathscr{Z}}^2 = \|\varphi\|_{L^2(\mathbb{R}^2)}^2 + \|\varphi_y\|_{L^2(\mathbb{R}^2)}^2 + \left\| D_x^{1/2} \varphi \right\|_{L^2(\mathbb{R}^2)}^2, \tag{1.2}$$

where $D_x^{1/2} \varphi$ denotes the fractional derivative of order $1/2$ with respect to x, defined via Fourier transform by $\widehat{D_x^{1/2} \varphi}(\xi_1, \xi_2) = |\xi_1|^{1/2} \widehat{\varphi}$.

The solitary waves we are interested in are of the form $u = \varphi(x - ct, y)$, where $u \in \mathscr{Z}$ and $c \neq 0$ is the wave speed; so, substituting this form of u in (1.1) and integrating once, we see that φ must satisfy

$$-c\varphi + \frac{1}{p+1} \varphi^{p+1} + \alpha \mathscr{H} \varphi_x + \varepsilon \varphi_{yy} = 0. \tag{1.3}$$

REMARK 1.1. *Note that we can assume that $|c| = 1$, since the scale change*

$$\psi(x, y) = |c|^{-1/p} \varphi\left(\frac{x}{|c|}, \frac{y}{\sqrt{|c|}} \right),$$

transforms (1.3) in φ, into the same in ψ, but with $|c| = 1$.

We begin our results classifying where solitary waves do not exist. We use Pohojaev type identities to prove that depending on p and on the signs of ε and α, solitary waves do not exist (see Theorem 2.1). Furthermore, we prove the existence of solitary wave solutions in some of the remaining cases. Our strategy is to consider a suitable minimization problem and use the concentration-compactness principle of Lions [27, 28] (see Theorem 2.2). To fix ideas, we prove for instance that for $c > 0$, $\alpha < 0$, $\varepsilon > 0$ and $0 < p < 4$, solitary waves do exist. Moreover, we prove that such solitary waves also are ground states (see Theorem 2.7). These results are similar to the ones given for KP equation in [13] and [14].

With the solitary waves at hand, the natural question arising is when such waves are or not orbitally stable. In this regard, by using the variational approach introduced by Cazenave and Lions [10] we show that for $\alpha\varepsilon < 0$, $c\alpha < 0$ and $0 < p < 4/3$ the solitary waves are stable. We note that in this case the value $p = 4/3$ is critical (as for the KP equation) in the sense that solitary waves are stable for $0 < p < 4/3$ and unstable for $4/3 < p < 4$. This last question has been addressed in [16].

At last, we prove some decay and regularity properties of the solitary waves. We show that such waves are positive, analytic and symmetric with respect to the transverse and propagation directions. Moreover, we prove that solitary waves decay exponentially in the transverse direction and algebraically in the propagation direction. We point out that Theorem 2.7 and Corollary 4.22 are very useful to prove our instability results in [16].

Concerning well-posedness results, we note that by the parabolic regularization theory, one can show that the initial value problem associated to (1.1) is locally well-posed in the Sobolev space $H^s(\mathbb{R}^2)$, $s > 2$. Improvement of this result will appear somewhere else (see [18]).

REMARK 1.2. *The scale-invariant spaces for the BO-ZK equations (1.1) are $\dot{H}^{s_1,s_2}(\mathbb{R}^2)$, $2s_1 + s_2 = \frac{3}{2} - \frac{2}{p}$ (see definitions below). Hence a reasonable framework for studying the local well-posedness of the BO-ZK equation (1.1) is the family of spaces $\dot{H}^{s_1,s_2}(\mathbb{R}^2)$, $2s_1 + s_2 \geq \frac{3}{2} - \frac{2}{p}$ (see [18]).*

REMARK 1.3. *The n-dimensional version of (1.1) is the equation*

$$u_t + u^p u_{x_1} + \alpha \mathscr{H} u_{x_1 x_1} + \sum_{i=2}^{n} \varepsilon_i u_{x_1 x_i x_i} = 0, \tag{1.4}$$

where $t \in \mathbb{R}^+$, $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and $\alpha, \varepsilon_i \in \mathbb{R}$, $i = 2, \ldots, n$. Existence and stability of solitary waves for (1.4) in the same spirit of this paper are addressed in [17].

Notation and Preliminaries. Throughout this paper we shall refer to equation (1.1) as BO-ZK equation. The exponent p in (1.1) will be a rational number of the form $p = k/m$, where m is odd and m and k are relatively prime. Function $\widehat{f}$ denotes the Fourier transform of $f = f(x, y)$, defined as

$$\widehat{f}(\xi_1, \xi_2) = \int_{\mathbb{R}^2} e^{-\mathrm{i}(x\xi_1 + y\xi_2)} f(x, y) \, dxdy.$$

For any $s \in \mathbb{R}$, space $H^s := H^s(\mathbb{R}^2)$ denotes the usual isotropic Sobolev space. Let $s_1, s_2 \in \mathbb{R}$. We define the anisotropic Sobolev spaces $H^{s_1,s_2} := H^{s_1,s_2}(\mathbb{R}^2)$ to be the set of all distributions f such that

$$\|f\|^2_{H^{s_1,s_2}} = \int_{\mathbb{R}^2} \left(1 + \xi_1^2\right)^{s_1} \left(1 + \xi_2^2\right)^{s_2} |\widehat{f}(\xi_1, \xi_2)|^2 \, d\xi_1 d\xi_2 < \infty.$$

We also define the fractional Sobolev-Liouville spaces $H_p^{(s_1,s_2)} := H_p^{(s_1,s_2)}(\mathbb{R}^2)$, $1 \leq p < \infty$, to be the set of all functions $f \in L^p(\mathbb{R}^2)$ such that

$$\|f\|_{H_p^{(s_1,s_2)}} = \|f\|_{L^p(\mathbb{R}^2)} + \sum_{i=1}^{2} \left\|D_{x_i}^{s_i} f\right\|_{L^p(\mathbb{R}^2)} < \infty,$$

where $D_{x_i}^{s_i} f$ denotes the Bessel derivative of order s_i with respect to x_i (see e.g. [25], [29]). For short, we denote $H_p^{(k)}(\mathbb{R}^2)$ as the space $H_p^{(k,k)}(\mathbb{R}^2)$ and $H^{(s_1,s_2)}(\mathbb{R}^2)$ as $p = 2$. Note that $H^{(s)} = H^s$.

REMARK 1.4. *Observe that* $\mathscr{Z} = H^{1/2,0}(\mathbb{R}^2) \cap H^{0,1}(\mathbb{R}^2) = H^{(1/2,1)}(\mathbb{R}^2)$.

It is easy to see that for $a \geq 0$,

$$\left(1 + x^2\right)^a \lesssim \left(1 + (x-y)^2\right)^a \left(1 + y^2\right)^a,$$

and

$$\left(1 + x^2\right)^a \lesssim \left(1 + (x-y)^2\right)^a + \left(1 + y^2\right)^a.$$

So,

REMARK 1.5. *Let* $2 \leq p < \infty$. *If* $1 - \frac{2}{p} \leq s \leq \min\{s_1, s_2\}$, *then the following embedding are continuous*

$$H^{s_1+s_2}(\mathbb{R}^2) \hookrightarrow H^{s_1,s_2}(\mathbb{R}^2) \hookrightarrow H^s(\mathbb{R}^2) \hookrightarrow L^p(\mathbb{R}^2).$$

Theorem 1 in [25] (see also [32, 33]) and Remark 1.5 imply the following embedding of $\mathscr{Z}$ in $L^p(\mathbb{R}^2)$ spaces:

$$\mathscr{Z} \hookrightarrow L^p(\mathbb{R}^2), \quad \text{for all} \ p \in [2,6]. \tag{1.5}$$

See also [34].

2. (Non)existence

This section is devoted to establish our existence and non-existence results of solitary waves. We begin with the non-existence one.

THEOREM 2.1. *The equation (1.3) do not admit any nontrivial solitary wave solution* $\varphi \in \mathscr{Z}$ *if none of the following cases occurs:*

(i) $\varepsilon = 1$, $c > 0$, $\alpha < 0$, $p < 4$;
(ii) $\varepsilon = -1$, $c < 0$, $\alpha > 0$, $p < 4$;
(iii) $\varepsilon = 1$, $c < 0$, $\alpha < 0$, $p > 4$;
(iv) $\varepsilon = -1$, $c > 0$, $\alpha > 0$, $p > 4$.

Proof. To prove the theorem, we apply a truncation argument to gain the regularity we need, then by using the Lebesgue dominated convergence theorem, we obtain some useful identities (see e.g. [13]). In fact, by multiplying the equation (1.3) by φ, $x\varphi_x$ and $y\varphi_y$, respectively; and integrating over $\mathbb{R}^2$, then by the properties of the Hilbert Transform (see [15]), we obtain the following relations:

$$\int_{\mathbb{R}^2} \left(-c\varphi^2 + \alpha\varphi\mathscr{H}\varphi_x - \varepsilon\varphi_y^2 + \frac{1}{p+1}\varphi^{p+2}\right) dxdy = 0, \tag{2.1}$$

$$\int_{\mathbb{R}^2} \left(c\varphi^2 + \varepsilon\varphi_y^2 - \frac{2}{(p+1)(p+2)}\varphi^{p+2}\right) dxdy = 0, \tag{2.2}$$

$$\int_{\mathbb{R}^2} \left(c\varphi^2 - \alpha\varphi\mathscr{H}\varphi_x - \varepsilon\varphi_y^2 - \frac{2}{(p+1)(p+2)}\varphi^{p+2}\right) dxdy = 0. \tag{2.3}$$

By adding (2.1) and (2.2), we get

$$\int_{\mathbb{R}^2} \left(\alpha\varphi\mathscr{H}\varphi_x + \frac{p}{(p+1)(p+2)}\varphi^{p+2}\right) dxdy = 0. \tag{2.4}$$

Also by adding (2.2) and (2.3) yields

$$\int_{\mathbb{R}^2} \left(c\varphi^2 - \frac{\alpha}{2}\varphi\mathscr{H}\varphi_x - \frac{2}{p+1}\varphi^{p+2} \right) \, dxdy = 0. \tag{2.5}$$

Eliminating φ^{p+2} from (2.4) and (2.5) leads to

$$\int_{\mathbb{R}^2} \left(2pc\varphi^2 + \alpha(4-p)\varphi\mathscr{H}\varphi_x \right) \, dxdy = 0. \tag{2.6}$$

On the other hand, adding (2.1) and (2.3) yields

$$\int_{\mathbb{R}^2} \left(2\varepsilon\varphi_y^2 - \frac{p}{(p+1)(p+2)}\varphi^{p+2} \right) \, dxdy = 0. \tag{2.7}$$

Plugging (2.2) in (2.7) we obtain

$$\int_{\mathbb{R}^2} \left(pc\varphi^2 + \varepsilon(p-4)\varphi_y^2 \right) \, dxdy = 0. \tag{2.8}$$

The proof follows from (2.6) and (2.8). $\blacksquare$

Now we prove the existence of solitary wave solutions of (1.1).

