SOLUTION SENSITIVITY FOR KARUSH–KUHN–TUCKER SYSTEMS WITH NONUNIQUE LAGRANGE MULTIPLIERS*

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ABSTRACT

This paper is devoted to quantitative stability of a given primal-dual solution of the Karush–Kuhn–Tucker system subject to parametric perturbations. We are mainly concerned with those cases when the dual solution associated to the base primal solution is nonunique. Starting with a review of known results regarding the Lipschitz-stable case, supplied by simple direct justifications based on piecewise analysis, we then proceed with new results for the cases of Hölder (square-root) stability. Our results include characterizations of asymptotic behavior and upper estimates of perturbed solutions, as well as some sufficient conditions for (the specific kinds of) stability of a given solution subject to directional perturbations. We argue that Lipschitz stability of strictly complementary multipliers is highly unlikely to occur, and we employ the recently introduced notion of a critical multiplier for dealing with Hölder stability.

Key words: Karush–Kuhn–Tucker system, mathematical programming, stability, sensitivity, constraint qualification, piecewise analysis, critical multipliers.
AMS subject classifications. 90C31

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1 Introduction

We consider the parametric *Karush–Kuhn–Tucker* (KKT) *system*

$$\begin{array}{l}\Phi(\sigma,\,x)+\left(\frac{\partial F}{\partial x}(\sigma,\,x)\right)^{\mathrm{T}}\lambda+\left(\frac{\partial G}{\partial x}(\sigma,\,x)\right)^{\mathrm{T}}\mu=0,\quad F(\sigma,\,x)=0,\\ \mu\geq 0,\quad G(\sigma,\,x)\leq 0,\quad \langle\mu,\,G(\sigma,\,x)\rangle=0,\end{array}\tag{1.1}$$

with respect to $(x,\,\lambda,\,\mu)\in\mathbf{R}^n\times\mathbf{R}^l\times\mathbf{R}^m$, where $\sigma\in\mathbf{R}^s$ is a parameter, and $\Phi\mathbf{R}^s\times\mathbf{R}^n\to\mathbf{R}^n$, $F:\mathbf{R}^s\times\mathbf{R}^n\to\mathbf{R}^l$, $G:\mathbf{R}^s\times\mathbf{R}^n\to\mathbf{R}^m$ are sufficiently smooth mappings.

If for some smooth function $f:\mathbf{R}^s\times\mathbf{R}^n\to\mathbf{R}$ it holds that

$$\Phi(\sigma,\,x)=\frac{\partial f}{\partial x}(\sigma,\,x),\quad \sigma\in\mathbf{R}^s,\,x\in\mathbf{R}^n,\tag{1.2}$$

then, as is well-known, (1.1) is the KKT optimality system for the parametric mathematical programming (MP) problem

$$\begin{array}{ll}\text{minimize} & f(\sigma,\,x)\\ \text{subject to} & F(\sigma,\,x)=0,\quad G(\sigma,\,x)\leq 0,\end{array}\tag{1.3}$$

characterizing stationary points and associated Lagrange multipliers of the latter problem. However, KKT systems with non-gradient Φ have some important applications beyond the field of MP (say, in the theory of variational inequalities).

Let $(\bar{x},\,\bar{\lambda},\,\bar{\mu})\in\mathbf{R}^n\times\mathbf{R}^l\times\mathbf{R}^m$ be a solution of (1.1) for some fixed (base) parameter value $\sigma=\bar{\sigma}\in\mathbf{R}^s$. Define the index sets

$$A=A(\bar{\sigma},\,\bar{x})=\{i=1,\,\ldots,\,m\mid G_i(\bar{\sigma},\,\bar{x})=0\},$$

$$N=N(\bar{\sigma},\,\bar{x})=\{i=1,\,\ldots,\,m\mid G_i(\bar{\sigma},\,\bar{x})<0\}.$$

Let us recall some standard constraint qualifications, to be used in the sequel. The Mangasarian–Fromovitz constraint qualification (MFCQ) consists of saying that rank $\frac{\partial F}{\partial x}(\bar{\sigma},\,\bar{x})=l$ and there exists $\bar{\xi}\in\ker\frac{\partial F}{\partial x}(\bar{\sigma},\,\bar{x})$ such that $\frac{\partial G_A}{\partial x}(\bar{\sigma},\,\bar{x})\bar{\xi}<0$. Here $\ker\Lambda$ stands for the kernel (null space) of a linear operator Λ, and for a vector z, z_I stands for its subvector with components z_i, $i\in I$. The strict Mangasarian–Fromovitz constraint qualification (SMFCQ) is a combination of MFCQ and the requirement that $(\bar{\lambda},\,\bar{\mu})$ is the unique dual solution (multiplier) of system (1.1) with $\sigma=\bar{\sigma}$, associated with the primal solution $\bar{x}$. The stronger linear independence constraint qualification (LICQ) consists of saying that

$$\mathrm{rank}\left(\begin{array}{c}\frac{\partial F}{\partial x}(\bar{\sigma},\,\bar{x})\\ \frac{\partial G_A}{\partial x}(\bar{\sigma},\,\bar{x})\end{array}\right)=l+|A|,$$

where $|I|$ stands for the cardinality of a finite set I.

We are interested in behavior of the primal-dual solution $(\bar{x},\,\bar{\lambda},\,\bar{\mu})$ subject to small parametric perturbations of (1.1), and of our main concern are the cases when $(\bar{\lambda},\,\bar{\mu})$ is possibly a nonunique multiplier associated with the primal solution $\bar{x}$. This means that we do not assume LICQ (or even SMFCQ) to hold at $\bar{x}$ (for $(\bar{\lambda},\,\bar{\mu})$) for the constraint system

$$F(\bar{\sigma},\,x)=0,\quad G(\bar{\sigma},\,x)\leq 0\tag{1.4}$$

(though some results provided below are meaningful and new in the case of a unique multiplier as well, and some even subsume uniqueness, or even SMFCQ). To this end, let $\mathcal{M} = \mathcal{M}(\bar{\sigma}, \bar{x})$ stand for the set of multipliers associated with $\bar{x}$, i.e., pairs $(\lambda, \mu) \in \mathbf{R}^l \times \mathbf{R}^m$ such that $(\bar{x}, \lambda, \mu)$ satisfies (1.1) for $\sigma = \bar{\sigma}$.

Let us note that in this work, we understand stability in a very weak sense: we say that the given primal-dual solution $(\bar{x}, \bar{\lambda}, \bar{\mu})$ (or the multiplier $(\bar{\lambda}, \bar{\mu})$ associated with the primal solution $\bar{x}$) is *stable* if it is persistent subject to *some* arbitrarily small perturbations. This means that there exists a sequence $\{\sigma^k\} \subset \mathbf{R}^s$ such that $\{\sigma^k\} \to \bar{\sigma}$, and for any k system (1.1) for $\sigma = \sigma^k$ has a solution (x^k, λ^k, μ^k) such that $\{x^k\} \to \bar{x}$, $\{\lambda^k\} \to \bar{\lambda}$, $\{\mu^k\} \to \bar{\mu}$. Moreover, we will often be talking about stability with respect to some subclass of possible perturbations, that is, for σ in some subset of a neighborhood of $\bar{\sigma}$.

Furthermore, define the index sets

$$A_+ = A_+(\bar{\sigma}, \bar{x}, \bar{\mu}) = \{i \in A \mid \bar{\mu}_i > 0\},$$

$$A_0 = A_0(\bar{\sigma}, \bar{x}, \bar{\mu}) = \{i \in A \mid \bar{\mu}_i = 0\}.$$

Condition $A_0 = \emptyset$ is known as the *strict complementarity condition*. One of the main messages of this work is that, surprisingly, *stability of strictly complementary multipliers is in a sense a more subtle issue than that of those multipliers that violate strict complementarity condition*, and in particular, one should not expect Lipschitz stability of strictly complementary multipliers (we emphasize again that we are concerned with the case when multiplier is not unique). Thus, the highly developed tools for treating the Lipschitzian behavior are not applicable in this context. At the same time, as the reader will see below, dealing with Hölder stable case requires much more burdensome constructions.

The basic tool of our analysis is the local piecewise decomposition of the solution set of (1.1). Denote by $\mathcal{A}_0$ the set of all partitions of A_0, that is, pairs (I_1, I_2) of index sets satisfying $I_1 \cup I_2 = A_0$, $I_1 \cap I_2 = \emptyset$. For each partition $(I_1, I_2) \in \mathcal{A}_0$, define the *branch system*

$$\begin{array}{l} \Phi(\sigma, x) + \left(\frac{\partial F}{\partial x}(\sigma, x)\right)^{\mathrm{T}} \lambda + \left(\frac{\partial G}{\partial x}(\sigma, x)\right)^{\mathrm{T}} \mu = 0, \quad F(\sigma, x) = 0, \quad G_{A_+}(\sigma, x) = 0, \\ \mu_{I_1} \geq 0, \quad G_{I_1}(\sigma, x) = 0, \quad \mu_{I_2} = 0, \quad G_{I_2}(\sigma, x) \leq 0, \quad \mu_N = 0. \end{array} \tag{1.5}$$

It can be easily checked that for $\sigma \in \mathbf{R}^s$ close enough to $\bar{\sigma}$, the solution set of (1.1) near $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is the union of the solution sets of these branch systems, and that $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is a solution of each branch system for $\sigma = \bar{\sigma}$. This piecewise decomposition allows for simple direct proofs of many sensitivity results, avoiding any use of multifunctions theory and/or generalized differentiation.

2 Lipschitz Stability

Define the mapping $\Psi : \mathbf{R}^s \times \mathbf{R}^n \times \mathbf{R}^l \times \mathbf{R}^m \to \mathbf{R}^n$,

$$\Psi(\sigma, x, \lambda, \mu) = \Phi(\sigma, x) + \left(\frac{\partial F}{\partial x}(\sigma, x)\right)^{\mathrm{T}} \lambda + \left(\frac{\partial G}{\partial x}(\sigma, x)\right)^{\mathrm{T}} \mu. \tag{2.1}$$

When (1.2) holds, Ψ is the gradient of the Lagrangian of problem (1.3) with respect to x.

In this section, which is mostly a survey, we discuss the cases when one can expect Lipschitz stability of the given primal-dual solution. To that end, we start with the following simple (though still very important) result on asymptotic behavior of solutions of the perturbed KKT systems.