THEOREM 2.2. *Let $\alpha\varepsilon, c\alpha < 0$ and $p = \frac{k}{m} < 4$, where $m \in \mathbb{N}$ is odd and m and k are relatively prime. Then the equation (1.3) admits a nontrivial solution $\varphi \in \mathscr{Z}$.*

Proof. The proof is based on the concentration-compactness principle [27, 28]. We suppose that $\alpha < 0$. The proof for $\alpha > 0$ is similar. Without loss of generality we assume that $\alpha = -1$ and $c = 1$. We consider the minimization problem

$$I_\lambda = \inf \left\{ I(\varphi) \; ; \; \varphi \in \mathscr{Z} \; , \; J(\varphi) = \int_{\mathbb{R}^2} \varphi^{p+2} \, dxdy = \lambda \right\}, \tag{2.9}$$

where $\lambda > 0$ and

$$I(\varphi) = \frac{1}{2}\int_{\mathbb{R}^2} \left(\varphi^2 + \varphi\mathscr{H}\varphi_x + \varphi_y^2 \right) \, dxdy.$$

Let $\{\varphi_n\} \subset \mathscr{Z}$ be a minimizing sequence of I_λ. By using the embedding (1.5), we obtain that

$$\lambda = \left| \int_{\mathbb{R}^2} \varphi^{p+2} \, dxdy \right| \leq C\|\varphi\|_{\mathscr{Z}}^{p+2} \leq CI_\lambda^{\frac{p+2}{2}},$$

for any $\varphi \in \mathscr{Z}$ and $p < 4$. Hence $I_\lambda < \infty$ and $I_\lambda > 0$ for any positive λ. Also, the fact that $I(\varphi) = \frac{1}{2}\|\varphi\|_{\mathscr{Z}}^2$ implies $\|\varphi_n\|_{\mathscr{Z}} < \infty$.

Now, for $r > 0$, we define the concentration functions

$$Q_n(r) = \sup_{(\widetilde{x},\widetilde{y})\in\mathbb{R}^2} \int_{B_r(\widetilde{x},\widetilde{y})} \rho_n \, dxdy,$$

where $\rho_n = |\varphi_n|^2 + \left| D_x^{1/2}\varphi_n \right|^2 + |\partial_y\varphi_n|^2$ and $B_r(x,y)$ denotes the ball of radius $r > 0$ centered at $(x,y) \in \mathbb{R}^2$. If the evanescence occurs, i.e., that for any $R > 0$,

$$\lim_{n\to+\infty} \sup_{(\widetilde{x},\widetilde{y})\in\mathbb{R}^2} \int_{B_r(\widetilde{x},\widetilde{y})} \rho_n \, dxdy = 0,$$

then by using the embedding (1.5), we obtain that $\lim_{n\to\infty} \|\varphi_n\|_{L^{p+2}} = 0$, which would contradict the constraint of the minimization problem. Now suppose that $\gamma \in (0, I_\lambda)$, where

$$\gamma = \lim_{r\to+\infty} \lim_{n\to+\infty} \sup_{(\widetilde{x},\widetilde{y})\in\mathbb{R}^2} \int_{B_r(\widetilde{x},\widetilde{y})} \rho_n \, dxdy.$$

By the definition of γ, for $\epsilon > 0$, there exist $r_1 \in \mathbb{R}$ and $N \in \mathbb{N}$ such that

$$\gamma - \epsilon < Q_n(r) \leq Q_n(2r) \leq \gamma < \gamma + \epsilon,$$

for any $r \geq r_1$ and $n \geq N$. Hence, there exists a sequence $\{(\widetilde{x}_n, \widetilde{y}_n)\} \subset \mathbb{R}^2$ such that

$$\int_{B_r(\widetilde{x}_n,\widetilde{y}_n)} \rho_n \, dxdy > \gamma - \epsilon, \qquad \int_{B_{2r}(\widetilde{x}_n,\widetilde{y}_n)} \rho_n \, dxdy < \gamma + \epsilon.$$

Let (ϕ, ψ) be in $(C_0^\infty(\mathbb{R}^2))^2$ satisfying

- supp $\phi \subset B_2(0,0)$, $\phi \equiv 1$ on $B_1(0,0)$ and $0 \leq \phi \leq 1$,
- supp $\psi \subset \mathbb{R}^2 \setminus B_2(0,0)$, $\psi \equiv 1$ on $\mathbb{R}^2 \setminus B_1(0,0)$ and $0 \leq \psi \leq 1$.

Now we define

$$g_n(x,y) = \phi_r((x,y) - (\widetilde{x}_n, \widetilde{y}_n))\varphi_n \quad \text{and} \quad h_n(x,y) = \psi_r((x,y) - (\widetilde{x}_n, \widetilde{y}_n))\varphi_n,$$

where

$$\phi_r(x,y) = \phi\left(\frac{(x,y)}{r}\right) \quad \text{and} \quad \psi_r(x,y) = \psi\left(\frac{(x,y)}{r}\right).$$

One can see that $g_n, h_n \in \mathscr{X}$. The following commutator estimate lemma is fruitful to obtain the splitting lemma below.

LEMMA 2.3 ([9, 11]). *Let $g \in C^\infty(\mathbb{R})$ with $g' \in L^\infty(\mathbb{R})$. Then $[\mathscr{H}, g]\partial_x$ is a bounded linear operator from $L^2(\mathbb{R})$ into $L^2(\mathbb{R})$ with*

$$\|[\mathscr{H}, g]\partial_x f\|_{L^2(\mathbb{R})} \leq C\|g'\|_{L^\infty(\mathbb{R})}\|f\|_{L^2(\mathbb{R})}.$$

The following splitting lemma enables us to rule out the dichotomy case in the concentration-compactness principle (see also [15]).

LEMMA 2.4. *Let $\{g_n\}_{n\in\mathbb{N}}$ and $\{h_n\}_{n\in\mathbb{N}}$ be as above. Then for every $\epsilon > 0$, there exist $\delta(\epsilon) > 0$ with $\lim_{\epsilon\to 0} \delta(\epsilon) = 0$, $\varrho \in (0, I_\lambda)$, $n_0 \in \mathbb{N}$ and $\rho \in (0, \lambda)$ satisfying the following for $n \geq n_0$:*

$$|I(\varphi_n) - I(g_n) - I(h_n)| \leq \delta(\epsilon), \tag{2.10}$$

$$|I(g_n) - \varrho| \leq \delta(\epsilon), \quad |I(h_n) - I_\lambda + \varrho| \leq \delta(\epsilon), \tag{2.11}$$

$$|J(\varphi_n) - J(g_n) - J(h_n)| \leq \delta(\epsilon), \tag{2.12}$$

$$|J(g_n) - \rho| \leq \delta(\epsilon), \quad |J(h_n) - \lambda + \rho| \leq \delta(\epsilon). \tag{2.13}$$

Proof. Obviously supp $g_n \cap$ supp $h_n = \emptyset$. For simplicity, we write $g_n = \phi_r\varphi_n$ and $h_n = \psi_r\varphi_n$. Thus, we have

$$2I(g_n) = \int_{\mathbb{R}^2} \phi_r^2 \left[\varphi_n^2 + \varphi_n\partial_x\mathscr{H}\varphi_n + \left(\partial_y^2\varphi_n\right)^2\right] \, dxdy + 2\int_{\mathbb{R}^2} \phi_r\varphi_n(\partial_y\phi_r)(\partial_y\varphi_n) \, dxdy$$

$$+ \int_{\mathbb{R}^2} \left[(\partial_y\phi_r)^2\varphi_n^2 + \varphi_n\phi_r\mathscr{H}(\varphi_n\partial_x\phi_r)\right] \, dxdy + \int_{\mathbb{R}^2} \varphi_n\phi_r[\mathscr{H}, \phi_r]\partial_x\varphi_n \, dxdy$$

and

$$2I(h_n) = \int_{\mathbb{R}^2} \psi_r^2 \left[\varphi_n^2 + \varphi_n \partial_x \mathscr{H} \varphi_n + \left(\partial_y^2 \varphi_n \right)^2 \right] \, dxdy + 2 \int_{\mathbb{R}^2} \psi_r \varphi_n (\partial_y \psi_r)(\partial_y \varphi_n) \, dxdy$$
$$+ \int_{\mathbb{R}^2} \left[(\partial_y \psi_r)^2 \varphi_n^2 + \varphi_n \psi_r \mathscr{H}(\varphi_n \partial_x \psi_r) \right] \, dxdy + \int_{\mathbb{R}^2} \varphi_n \psi_r [\mathscr{H}, \psi_r] \partial_x \varphi_n \, dxdy.$$

Since $\|\phi_r\|_{L^\infty} = \|\psi_r\|_{L^\infty} = 1$, $\|\nabla \phi_r\|_{L^\infty} \le \frac{1}{r}\|\nabla\phi\|_{L^\infty}$ and $\|\nabla\psi_r\|_{L^\infty} \le \frac{1}{r}\|\nabla\psi\|_{L^\infty}$, it follows from Lemma 2.3 that

$$\left| I(g_n) - \frac{1}{2} \int_{\mathbb{R}^2} \phi_r^2 \left[\varphi_n^2 + \varphi_n \partial_x \mathscr{H} \varphi_n + \left(\partial_y^2 \varphi_n \right)^2 \right] \, dxdy \right| \le \frac{1}{2}\delta(\epsilon);$$

and

$$\left| I(h_n) - \frac{1}{2} \int_{\mathbb{R}^2} \psi_r^2 \left[\varphi_n^2 + \varphi_n \partial_x \mathscr{H} \varphi_n + \left(\partial_y^2 \varphi_n \right)^2 \right] \, dxdy \right| \le \frac{1}{2}\delta(\epsilon).$$

The above inequalities imply (2.10). From this, one infers that there exists $\varrho(\epsilon) \in [0, I_\lambda]$ (and taking subsequences if necessary) such that $\lim_{n\to\infty} I(g_n) = \varrho(\epsilon)$, and thus

$$|I(g_n) - I_\lambda + \varrho| \le \delta(\epsilon).$$

Analogously, from (2.10), the fact that $\operatorname{supp} g_n \cap \operatorname{supp} h_n = \emptyset$ and the embedding (1.5), one can obtain

$$|J(\varphi_n) - J(g_n) - J(h_n)| \le \delta(\varepsilon).$$

Therefore we assume that

$$\lim_{n\to+\infty} J(g_n) = \rho(\epsilon), \qquad \lim_{n\to+\infty} J(h_n) = \widetilde{\rho}(\epsilon)$$

with $|\lambda - \rho(\epsilon) - \widetilde{\rho}(\epsilon)| \le \delta(\epsilon)$. If $\lim_{\epsilon\to 0} \rho(\epsilon) = 0$, then choosing ϵ sufficiently small, we have $J(h_n) > 0$, for n large enough. Hence by considering $(\widetilde{\rho}(\epsilon)J(h_n))^{\frac{1}{p+2}} h_n$, we obtain that (note that $J\left((\widetilde{\rho}(\epsilon)J(h_n))^{\frac{1}{p+2}} h_n \right) = \widetilde{\rho}(\epsilon)$)

$$I_{\widetilde{\rho}(\epsilon)} \le \liminf_{n\to+\infty} I(h_n) \le I_\lambda - \gamma + \delta(\epsilon),$$

which leads to a contradiction since $\lim_{\epsilon\to 0} \widetilde{\rho}(\epsilon) = \lambda$. Thus $\rho = \lim_{\epsilon\to 0} \rho(\epsilon) > 0$. Necessarily $\rho < \lambda$, because the case $\rho = \lambda$ is ruled out in the same manner with h_n instead of g_n. Since $\rho \in (0, \lambda)$, one infers that necessarily $\varrho = \lim_{\epsilon\to+\infty} \varrho(\epsilon) \in (0, I_\lambda)$. This completes the proof of the lemma. ∎

Now, we come back to the proof of Theorem 2.2. The previous lemma implies that

$$I_\lambda \ge I_\rho + I_{\lambda-\rho}. \tag{2.14}$$

This inequality contradicts the subadditivity condition of I_λ coming from $I_\lambda = \lambda^{2/(p+2)} I_1$. Hence we rule out the dichotomy.

Therefore the remaining case in the concentration-compactness principle is locally compactness. Then there exists a sequence $\{(x_n, y_n)\}_{n\in\mathbb{N}} \subset \mathbb{R}^2$, such that for all $\epsilon > 0$, there exist finite $R > 0$ and $n_0 > 0$, with

$$\int_{B_R(x_n, y_n)} \rho_n \, dxdy \ge \iota_\lambda - \epsilon,$$

for $n \geq n_0$, where

$$\iota_\lambda = \lim_{n \to +\infty} \int_{\mathbb{R}^2} \rho_n \; dxdy.$$

This implies that for n large enough

$$\int_{B_R(x_n, y_n)} |\varphi_n|^2 \; dxdy \geq \int_{\mathbb{R}^2} |\varphi_n|^2 \; dxdy - 2\epsilon.$$

Since φ_n is bounded in the Hilbert space $\mathscr{Z}$, there exists $\varphi \in \mathscr{Z}$ such that a subsequence of $\{\varphi_n(\cdot - (x_n, y_n))\}_{n \in \mathbb{N}}$ (denoted by the same) converges weakly in $\mathscr{Z}$. We then have

$$\int_{\mathbb{R}^2} |\varphi|^2 \; dxdy \leq \liminf_{n \to +\infty} \int_{\mathbb{R}^2} |\varphi_n|^2 \; dxdy \leq \liminf_{n \to +\infty} \int_{B_R(x_n, y_n)} |\varphi_n|^2 \; dxdy + 2\epsilon.$$

But we know the compactness embedding $\mathscr{Z}$ into L^2 on bounded sets. Consequently $\{u_n(\cdot - (x_n, y_n))\}_{n \in \mathbb{N}}$ converges strongly in $L^2_{loc}(\mathbb{R}^2)$. But the last inequality above implies that this strong convergence also takes place in $L^2(\mathbb{R}^2)$. Thus by the embedding (1.5), $\{\varphi_n(\cdot - (x_n, y_n))\}_{n \in \mathbb{N}}$ also converges to φ strongly in $L^{p+2}(\mathbb{R}^2)$ so that $J(\varphi) = \lambda$ and

$$I_\lambda = \lim_{n \to +\infty} I(\varphi_n) = I(\varphi),$$

that is, φ is a solution of I_λ.

Now by using the Lagrange multiplier theorem, there exists $\theta \in \mathbb{R}$ such that

$$\varphi - \alpha \mathscr{H} \varphi_x - \varphi_{yy} = \theta(p+2)\varphi^{p+1}, \tag{2.15}$$

in $\mathscr{Z}'$ (the dual space of $\mathscr{Z}$ in L^2−duality). By a scale change, φ satisfies (1.3). ∎

REMARK 2.5. *Theorem 2.2 shows the existence of solitary wave solutions of* (1.1) *when* (i) *and* (ii) *occur in Theorem 2.1. Unfortunately, we do not know existence or nonexistence of solitary wave in the cases* (iii) *and* (iv) .

Now we are going to prove that the minima of Theorem 2.2 are exactly the *ground state* solutions of the equation (1.3). The proof is close to Lemma 2.1 in [14].

DEFINITION 2.6. *A solution φ of the equation (1.3) is called a ground state, if φ minimizes the action*

$$S(u) = E(u) + c\mathscr{F}(u)$$

among all solutions of the equation (1.3).

THEOREM 2.7. *There is a real number λ^* such that for $u^* \in \mathscr{Z}$ the following affirmations are equivalent modulo a scale change:*

(i) *$J(u^*) = \lambda^*$ and u^* is a minimum of I_{λ^*},*
(ii) *$\mathcal{K}(u^*) = 0$ and*

$$\inf \left\{ \int_{\mathbb{R}^2} u\mathscr{H} u_x \; dxdy, \; u \in \mathscr{Z}, \; u \neq 0, \; \mathcal{K}(u) = 0 \right\} = \int_{\mathbb{R}^2} u^* \mathscr{H} u_x^* \; dxdy,$$

(iii) *u^* is a ground state,*

(iv) $\mathcal{K}(u^*) = 0$ *and*

$$\inf\left\{\mathcal{K}(u),\ u \in \mathscr{Z},\ u \neq 0,\ \int_{\mathbb{R}^2} u\mathscr{H}u_x\ dxdy = \int_{\mathbb{R}^2} u^*\mathscr{H}u_x^*\ dxdy\right\} = 0,$$

where

$$\mathcal{K}(u) = \frac{1}{2}\int_{\mathbb{R}^2} cu^2 + u_y^2\ dxdy - \frac{1}{(p+1)(p+2)}J(u).$$

Proof. We set $\lambda^* = (2(p+1)I_1)^{\frac{p+2}{p}}$.

(i) $\Rightarrow$ (ii) : Assume that u^* satisfies (i). Let $u \in \mathscr{Z}$ with $u \neq 0$ and $\mathcal{K}(u) = 0$, which implies that $J(u) > 0$. Thus we set

$$u_\mu(x,y) = u\left(\frac{x}{\mu},y\right), \quad\text{with}\quad \mu = \frac{J(u^*)}{J(u)},$$

so that $J(u_\mu) = J(u^*)$ and $\mathcal{K}(u_\mu) = 0$. Since u^* is a minimum of I_{λ^*}, one can see that $\mathcal{K}(u^*) = 0$ and

$$\mathcal{K}(u^*) + C_p J(u^*) + \frac{1}{2}\int_{\mathbb{R}^2} u^*\mathscr{H}u_x^*\ dxdy \leq \mathcal{K}(u_\mu) + C_p J(u_\mu) + \frac{1}{2}\int_{\mathbb{R}^2} u_\mu\mathscr{H}(u_\mu)_x\ dxdy,$$

where $C_p = \frac{1}{(p+1)(p+2)}$; this implies that

$$\int_{\mathbb{R}^2} u^*\mathscr{H}u_x^*\ dxdy \leq \int_{\mathbb{R}^2} u\mathscr{H}u_x\ dxdy,$$

and (ii) holds.

(ii) $\Rightarrow$ (iii) : If u^* satisfies (ii), then there is a Lagrange multiplier θ such that

$$cu^* - u_{yy}^* + \theta\mathscr{H}u_x^* - \frac{1}{p+1}(u^*)^{p+1} = 0.$$

By multiplying the above equation by u^*, integrating by parts and using $\mathcal{K}(u^*) = 0$, we can see that θ is positive. Hence the scale change $u_*(x,y) = u^*(\theta x,y)$ satisfies the equation (1.3).

On the other hand, the identity $S(u) = \mathcal{K}(u) + \frac{1}{2}\int_{\mathbb{R}^2} u\mathscr{H}u_x\ dxdy$ shows that if u is a solution of (1.3), then

$$S(u) = \frac{1}{2}\int_{\mathbb{R}^2} u\mathscr{H}u_x\ dxdy \geq \frac{1}{2}\int_{\mathbb{R}^2} u^*\mathscr{H}u_x^*\ dxdy = \frac{1}{2}\int_{\mathbb{R}^2} u_*\mathscr{H}(u_*)_x\ dxdy = S(u_*);$$

hence u_* is a ground state.

(iii) $\Rightarrow$ (i) : By using the proof of Theorem 2.1, one can see that if u is a solution of (1.3), then $\mathcal{K}(u) = 0$ and

$$I(u) = \frac{1}{2}\left(1 + \frac{3}{p}\right)\int_{\mathbb{R}^2} u\mathscr{H}u_x\ dxdy. \tag{2.16}$$

Hence if u^* is a ground state, then u^* minimizes both $I(u)$ and $\int_{\mathbb{R}^2} u\mathscr{H}u_x\ dxdy$ among all the solutions of (1.3). Let $\lambda = J(u)$ and $\widetilde{u}$ be a minimum of I_λ. Then

$$I_\lambda = I(\widetilde{u}) \leq I(u^*)$$

and there is a positive number θ such that

$$c\widetilde{u} - \widetilde{u}_{yy} + \mathscr{H}\widetilde{u}_x = \frac{\theta}{p+1}\widetilde{u}^{p+1}.$$

Using the equations satisfied by $\widetilde{u}$ and u^*, the preceding inequality is written as

$$I_\lambda = \frac{\lambda\theta}{p+1} \leq \frac{\lambda}{p+1};$$

hence $\theta \leq 1$. On the other hand, $u_\star = \theta^p\widetilde{u}$ satisfies the equation (1.3), and since u^* is a ground state, we have

$$I(u^*) \leq I(u_\star) \leq \theta^{2p}I(\widetilde{u}),$$

so that $\theta \geq 1$. Thusly $u^* = \widetilde{u}$ is a minimum of I_λ with $\lambda = \lambda^*$.

(ii) $\Rightarrow$ (iv) : Let $u \in \mathscr{Z}$ with $\int_{\mathbb{R}^2} u\mathscr{H}u_x \, dxdy = \int_{\mathbb{R}^2} u^*\mathscr{H}u_x^* \, dxdy$. Suppose that $\mathcal{K}(u) < 0$. Since $\mathcal{K}(\tau u) > 0$ for $\tau > 0$ sufficiently small, then there is a $\tau_0 \in (0,1)$ such that $\mathcal{K}(\tau_0 u) = 0$. Thus by setting $\widetilde{u} = \tau_0 u$, one has $\widetilde{u} \in \mathscr{Z}$, $\mathcal{K}(\widetilde{u}) = 0$ and

$$\int_{\mathbb{R}^2} \widetilde{u}\mathscr{H}\widetilde{u}_x \, dxdy < \int_{\mathbb{R}^2} u\mathscr{H}u_x \, dxdy = \int_{\mathbb{R}^2} u^*\mathscr{H}u_x^* \, dxdy,$$

which contradicts (ii) and shows that u^* satisfies (iv) because $\mathcal{K}(u^*) = 0$.