Theorem 2.1 *Let $(\bar{x}, \bar{\lambda}, \bar{\mu}) \in \mathbf{R}^n \times \mathbf{R}^l \times \mathbf{R}^m$ be a solution of system* (1.1) *for $\sigma = \bar{\sigma} \in \mathbf{R}^s$. Let sequences $\{\sigma^k\} \subset \mathbf{R}^s$, $\{x^k\} \subset \mathbf{R}^n$, $\{\lambda^k\} \subset \mathbf{R}^l$, $\{\mu^k\} \subset \mathbf{R}^m$ and $\{t_k\} \subset \mathbf{R}_+ \setminus \{0\}$ be such that $\{\sigma^k\} \to \bar{\sigma}$, $\{x^k\} \to \bar{x}$, $\{\lambda^k\} \to \bar{\lambda}$, $\{\mu^k\} \to \bar{\mu}$, $\{t_k\} \to 0$, and such that for each k the point (x^k, λ^k, μ^k) is a solution of system* (1.1) *for $\sigma = \sigma^k$.*

Then any limit point $(d, \xi, \eta, \zeta) \in \mathbf{R}^s \times \mathbf{R}^n \times \mathbf{R}^l \times \mathbf{R}^m$ of the sequence $\{(\sigma^k - \bar{\sigma}, x^k - \bar{x}, \lambda^k - \bar{\lambda}, \mu^k - \bar{\mu})/t_k\}$ satisfies the system

$$\frac{\partial \Psi}{\partial \sigma}(\bar{\sigma}, \bar{x}, \bar{\lambda}, \bar{\mu})d + \frac{\partial \Psi}{\partial x}(\bar{\sigma}, \bar{x}, \bar{\lambda}, \bar{\mu})\xi + \left(\frac{\partial F}{\partial x}(\bar{\sigma}, \bar{x})\right)^{\mathrm{T}} \eta + \left(\frac{\partial G}{\partial x}(\bar{\sigma}, \bar{x})\right)^{\mathrm{T}} \zeta = 0, \tag{2.2}$$

$$\frac{\partial F}{\partial \sigma}(\bar{\sigma}, \bar{x})d + \frac{\partial F}{\partial x}(\bar{\sigma}, \bar{x})\xi = 0, \tag{2.3}$$

$$\frac{\partial G_{A_+}}{\partial \sigma}(\bar{\sigma}, \bar{x})d + \frac{\partial G_{A_+}}{\partial x}(\bar{\sigma}, \bar{x})\xi = 0, \tag{2.4}$$

$$\zeta_{A_0} \geq 0, \ \frac{\partial G_{A_0}}{\partial \sigma}(\bar{\sigma}, \bar{x})d + \frac{\partial G_{A_0}}{\partial x}(\bar{\sigma}, \bar{x})\xi \leq 0, \tag{2.5}$$

$$\zeta_i \left(\left\langle \frac{\partial G_i}{\partial \sigma}(\bar{\sigma}, \bar{x}), d \right\rangle + \left\langle \frac{\partial G_i}{\partial x}(\bar{\sigma}, \bar{x}), \xi \right\rangle \right) = 0, \ i \in A_0, \tag{2.6}$$

$$\zeta_N = 0. \tag{2.7}$$

This result can be found in the literature in various forms, and with various levels of generality; see, e.g., [14], [2, Theorem 5.10], [16], and the earlier works [12, 13, 15]. Perhaps the simplest way to derive this result is to employ the piecewise decomposition of system (1.1); see [6]. (For each partition $(I_1, I_2) \in \mathcal{A}_0$, consider the branch system (1.5) linearized at $(\bar{\sigma}, \bar{x}, \bar{\lambda}, \bar{\mu})$, and take the union of solution cones of such linearized systems. This will result in (2.2)–(2.7). A more subtle version of this argument will be employed below in Theorem 3.1.)

Note that Theorem 2.1 does not establish neither the existence of solutions of perturbed KKT systems nor the existence of limit points for the sequence $\{(\sigma^k - \bar{\sigma}, x^k - \bar{x}, \lambda^k - \bar{\lambda}, \mu^k - \bar{\mu})/t_k\}$. The latter can be guaranteed for $t_k = \|\sigma^k - \bar{\sigma}\|$ if the local Lipschitz upper estimate of the distance from $(\bar{x}, \bar{\lambda}, \bar{\mu})$ to the solution set of the perturbed KKT system (1.1) holds. Sufficient conditions for this property (known in the literature under various names; see [14]) are contained in the next theorem which can be found, e.g., in [9], [2, Theorem 5.9], [14], [11, Theorem 8.11 and Corollary 8.13], [16]. Consider the system

$$\frac{\partial \Psi}{\partial x}(\bar{\sigma}, \bar{x}, \bar{\lambda}, \bar{\mu})\xi + \left(\frac{\partial F}{\partial x}(\bar{\sigma}, \bar{x})\right)^{\mathrm{T}} \eta + \left(\frac{\partial G}{\partial x}(\bar{\sigma}, \bar{x})\right)^{\mathrm{T}} \zeta = 0, \tag{2.8}$$

$$\frac{\partial F}{\partial x}(\bar{\sigma}, \bar{x})\xi = 0, \tag{2.9}$$

$$\frac{\partial G_{A_+}}{\partial x}(\bar{\sigma},\, \bar{x})\xi = 0, \tag{2.10}$$

$$\zeta_{A_0} \geq 0,\ \frac{\partial G_{A_0}}{\partial x}(\bar{\sigma},\, \bar{x})\xi \leq 0, \tag{2.11}$$

$$\zeta_i \left\langle \frac{\partial G_i}{\partial x}(\bar{\sigma},\, \bar{x}),\, \xi \right\rangle = 0,\ i \in A_0, \tag{2.12}$$

$$\zeta_N = 0, \tag{2.13}$$

which is just system (2.2)–(2.7) for $d = 0$.

Theorem 2.2 *Let $(\bar{x},\, \bar{\lambda},\, \bar{\mu}) \in \mathbf{R}^n \times \mathbf{R}^l \times \mathbf{R}^m$ be a solution of system* (1.1) *for $\sigma = \bar{\sigma} \in \mathbf{R}^s$. Suppose that system* (2.8)–(2.13) *has only the trivial solution $(\xi,\, \eta,\, \zeta) = (0,\, 0,\, 0)$.*

Then for each $\sigma \in \mathbf{R}^s$ close enough to $\bar{\sigma}$, any solution $(x(\sigma),\, \lambda(\sigma),\, \mu(\sigma))$ of system (1.1) *with $x(\sigma)$ close enough to $\bar{x}$ satisfies the estimate*

$$\|x(\sigma) - \bar{x}\| + \|\lambda(\sigma) - \bar{\lambda}\| + \|\mu(\sigma) - \bar{\mu}\| = O(\|\sigma - \bar{\sigma}\|). \tag{2.14}$$

This theorem follows immediately from Theorem 2.1 (applied with $t_k = \|x(\sigma^k) - \bar{x}\| + \|\lambda(\sigma^k) - \bar{\lambda}\| + \|\mu(\sigma^k) - \bar{\mu}\|$ for a given sequence $\{\sigma^k\} \subset \mathbf{R}^s$ convergent to $\bar{\sigma}$) and a simple additional argument showing that if system (2.8)–(2.13) has only the trivial solution then closedeness of σ to $\bar{\sigma}$ and $x(\sigma)$ to $\bar{x}$ necessarily implies closeness of $(\lambda(\sigma),\, \mu(\sigma))$ to $(\bar{\lambda},\, \bar{\mu})$.

We stress again that Theorem 2.2 still does not establish the existence of solutions of perturbed KKT systems.

Evidently, (2.8)–(2.13) can be regarded as the KKT system for the QP problem

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}\frac{\partial \Psi}{\partial x}(\bar{\sigma},\, \bar{x},\, \bar{\lambda},\, \bar{\mu})[\xi,\, \xi] \\ \text{subject to} & \xi \in C, \end{array} \tag{2.15}$$

where

$$C = C(\bar{\sigma},\, \bar{x}) = \left\{ \xi \in \mathbf{R}^n \,\middle|\, \frac{\partial F}{\partial x}(\bar{\sigma},\, \bar{x})\xi = 0,\ \frac{\partial G_{A_+}}{\partial x}(\bar{\sigma},\, \bar{x})\xi = 0,\ \frac{\partial G_{A_0}}{\partial x}(\bar{\sigma},\, \bar{x})\xi \leq 0 \right\}$$

is the *critical cone* of the KKT system (1.1) for $\sigma = \bar{\sigma}$ at $\bar{x}$. Hence, system (2.8)–(2.13) has only the trivial solution if and only if for any stationary point ξ of problem (2.15) and any associated Lagrange multiplier $(\eta,\, \zeta_{A_+},\, \zeta_{A_0})$, it holds that $(\xi,\, \eta,\, \zeta_{A_+},\, \zeta_{A_0}) = (0,\, 0,\, 0,\, 0)$ (this fact was pointed out in [11, Corollary 8.18], in a somewhat different form).

Note, however, that estimate (2.14) implies that $(\bar{x},\, \bar{\lambda},\, \bar{\mu})$ is an isolated solution of system (1.1) for $\sigma = \bar{\sigma}$. Moreover, it is evident that if the system (2.8)–(2.13) has only the trivial solution then SMFCQ holds at $\bar{x}$ for $(\bar{\lambda},\, \bar{\mu})$, and this is not the main case of interest in this work. To this end, we state the following theorem.

Theorem 2.3 *Let $(\bar{x},\, \bar{\lambda},\, \bar{\mu}) \in \mathbf{R}^n \times \mathbf{R}^l \times \mathbf{R}^m$ be a solution of system* (1.1) *for $\sigma = \bar{\sigma} \in \mathbf{R}^s$. Suppose that for any solution $(\xi,\, \eta,\, \zeta)$ of system* (2.8)–(2.13) *it holds that $\xi = 0$.*

Then for each $\sigma \in \mathbf{R}^s$ close enough to $\bar{\sigma}$, any solution $(x(\sigma),\, \lambda(\sigma),\, \mu(\sigma))$ of system (1.1) *close enough to $(\bar{x},\, \bar{\lambda},\, \bar{\mu})$ satisfies the estimate*

$$\|x(\sigma) - \bar{x}\| + \operatorname{dist}((\lambda(\sigma),\, \mu(\sigma)),\, \mathcal{M}) = O(\|\sigma - \bar{\sigma}\|). \tag{2.16}$$

This theorem can be derived from the results in [3]. Let us however provide a simple direct proof, based on piecewise decomposition.

Proof. We first prove the estimate

$$\|x(\sigma) - \bar{x}\| = O(\|\sigma - \bar{\sigma}\|). \tag{2.17}$$

Suppose that this is not the case. Then there exist sequences $\{\sigma^k\} \subset \mathbf{R}^s \setminus \{\bar{\sigma}\}$, $\{x^k\} \subset \mathbf{R}^n$, $\{\lambda^k\} \subset \mathbf{R}^l$ and $\{\mu^k\} \subset \mathbf{R}^m$ such that $\{\sigma^k\} \to \bar{\sigma}$, $\{x^k\} \to \bar{x}$, $\{\lambda^k\} \to \bar{\lambda}$, $\{\mu^k\} \to \bar{\mu}$, for each k the point (x^k, λ^k, μ^k) is a solution of system (1.1) for $\sigma = \sigma^k$, and

$$\frac{\|x^k - \bar{x}\|}{\|\sigma^k - \bar{\sigma}\|} \to \infty, \tag{2.18}$$

or the other way round,

$$\|\sigma^k - \bar{\sigma}\| = o(\|x^k - \bar{x}\|) \tag{2.19}$$

((2.18) certainly implies that $x^k \neq \bar{x}$ for all k large enough).

We may suppose that there exists $(I_1, I_2) \in \mathcal{A}_0$ such that the point (x^k, λ^k, μ^k) is a solution of the branch system (1.5) for $\sigma = \sigma^k$, for each k. Employing notation (2.1), and taking into account (2.19) and the definition of the index sets involved, we then have

$$\begin{aligned}
0 &= \Phi(\sigma^k, x^k) + \left(\frac{\partial F}{\partial x}(\sigma^k, x^k)\right)^{\mathrm{T}} \lambda^k + \left(\frac{\partial G}{\partial x}(\sigma^k, x^k)\right)^{\mathrm{T}} \mu^k \\
&= \Psi(\sigma^k, x^k, \bar{\lambda}, \bar{\mu}) + \left(\frac{\partial F}{\partial x}(\sigma^k, x^k)\right)^{\mathrm{T}} (\lambda^k - \bar{\lambda}) + \left(\frac{\partial G}{\partial x}(\sigma^k, x^k)\right)^{\mathrm{T}} (\mu^k - \bar{\mu}) \\
&= \Psi(\bar{\sigma}, x^k, \bar{\lambda}, \bar{\mu}) + \left(\frac{\partial F}{\partial x}(\bar{\sigma}, x^k)\right)^{\mathrm{T}} (\lambda^k - \bar{\lambda}) + \left(\frac{\partial G}{\partial x}(\bar{\sigma}, x^k)\right)^{\mathrm{T}} (\mu^k - \bar{\mu}) + O(\|\sigma^k - \bar{\sigma}\|) \\
&= \frac{\partial \Psi}{\partial x}(\bar{\sigma}, \bar{x}, \bar{\lambda}, \bar{\mu})(x^k - \bar{x}) + \left(\frac{\partial F}{\partial x}(\bar{\sigma}, \bar{x})\right)^{\mathrm{T}} (\lambda^k - \bar{\lambda}) \\
&\quad + \left(\frac{\partial G_{A_+}}{\partial x}(\bar{\sigma}, \bar{x})\right)^{\mathrm{T}} (\mu^k - \bar{\mu})_{A_+} + \left(\frac{\partial G_{I_1}}{\partial x}(\bar{\sigma}, \bar{x})\right)^{\mathrm{T}} \mu^k_{I_1} + o(\|x^k - \bar{x}\|),
\end{aligned} \tag{2.20}$$