(iv) $\Rightarrow$ (ii) : Let $u \in \mathscr{Z}$ with $\mathcal{K}(u) = 0$ and $u \neq 0$. Suppose that

$$\int_{\mathbb{R}^2} u\mathscr{H}u_x \, dxdy < \int_{\mathbb{R}^2} u^*\mathscr{H}u_x^* \, dxdy.$$

But since $\mathcal{K}(\tau u) < 0$ for $\tau > 1$, then one can find $\tau_0 > 1$ with

$$\int_{\mathbb{R}^2} (\tau_0 u)\mathscr{H}(\tau_0 u)_x \, dxdy = \int_{\mathbb{R}^2} u^*\mathscr{H}u_x^* \, dxdy$$

and $\mathcal{K}(\tau_0 u) < 0$. This contradicts (iv). Hence $\int_{\mathbb{R}^2} u\mathscr{H}u_x \, dxdy \geq \int_{\mathbb{R}^2} u^*\mathscr{H}u_x^* \, dxdy$ and (ii) holds. ∎

REMARK 2.8. *Note that the proof of the above theorem shows that, indeed,* (i) *and* (iii) *are equivalent and imply* (ii) *and* (iv)*; which are also equivalent; but the converse holds modulo a scale change.*

REMARK 2.9. *It has been shown that these ground state solutions are exactly the minimizers of $E(u)$ under a suitable constraint $\mathscr{F}(u) = \mu^*$ (see [16, 17]).*

3. Stability

We start by defining our notion of orbital stability:

DEFINITION 3.1. *Let φ_c be a solitary wave solution of (1.1). We say that φ_c is orbitally stable if for all $\eta > 0$, there is a $\delta > 0$ such that for any $u_0 \in H^s\left(\mathbb{R}^2\right)$, $s > 2$, with $\|u_0 - \varphi_c\|_{\mathscr{X}} \leq \delta$, the corresponding solution $u(t)$ of (1.1) with $u(0) = u_0$ satisfies*

$$\sup_{t \geq 0} \inf_{r \in \mathbb{R}^2} \|u(t) - \varphi_c(\cdot - r)\|_{\mathscr{X}} \leq \eta.$$

The following theorem is a consequence of Theorem 2.2 and it will be main key to obtain our stability results of the solitary waves for the BO-ZK equations. Hereafter, without loss of generality we assume that $\alpha = -1$ and $c > 0$.

THEOREM 3.2. *Let $\lambda > 0$. Then,*

(i) *every minimizing sequence to I_λ converges, up to a translation, in $\mathscr{Z}$ to an element of the minimizers set*

$$M_\lambda = \{\varphi \in \mathscr{Z};\ I(\varphi) = I_\lambda,\ J(\varphi) = \lambda\}.$$

(ii) *Let $\{\varphi_n\}$ be a minimizing sequence for I_λ. Then we have*

$$\lim_{n \to +\infty} \inf_{\psi \in M_\lambda,\ z \in \mathbb{R}^2} \|\varphi_n(\cdot + z) - \psi\|_{\mathscr{Z}} = 0, \tag{3.1}$$

$$\lim_{n \to +\infty} \inf_{\psi \in M_\lambda} \|\varphi_n - \psi\|_{\mathscr{Z}} = 0. \tag{3.2}$$

Proof. (i)   Let $\{\varphi_n\}$ be a minimizing sequence for I_λ. By defining

$$\psi_n = \frac{\lambda^{\frac{1}{p+2}}}{\|\varphi_n\|_{L^{p+2}}} \varphi_n,$$

we obtain that $J(\psi_n) = \lambda$ and

$$I_\lambda \leq I(\psi_n) \leq I(\varphi_n).$$

This implies that $\{\psi_n\}$ is a minimizing sequence for I_λ. Then there is a subsequence of $\{\psi_n\}$, denoted by the same, and a sequence $\{r_n\} \subset \mathbb{R}^2$ such that

$$\psi_n(\cdot + r_n) \longrightarrow \psi,$$

as $n \to \infty$ in $\mathscr{Z}-$norm and $\psi \in M_\lambda$. On the other hand, we have that the sequence

$$\left\{ \frac{\lambda^{2/(p+2)}}{\|\varphi_n\|_{L^{p+2}}} \right\}$$

converges, up to a subsequence, to a real number $\ell \in [0, 1]$. Using the equality

$$M_\lambda = \lim_{n \to \infty} I(\psi_n) = \ell I_\lambda,$$

we obtain that $\ell = 1$. Now we will show that $\{\varphi_n(\cdot + r_n)\}$ converges strongly in $\mathscr{Z}$ to an element of M_λ. Indeed, we have

$$\lim_{n \to \infty} \varphi_n(\cdot + r_n) = \lim_{n \to \infty} \psi_n(\cdot + r_n) = \psi,$$

in $\mathscr{Z}$. By the embedding (1.5), it follows that $\varphi_n(\cdot + r_n) \to \psi$ in L^{p+2} as $n \to \infty$. Hence

$$J(\psi) = \lim_{n \to \infty} J(\varphi_n(\cdot + r_n)) = \lim_{n \to \infty} J(\varphi_n) = \lambda.$$

Finally, since $I(\psi) = I_\lambda$, we have that $\psi \in M_\lambda$. This shows the first part of the theorem.

(ii)   If (ii) does not hold, then there exists a subsequence of $\{\varphi_n\}$, denoted by the same, and $\epsilon > 0$ such that

$$\varpi = \inf_{\psi \in M_\lambda, r \in \mathbb{R}^2} \|\varphi_n(\cdot + r) - \psi\|_{\mathscr{Z}} \geq \epsilon,$$

for all n. Since $\{\varphi_n\}$ is a minimizing sequence for I_λ, by using (i), we have that there exists a sequence $\{r_n\} \subset \mathbb{R}^2$ such that, up to a subsequence, $\varphi_n(\cdot + r_n) \to \varphi$ in $\mathscr{Z}$ as $n \to \infty$. So for n large enough, we have

$$\frac{\epsilon}{2} \geq \|\varphi_n(\cdot + r) - \psi\|_{\mathscr{Z}} \geq \varpi \geq \epsilon.$$

This contradiction shows (3.1).

The proof of (3.2) follows from the fact that if $\psi \in M_\lambda$, then $\psi(\cdot + r) \in I_\lambda$ for all $r \in \mathbb{R}^2$, and the following equality:

$$\inf_{\psi \in M_\lambda} \|\varphi_n - \psi\|_{\mathscr{X}} = \inf_{\psi \in M_\lambda, r \in \mathbb{R}^2} \inf_{\psi \in M_\lambda} \|\varphi_n - \psi(\cdot - r)\|_{\mathscr{X}} = \inf_{\psi \in M_\lambda} \|\varphi_n(\cdot + r) - \psi\|_{\mathscr{X}}.$$

This completes the proof of the theorem. ∎

The following lemma easily shows that there exists a $\lambda > 0$ such that every element in the set of minimizers satisfies (1.3).

LEMMA 3.3. *For $\lambda = (2(p+1)I_1)^{\frac{p+2}{p}}$ in our minimization problem, we have that if $\varphi \in M_\lambda$, then φ is a solitary wave solution for the BO-ZK equation (1.3).*

Now for λ in the above lemma, we define the set

$$\mathcal{N}_c = \{\varphi \in \mathscr{Z};\ J(\varphi) = 2(p+1)I(\varphi) = \lambda\}.$$

One can see that $M_\lambda = \mathcal{N}_c$. Next for any $c > 0$ and any $\varphi \in \mathcal{N}_c$, we define the function

$$d(c) = E(\varphi) + c\mathscr{F}(\varphi).$$

LEMMA 3.4. *$d(c)$ is constant on $\mathcal{N}_c$, differentiable and strictly increasing for $c > 0$ and $p < \frac{4}{3}$. Moreover, $d''(c) > 0$ if and only if $p < \frac{4}{3}$.*

Proof. It is easy to see that

$$d(c) = I(\varphi) - \frac{1}{(p+1)(p+2)}J(\varphi) = \frac{p}{2(p+1)(p+2)}J(\varphi) = \frac{p(2(p+1))^{\frac{2}{p}}}{p+2}I_1^{\frac{p+2}{p}}.$$

Therefore,

$$d(c) = \frac{p}{2(p+1)(p+2)}c^{\frac{2}{p} - \frac{1}{2}}J(\psi),$$

where $\psi(x, y) = c^{-\frac{1}{p}}\varphi\left(\frac{x}{c}, \frac{y}{\sqrt{c}}\right)$. Note that ψ satisfies (1.3), with $c = 1$. But we know that

$$\frac{1}{(p+1)(p+2)}J(\varphi) = \frac{4c}{4-p}\mathscr{F}(\varphi).$$

Thusly, we obtain that

$$d''(c) = \left(\frac{2}{p} - \frac{3}{2}\right)c^{\frac{2}{p} - \frac{5}{2}}\mathscr{F}(\psi).$$

This proves the lemma. ∎

Now we are going to study the behavior of d in a neighborhood of the set $\mathcal{N}_c$.

LEMMA 3.5. *Let $c > 0$. Then there exists a small positive number ϵ and a C^1-map $v : \mathcal{B}_\epsilon(\mathcal{N}_c) \to (0, +\infty)$ defined by*

$$v(u) = d^{-1}\left(\frac{p}{2(p+1)(p+2)}J(\varphi)\right),$$

such that $v(\varphi) = c$ for every $\varphi \in \mathcal{N}_c$, where

$$\mathcal{B}_\epsilon(\mathcal{N}_c) = \left\{\varphi \in \mathscr{Z};\ \inf_{\psi \in \mathcal{N}_c}\|\varphi - \psi\|_{\mathscr{X}} < \epsilon\right\}.$$

Proof. Without loss of generality we assume that $c = 1$. It is easy to see that $\mathcal{N}_c$ is a bounded set in $\mathscr{Z}$. Moreover

$$\mathcal{N}_c \subset B(0, r) \subset \mathscr{Z},$$

where $r = (2(p+1))^{\frac{2}{p}} I_1^{\frac{p+2}{p}}$ and $B(0, r)$ is the ball of radius $r > 0$ centered at the origin in $\mathscr{Z}$. Let $\rho > 0$ be sufficiently large such that $\mathcal{N}_c \subset B(0, \rho) \subset \mathscr{Z}$. Since the function $u \to J(u)$ is uniformly continuous on bounded sets, there exists $\epsilon > 0$ such that if $u, v \in B(0, \rho)$ and $\|u - v\|_{\mathscr{Z}} < 2\epsilon$ then $|J(u) - J(v)| < \rho$. Considering the neighborhoods $\mathscr{I} = (d(c) - \rho, d(c) + \rho)$ and $\mathcal{B}_\epsilon(\mathcal{N}_c)$ of $d(c)$ and $\mathcal{N}_c$, respectively, we have that if $u \in \mathcal{B}_\epsilon(\mathcal{N}_c)$ then $J(u) \in \mathscr{I}$. Therefore v is well defined on $\mathcal{B}_\epsilon(\mathcal{N}_c)$ and satisfies $v(\varphi) = c$, for all $\varphi \in \mathcal{N}_c$. ∎

Next, we establish the main inequality in our study of stability.

LEMMA 3.6. *Let $c > 0$ and suppose that $d''(c) > 0$. Then for all $u \in \mathcal{B}_\epsilon(\mathcal{N}_c)$ and any $\varphi \in \mathcal{N}_c$,*

$$E(u) - E(\varphi) + v(u)\left(\mathscr{F}(u) - \mathscr{F}(\varphi)\right) \geq \frac{1}{4} d''(c)|v(u) - c|^2.$$

Proof. Let

$$I_\omega(\varphi) = \frac{1}{2} \int_{\mathbb{R}^2} \left(\omega\varphi^2 + \varphi\mathscr{H}\varphi_x + \varphi_y^2\right) \, dxdy$$

and φ_ω any element of $\mathcal{N}_\omega$. Then, we have

$$E(u) + v(u)\mathscr{F}(u) = I_{v(u)}(u) - \frac{1}{(p+1)(p+2)} J(u).$$

On the other hand, we have $J(u) = J\left(\varphi_{v(u)}\right)$, since $d(v(u)) = \frac{p}{2(p+1)(p+2)} J(u)$ for $u \in \mathcal{B}_\epsilon(\mathcal{N}_c)$ and $d(v(u)) = \frac{p}{2(p+1)(p+2)} J\left(\varphi_{v(u)}\right)$. Thusly,

$$I_{v(u)}(u) \geq I_{v(u)}\left(\varphi_{v(u)}\right).$$

Therefore by using the Taylor expansion of d at c, we obtain that

$$E(u) + v(u)\mathscr{F}(u) \geq I_{v(u)}\left(\varphi_{v(u)}\right) - \frac{1}{(p+1)(p+2)} J\left(\varphi_{v(u)}\right)$$

$$= d(v(u)) \geq d(c) + \mathscr{F}(\varphi)(v(u) - c) + \frac{1}{4} d''(c)|v(u) - c|^2$$

$$= E(\varphi) + v(u)\mathscr{F}(\varphi) + \frac{1}{4} d''(c)|v(u) - c|^2.$$

This completes the lemma. ∎

Before proving our stability result, we state a well-posedness result for (1.1); which can be proved by using the parabolic regularization theory (see [20]).

THEOREM 3.7. *Let $s > 2$. Then for any $u_0 \in H^s(\mathbb{R}^2)$, there exist $T = T(\|u_0\|_{H^s}) > 0$ and a unique solution $u \in C([0, T]; H^s(\mathbb{R}^2))$ of the equation (1.1) with $u(0) = u_0$ and $u(t)$ depends on u_0 continuously in the H^s-norm. In addition, $u(t)$ satisfies $E(u(t)) = E(u_0)$, $\mathscr{F}(u(t)) = \mathscr{F}(u_0)$, for all $t \in [0, T]$.*

Now we will prove our nonlinear stability result of the set $\mathcal{N}_c$ in $\mathscr{Z}$.

THEOREM 3.8. *Let $c > 0$ and $\lambda = (2(p+1)I_1)^{\frac{p+2}{p}}$. Then the set $\mathscr{N}_c = M_\lambda$ is $\mathscr{Z}$-stable with regard to the flow of the BO-ZK equation if $p < 4/3$, that is, for all positive ϵ, there is a positive δ such that if $u_0 \in H^s$, $s > 2$, and $\|u_0 - \varphi\|_{\mathscr{Z}} \leq \delta$, then the solution $u(t)$ of (1.1) with $u(0) = u_0$ satisfies*

$$\sup_{t \geq 0} \inf_{\psi \in \mathscr{N}_c} \|u(t) - \psi\|_{\mathscr{Z}} \leq \epsilon.$$

Proof. Assume that $\mathscr{N}_c$ is $\mathscr{Z}$-unstable with regard to the flow of the BO-ZK equation. Then there is a sequence of initial data $u_k(0) \in \mathcal{B}_{\frac{1}{k}}(\mathscr{N}_c) \cap H^s(\mathbb{R}^2)$, $s > 2$, such that

$$\sup_{t \in [0,T)} \inf_{\varphi \in \mathscr{N}_c} \|u_k(t) - \varphi\|_{\mathscr{Z}} \geq \epsilon, \tag{3.3}$$

where $u_k(t)$ is the solution of (1.1) with initial data $u_k(0)$. So we can find, for k large enough, a time t_k such that

$$\inf_{\varphi \in \mathscr{N}_c} \|u_k(t_k) - \varphi\|_{\mathscr{Z}} = \frac{\epsilon}{2}, \tag{3.4}$$

by continuity in t. Now since E and $\mathscr{F}$ are conserved, we can find $\varphi_k \in \mathscr{N}_c$ such that

$$|E(u_k(t_k)) - E(\varphi_k)| = |E(u_k(0)) - E(\varphi_k)| \to 0, \tag{3.5}$$

$$|\mathscr{F}(u_k(t_k)) - \mathscr{F}(\varphi_k)| = |\mathscr{F}(u_k(0)) - \mathscr{F}(\varphi_k)| \to 0, \tag{3.6}$$

as $k \to +\infty$. By using Lemma 3.6, we have

$$E(u_k(t_k)) - E(\varphi_k) + v(u_k(t_k))(\mathscr{F}(u_k(t_k)) - \mathscr{F}(\varphi_k)) \geq \frac{1}{4}d''(c)|v(u_k(t_k)) - c|^2,$$

by choosing k large enough. This implies that $v(u_k(t_k)) \to c$ as $k \to +\infty$, since $u_k(t_k)$ is uniformly bounded in k. Hence, by the definition of v and continuity of d, we have

$$\lim_{k \to +\infty} J(u_k(t_k)) = \frac{2(p+1)(p+2)}{p} d(c). \tag{3.7}$$

On the other hand, by Lemma 3.4, we have

$$I(u_k(t_k)) = E(u_k(t_k)) + c\mathscr{F}(u_k(t_k)) + \frac{1}{(p+1)(p+2)}J(u_k(t_k))$$

$$= d(c) + E(u_k(t_k)) - E(\varphi_k) + c(\mathscr{F}(u_k(t_k)) - \mathscr{F}(\varphi_k)) + \frac{1}{(p+1)(p+2)}J(u_k(t_k)).$$

Then by (3.7), we obtain that

$$\lim_{k \to +\infty} I(u_k(t_k)) = \frac{p+2}{p}d(c) = (2(p+1))^{\frac{2}{p}} I_1^{\frac{p+2}{p}}. \tag{3.8}$$

By defining

$$\vartheta_k(t_k) = (J(u_k(t_k)))^{-\frac{1}{p+2}} u_k(t_k),$$

in $\mathscr{Z}$, we obtain that $J(\vartheta_k(t_k)) = 1$. Therefore by using (3.7), (3.8) and Lemma 3.4, we obtain that

$$\lim_{k \to +\infty} I(\vartheta_k(t_k)) = I_1. \tag{3.9}$$

Hence $\{\vartheta_k(t_k)\}$ is a minimizing sequence of I_1, so, from Theorem 3.2, there exists a sequence $\psi_k \subset M_1$ such that

$$\lim_{k \to +\infty} \|\vartheta_k(t_k) - \psi_k\|_{\mathscr{Z}} = 0. \tag{3.10}$$

On the other hand, from the Lagrange multiplier theorem, there exist $\theta_k \in \mathbb{R}$ such that

$$\mathscr{H}(\psi_k)_x + c\psi_k - (\psi_k)_{yy} = \theta_k(p+2)\psi_k^{p+1}, \tag{3.11}$$

so $2I_1 = \theta_k(p+2)$, which implies $\theta_k = \theta$ for all k. By scaling $\varphi_k = \mu\psi_k$ with

$$\mu^p = \theta(p+1)(p+2) = 2(p+1)I_1,$$

we obtain that φ_k satisfy (1.3) and $2(p+1)I(\varphi_k) = J(\varphi_k) = \mu^{p+2}$, which implies that $\varphi_k \in \mathscr{N}_c$ for all k. Also, by (3.7)-(3.10) and Lemma 3.4, we have

$$\|u_k(t_k) - \varphi_k\|_{\mathscr{Z}} = (J(u_k(t_k)))^{\frac{1}{p+2}} \left\| (J(u_k(t_k)))^{-\frac{1}{p+2}} (u_k(t_k) - \varphi_k) \right\|_{\mathscr{Z}}$$

$$\leq (J(u_k(t_k)))^{\frac{1}{p+2}} \left(\left\| \vartheta_k(t_k) - \mu^{-1}\varphi_k \right\|_{\mathscr{Z}} + \mu^{-1}\|\varphi_k\|_{\mathscr{Z}} - (J(u_k(t_k)))^{-\frac{1}{p+2}} \right).$$

This implies that

$$\lim_{k \to +\infty} \|u_k(t_k) - \varphi_k\|_{\mathscr{Z}} = 0;$$

which contradicts (3.4); and the proof is complete. ∎

4. Decay and Regularity

In order to investigate the regularity and the decaying properties of the solitary wave solutions of (1.1), we need to study the *kernel* of (1.3). So we remind the reader some properties of the anisotropic Sobolev spaces.

LEMMA 4.1. *If $s_i > \frac{1}{2}$, for $i = 1, 2$, then H^{s_1,s_2} is an algebra.*

One can prove the following interpolation in the anisotropic spaces.