and similarly

$$\begin{aligned}
0 &= F(\sigma^k, x^k) \\
&= \frac{\partial F}{\partial x}(\bar{\sigma}, \bar{x})(x^k - \bar{x}) + o(\|x^k - \bar{x}\|),
\end{aligned} \tag{2.21}$$

$$\begin{aligned}
0 &= G_{A_+ \cup I_1}(\sigma^k, x^k) \\
&= \frac{\partial G_{A_+ \cup I_1}}{\partial x}(\bar{\sigma}, \bar{x})(x^k - \bar{x}) + o(\|x^k - \bar{x}\|),
\end{aligned} \tag{2.22}$$

$$\begin{aligned}
0 &\geq G_{I_2}(\sigma^k, x^k) \\
&= \frac{\partial G_{I_2}}{\partial x}(\bar{\sigma}, \bar{x})(x^k - \bar{x}) + o(\|x^k - \bar{x}\|),
\end{aligned} \tag{2.23}$$

$$\mu^k_{I_1} \geq 0, \quad \mu^k_{I_2 \cup N} = 0. \tag{2.24}$$

Let $\mathrm{im}\,\Lambda$ stand for the image (range space) of a linear operator Λ. Taking into account the inequality in (2.24), relation (2.20) implies the inclusion

$$\begin{aligned} -\,\mathrm{im}\left(\frac{\partial F}{\partial x}(\bar{\sigma},\, \bar{x})\right)^{\mathrm{T}} - \mathrm{im}\left(\frac{\partial G_{A_+}}{\partial x}(\bar{\sigma},\, \bar{x})\right)^{\mathrm{T}} & \\ -\left(\frac{\partial G_{I_1}}{\partial x}(\bar{\sigma},\, \bar{x})\right)^{\mathrm{T}}\left(\mathbf{R}^{|I_1|}_+\right) &\ni \frac{\partial \Psi}{\partial x}(\bar{\sigma},\, \bar{x},\, \bar{\lambda},\, \bar{\mu})(x^k - \bar{x}) \\ &\quad + o(\|x^k - \bar{x}\|), \end{aligned} \tag{2.25}$$

where the set in the left-hand side is a closed cone (as a sum of two linear subspaces and a polyhedral cone).

Suppose further that the entire sequence $\{(x^k - \bar{x})/\|x^k - \bar{x}\|\}$ converges to some $\xi \in \mathbf{R}^n$, $\|\xi\| = 1$. Dividing (2.25) and (2.21)–(2.23) by $\|x^k - \bar{x}\|$, and passing onto the limit as $k \to \infty$, we then obtain

$$\frac{\partial \Psi}{\partial x}(\bar{\sigma},\, \bar{x},\, \bar{\lambda},\, \bar{\mu})\xi \in -\,\mathrm{im}\left(\frac{\partial F}{\partial x}(\bar{\sigma},\, \bar{x})\right)^{\mathrm{T}} - \mathrm{im}\left(\frac{\partial G_{A_+}}{\partial x}(\bar{\sigma},\, \bar{x})\right)^{\mathrm{T}} - \left(\frac{\partial G_{I_1}}{\partial x}(\bar{\sigma},\, \bar{x})\right)^{\mathrm{T}}\left(\mathbf{R}^{|I_1|}_+\right), \tag{2.26}$$

$$\frac{\partial F}{\partial x}(\bar{\sigma},\, \bar{x})\xi = 0, \quad \frac{\partial G_{A_+ \cup I_1}}{\partial x}(\bar{\sigma},\, \bar{x})\xi = 0, \quad \frac{\partial G_{I_2}}{\partial x}(\bar{\sigma},\, \bar{x})\xi \leq 0. \tag{2.27}$$

Inclusion (2.26) means that there exist $\eta \in \mathbf{R}^l$ and $\zeta \in \mathbf{R}^m$ satisfying (2.8), and such that

$$\zeta^k_{I_1} \geq 0, \quad \zeta^k_{I_2 \cup N} = 0.$$

Combining this with (2.27) we obtain that $(\xi,\, \eta,\, \zeta)$ satisfies (2.8)–(2.13), which is a contradiction, since $\xi \neq 0$. We thus proved (2.17).

In order to establish the remaining estimate

$$\mathrm{dist}((\lambda(\sigma),\, \mu(\sigma)),\, \mathcal{M}) = O(\|\sigma - \bar{\sigma}\|), \tag{2.28}$$

observe that $\mathcal{M}$ is the solution set of the linear system

$$\left(\frac{\partial F}{\partial x}(\bar{\sigma},\, \bar{x})\right)^{\mathrm{T}}\lambda + \left(\frac{\partial G}{\partial x}(\bar{\sigma},\, \bar{x})\right)^{\mathrm{T}}\mu = -\Phi(\bar{\sigma},\, \bar{x}), \quad \mu_A \geq 0, \quad \mu_N = 0.$$

Employing Hoffman's lemma (see, e.g., [2, Theorem 2.200]) and (1.1), for $\sigma \in \mathbf{R}^n$ close enough to $\bar{\sigma}$ we now obtain that

$$\begin{aligned} \mathrm{dist}((\lambda(\sigma),\, \mu(\sigma)),\, \mathcal{M}) &= O\left(\|\Psi(\bar{\sigma},\, \bar{x},\, \lambda(\sigma),\, \mu(\sigma))\| + \sum_{i \in A}\min\{0,\, \mu_i(\sigma)\} + \|\mu_N(\sigma)\|\right) \\ &= O(\|\Psi(\sigma,\, x(\sigma),\, \lambda(\sigma),\, \mu(\sigma)) - \Psi(\bar{\sigma},\, \bar{x},\, \lambda(\sigma),\, \mu(\sigma))\|) \\ &= O(\|\sigma - \bar{\sigma}\|) + O(\|x(\sigma) - \bar{x}\|), \end{aligned} \tag{2.29}$$

and (2.28) now follows by the above-proved estimate (2.17). This completes the proof. ■

Note that the condition that $\xi = 0$ for any solution $(\xi,\, \eta,\, \zeta)$ of system (2.8)–(2.13) can be expressed as follows: $\xi = 0$ is a unique stationary point of problem (2.15). As a sufficient condition for these equivalent properties, let us mention the following one (pointed out in [9]; see also [11, Theorem 8.24]):

$$\frac{\partial \Psi}{\partial x}(\bar{\sigma},\, \bar{x},\, \bar{\lambda},\, \bar{\mu})[\xi,\, \xi] \neq 0 \quad \forall\, \xi \in C \setminus \{0\}.$$

This follows immediately by multiplying (2.8) by ξ and employing (2.9)–(2.13).

Let us for a moment turn our attention to the parametric KKT system with *canonical perturbations*:

$$\begin{array}{l} \Phi(\sigma,\, x) + \left(\frac{\partial F}{\partial x}(\sigma,\, x)\right)^{\mathrm{T}} \lambda + \left(\frac{\partial G}{\partial x}(\sigma,\, x)\right)^{\mathrm{T}} \mu = a, \quad F(\sigma,\, x) = b, \\ \mu \geq 0, \quad G(\sigma,\, x) \leq c, \quad \langle \mu,\, G(\sigma,\, x) - c\rangle = 0, \end{array} \tag{2.30}$$

where $a \in \mathbf{R}^n$, $b \in \mathbf{R}^l$ and $c \in \mathbf{R}^m$ are additional (canonical) parameters. It is well known that in this case, the outer estimate of limiting directions presented in Theorem 2.1 is exact: each tuple $(d,\, \xi,\, \eta,\, \zeta)$ satisfying (2.2)–(2.7) is in fact a limit point of the sequence $\{(\sigma^k - \bar{\sigma},\, x^k - \bar{x},\, \lambda^k - \bar{\lambda},\, \mu^k - \bar{\mu})/t_k\}$ for some $\{\sigma^k\}$, $\{x^k\}$, $\{\lambda^k\}$, $\{\mu^k\}$ and $\{t_k\}$ with the properties specified in Theorem 2.1. Moreover, the sufficient condition for the estimate (2.14), established in Theorem 2.2, is also necessary in this case (see [10, 9, 16], and [11, Theorem 8.11]). Moreover, the sufficient condition for the estimate (2.16) established in Theorem 2.3, is also necessary in this case (it is actually necessary for the estimate (2.17)). These facts can be proved by explicitly constructing the needed perturbations; see the related discussion in Remark 3.3 below.

The results presented above still do not answer the question about solvability of perturbed KKT systems. The next result pointed out in [6] establishes sufficient conditions for Lipschitz stability of $(\bar{x},\, \bar{\lambda},\, \bar{\mu})$ with a *given* $(\bar{\lambda},\, \bar{\mu}) \in \mathcal{M}$ subject to directional perturbations. More precisely, for a given tuple $(d,\, \xi,\, \eta,\, \zeta)$, and a given mapping $\rho : \mathbf{R}_+ \to \mathbf{R}^s$ such that $\rho(t) = o(t)$, we consider the arc $\sigma(t) = \bar{\sigma} + td + \rho(t)$ in the space of parameter values, and we are searching for solutions of the form $(\bar{x} + t\xi,\, \bar{\lambda} + t\eta,\, \bar{\mu} + t\zeta) + o(t)$ of system (1.1), $t \geq 0$. Recall that, according to Theorem 2.1, we can expect this to hold only for tuples $(d,\, \xi,\, \eta,\, \zeta)$ satisfying (2.2)–(2.7). Moreover, Theorem 2.1 implies that for a given d, we can expect Lipschitz stability of $(\bar{x},\, \bar{\lambda},\, \bar{\mu})$ with some specific $(\bar{\lambda},\, \bar{\mu}) \in \mathcal{M}$ subject to perturbations of the kind specified above only provided the corresponding system (2.2)–(2.7) with respect to $(\xi,\, \eta,\, \zeta)$ is solvable.

Define the index sets

$$A_0^+ = A_0^+(\bar{\sigma},\, \bar{x},\, \bar{\mu};\, d,\, \xi,\, \zeta) = \left\{ i \in A_0 \,\middle|\, \left\langle \frac{\partial G_i}{\partial \sigma}(\bar{\sigma},\, \bar{x}),\, d \right\rangle + \left\langle \frac{\partial G_i}{\partial x}(\bar{\sigma},\, \bar{x}),\, \xi \right\rangle = 0,\ \zeta_i > 0 \right\},$$

$$A_0^0 = A_0^0(\bar{\sigma},\, \bar{x},\, \bar{\mu};\, d,\, \xi,\, \zeta) = \left\{ i \in A_0 \,\middle|\, \left\langle \frac{\partial G_i}{\partial \sigma}(\bar{\sigma},\, \bar{x}),\, d \right\rangle + \left\langle \frac{\partial G_i}{\partial x}(\bar{\sigma},\, \bar{x}),\, \xi \right\rangle = 0,\ \zeta_i = 0 \right\},$$

$$A_0^N = A_0^N(\bar{\sigma},\, \bar{x},\, \bar{\mu};\, d,\, \xi,\, \zeta) = \left\{ i \in A_0 \,\middle|\, \left\langle \frac{\partial G_i}{\partial \sigma}(\bar{\sigma},\, \bar{x}),\, d \right\rangle + \left\langle \frac{\partial G_i}{\partial x}(\bar{\sigma},\, \bar{x}),\, \xi \right\rangle < 0,\ \zeta_i = 0 \right\}.$$

Let $\mathcal{A}_0^0$ stand for the set of all partitions of A_0^0, that is, pairs $(I_0^1,\, I_0^2)$ of index sets satisfying $I_0^1 \cup I_0^2 = A_0^0$, $I_0^1 \cap I_0^2 = \emptyset$. Note that for an arbitrary partition $(I_0^1,\, I_0^2) \in \mathcal{A}_0^0$, the pair $(I_1,\, I_2)$ of index sets defined by $I_1 = I_0^1 \cup A_0^+$ and $I_2 = I_0^2 \cup A_0^N$ belongs to $\mathcal{A}_0$.