LEMMA 4.2. *If $s_{1,i} \leq \varrho_i \leq s_{2,i}$, $i = 1, 2$, with*

$$(\varrho_1, \varrho_2) = \theta(s_{1,1}, s_{1,2}) + (1-\theta)(s_{2,1}, s_{2,2})$$

and $\theta \in [0,1]$, then

$$\|f\|_{H^{\varrho_1,\varrho_2}} \leq \|f\|_{H^{s_{1,1},s_{1,2}}}^{\theta} \|f\|_{H^{s_{2,1},s_{2,2}}}^{1-\theta}. \tag{4.1}$$

REMARK 4.3. *Note that by using the Residue theorem, the kernel of the solution of (1.3) can be written in the following form*

$$K(x,y) = C \int_0^{+\infty} \frac{|\alpha|\sqrt{t}}{\alpha^2 t^2 + x^2} e^{-\left(ct + \frac{y^2}{4t}\right)} dt, \tag{4.2}$$

where $C > 0$ is independent of α, x and y, and $\widehat{K}(\xi,\eta) = \dfrac{1}{c - \alpha|\xi| + \eta^2}$. Also by Fubini's theorem, we obtain that

$$\|K\|_{L^1} = C \int_0^{+\infty} \int_{\mathbb{R}^2} \frac{|\alpha|\sqrt{t}}{\alpha^2 t^2 + x^2} e^{-\left(ct + \frac{y^2}{4t}\right)} dxdydt = C(\alpha) \int_0^{+\infty} e^{-ct} dt.$$

Therefore,

LEMMA 4.4. *K is an even (in x and y) positive function, decreases in the $x-$direction and $y-$direction, tends to zero at infinity and belongs to $C^\infty\left(\mathbb{R}^2 \setminus \{(0,0)\}\right)$. Furthermore, $\widehat{K} \in L^p(\mathbb{R}^2)$, for any $p \in (3/2, +\infty]$ and $K \in L^p(\mathbb{R}^2)$, for any $p \in [1, +\infty)$.*

THEOREM 4.5. *Any solitary wave solution φ of (1.3), with $p \in \mathbb{N}$, belongs to $H_r^{(k)}$, for all $k \in \mathbb{N}$ and all $r \in [1, +\infty]$. Furthermore, if $0 < p < 4$, any solitary wave solution φ is continuous, belongs to $L^\infty(\mathbb{R}^2)$ and tends to zero at infinity.*

Proof. Setting $g \equiv -\dfrac{\varphi^{p+1}}{p+1}$, (1.3) yields

$$\widehat{\varphi} = \frac{\widehat{g}}{c - \alpha|\xi| + \varepsilon\eta^2}. \tag{4.3}$$

This implies that $\varphi \in H^{\frac{1}{2},1}(\mathbb{R}^2) \cap H^{0,2}(\mathbb{R}^2) \cap H^{1,0}(\mathbb{R}^2)$. By using Lemma 4.2 and the embedding (1.5), we obtain that $\varphi \in H^{s,2(1-s)}(\mathbb{R}^2)$, for any $s \in [0,1]$. By a bootstrapping argument and using the Lemmata 4.2 and 4.1, the proof of first part will be complete. The second part follows from the embedding (1.5), the Young inequality and the properties of K in Lemma 4.4. ∎

Now, we are going to study the symmetry properties of the solitay wave solutions of (1.1). Here, for $u : \mathbb{R}^2 \to \mathbb{R}$, $u^\sharp$ will represent the Steiner symmetrization of u with respect to $\{x = 0\}$ and $u^\star$ the Steiner symmetrization of u with respect to $\{y = 0\}$ (see for example [7, 22, 36]).

LEMMA 4.6. *If $f \in \mathscr{Z}$, then $f^\star, f^\sharp, |f| \in \mathscr{Z}$.*

Proof. Remember the kernel of K in (4.2). By setting $g = |f|$, then we have

$$\langle f, K * f \rangle \leq \langle g, K * g \rangle,$$

for every $c > 0$. Therefore

$$\int_{\mathbb{R}^2} \widehat{K}(\xi, \eta) \left|\widehat{f}(\xi, \eta)\right|^2 d\xi d\eta = \langle f, K * f \rangle \leq \langle g, K * g \rangle = \int_{\mathbb{R}^2} \widehat{K}(\xi, \eta) \left|\widehat{g}(\xi, \eta)\right|^2 d\xi d\eta.$$

So, we have

$$\int_{\mathbb{R}^2} c\left(1 - c\widehat{K}\right) |\widehat{g}(\xi, \eta)|^2 d\xi d\eta \leq \int_{\mathbb{R}^2} c\left(1 - c\widehat{K}\right) \left|\widehat{f}(\xi, \eta)\right|^2 d\xi d\eta, \tag{4.4}$$

since $\left\|\widehat{f}\right\|_{L^2} = \|\widehat{g}\|_{L^2}$. By taking the limit as $c \to +\infty$ on both sides of (4.4) and using the Monotone Convergence Theorem, we obtain that

$$\int_{\mathbb{R}^2} \left(|\xi| + \eta^2\right) |\widehat{g}(\xi, \eta)|^2 d\xi d\eta \leq \int_{\mathbb{R}^2} \left(|\xi| + \eta^2\right) \left|\widehat{f}(\xi, \eta)\right|^2 d\xi d\eta,$$

which shows that $|f| \in \mathscr{Z}$.

Let us prove that $f^\sharp \in \mathscr{Z}$. One can see that $K^\sharp = K = K^\star$. Then the Reisz-Sobolev rearrangement inequality (see [7, 22, 36]) implies that

$$\int_{\mathbb{R}^4} f(x,y)f(s,t)K(x-s, y-t) \, ds dt \, dx dy \leq \int_{\mathbb{R}^4} f^\sharp(x,y)f^\sharp(s,t)K(x-s, y-t) \, ds dt \, dx dy.$$

Then it follows that

$$\int_{\mathbb{R}^2} \widehat{K}(\xi, \eta) \left|\widehat{f}(\xi, \eta)\right|^2 \leq \int_{\mathbb{R}^2} \widehat{K}(\xi, \eta) \left|\widehat{f^\sharp}(\xi, \eta)\right|^2.$$

On the other hand, by using the fact that

$$\left\|\widehat{f}\right\|_{L^2(\mathbb{R}^2)} = \|f\|_{L^2(\mathbb{R}^2)} = \|f^\sharp\|_{L^2(\mathbb{R}^2)} = \left\|\widehat{f^\sharp}\right\|_{L^2(\mathbb{R}^2)},$$

a similar analysis as in the preceding proof shows that $f^\sharp \in \mathscr{Z}$. Analogously, one can prove that $f^\star \in \mathscr{Z}$. ∎

REMARK 4.7. *Note that the function $K(x, y)$ is not radial.*

LEMMA 4.8. *If $\varphi \in M_\lambda$, then $\varphi^\sharp, \varphi^\star \in M_\lambda$.*

Proof. Since Steiner symmetrization preserves the $L^{p+2}-$norm, it follows that $J(\varphi) = J(\varphi^\sharp)$. So, by using Lemma 4.6, we get

$$I_\lambda \le I\left(\varphi^\sharp\right) \le I(\varphi) = I_\lambda.$$

Therefore, we have that $\varphi^\sharp \in M_\lambda$. Similarly, $\varphi^\star \in M_\lambda$. ∎

Now, we prove our theorem concerning the symmetry properties of the solitary wave solutions of the equation (1.1).

THEOREM 4.9. *The solitary wave solutions of the equation (1.1) are radially symmetric with respect to the transverse direction and the propagation direction.*

Proof. By Theorems 2.2 and 4.5, there is the function ψ satisfying (1.3). By choosing $\varphi = \psi^{\sharp\star}$, we have that φ is a solitary wave solution of the equation (1.1) which is symmetric with respect to $\{x = 0\}$ and $\{y = 0\}$. ∎

REMARK 4.10. *Note that by the definition of the Steiner rearrangement, we have that $\varphi^{\star\sharp} \equiv \varphi^{\sharp\star}$.*

REMARK 4.11. *One may also obtain the symmetry properties of the solitary wave solutions of (1.1) by using the reflection method and a unique continuation result (see [30] and [18]).*

Now, we are going to establish the positivity of the solitary wave solutions of (1.1).

THEOREM 4.12. *The solitary wave solution φ obtained in Theorem 2.2 is positive.*

Proof. The proof follows from the proof of the Theorem 2.2, Lemmata 4.4 and 4.6, Theorem 4.5 and the following identity

$$\varphi(x, y) = \frac{1}{p+1} K * \varphi^{p+1}(x, y). \tag{4.5}$$

∎

Regarding on the decay properties of the solitary wave solutions of (1.1), one can prove easily the following properties of the kernel K.

LEMMA 4.13. $K \in H^{s_1,0}\left(\mathbb{R}^2\right) \cap H^{0,s_2}\left(\mathbb{R}^2\right)$, *for any $s_1 < \frac{1}{4}$ and $s_2 < \frac{1}{2}$. Moreover, $K \in H^{r,s}\left(\mathbb{R}^2\right) \cap H^{(s_1,s_2)}\left(\mathbb{R}^2\right)$, where $rs_2 + ss_1 = s_1 s_2$ and $r \in [0, 1]$.*

LEMMA 4.14. (i) $\widehat{K} \in H^{s_1,0}\left(\mathbb{R}^2\right) \cap H^{0,s_2}\left(\mathbb{R}^2\right)$, *for any $s_1 < \frac{3}{2}$ and $s_2 \in \mathbb{R}$. Moreover, $\widehat{K} \in H^{r,s}\left(\mathbb{R}^2\right) \cap H^{(s_1,s_2)}\left(\mathbb{R}^2\right)$, where $rs_2 + ss_1 = s_1 s_2$ and $r \in [0, 1]$.*

(ii) $\widehat{K} \in H_p^{(s_1, s_2)}\left(\mathbb{R}^2\right)$, for any $s_1 < 1 + \dfrac{1}{p}$, $p \geq 2$ and $s_2 \in \mathbb{R}$.

(iii) $|x|^{s_1}|y|^{s_2}K \in L^p\left(\mathbb{R}^2\right)$, for any $s_1 < 2 - \dfrac{1}{p}$, $2s_1 + s_2 \geq 3\left(1 - \dfrac{1}{p}\right)$ and $p \geq 1$.

LEMMA 4.15. *Let ℓ and m be two constants satisfying $0 < \ell < m - 2$. Then there exists $C > 0$, depending only on ℓ and m, such that for all $\epsilon > 0$, we have*

$$\int_{\mathbb{R}^2} \frac{|y|^\ell}{(1 + \epsilon|y|)^m(1 + |x - y|)^m}\, dy \leq \frac{C\,|x|^\ell}{(1 + \epsilon|x|)^m}, \qquad \forall |x| \geq 1, \qquad (4.6)$$

$$\int_{\mathbb{R}^2} \frac{1}{(1 + \epsilon|y|)^m(1 + |x - y|)^m}\, dy \leq \frac{C}{(1 + \epsilon|x|)^m}, \qquad \forall x \in \mathbb{R}^2. \qquad (4.7)$$

The proof of Lemma 4.15 is elementary and is essentially the same as the proof of Lemma 3.1.1 in [6].

THEOREM 4.16. *For any solitary wave solution of (1.3), we have*

(i) $|x|^\ell|y|^\varrho\varphi(x, y) \in L^p\left(\mathbb{R}^2\right)$ for all $p \in (1, +\infty)$, any $\ell \in [0, 1)$ and any $\varrho \geq 0$,

(ii) $|(x, y)|^\theta\varphi(x, y) \in L^p\left(\mathbb{R}^2\right)$ for all $p \in (1, +\infty)$ and any $\theta \in [0, 1)$,

(iii) $\varphi \in L^1\left(\mathbb{R}^2\right)$.

Proof. (i)   Choose $\ell \in \left[0, s_1 - 1 + \dfrac{1}{p}\right)$ and $p > 1$, where $s_1 < 2 - \dfrac{1}{p}$. For $0 < \epsilon < 1$, we denote

$$h_\epsilon(x, y) = \mathcal{A}_\epsilon(x, y)\,\varphi(x, y),$$

where

$$\mathcal{A}(x, y) = \frac{|x|^\ell|y|^\varrho}{(1 + \epsilon|x|)^{s_1}(1 + \epsilon|y|)^{s_2}}$$

and $s_2 \geq 3\left(1 - \dfrac{1}{p}\right)$. Then $h_\epsilon \in L^{p'}\left(\mathbb{R}^2\right)$, where $p' = \frac{p}{p-1}$. Using Hölder's inequality, we obtain that

$$|\varphi(x, y)| \leq C(s_1, s_2, p)\left(\int_{\mathbb{R}^2} |\mathcal{G}(z, w)|^{p'}\, dzdw\right)^{\frac{1}{p'}},$$

where

$$\mathcal{G}(z, w) = \frac{g(\varphi)(z, w)}{(1 + |x - z|)^{s_1}(1 + |y - w|)^{s_2}},$$

$g(t) = \dfrac{t^{p+1}}{p + 1}$ and

$$C(s_1, s_2, p) = \left\|(1 + |x|)^{s_1}(1 + |y|)^{s_2}K\right\|_{L^p(\mathbb{R}^2)}.$$

Note that the fact that $\varphi \to 0$ as $|(x, y)| \to +\infty$ implies that for every $\delta > 0$, there exists $R_\delta > 1$ such that if $|(x, y)| \geq R_\delta$, we have

$$\left|g(\varphi)(x, y)\right| \leq \delta|\varphi(x, y)|.$$

By using Hölder's inequality, we obtain that

$$
\int_{\mathbb{R}^2 \setminus B(0,R_\delta)} |\hbar_\epsilon(x,y)|^{p'} \, dxdy = \int_{\mathbb{R}^2 \setminus B(0,R_\delta)} |\hbar_\epsilon(x,y)|^{p'-r} \mathcal{A}^r(x,y) |\varphi(x,y)|^r \, dxdy
$$

$$
\leq C(s_1,s_2,p)^r \int_{\mathbb{R}^2 \setminus B(0,R_\delta)} |\hbar_\epsilon(x,y)|^{p'-r} \mathcal{A}^r(x,y) \, \|\mathcal{G}\|^r_{L^{p'}(\mathbb{R}^2)}(x,y) \, dxdy
$$

$$
\leq C(s_1,s_2,p)^r \|\hbar_\epsilon\|^{p'-r}_{L^{p'}(\mathbb{R}^2 \setminus B(0,R_\delta))} \left\| \mathcal{A} \, \|\mathcal{G}\|_{L^{p'}(\mathbb{R}^2)} \right\|^r_{L^{p'}(\mathbb{R}^2 \setminus B(0,R_\delta))}
$$

Thusly,

$$
\int_{\mathbb{R}^2 \setminus B(0,R_\delta)} |\hbar_\epsilon(x,y)|^{p'} \, dxdy \leq C(s_1,s_2,p)^{p'} \int_{\mathbb{R}^2 \setminus B(0,R_\delta)} \mathcal{A}^{p'}(x,y) \|\mathcal{G}\|^{p'}_{L^{p'}(\mathbb{R}^2)} \, dxdy.
$$

Using Fubini's theorem and Lemma 4.15, we obtain

$$
\int_{\mathbb{R}^2 \setminus B(0,R_\delta)} \mathcal{A}^{p'}(x,y) \|\mathcal{G}\|^{p'}_{L^{p'}(\mathbb{R}^2)}(x,y) \, dxdy
$$

$$
= \int_{\mathbb{R}^2} |g(\varphi)(z,w)|^{p'} \left(\int_{\mathbb{R}^2 \setminus B(0,R_\delta)} \frac{\mathcal{A}^{p'}(x,y)}{(1+|x-z|)^{p's_1}(1+|y-w|)^{p's_2}} \, dxdy \right) dzdw
$$

$$
\leq C \int_{\mathbb{R}^2 \setminus B(0,R_\delta)} |g(\varphi)(z,w)|^{p'} \mathcal{A}^{p'}(z,w) \, dzdw
$$

$$
+ \int_{B(0,R_\delta)} |g(\varphi)(z,w)|^{p'} \left(\int_{\mathbb{R}^2 \setminus B(0,R_\delta)} \frac{\mathcal{A}^{p'}(x,y)}{(1+|x-z|)^{p's_1}(1+|y-w|)^{p's_2}} \, dxdy \right) dzdw.
$$

The last integral is bounded by a constant $\mathcal{C}'$ depending on φ and R_δ and independent of ϵ. Therefore, by using the fact that $|g(\varphi)(x,y)| \leq \delta|\varphi(x,y)|$ on $\mathbb{R}^2 \setminus B(0,R_\delta)$, we get

$$
\int_{\mathbb{R}^2 \setminus B(0,R_\delta)} |\hbar_\epsilon(x,y)|^{p'} \, dxdy \leq C(s_1,s_2,p)^{p'} \left(C\delta^{p'} \int_{\mathbb{R}^2 \setminus B(0,R_\delta)} |\hbar_\epsilon(x,y)|^{p'} \, dxdy + \mathcal{C}' \right).
$$

Choosing δ such that $C(s_1,s_2,p)\delta \mathcal{C}^{\frac{1}{p'}} < 1$, from the last inequality, we deduce that

$$
\int_{\mathbb{R}^2 \setminus B(0,R_\delta)} |\hbar_\epsilon(x,y)|^{p'} \, dxdy \leq \mathcal{C}, \tag{4.8}
$$

where $\mathcal{C}$ is a constant independent of ϵ. Now, we let $\epsilon \to 0$ in (4.8) and apply Fatou's lemma to obtain that

$$
\int_{\mathbb{R}^2 \setminus B(0,R_\delta)} |x|^{\ell p'} |y|^{\varrho p'} |\varphi(x,y)|^{p'} \, dxdy \leq \mathcal{C}.
$$

Hence $|x|^\ell |y|^\varrho \varphi(x,y) \in L^{p'}\left(\mathbb{R}^2\right)$, for $p' = \dfrac{p}{p-1}$.

Now by taking the limits $p \to 1$ and $p \to +\infty$, we obtain that $\ell \to 1$ and $p' \in (1,+\infty)$. This proves part (i) of the theorem.

(ii) The proof follows from (i).

(iii) Let $s > 1$ and g, δ and R_δ be the same above in (i). Define

$$
\mathcal{A}_\epsilon(x,y) = \frac{1}{(1+\epsilon|(x,y)|)^s}.
$$

Therefore, we have

$$\int_{\mathbb{R}^2 \setminus B(0,R_\delta)} |\varphi(x,y)| \mathcal{A}_\epsilon(x,y) \, dxdy$$

$$\leq \int_{\mathbb{R}^2} |g(\varphi)(z,w)| \left(\int_{\mathbb{R}^2 \setminus B(0,R_\delta)} \mathcal{A}_\epsilon(x,y) K(x-z,y-w) \, dxdy \right) dzdw$$

$$\leq \int_{\mathbb{R}^2} |g(\varphi)(z,w)| \left(\int_{\mathbb{R}^2 \setminus B(0,R_\delta)} \mathcal{A}_1^{-2}(x-z,y-w) K^2(x-z,y-w) \, dxdy \right)^{\frac{1}{2}}$$

$$\cdot \left(\int_{\mathbb{R}^2 \setminus B(0,R_\delta)} \mathcal{A}_1^2(x-z,y-w) \mathcal{A}_\epsilon^2(x,y) \, dxdy \right)^{\frac{1}{2}} dzdw$$

$$\leq C(s) \mathcal{C}^{\frac{1}{2}} \int_{\mathbb{R}^2} |g(\varphi)(z,w)| \, \mathcal{A}_\epsilon(z,w) \, dzdw$$

$$\leq C(s) \mathcal{C}^{\frac{1}{2}} \delta \int_{\mathbb{R}^2 \setminus B(0,R_\delta)} |\varphi(z,w)| \mathcal{A}_\epsilon(z,w) \, dzdw$$

$$+ C(s) \mathcal{C}^{\frac{1}{2}} \int_{B(0,R_\delta)} |g(\varphi)(z,w)| \, dzdw,$$

by using Fubini's theorem, Lemma 4.15 and the fact that φ, $\mathcal{A}_\epsilon \in L^2 \left(\mathbb{R}^2 \right)$ and $\varphi \mathcal{A}_\epsilon \in L^1 \left(\mathbb{R}^2 \right)$. Hence by the restriction on δ, and using Fatou's lemma as $\epsilon \to 0$, we conclude that $\varphi \in L^1 \left(\mathbb{R}^2 \right)$. ∎

The following corollary is an immediate consequence of (4.5), Theorem 4.16 and the inequality

$$|t|^\theta \leq C \left(|t-s|^\theta + |s|^\theta \right), \quad \text{for } \theta \geq 0. \tag{4.9}$$

Corollary 4.17. *Suppose that $\varphi \in L^\infty \left(\mathbb{R}^2 \right)$ satisfies (1.3) and $\varphi \to 0$ at infinity. Then*

(i) $|x|^\ell |y|^\varrho \varphi(x,y) \in L^\infty \left(\mathbb{R}^2 \right)$, *for all $\ell \in [0,1)$ and any $\varrho \geq 0$,*

(ii) $|(x,y)|^\theta \varphi(x,y) \in L^\infty \left(\mathbb{R}^2 \right)$, *for all $\theta \in [0,1)$.*

LEMMA 4.18. $|x|^\ell |y|^\varrho K \in L^\infty \left(\mathbb{R}^2 \right)$, *for any $\ell \leq 2$ and any $\varrho \geq 0$.*

Proof. Suppose that $|x| \geq 1$, so we have

$$K(x,y) < C(\alpha) \int_0^{+\infty} \frac{e^{-ct\sqrt{t}}}{x^2} e^{-\frac{y^2}{4t}} \, dt \leq C(\alpha) \int_0^{+\infty} \frac{e^{-ct\sqrt{t}}}{x^2} \left(\frac{4t}{y^2} \right)^\nu \, dt = \frac{C(\alpha)}{x^2 |y|^{2\nu}},$$

for any $\nu \geq 0$. On the other hand, for $0 < |x| \leq 1$, by a change of variables, we have that

$$K(x,y) \leq \frac{C(\alpha)}{\sqrt{|x|}} \int_0^{+\infty} \frac{e^{-t|x|}\sqrt{t}}{1+\alpha^2 t^2} e^{-\frac{y^2}{4|x|t}} \, dt$$

$$\leq \frac{C(\alpha)}{\sqrt{|x|}} \int_0^{+\infty} \frac{e^{-t|x|}\sqrt{t}}{1+\alpha^2 t^2} \left(\frac{4|x|t}{y^2} \right)^\nu \, dt \leq \frac{C(\alpha)}{|y|^{2\nu}},$$

for any $\nu \geq 0$. This completes the proof. ∎

Corollary 4.19. $|x|^\ell |y|^\varrho \varphi(x, y) \in L^\infty(\mathbb{R}^2)$, for any $\ell \leq 2$ and any $\varrho \geq 0$.