For each index set $I \subset A$ define the matrix

$$D_I = D_I(\bar{\sigma},\, \bar{x},\, \bar{\lambda},\, \bar{\mu}) = \begin{pmatrix} \frac{\partial \Psi}{\partial x}(\bar{\sigma},\, \bar{x},\, \bar{\lambda},\, \bar{\mu}) & \left(\frac{\partial F}{\partial x}(\bar{\sigma},\, \bar{x})\right)^{\mathrm{T}} & \left(\frac{\partial G_I}{\partial x}(\bar{\sigma},\, \bar{x})\right)^{\mathrm{T}} \\ \frac{\partial F}{\partial x}(\bar{\sigma},\, \bar{x}) & 0 & 0 \\ \frac{\partial G_I}{\partial x}(\bar{\sigma},\, \bar{x}) & 0 & 0 \end{pmatrix}. \tag{2.31}$$

The next theorem is a direct corollary of [2, Theorem 4.9 (iii)]; it employs Gollan's condition for the branch system (1.5) for $\sigma = \bar{\sigma}$ at $(\bar{\sigma},\, \bar{x},\, \bar{\lambda},\, \bar{\mu})$ in a direction d, which can be stated as follows:

$$\det D_{A_+ \cup I_1} \neq 0, \tag{2.32}$$

and there exists a tuple $(\bar{\xi},\, \bar{\eta},\, \bar{\zeta}) \in \mathbf{R}^n \times \mathbf{R}^l \times \mathbf{R}^m$ such that

$$\frac{\partial \Psi}{\partial \sigma}(\bar{\sigma},\, \bar{x},\, \bar{\lambda},\, \bar{\mu})d + \frac{\partial \Psi}{\partial x}(\bar{\sigma},\, \bar{x},\, \bar{\lambda},\, \bar{\mu})\bar{\xi} + \left(\frac{\partial F}{\partial x}(\bar{\sigma},\, \bar{x})\right)^{\mathrm{T}} \bar{\eta} + \left(\frac{\partial G}{\partial x}(\bar{\sigma},\, \bar{x})\right)^{\mathrm{T}} \bar{\zeta} = 0, \tag{2.33}$$

$$\frac{\partial F}{\partial \sigma}(\bar{\sigma},\, \bar{x})d + \frac{\partial F}{\partial x}(\bar{\sigma},\, \bar{x})\bar{\xi} = 0, \quad \frac{\partial G_{A_+}}{\partial \sigma}(\bar{\sigma},\, \bar{x})d + \frac{\partial G_{A_+}}{\partial x}(\bar{\sigma},\, \bar{x})\bar{\xi} = 0, \tag{2.34}$$

$$\frac{\partial G_{I_1}}{\partial \sigma}(\bar{\sigma},\, \bar{x})d + \frac{\partial G_{I_1}}{\partial x}(\bar{\sigma},\, \bar{x})\bar{\xi} = 0, \quad \bar{\zeta}_{I_1} > 0, \tag{2.35}$$

$$\frac{\partial G_{I_2}}{\partial \sigma}(\bar{\sigma},\, \bar{x})d + \frac{\partial G_{I_2}}{\partial x}(\bar{\sigma},\, \bar{x})\bar{\xi} < 0, \quad \bar{\zeta}_{I_2} = 0, \tag{2.36}$$

$$\bar{\zeta}_N = 0. \tag{2.37}$$

Theorem 2.4 *Let $(\bar{x},\, \bar{\lambda},\, \bar{\mu}) \in \mathbf{R}^n \times \mathbf{R}^l \times \mathbf{R}^m$ be a solution of system* (1.1) *for $\sigma = \bar{\sigma} \in \mathbf{R}^s$. Suppose that a tuple $(d,\, \xi,\, \eta,\, \zeta) \in \mathbf{R}^s \times \mathbf{R}^n \times \mathbf{R}^l \times \mathbf{R}^m$ satisfies* (2.2)–(2.7)*, and there exists a partition $(I_0^1,\, I_0^2) \in \mathcal{A}_0^0$ such that for $I_1 = I_0^1 \cup A_0^+$ and $I_2 = I_0^2 \cup A_0^N$, condition* (2.32) *holds, and there exists $(\bar{\xi},\, \bar{\eta},\, \bar{\zeta}) \in \mathbf{R}^n \times \mathbf{R}^l \times \mathbf{R}^m$ satisfying* (2.33)–(2.37).

Then for any mapping $\rho : \mathbf{R}_+ \to \Sigma$ such that $\rho(t) = o(t)$, and any $t \geq 0$ small enough, system (1.1) *for $\sigma = \bar{\sigma} + td + \rho(t)$ has a solution of the form $(\bar{x} + t\xi,\, \bar{\lambda} + t\eta,\, \bar{\mu} + t\zeta) + o(t)$.*

It is interesting to note that if $A_0 \neq \emptyset$ then Gollan's condition for the branch system does not imply not only LICQ but even MFCQ for the constrains (1.4) at $\bar{x}$.

Example 2.1 Let $s = 4$, $n = 2$, $l = 0$, $m = 2$, $f(\sigma,\, x) = x_2^2/2 + \sigma_1 x_1 + \sigma_2 x_2$, $G(\sigma,\, x) = (-x_1 + x_2^2/2 - \sigma_3,\, x_1 - x_2^2 - \sigma_4)$. Then $\bar{x} = 0$ is a solution of problem (1.3) for $\sigma = \bar{\sigma} = 0$, and moreover, $\bar{x}$ paired with any element of $\mathcal{M} = \{\mu \in \mathbf{R}^2 \mid \mu_1 = \mu_2 \geq 0\}$ satisfies the corresponding system (1.1) with Φ defined according to (1.2), even though MFCQ does not hold at $\bar{x}$ for the constraints $G(\bar{\sigma},\, x) \leq 0$.

Take $\bar{\mu} = 0$ (the unique multiplier in $\mathcal{M}$ violating the strict complementarity condition). It can be easily checked that for any $d \in \mathbf{R}^4$ such that $d_1 > 0$, $d_3 + d_4 \geq 0$, Theorem 2.4 is applicable with the corresponding $(\xi,\, \zeta) \in \mathbf{R}^2 \times \mathbf{R}^2$.

Thus, Theorem 2.4 is an appropriate tool for dealing with multipliers violating strict complementarity, and in particular, one can expect Lipschitz stability of (some of) such multipliers.

But what about strictly complementary multipliers: can they be stable, what kind of quantitative stability can be expected for them, and what sufficient conditions for this can be suggested? Theorem 2.4 is not appropriate for answering these questions. Indeed, if $A_0 = \emptyset$ then Gollan's condition for the branch system implies LICQ, the case we do not deal with in this work.

Moreover, according to [6, Lemma 4.1], if for a given pair $(d, \xi) \in \mathbf{R}^s \times \mathbf{R}^n$ there exist $\eta \in \mathbf{R}^l$ and $\zeta \in \mathbf{R}^m$ satisfying (2.2)–(2.7) then $(\bar{\lambda}, \bar{\mu})$ is a solution of the LP problem

$$\begin{array}{ll} \text{maximize} & \left\langle \lambda,\, \frac{\partial F}{\partial \sigma}(\bar{\sigma},\, \bar{x})d \right\rangle + \left\langle \mu,\, \frac{\partial G}{\partial \sigma}(\bar{\sigma},\, \bar{x})d \right\rangle \\ \text{subject to} & \left(\frac{\partial F}{\partial x}(\bar{\sigma},\, \bar{x})\right)^{\mathrm{T}} \lambda + \left(\frac{\partial G}{\partial x}(\bar{\sigma},\, \bar{x})\right)^{\mathrm{T}} \mu = -\frac{\partial f}{\partial x}(\bar{\sigma},\, \bar{x}), \quad \mu_A \geq 0, \quad \mu_N = 0. \end{array} \tag{2.38}$$

For generic $\frac{\partial F}{\partial \sigma}(\bar{\sigma},\, \bar{x})$ and $\frac{\partial G}{\partial \sigma}(\bar{\sigma},\, \bar{x})$, and for a typical $d \in \mathbf{R}^n$, one should expect that solutions of this LP problem will belong to the relative boundary of its feasible set (which is $\mathcal{M}$), and the relative boundary is comprised by multipliers violating strict complementarity. More precisely, it follows from (2.3), (2.4) that the strictly complementary multipliers may exist only provided

$$\frac{\partial \tilde{F}_A}{\partial \sigma}(\bar{\sigma},\, \bar{x})d \in \operatorname{im} \frac{\partial \tilde{F}_A}{\partial x}(\bar{\sigma},\, \bar{x}), \tag{2.39}$$

where for any index set $I \subset A$, the aggregated constraint mapping $\tilde{F}_I : \mathbf{R}^s \times \mathbf{R}^n \to \mathbf{R}^l \times \mathbf{R}^{|I|}$ is given by

$$\tilde{F}_I(\sigma,\, x) = (F(\sigma,\, x),\, G_I(\sigma,\, x)). \tag{2.40}$$

The set in the right-hand side of (2.39) is a proper subspace in $\mathbf{R}^l \times \mathbf{R}^{|A|}$, unless LICQ holds, and this is a very severe restriction on the left-hand side of (2.39).

According to Theorem 2.1, this means that, generically, one should not expect Lipschitz stability of strictly complementary multipliers. However, as will be demonstrated in the next section, this does not mean that all strictly complementary multipliers are generically unstable. We will specify the subclass of such multipliers which can be expected to be stable (though not Lipschitz-stable).

In Example 2.1, if $d_3 + d_4 > 0$ then $\bar{\lambda} = 0$ is the unique solution of problem (2.38); if $d_3 + d_4 < 0$ then (2.38) has no solutions; and only if $d_3 + d_4 = 0$ then (2.39) holds, and any point in $\mathcal{M}$ is a solution of (2.38).

3 Hölder Stability

In Theorem 3.4 below we provide a sufficient condition for Hölder (square-root) stability of a strictly complementary multiplier. Let us, however, first present the result on asymptotic behavior of Hölder-stable solutions of the perturbed KKT systems, and the results on local Hölder upper estimate of the distance from $(\bar{x},\, \bar{\lambda},\, \bar{\mu})$ and from $\bar{x} \times \mathcal{M}$ to the solution set of the perturbed KKT system (1.1). These results do not assume strict complementarity of the multiplier in question.

Theorem 3.1 *Let* $(\bar{x}, \bar{\lambda}, \bar{\mu}) \in \mathbf{R}^n \times \mathbf{R}^l \times \mathbf{R}^m$ *be a solution of system* (1.1) *for* $\sigma = \bar{\sigma} \in \mathbf{R}^s$. *Let sequences* $\{\sigma^k\} \subset \mathbf{R}^s$, $\{x^k\} \subset \mathbf{R}^n$, $\{\lambda^k\} \subset \mathbf{R}^l$, $\{\mu^k\} \subset \mathbf{R}^m$ *and* $\{t_k\} \subset \mathbf{R}_+ \setminus \{0\}$ *be such that* $\{\sigma^k\} \to \bar{\sigma}$, $\{x^k\} \to \bar{x}$, $\{\lambda^k\} \to \bar{\lambda}$, $\{\mu^k\} \to \bar{\mu}$, $\{t_k\} \to 0$, *and such that for each* k *the point* (x^k, λ^k, μ^k) *is a solution of system* (1.1) *for* $\sigma = \sigma^k$.

Then any limit point $(d, \xi, \eta, \zeta) \in \mathbf{R}^s \times \mathbf{R}^n \times \mathbf{R}^l \times \mathbf{R}^m$ *of the sequence* $\{((\sigma^k - \bar{\sigma})/t_k^2, (x^k - \bar{x})/t_k, (\lambda^k - \bar{\lambda})/t_k, (\mu^k - \bar{\mu})/t_k)\}$ *satisfies the system* (2.8)–(2.11), (2.13), *and there exist* $x \in \mathbf{R}^n$, $\lambda \in \mathbf{R}^l$ *and* $\mu \in \mathbf{R}^m$ *such that*

$$\begin{aligned} &\frac{\partial \Psi}{\partial \sigma}(\bar{\sigma}, \bar{x}, \bar{\lambda}, \bar{\mu})d + \frac{1}{2}\frac{\partial^2 \Psi}{\partial x^2}(\bar{\sigma}, \bar{x}, \bar{\lambda}, \bar{\mu})[\xi, \xi] \\ &+ \left(\frac{\partial^2 F}{\partial x^2}(\bar{\sigma}, \bar{x})[\xi]\right)^{\mathrm{T}} \eta + \left(\frac{\partial^2 G}{\partial x^2}(\bar{\sigma}, \bar{x})[\xi]\right)^{\mathrm{T}} \zeta = -\frac{\partial \Psi}{\partial x}(\bar{\sigma}, \bar{x}, \bar{\lambda}, \bar{\mu})x \\ &\qquad - \left(\frac{\partial F}{\partial x}(\bar{\sigma}, \bar{x})\right)^{\mathrm{T}} \lambda - \left(\frac{\partial G}{\partial x}(\bar{\sigma}, \bar{x})\right)^{\mathrm{T}} \mu, \end{aligned} \tag{3.1}$$

$$\frac{\partial F}{\partial \sigma}(\bar{\sigma}, \bar{x})d + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(\bar{\sigma}, \bar{x})[\xi, \xi] = -\frac{\partial F}{\partial x}(\bar{\sigma}, \bar{x})x, \tag{3.2}$$

$$\frac{\partial G_{A_+}}{\partial \sigma}(\bar{\sigma}, \bar{x})d + \frac{1}{2}\frac{\partial^2 G_{A_+}}{\partial x^2}(\bar{\sigma}, \bar{x})[\xi, \xi] = -\frac{\partial G_{A_+}}{\partial x}(\bar{\sigma}, \bar{x})x, \tag{3.3}$$

$$\mu_{A_0} \geq 0, \quad \frac{\partial G_{A_0}}{\partial \sigma}(\bar{\sigma}, \bar{x})d + \frac{1}{2}\frac{\partial^2 G_{A_0}}{\partial x^2}(\bar{\sigma}, \bar{x})[\xi, \xi] \leq -\frac{\partial G_{A_0}}{\partial x}(\bar{\sigma}, \bar{x})x, \tag{3.4}$$

$$\begin{aligned} &(\zeta_i + \mu_i)\left(\left\langle \frac{\partial G_i}{\partial x}(\bar{\sigma}, \bar{x}), \xi \right\rangle + \left\langle \frac{\partial G_i}{\partial \sigma}(\bar{\sigma}, \bar{x}), d \right\rangle \right. \\ &\qquad \left. + \frac{1}{2}\frac{\partial^2 G_i}{\partial x^2}(\bar{\sigma}, \bar{x})[\xi, \xi] + \left\langle \frac{\partial G_i}{\partial x}(\bar{\sigma}, \bar{x}), x \right\rangle \right) = 0, \ i \in A_0, \end{aligned} \tag{3.5}$$

$$\mu_N = 0. \tag{3.6}$$

Proof. Evidently, we can split the sequence $\{(\sigma^k, x^k, \lambda^k, \mu^k)\}$ into a finite number of subsequences corresponding to different partitions $(I_1, I_2) \in \mathcal{A}_0$, so that all points of each subsequence satisfy (1.5) for the same (I_1, I_2).

Fix some $(I_1, I_2) \in \mathcal{A}_0$, and suppose for convenience that the point (x^k, λ^k, μ^k) is a solution of the branch system (1.5) for $\sigma = \sigma^k$, for each k. Suppose further that the entire sequence $\{((\sigma^k - \bar{\sigma})/t_k^2, (x^k - \bar{x})/t_k, (\lambda^k - \bar{\lambda})/t_k, (\mu^k - \bar{\mu})/t_k)\}$ converges to (d, ξ, η, ζ). Then this sequence is bounded, which implies the following relations:

$$\|\sigma^k - \bar{\sigma}\| = O(t_k^2), \quad \|x^k - \bar{x}\| = O(t_k), \quad \|\lambda^k - \bar{\lambda}\| = O(t_k), \quad \|\mu^k - \bar{\mu}\| = O(t_k). \tag{3.7}$$

Employing notation (2.1), by (1.5), (3.7) and the definition of the index sets involved, we then obtain

$$\begin{aligned}
0 &= \Phi(\sigma^k,\, x^k) + \left(\frac{\partial F}{\partial x}(\sigma^k,\, x^k)\right)^{\mathrm{T}} \lambda^k + \left(\frac{\partial G}{\partial x}(\sigma^k,\, x^k)\right)^{\mathrm{T}} \mu^k \\
&= \Psi(\sigma^k,\, x^k,\, \bar{\lambda},\, \bar{\mu}) + \left(\frac{\partial F}{\partial x}(\sigma^k,\, x^k)\right)^{\mathrm{T}} (\lambda^k - \bar{\lambda}) + \left(\frac{\partial G}{\partial x}(\sigma^k,\, x^k)\right)^{\mathrm{T}} (\mu^k - \bar{\mu}) \\
&= \frac{\partial \Psi}{\partial \sigma}(\bar{\sigma},\, \bar{x},\, \bar{\lambda},\, \bar{\mu})(\sigma^k - \bar{\sigma}) + \frac{\partial \Psi}{\partial x}(\bar{\sigma},\, \bar{x},\, \bar{\lambda},\, \bar{\mu})(x^k - \bar{x}) \\
&\quad + \left(\frac{\partial F}{\partial x}(\bar{\sigma},\, \bar{x})\right)^{\mathrm{T}} (\lambda^k - \bar{\lambda}) + \left(\frac{\partial G_{A_+}}{\partial x}(\bar{\sigma},\, \bar{x})\right)^{\mathrm{T}} (\mu^k - \bar{\mu})_{A_+} + \left(\frac{\partial G_{I_1}}{\partial x}(\bar{\sigma},\, \bar{x})\right)^{\mathrm{T}} \mu^k_{I_1} \\
&\quad + \frac{1}{2}\frac{\partial^2 \Psi}{\partial x^2}(\bar{\sigma},\, \bar{x},\, \bar{\lambda},\, \bar{\mu})[x^k - \bar{x},\, x^k - \bar{x}] + \left(\frac{\partial^2 F}{\partial x^2}(\bar{\sigma},\, \bar{x})[x^k - \bar{x}]\right)^{\mathrm{T}} (\lambda^k - \bar{\lambda}) \\
&\quad + \left(\frac{\partial^2 G_{A_+}}{\partial x^2}(\bar{\sigma},\, \bar{x})[x^k - \bar{x}]\right)^{\mathrm{T}} (\mu^k - \bar{\mu})_{A_+} + \left(\frac{\partial^2 G_{I_1}}{\partial x^2}(\bar{\sigma},\, \bar{x})[x^k - \bar{x}]\right)^{\mathrm{T}} \mu^k_{I_1} \\
&\quad + o(t_k^2),
\end{aligned} \tag{3.8}$$

and similarly

$$\begin{aligned}
0 &= F(\sigma^k,\, x^k) \\
&= \frac{\partial F}{\partial \sigma}(\bar{\sigma},\, \bar{x})(\sigma^k - \bar{\sigma}) + \frac{\partial F}{\partial x}(\bar{\sigma},\, \bar{x})(x^k - \bar{x}) \\
&\quad + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(\bar{\sigma},\, \bar{x})[x^k - \bar{x},\, x^k - \bar{x}] + o(t_k^2),
\end{aligned} \tag{3.9}$$

$$\begin{aligned}
0 &= G_{A_+ \cup I_1}(\sigma^k,\, x^k) \\
&= \frac{\partial G_{A_+ \cup I_1}}{\partial \sigma}(\bar{\sigma},\, \bar{x})(\sigma^k - \bar{\sigma}) + \frac{\partial G_{A_+ \cup I_1}}{\partial x}(\bar{\sigma},\, \bar{x})(x^k - \bar{x}) \\
&\quad + \frac{1}{2}\frac{\partial^2 G_{A_+ \cup I_1}}{\partial x^2}(\bar{\sigma},\, \bar{x})[x^k - \bar{x},\, x^k - \bar{x}] + o(t_k^2),
\end{aligned} \tag{3.10}$$

$$\begin{aligned}
0 &\geq G_{I_2}(\sigma^k,\, x^k) \\
&= \frac{\partial G_{I_2}}{\partial \sigma}(\bar{\sigma},\, \bar{x})(\sigma^k - \bar{\sigma}) + \frac{\partial G_{I_2}}{\partial x}(\bar{\sigma},\, \bar{x})(x^k - \bar{x}) \\
&\quad + \frac{1}{2}\frac{\partial^2 G_{I_2}}{\partial x^2}(\bar{\sigma},\, \bar{x})[x^k - \bar{x},\, x^k - \bar{x}] + o(t_k^2),
\end{aligned} \tag{3.11}$$

$$\mu^k_{I_1} \geq 0, \quad \mu^k_{I_2 \cup N} = 0. \tag{3.12}$$

Dividing (3.8)–(3.12) by t_k, and passing onto the limit as $k \to \infty$, we obtain (2.8)–(2.10), (2.13) and the relations

$$\zeta_{I_1} \geq 0, \quad \frac{\partial G_{I_1}}{\partial x}(\bar{\sigma},\, \bar{x})\xi = 0, \quad \zeta_{I_2} = 0, \quad \frac{\partial G_{I_2}}{\partial x}(\bar{\sigma},\, \bar{x})\xi \leq 0. \tag{3.13}$$

Define the matrix

$$D = D(\bar\sigma,\, \bar x,\, \bar\lambda,\, \bar\mu) = \begin{pmatrix} \frac{\partial \Psi}{\partial x}(\bar\sigma,\, \bar x,\, \bar\lambda,\, \bar\mu) & \left(\frac{\partial F}{\partial x}(\bar\sigma,\, \bar x)\right)^{\mathrm{T}} & \left(\frac{\partial G_{A_+}}{\partial x}(\bar\sigma,\, \bar x)\right)^{\mathrm{T}} & \left(\frac{\partial G_{I_1}}{\partial x}(\bar\sigma,\, \bar x)\right)^{\mathrm{T}} \\ \frac{\partial F}{\partial x}(\bar\sigma,\, \bar x) & 0 & 0 & 0 \\ \frac{\partial G_{A_+}}{\partial x}(\bar\sigma,\, \bar x) & 0 & 0 & 0 \\ \frac{\partial G_{I_1}}{\partial x}(\bar\sigma,\, \bar x) & 0 & 0 & 0 \\ \frac{\partial G_{I_2}}{\partial x}(\bar\sigma,\, \bar x) & 0 & 0 & 0 \end{pmatrix} \tag{3.14}$$

and the vector

$$\omega = \begin{pmatrix} \frac{\partial \Psi}{\partial \sigma}(\bar\sigma,\, \bar x,\, \bar\lambda,\, \bar\mu)d + \frac{1}{2}\frac{\partial^2 \Psi}{\partial x^2}(\bar\sigma,\, \bar x,\, \bar\lambda,\, \bar\mu)[\xi,\, \xi] + \left(\frac{\partial^2 F}{\partial x^2}(\bar\sigma,\, \bar x)[\xi]\right)^{\mathrm{T}}\eta \\ + \left(\frac{\partial^2 G_{A_+}}{\partial x^2}(\bar\sigma,\, \bar x)[\xi]\right)^{\mathrm{T}}\zeta_{A_+} + \left(\frac{\partial^2 G_{I_1}}{\partial x^2}(\bar\sigma,\, \bar x)[\xi]\right)^{\mathrm{T}}\zeta_{I_1} \\ \frac{\partial F}{\partial \sigma}(\bar\sigma,\, \bar x)d + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(\bar\sigma,\, \bar x)[\xi,\, \xi] \\ \frac{\partial G_{A_+}}{\partial \sigma}(\bar\sigma,\, \bar x)d + \frac{1}{2}\frac{\partial^2 G_{A_+}}{\partial x^2}(\bar\sigma,\, \bar x)[\xi,\, \xi] \\ \frac{\partial G_{I_1}}{\partial \sigma}(\bar\sigma,\, \bar x)d + \frac{1}{2}\frac{\partial^2 G_{I_1}}{\partial x^2}(\bar\sigma,\, \bar x)[\xi,\, \xi] \\ \frac{\partial G_{I_2}}{\partial \sigma}(\bar\sigma,\, \bar x)d + \frac{1}{2}\frac{\partial^2 G_{I_2}}{\partial x^2}(\bar\sigma,\, \bar x)[\xi,\, \xi] \end{pmatrix}.$$