Proof. Without loss of generality, we assume that $\varrho = 0$. Let $\ell < 1$ and $\gamma = \min\{2, (p+1)\ell\}$. Because

$$|x|^\gamma \left|\varphi^{p+1}(x, y)\right| \leq \left(|\varphi(x, y)||x|^{\frac{\gamma}{p+1}}\right) \left(|\varphi(x, y)||x|^{\frac{\gamma}{p+1}}\right)^p,$$

then by using (4.9), Corollary 4.17, Lemma 4.18 and Theorem 4.16, we obtain that $|x|^\gamma \varphi(x, y) \in L^\infty(\mathbb{R}^2)$. If $\gamma = (p+1)\ell$, one may use the above argument to show that $|x|^\gamma \varphi(x, y) \in L^\infty(\mathbb{R}^2)$ for $\gamma = \min\{2, (p+1)^2\ell\}$. Then repeating this argument at most finitely many times leads to the conclusion. ∎

The following corollary follows from (4.9), Corollary 4.17 and Theorem 4.16.

Corollary 4.20. (i) $|x|^\ell |y|^\varrho \varphi(x, y) \in L^1(\mathbb{R}^2)$, for all $\ell \in [0, 1)$ and any $\varrho \geq 0$,
(ii) $|(x, y)|^\theta \varphi(x, y) \in L^1(\mathbb{R}^2)$, for all $\theta \in [0, 1)$.

LEMMA 4.21. There exists $\sigma_0 > 0$ such that for any $\sigma < \sigma_0$ and any $s < \frac{3}{2}$, we have
(i) $|x|^s e^{\sigma|y|} K \in L^2(\mathbb{R}^2)$,
(ii) $K \in L_y^p L_x^1(\mathbb{R}^2)$; for any $1 \leq p \leq \infty$,
(iii) $|x|^s e^{\sigma|y|} K \in L_y^2 L_x^1(\mathbb{R}^2)$; where $\|\cdot\|_{L_y^q L_x^p(\mathbb{R}^2)} = \left\| \|\cdot\|_{L_x^p(\mathbb{R})} \right\|_{L_y^q(\mathbb{R})}$.

Proof. By a change of variables, K can be written in the following form

$$K(x, y) = |\alpha| \int_0^{+\infty} \frac{e^{-c|x|t}}{1 + \alpha^2 t^2} \left(\frac{t}{|x|}\right)^{\frac{1}{2}} e^{-\frac{y^2}{4|x|t}} \, dt.$$

Hence,

$$\left\| |x|^s e^{\sigma|y|} K \right\|_{L^2(\mathbb{R}^2)} \leq \alpha \int_0^{+\infty} \frac{\sqrt{t}}{1 + \alpha^2 t^2} \left(\int_\mathbb{R} |x|^{2s-1} e^{-2c|x|t} \int_\mathbb{R} e^{2\sigma|y|} e^{-\frac{y^2}{2|x|t}} \, dydx\right)^{\frac{1}{2}} dt$$

$$= \alpha \int_0^{+\infty} \frac{t^{\frac{3}{4}}}{1 + \alpha^2 t^2} \left(\int_\mathbb{R} |x|^{2s-\frac{1}{2}} e^{-2|x|t(c-\sigma^2)} \int_{-\sigma\sqrt{|x|t}}^{+\infty} e^{-2y^2} \, dydx\right)^{\frac{1}{2}} dt$$

$$\leq C(\alpha) \int_0^{+\infty} \frac{t^{\frac{3}{4}}}{1 + \alpha^2 t^2} \left(\int_\mathbb{R} |x|^{2s-\frac{1}{2}} e^{-2|x|t(c-\sigma^2)} dx\right)^{\frac{1}{2}} dt$$

$$= C(\alpha) \frac{\sqrt{\Gamma\left(2s + \frac{1}{2}\right)}}{(2(c - \sigma^2))^{s+\frac{1}{4}}} \int_0^{+\infty} \frac{t^{\frac{1}{2}-s}}{1 + \alpha^2 t^2} \, dt,$$

which is finite for any $\sigma < \sqrt{c}$ and any $s < \frac{3}{2}$.

The proof of (ii) follows from the following identity:

$$\|K\|_{L_x^1} = C(\alpha) \int_0^\infty \frac{e^{-ct-\frac{y^2}{4t}}}{\sqrt{t}} \, dt = \frac{C(\alpha)}{\sqrt{c}} e^{-\sqrt{c}|y|}.$$

The proof of (iii) is similar. ∎

The following corollary is a consequence of the Young inequality:

$$\|f * g\|_{L_y^q L_x^p(\mathbb{R}^2)} \leq \|f\|_{L_y^{q_1} L_x^{p_1}(\mathbb{R}^2)} \|g\|_{L_y^{q_2} L_x^{p_2}(\mathbb{R}^2)},$$

where $1 \leq p, q, p_1, q_1, p_2, q_2 \leq \infty$, $1 + \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and $1 + \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$.

Corollary 4.22. $\varphi \in L_y^p L_x^1 \left(\mathbb{R}^2\right)$, for any $1 \leq p \leq \infty$.

Now we state our main result of decaying of the solitary wave solutions.

THEOREM 4.23. *Let* $\sigma_0 > 0$ *be in Lemma 4.21. Then for any* $\sigma \in [0, \sigma_0)$ *and any* $s < \frac{3}{2}$*, we have that* $|x|^s e^{\sigma|y|} \varphi(x, y) \in L^1\left(\mathbb{R}^2\right) \cap L^\infty\left(\mathbb{R}^2\right)$.

Proof. Without loss of generality we assume that $s = 0$. By using Lemma 4.21 and the proof of Corollary 3.14 in [6], with natural modifications, there exists $\widetilde{\sigma} \geq \sigma_0$ such that $e^{\sigma|y|}\varphi(x, y) \in L^1\left(\mathbb{R}^2\right)$, for any $\sigma < \widetilde{\sigma}$. Now by using the following inequality:

$$|\varphi(x,y)|e^{\sigma|y|} \leq \int_{\mathbb{R}^2} |K(x-z, y-w)|e^{\sigma|y-w|}|\varphi(z,w)|e^{\sigma|w|}|\varphi(z,w)|^p \, dzdw, \quad (4.10)$$

and the facts $\varphi(x,y)e^{\sigma|y|} \in L^1\left(\mathbb{R}^2\right)$, $\varphi \in L^\infty\left(\mathbb{R}^2\right)$ and $K(x,y)e^{\sigma|y|} \in L^2\left(\mathbb{R}^2\right)$, for any $\sigma < \sigma_0$, we obtain that $\varphi(x,y)e^{\sigma|y|} \in L^\infty\left(\mathbb{R}^2\right)$, for any $\sigma < \sigma_0$. ∎

Finally, the following theorem shows that analyticity of our solitary wave solutions.

THEOREM 4.24. *Let* $1 \leq p < 4$ *be integer. Then there exist* $\sigma > 0$ *and an holomorphic function* f *of two variables* z_1 *and* z_2*, defined in the domain*

$$\mathcal{H}_\sigma = \left\{(z_1, z_2) \in \mathbb{C}^2 \; ; \; |\mathrm{Im}(z_1)| < \sigma, \; |\mathrm{Im}(z_2)| < \sigma\right\}$$

such that $f(x, y) = \varphi(x, y)$ *for all* $(x, y) \in \mathbb{R}^2$.

Proof. By the Cauchy-Schwarz inequality, we have that $\widehat{\varphi} \in L^1\left(\mathbb{R}^2\right)$. The equation (1.3) implies that

$$|\xi| \, |\widehat{\varphi}|\, (\xi, \eta) \leq \overbrace{|\widehat{\varphi}| * \cdots * |\widehat{\varphi}|}^{p+1}(\xi, \eta), \qquad (4.11)$$

$$|\eta| \, |\widehat{\varphi}|\, (\xi, \eta) \leq \underbrace{|\widehat{\varphi}| * \cdots * |\widehat{\varphi}|}_{p+1}(\xi, \eta). \qquad (4.12)$$

We denote $\mathscr{T}_1(|\widehat{\varphi}|) = |\widehat{\varphi}|$ and for $m \geq 1$, $\mathscr{T}_{m+1}(|\widehat{\varphi}|) = \mathscr{T}_m(|\widehat{\varphi}|) * |\widehat{\varphi}|$. It can be easily seen by induction that

$$r^m|\widehat{\varphi}|(\xi, \eta) \leq (m-1)! \, (p+1)^{m-1} \mathscr{T}_{mp+1}(|\widehat{\varphi}|)(\xi, \eta), \qquad (4.13)$$

where $r = |(\xi, \eta)|$. Then we have

$$r^m|\widehat{\varphi}|(\xi, \eta) \leq (m-1)! \, (m+1)^{m-1} \left\|\mathscr{T}_{mp+1}(|\widehat{\varphi}|)\right\|_{L^\infty(\mathbb{R}^2)}$$

$$\leq (m-1)! \, (m+1)^{m-1} \left\|\mathscr{T}_{mp}(|\widehat{\varphi}|)\right\|_{L^2(\mathbb{R}^2)} \|\widehat{\varphi}\|_{L^2(\mathbb{R}^2)}$$

$$\leq (m-1)! \, (m+1)^{m-1} \|\widehat{\varphi}\|_{L^1(\mathbb{R}^2)}^{mp} \|\widehat{\varphi}\|_{L^2(\mathbb{R}^2)}^2.$$

Let

$$a_m = \frac{(m-1)! \, (m+1)^{m-1} \|\widehat{\varphi}\|_{L^1(\mathbb{R}^2)}^{mp} \|\widehat{\varphi}\|_{L^2(\mathbb{R}^2)}^2}{m!},$$

then it is clear that

$$\frac{a_{m+1}}{a_m} \longrightarrow (p+1)\|\widehat{\varphi}\|_{L^1(\mathbb{R}^2)}^p,$$

as $m \to +\infty$. Therefore the series $\sum_{m=0}^{\infty} t^m r^m |\widehat{\varphi}|(\xi, \eta)/m!$ converges uniformly in $L^\infty(\mathbb{R}^2)$, if $0 < t < \sigma = \frac{1}{p+1} \|\widehat{\varphi}\|_{L^1(\mathbb{R}^2)}^{-p}$. Hence $e^{tr} \widehat{\varphi}(\xi, \eta) \in L^\infty(\mathbb{R}^2)$, for $t < \sigma$.

We define the function

$$f(z_1, z_2) = \int_{\mathbb{R}^2} e^{i(\xi z_1 + \eta z_2)} \widehat{\varphi}(\xi, \eta) \, d\xi d\eta.$$

By the Paley-Wiener Theorem, f is well defined and analytic in $\mathcal{H}_\sigma$; and by Plancherel's Theorem, we have $f(x, y) = \varphi(x, y)$ for all $(x, y) \in \mathbb{R}^2$. This proves the theorem. ∎

Acknowledgments

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