Furthermore, for each k define the vector

$$w^k = \begin{pmatrix} \frac{\partial \Psi}{\partial \sigma}(\bar\sigma,\, \bar x,\, \bar\lambda,\, \bar\mu)(\sigma^k - \bar\sigma) + \frac{1}{2}\frac{\partial^2 \Psi}{\partial x^2}(\bar\sigma,\, \bar x,\, \bar\lambda,\, \bar\mu)[x^k - \bar x,\, x^k - \bar x] \\ + \left(\frac{\partial^2 F}{\partial x^2}(\bar\sigma,\, \bar x)[x^k - \bar x]\right)^{\mathrm{T}}(\lambda^k - \bar\lambda) \\ + \left(\frac{\partial^2 G_{A_+}}{\partial x^2}(\bar\sigma,\, \bar x)[x^k - \bar x]\right)^{\mathrm{T}}(\mu^k - \bar\mu)_{A_+} + \left(\frac{\partial^2 G_{I_1}}{\partial x^2}(\bar\sigma,\, \bar x)[x^k - \bar x]\right)^{\mathrm{T}}\mu^k_{I_1} \\ \frac{\partial F}{\partial \sigma}(\bar\sigma,\, \bar x)(\sigma^k - \bar\sigma) + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(\bar\sigma,\, \bar x)[x^k - \bar x,\, x^k - \bar x] \\ \frac{\partial G_{A_+}}{\partial \sigma}(\bar\sigma,\, \bar x)(\sigma^k - \bar\sigma) + \frac{1}{2}\frac{\partial^2 G_{A_+}}{\partial x^2}(\bar\sigma,\, \bar x)[x^k - \bar x,\, x^k - \bar x] \\ \frac{\partial G_{I_1}}{\partial \sigma}(\bar\sigma,\, \bar x)(\sigma^k - \bar\sigma) + \frac{1}{2}\frac{\partial^2 G_{I_1}}{\partial x^2}(\bar\sigma,\, \bar x)[x^k - \bar x,\, x^k - \bar x] \\ \frac{\partial G_{I_2}}{\partial \sigma}(\bar\sigma,\, \bar x)(\sigma^k - \bar\sigma) + \frac{1}{2}\frac{\partial^2 G_{I_2}}{\partial x^2}(\bar\sigma,\, \bar x)[x^k - \bar x,\, x^k - \bar x] \end{pmatrix}$$

Taking into account the inequality in (3.12), relations (3.8)–(3.11) imply the inclusion

$$\begin{aligned} &-D\left(\mathbf{R}^n \times \mathbf{R}^l \times \mathbf{R}^{|A_+|} \times \mathbf{R}^{|I_1|}_+\right) \\ &-\{0\} \times \{0\} \times \{0\} \times \{0\} \times \mathbf{R}^{|I_2|}_+ \quad \ni \quad w^k + o(t_k^2), \end{aligned} \tag{3.15}$$

where the set in the left-hand side is a closed cone (as a sum of polyhedral cones).

Note that $\{w^k/t_k^2\} \to \omega$. Dividing (3.15) by t_k^2, and passing onto the limit as $k \to \infty$, we thus obtain

$$\omega \in -D\left(\mathbf{R}^n \times \mathbf{R}^l \times \mathbf{R}^{|A_+|} \times \mathbf{R}^{|I_1|}_+\right) - \{0\} \times \{0\} \times \{0\} \times \{0\} \times \mathbf{R}^{|I_2|}_+. \tag{3.16}$$

Inclusion (3.16) means that there exist $x \in \mathbf{R}^n$, $\lambda \in \mathbf{R}^l$ and $\mu \in \mathbf{R}^m$ satisfying (3.1)–(3.3), (3.6), and the relations

$$\mu_{I_1} \geq 0, \quad \frac{\partial G_{I_1}}{\partial \sigma}(\bar\sigma,\, \bar x)d + \frac{1}{2}\frac{\partial^2 G_{I_1}}{\partial x^2}(\bar\sigma,\, \bar x)[\xi,\, \xi] = -\frac{\partial G_{I_1}}{\partial x}(\bar\sigma,\, \bar x)x,$$

$$\mu_{I_2} = 0, \quad \frac{\partial G_{I_2}}{\partial \sigma}(\bar{\sigma},\, \bar{x})d + \frac{1}{2}\frac{\partial^2 G_{I_2}}{\partial x^2}(\bar{\sigma},\, \bar{x})[\xi,\, \xi] \leq -\frac{\partial G_{I_2}}{\partial x}(\bar{\sigma},\, \bar{x})x.$$

These relations combined with (3.13) imply (2.11), (3.4) and (3.5). This completes the proof. ∎

For $d = 0$, relations (3.1)–(3.6) take the form

$$\begin{aligned} &\frac{1}{2}\frac{\partial^2 \Psi}{\partial x^2}(\bar{\sigma},\, \bar{x},\, \bar{\lambda},\, \bar{\mu})[\xi,\, \xi] \\ &+ \left(\frac{\partial^2 F}{\partial x^2}(\bar{\sigma},\, \bar{x})[\xi]\right)^{\mathrm{T}} \eta + \left(\frac{\partial^2 G}{\partial x^2}(\bar{\sigma},\, \bar{x})[\xi]\right)^{\mathrm{T}} \zeta \quad = \quad -\frac{\partial \Psi}{\partial x}(\bar{\sigma},\, \bar{x},\, \bar{\lambda},\, \bar{\mu})x \\ &\qquad\qquad - \left(\frac{\partial F}{\partial x}(\bar{\sigma},\, \bar{x})\right)^{\mathrm{T}} \lambda - \left(\frac{\partial G}{\partial x}(\bar{\sigma},\, \bar{x})\right)^{\mathrm{T}} \mu, \end{aligned} \tag{3.17}$$

$$\frac{1}{2}\frac{\partial^2 F}{\partial x^2}(\bar{\sigma},\, \bar{x})[\xi,\, \xi] = -\frac{\partial F}{\partial x}(\bar{\sigma},\, \bar{x})x, \tag{3.18}$$

$$\frac{1}{2}\frac{\partial^2 G_{A_+}}{\partial x^2}(\bar{\sigma},\, \bar{x})[\xi,\, \xi] = -\frac{\partial G_{A_+}}{\partial x}(\bar{\sigma},\, \bar{x})x, \tag{3.19}$$

$$\mu_{A_0} \geq 0, \quad \frac{1}{2}\frac{\partial^2 G_{A_0}}{\partial x^2}(\bar{\sigma},\, \bar{x})[\xi,\, \xi] \leq -\frac{\partial G_{A_0}}{\partial x}(\bar{\sigma},\, \bar{x})x, \tag{3.20}$$

$$(\zeta_i + \mu_i)\left(\left\langle \frac{\partial G_i}{\partial x}(\bar{\sigma},\, \bar{x}),\, \xi \right\rangle + \frac{1}{2}\frac{\partial^2 G_i}{\partial x^2}(\bar{\sigma},\, \bar{x})[\xi,\, \xi] + \left\langle \frac{\partial G_i}{\partial x}(\bar{\sigma},\, \bar{x}),\, x \right\rangle\right) = 0,\ i \in A_0, \tag{3.21}$$

$$\mu_N = 0. \tag{3.22}$$

Theorem 3.2 *Let $(\bar{x},\, \bar{\lambda},\, \bar{\mu}) \in \mathbf{R}^n \times \mathbf{R}^l \times \mathbf{R}^m$ be a solution of system* (1.1) *for $\sigma = \bar{\sigma} \in \mathbf{R}^s$. Suppose that for any solution $(\xi,\, \eta,\, \zeta)$ of system* (2.8)*–*(2.11)*,* (2.13) *such that there exist $x \in \mathbf{R}^n$, $\lambda \in \mathbf{R}^l$ and $\mu \in \mathbf{R}^m$ satisfying* (3.17)*–*(3.22)*, it holds that $(\xi,\, \eta,\, \zeta) = (0,\, 0,\, 0)$.*

Then for each $\sigma \in \mathbf{R}^s$ close enough to $\bar{\sigma}$, any solution $(x(\sigma),\, \lambda(\sigma),\, \mu(\sigma))$ of system (1.1) *close enough to $(\bar{x},\, \bar{\lambda},\, \bar{\mu})$ satisfies the estimate*

$$\|x(\sigma) - \bar{x}\| + \|\lambda(\sigma) - \bar{\lambda}\| + \|\mu(\sigma) - \bar{\mu}\| = O(\|\sigma - \bar{\sigma}\|^{1/2}). \tag{3.23}$$

This theorem is an immediate consequence of Theorem 3.1 applied with $t_k = \|x(\sigma^k) - \bar{x}\| + \|\lambda(\sigma^k) - \bar{\lambda}\| + \|\mu(\sigma^k) - \bar{\mu}\|$ for a given sequence $\{\sigma^k\} \subset \mathbf{R}^s$ convergent to $\bar{\sigma}$.

System (2.8)–(2.11), (2.13), (3.17)–(3.22) is similar to the KKT system for the MP problem

$$\begin{aligned} \text{minimize} \quad & \tfrac{\partial \Psi}{\partial x}(\bar{\sigma},\, \bar{x},\, \bar{\lambda},\, \bar{\mu})[\xi,\, x] + \tfrac{1}{3!}\tfrac{\partial^2 \Psi}{\partial x^2}(\bar{\sigma},\, \bar{x},\, \bar{\lambda},\, \bar{\mu})[\xi,\, \xi,\, \xi] \\ \text{subject to} \quad & \tfrac{\partial F}{\partial x}(\bar{\sigma},\, \bar{x})x + \tfrac{1}{2}\tfrac{\partial^2 F}{\partial x^2}(\bar{\sigma},\, \bar{x})[\xi,\, \xi] = 0, \quad \tfrac{\partial G_{A_+}}{\partial x}(\bar{\sigma},\, \bar{x})x + \tfrac{1}{2}\tfrac{\partial^2 G_{A_+}}{\partial x^2}(\bar{\sigma},\, \bar{x})[\xi,\, \xi] = 0, \\ & \tfrac{\partial G_{A_0}}{\partial x}(\bar{\sigma},\, \bar{x})x + \tfrac{1}{2}\tfrac{\partial^2 G_{A_0}}{\partial x^2}(\bar{\sigma},\, \bar{x})[\xi,\, \xi] \leq 0, \\ & \tfrac{\partial F}{\partial x}(\bar{\sigma},\, \bar{x})\xi = 0, \quad \tfrac{\partial G_{A_+}}{\partial x}(\bar{\sigma},\, \bar{x})\xi = 0, \quad \tfrac{\partial G_{A_0}}{\partial x}(\bar{\sigma},\, \bar{x})\xi \leq 0 \end{aligned} \tag{3.24}$$

in variables ξ, $x \in \mathbf{R}^n$, though (3.21) is stronger than the usual complementary slackness condition for this problem. In particular, if for any stationary point (ξ, x) of problem (3.24) and any associated Lagrange multiplier $(\eta, \zeta_{A_+}, \zeta_{A_0}, \lambda, \mu_{A_+}, \mu_{A_0})$ it holds that $(\xi, \eta, \zeta) = (0, 0, 0)$ then Theorem 3.2 is applicable.

The use of Theorem 3.2 is demonstrated by the following

Example 3.1 Let $s = 3$, $n = 2$, $l = 1$, $m = 0$, $\Phi(\sigma, x) = (1 + x_2 + \sigma_1, x_2^2 + \sigma_2)$, $F(\sigma, x) = x_1 - \sigma_3$. Then $(\bar{x}, \bar{\lambda}) = (0, -1)$ is the unique solution of system (1.1) for $\bar{\sigma} = 0$.

For any $\sigma \in \mathbf{R}^3$ with $\sigma_2 < 0$, system (1.1) has two solution of the form

$$(x(\sigma), \lambda(\sigma)) = ((\sigma_3, \pm\sqrt{-\sigma_2}), -1 \mp \sqrt{-\sigma_2} - \sigma_1),$$

and thus $(\bar{x}, \bar{\lambda})$ is only Hölder stable.

It can be easily checked that Theorem 3.2 is applicable in this example.

Remark 3.1 Along with Theorem 3.1, the same way one can derive some other asymptotic results, e.g., for sequences of the form $\{((\sigma^k - \bar{\sigma})/t_k^2, (x^k - \bar{x})/t_k, (\lambda^k - \bar{\lambda})/t_k^2, (\mu^k - \bar{\mu})/t_k^2)\}$. Accordingly, along with Theorem 3.2, one can derive sufficient conditions for the (2.17) and the stronger estimate for multipliers:

$$\|\lambda(\sigma) - \bar{\lambda}\| + \|\mu(\sigma) - \bar{\mu}\| = O(\|\sigma - \bar{\sigma}\|).$$

We skip this statement, in order to save space, and because the reader can easily restore it.

Example 3.1 demonstrates the difference between the unmodified and thus modified versions of Theorem 3.2: the modified Theorem 3.2 will work if we remove the term x_2 in the first component of Φ.

Note, however, that estimate (3.23), though weaker than (2.14), still implies that $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is an isolated solution of system (1.1) for $\sigma = \bar{\sigma}$, and hence, $(\bar{\lambda}, \bar{\mu})$ is a unique multiplier associated with $\bar{x}$. Moreover, it is evident that if the system comprised by (2.8)–(2.11), (2.13) and (3.17)–(3.22) with $x = 0$, $\lambda = 0$ and $\mu = 0$ has only the trivial solution then SMFCQ still holds at $\bar{x}$ for $(\bar{\lambda}, \bar{\mu})$.

Theorem 3.3 *Let $(\bar{x}, \bar{\lambda}, \bar{\mu}) \in \mathbf{R}^n \times \mathbf{R}^l \times \mathbf{R}^m$ be a solution of system* (1.1) *for $\sigma = \bar{\sigma} \in \mathbf{R}^s$. Suppose that:*

(i) For any solution (ξ, η, ζ) of system (2.8)–(2.11), (2.13) *such that there exist $x \in \mathbf{R}^n$, $\lambda \in \mathbf{R}^l$ and $\mu \in \mathbf{R}^m$ satisfying* (3.17)–(3.22) *it holds that $\xi = 0$.*

(ii) For any (ξ, η, ζ) such that there exist $x, \tilde{x} \in \mathbf{R}^n$, $\lambda, \tilde{\lambda} \in \mathbf{R}^l$ and $\mu, \tilde{\mu} \in \mathbf{R}^m$ satisfying

$$\begin{aligned}\left(\frac{\partial^2 F}{\partial x^2}(\bar{\sigma}, \bar{x})[\xi]\right)^{\mathrm{T}} \eta + \left(\frac{\partial^2 G}{\partial x^2}(\bar{\sigma}, \bar{x})[\xi]\right)^{\mathrm{T}} \zeta &= -\frac{\partial \Psi}{\partial x}(\bar{\sigma}, \bar{x}, \bar{\lambda}, \bar{\mu})\tilde{x} \\ &\quad - \left(\frac{\partial F}{\partial x}(\bar{\sigma}, \bar{x})\right)^{\mathrm{T}} \tilde{\lambda} - \left(\frac{\partial G}{\partial x}(\bar{\sigma}, \bar{x})\right)^{\mathrm{T}} \tilde{\mu},\end{aligned} \tag{3.25}$$

$$\left(\frac{\partial F}{\partial x}(\bar{\sigma},\, \bar{x})\right)^{\mathrm{T}} \eta + \left(\frac{\partial G}{\partial x}(\bar{\sigma},\, \bar{x})\right)^{\mathrm{T}} \zeta = 0, \tag{3.26}$$

$$\frac{\partial \Psi}{\partial x}(\bar{\sigma},\, \bar{x},\, \bar{\lambda},\, \bar{\mu})\xi = -\left(\frac{\partial F}{\partial x}(\bar{\sigma},\, \bar{x})\right)^{\mathrm{T}} \lambda - \left(\frac{\partial G}{\partial x}(\bar{\sigma},\, \bar{x})\right)^{\mathrm{T}} \mu, \tag{3.27}$$

$$\frac{1}{2}\frac{\partial^2 F}{\partial x^2}(\bar{\sigma},\, \bar{x})[\xi,\, \xi] = -\frac{\partial F}{\partial x}(\bar{\sigma},\, \bar{x})x, \tag{3.28}$$

$$\frac{1}{2}\frac{\partial^2 G_{A_+}}{\partial x^2}(\bar{\sigma},\, \bar{x})[\xi,\, \xi] = -\frac{\partial G_{A_+}}{\partial x}(\bar{\sigma},\, \bar{x})x, \tag{3.29}$$

$$\zeta_{A_0} \geq 0, \quad \frac{1}{2}\frac{\partial^2 G_{A_0}}{\partial x^2}(\bar{\sigma},\, \bar{x})[\xi,\, \xi] \leq -\frac{\partial G_{A_0}}{\partial x}(\bar{\sigma},\, \bar{x})x, \tag{3.30}$$

$$\zeta_N = 0, \tag{3.31}$$

$$\frac{\partial F}{\partial x}(\bar{\sigma},\, \bar{x})\tilde{x} = 0, \tag{3.32}$$

$$\frac{\partial G_{A_+}}{\partial x}(\bar{\sigma},\, \bar{x})\tilde{x} = 0, \tag{3.33}$$

$$\mu_{A_0} \geq 0, \quad \frac{\partial G_{A_0}}{\partial x}(\bar{\sigma},\, \bar{x})\tilde{x} \leq 0, \tag{3.34}$$

$$\mu_N = 0, \tag{3.35}$$

$$\frac{\partial F}{\partial x}(\bar{\sigma},\, \bar{x})\xi = 0, \tag{3.36}$$

$$\frac{\partial G_{A_+}}{\partial x}(\bar{\sigma},\, \bar{x})\xi = 0, \tag{3.37}$$

$$\tilde{\mu}_{A_0} \geq 0, \quad \frac{\partial G_{A_0}}{\partial x}(\bar{\sigma},\, \bar{x})\xi \leq 0, \tag{3.38}$$

$$\tilde{\mu}_N = 0, \tag{3.39}$$

$$\begin{aligned} & (\zeta_i + \mu_i + \tilde{\mu}_i)\Bigg(\frac{1}{2}\frac{\partial^2 G_i}{\partial x^2}(\bar{\sigma},\, \bar{x})[\xi,\, \xi] \\ & + \left\langle \frac{\partial G_i}{\partial x}(\bar{\sigma},\, \bar{x}),\, x \right\rangle + \left\langle \frac{\partial G_i}{\partial x}(\bar{\sigma},\, \bar{x}),\, \xi \right\rangle + \left\langle \frac{\partial G_i}{\partial x}(\bar{\sigma},\, \bar{x}),\, \tilde{x} \right\rangle\Bigg) = 0,\ i \in A_0, \end{aligned} \tag{3.40}$$

it holds that $\xi = 0$ *or* $(\eta,\, \zeta) = (0,\, 0)$.

Then for each $\sigma \in \mathbf{R}^s$ *close enough to* $\bar{\sigma}$, *any solution* $(x(\sigma),\, \lambda(\sigma),\, \mu(\sigma))$ *of system* (1.1) *close enough to* $(\bar{x},\, \bar{\lambda},\, \bar{\mu})$ *satisfies the estimate*

$$\|x(\sigma) - \bar{x}\| + \mathrm{dist}((\lambda(\sigma),\, \mu(\sigma)),\, \mathcal{M}) = O(\|\sigma - \bar{\sigma}\|^{1/2}). \tag{3.41}$$

Proof. We need to prove the estimate

$$\|x(\sigma) - \bar{x}\| = O(\|\sigma - \bar{\sigma}\|^{1/2}). \tag{3.42}$$

The remaining estimate

$$\mathrm{dist}((\lambda(\sigma),\, \mu(\sigma)),\, \mathcal{M}) = O(\|\sigma - \bar{\sigma}\|^{1/2}),$$

will then follow from (2.29) (obtained the same way as in Theorem 2.3) and (3.42).

Suppose that (3.42) does not hold. Then there exist sequences $\{\sigma^k\} \subset \mathbf{R}^s \setminus \{\bar{\sigma}\}$, $\{x^k\} \subset \mathbf{R}^n$, $\{\lambda^k\} \subset \mathbf{R}^l$ and $\{\mu^k\} \subset \mathbf{R}^m$ such that $\{\sigma^k\} \to \bar{\sigma}$, $\{x^k\} \to \bar{x}$, $\{\lambda^k\} \to \bar{\lambda}$, $\{\mu^k\} \to \bar{\mu}$, for each k the point $(x^k,\, \lambda^k,\, \mu^k)$ is a solution of system (1.1) for $\sigma = \sigma^k$, and

$$\frac{\|x^k - \bar{x}\|}{\|\sigma^k - \bar{\sigma}\|^{1/2}} \to \infty, \tag{3.43}$$

or the other way round,

$$\|\sigma^k - \bar{\sigma}\| = o(\|x^k - \bar{x}\|^2) \tag{3.44}$$

((3.43) implies that $x^k \neq \bar{x}$ for all k large enough).

Suppose further that the entire sequence $\{(x^k - \bar{x})/\|x^k - \bar{x}\|\}$ converges to some $\xi \in \mathbf{R}^n$, $\|\xi\| = 1$. Consider first the case when

$$\|(\lambda^k - \bar{\lambda},\, \mu^k - \bar{\mu})\| = O(\|x^k - \bar{x}\|). \tag{3.45}$$

Then we may suppose that the entire sequence $\{(\lambda^k - \bar{\lambda},\, \mu^k - \bar{\mu})/\|x^k - \bar{x}\|\}$ converges to some $(\eta,\, \zeta) \in \mathbf{R}^l \times \mathbf{R}^m$. Applying Theorem 3.1 with $t_k = \|x^k - \bar{x}\|$, and taking into account (3.44), we conclude that $(\xi,\, \eta,\, \zeta)$ satisfies (2.8)–(2.11), (2.13) and (3.17)–(3.22) with some $x \in \mathbf{R}^n$, $\lambda \in \mathbf{R}^l$ and $\mu \in \mathbf{R}^m$, which is a contradiction, since $\xi \neq 0$.

Let now (3.45) be violated. Then we may assume that

$$\|x^k - \bar{x}\| = o(\|(\lambda^k - \bar{\lambda},\, \mu^k - \bar{\mu})\|). \tag{3.46}$$

Perhaps, some appropriate generalizations of Theorem 3.1 may help in this case as well, but we prefer to give a direct proof.

By the same argument as in Theorem 3.1, we may suppose that there exists $(I_1,\, I_2) \in \mathcal{A}_0$ such that for all k

$$\begin{aligned}
0 &= \Phi(\sigma^k,\, x^k) + \left(\frac{\partial F}{\partial x}(\sigma^k,\, x^k)\right)^{\mathrm{T}} \lambda^k + \left(\frac{\partial G}{\partial x}(\sigma^k,\, x^k)\right)^{\mathrm{T}} \mu^k \\
&= \Psi(\sigma^k,\, x^k,\, \bar{\lambda},\, \bar{\mu}) + \left(\frac{\partial F}{\partial x}(\sigma^k,\, x^k)\right)^{\mathrm{T}} (\lambda^k - \bar{\lambda}) + \left(\frac{\partial G}{\partial x}(\sigma^k,\, x^k)\right)^{\mathrm{T}} (\mu^k - \bar{\mu}) \\
&= \Psi(\bar{\sigma},\, x^k,\, \bar{\lambda},\, \bar{\mu}) + \left(\frac{\partial F}{\partial x}(\bar{\sigma},\, x^k)\right)^{\mathrm{T}} (\lambda^k - \bar{\lambda}) + \left(\frac{\partial G}{\partial x}(\bar{\sigma},\, x^k)\right)^{\mathrm{T}} (\mu^k - \bar{\mu}) + O(\|\sigma^k - \bar{\sigma}\|) \\
&= \frac{\partial \Psi}{\partial x}(\bar{\sigma},\, \bar{x},\, \bar{\lambda},\, \bar{\mu})(x^k - \bar{x})
\end{aligned}$$

$$
\begin{aligned}
&+\left(\frac{\partial F}{\partial x}(\bar{\sigma},\,\bar{x})\right)^{\mathrm{T}}(\lambda^k-\bar{\lambda})+\left(\frac{\partial G_{A_+}}{\partial x}(\bar{\sigma},\,\bar{x})\right)^{\mathrm{T}}(\mu^k-\bar{\mu})_{A_+}+\left(\frac{\partial G_{I_1}}{\partial x}(\bar{\sigma},\,\bar{x})\right)^{\mathrm{T}}\mu^k_{I_1}\\
&+\frac{1}{2}\frac{\partial^2\Psi}{\partial x^2}(\bar{\sigma},\,\bar{x},\,\bar{\lambda},\,\bar{\mu})[x^k-\bar{x},\,x^k-\bar{x}]+\left(\frac{\partial^2 F}{\partial x^2}(\bar{\sigma},\,\bar{x})[x^k-\bar{x}]\right)^{\mathrm{T}}(\lambda^k-\bar{\lambda})\\
&+\left(\frac{\partial^2 G_{A_+}}{\partial x^2}(\bar{\sigma},\,\bar{x})[x^k-\bar{x}]\right)^{\mathrm{T}}(\mu^k-\bar{\mu})_{A_+}+\left(\frac{\partial^2 G_{I_1}}{\partial x^2}(\bar{\sigma},\,\bar{x})[x^k-\bar{x}]\right)^{\mathrm{T}}\mu^k_{I_1}\\
&+o(\|x^k-\bar{x}\|^2),
\end{aligned}
\tag{3.47}
$$

where (3.46) was taken into account, and similarly

$$
\begin{aligned}
0 &= F(\sigma^k,\,x^k)\\
&= \frac{\partial F}{\partial x}(\bar{\sigma},\,\bar{x})(x^k-\bar{x})+\frac{1}{2}\frac{\partial^2 F}{\partial x^2}(\bar{\sigma},\,\bar{x})[x^k-\bar{x},\,x^k-\bar{x}]+o(\|x^k-\bar{x}\|^2),
\end{aligned}
\tag{3.48}
$$

$$
\begin{aligned}
0 &= G_{A_+\cup I_1}(\sigma^k,\,x^k)\\
&= \frac{\partial G_{A_+\cup I_1}}{\partial x}(\bar{\sigma},\,\bar{x})(x^k-\bar{x})+\frac{1}{2}\frac{\partial^2 G_{A_+\cup I_1}}{\partial x^2}(\bar{\sigma},\,\bar{x})[x^k-\bar{x},\,x^k-\bar{x}]+o(\|x^k-\bar{x}\|^2),
\end{aligned}
\tag{3.49}
$$

$$
\begin{aligned}
0 &\geq G_{I_2}(\sigma^k,\,x^k)\\
&= \frac{\partial G_{I_2}}{\partial x}(\bar{\sigma},\,\bar{x})(x^k-\bar{x})+\frac{1}{2}\frac{\partial^2 G_{I_2}}{\partial x^2}(\bar{\sigma},\,\bar{x})[x^k-\bar{x},\,x^k-\bar{x}]+o(\|x^k-\bar{x}\|^2),
\end{aligned}
\tag{3.50}
$$

$$
\mu^k_{I_1}\geq 0,\quad \mu^k_{I_2\cup N}=0.
\tag{3.51}
$$

From (3.47) and the inequality in (3.51), we obtain the inclusion

$$
\begin{aligned}
&-\operatorname{im}\left(\frac{\partial F}{\partial x}(\bar{\sigma},\,\bar{x})\right)^{\mathrm{T}}-\operatorname{im}\left(\frac{\partial G_{A_+}}{\partial x}(\bar{\sigma},\,\bar{x})\right)^{\mathrm{T}}\\
&\qquad-\left(\frac{\partial G_{I_1}}{\partial x}(\bar{\sigma},\,\bar{x})\right)^{\mathrm{T}}\left(\mathbf{R}^{|I_1|}_+\right)\ \ni\ \frac{\partial\Psi}{\partial x}(\bar{\sigma},\,\bar{x},\,\bar{\lambda},\,\bar{\mu})(x^k-\bar{x})+o(\|x^k-\bar{x}\|),
\end{aligned}
$$

with the closed cone in the left-hand side. Dividing this inclusion and (3.48)–(3.50) by $\|x^k-\bar{x}\|$, and passing onto the limit as $k\to\infty$, we obtain the inclusion

$$
\begin{aligned}
\frac{\partial\Psi}{\partial x}(\bar{\sigma},\,\bar{x},\,\bar{\lambda},\,\bar{\mu})\xi \ \in\ &-\operatorname{im}\left(\frac{\partial F}{\partial x}(\bar{\sigma},\,\bar{x})\right)^{\mathrm{T}}-\operatorname{im}\left(\frac{\partial G_{A_+}}{\partial x}(\bar{\sigma},\,\bar{x})\right)^{\mathrm{T}}\\
&-\left(\frac{\partial G_{I_1}}{\partial x}(\bar{\sigma},\,\bar{x})\right)^{\mathrm{T}}\left(\mathbf{R}^{|I_1|}_+\right)
\end{aligned}
\tag{3.52}
$$

and the relations (3.36), (3.37), and

$$
\frac{\partial G_{I_1}}{\partial x}(\bar{\sigma},\,\bar{x})\xi=0,\quad \frac{\partial G_{I_2}}{\partial x}(\bar{\sigma},\,\bar{x})\xi\leq 0.
\tag{3.53}
$$

Inclusion (3.52) means that there exist $\lambda \in \mathbf{R}^l$ and $\mu \in \mathbf{R}^m$ satisfying (3.27), (3.35) and such that

$$\mu_{I_1} \geq 0, \quad \mu_{I_2} = 0. \tag{3.54}$$

Furthermore, relations (3.48)–(3.50) can be re-written in the form

$$-\operatorname{im}\left(\begin{array}{c} \frac{\partial F}{\partial x}(\bar{\sigma},\, \bar{x}) \\ \frac{\partial G_{A_+}}{\partial x}(\bar{\sigma},\, \bar{x}) \\ \frac{\partial G_{I_1}}{\partial x}(\bar{\sigma},\, \bar{x}) \\ \frac{\partial G_{I_2}}{\partial x}(\bar{\sigma},\, \bar{x}) \end{array}\right)$$

$$-\{0\} \times \{0\} \times \{0\} \times \mathbf{R}_+^{|I_2|} \quad \ni \quad \frac{1}{2}\left(\begin{array}{c} \frac{\partial^2 F}{\partial x^2}(\bar{\sigma},\, \bar{x})[x^k - \bar{x},\, x^k - \bar{x}] \\ \frac{\partial^2 G_{A_+}}{\partial x^2}(\bar{\sigma},\, \bar{x})[x^k - \bar{x},\, x^k - \bar{x}] \\ \frac{\partial^2 G_{I_1}}{\partial x^2}(\bar{\sigma},\, \bar{x})[x^k - \bar{x},\, x^k - \bar{x}] \\ \frac{\partial^2 G_{I_2}}{\partial x^2}(\bar{\sigma},\, \bar{x})[x^k - \bar{x},\, x^k - \bar{x}] \end{array}\right) + o(\|x^k - \bar{x}\|^2),$$

with the closed cone in the left-hand side. Dividing this inclusion by $\|x^k - \bar{x}\|^2$, and passing onto the limit as $k \to \infty$, we obtain the inclusion

$$\frac{1}{2}\left(\begin{array}{c} \frac{\partial^2 F}{\partial x^2}(\bar{\sigma},\, \bar{x})[\xi,\, \xi] \\ \frac{\partial^2 G_{A_+}}{\partial x^2}(\bar{\sigma},\, \bar{x})[\xi,\, \xi] \\ \frac{\partial^2 G_{I_1}}{\partial x^2}(\bar{\sigma},\, \bar{x})[\xi,\, \xi] \\ \frac{\partial^2 G_{I_2}}{\partial x^2}(\bar{\sigma},\, \bar{x})[\xi,\, \xi] \end{array}\right) \in -\operatorname{im}\left(\begin{array}{c} \frac{\partial F}{\partial x}(\bar{\sigma},\, \bar{x}) \\ \frac{\partial G_{A_+}}{\partial x}(\bar{\sigma},\, \bar{x}) \\ \frac{\partial G_{I_1}}{\partial x}(\bar{\sigma},\, \bar{x}) \\ \frac{\partial G_{I_2}}{\partial x}(\bar{\sigma},\, \bar{x}) \end{array}\right) - \{0\} \times \{0\} \times \{0\} \times \mathbf{R}_+^{|I_2|},$$

which means the existence of $x \in \mathbf{R}^n$ satisfying (3.28), (3.29) and the relations

$$\frac{1}{2}\frac{\partial^2 G_{I_1}}{\partial x^2}(\bar{\sigma},\, \bar{x})[\xi,\, \xi] = -\frac{\partial G_{I_1}}{\partial x}(\bar{\sigma},\, \bar{x})x, \quad \frac{1}{2}\frac{\partial^2 G_{I_2}}{\partial x^2}(\bar{\sigma},\, \bar{x})[\xi,\, \xi] \leq -\frac{\partial G_{I_2}}{\partial x}(\bar{\sigma},\, \bar{x})x. \tag{3.55}$$

Suppose now that the entire sequence $\{(\lambda^k - \bar{\lambda},\, \mu^k - \bar{\mu})/\|(\lambda^k - \bar{\lambda},\, \mu^k - \bar{\mu})\|\}$ converges to some $(\eta,\, \zeta) \in \mathbf{R}^l \times \mathbf{R}^m$, $\|(\eta,\, \zeta)\| = 1$.

Dividing (3.47), (3.51) by $\|(\lambda^k - \bar{\lambda},\, \mu^k - \bar{\mu})\|$, employing (3.46), and passing onto the limit as $k \to \infty$, we obtain (3.26), (3.31) and the relations

$$\zeta_{I_1} \geq 0, \quad \zeta_{I_2} = 0. \tag{3.56}$$

Define the vector $\omega \in \mathbf{R}^n \times \mathbf{R}^l \times \mathbf{R}^{|A_+|} \times \mathbf{R}^{|I_1|} \times \mathbf{R}^{|I_2|}$,

$$\omega = \left(\begin{array}{c} \left(\frac{\partial^2 F}{\partial x^2}(\bar{\sigma},\, \bar{x})[\xi]\right)^{\mathrm{T}} \eta + \left(\frac{\partial^2 G_{A_+}}{\partial x^2}(\bar{\sigma},\, \bar{x})[\xi]\right)^{\mathrm{T}} \zeta_{A_+} + \left(\frac{\partial^2 G_{I_1}}{\partial x^2}(\bar{\sigma},\, \bar{x})[\xi]\right)^{\mathrm{T}} \zeta_{I_1} \\ 0 \\ 0 \\ 0 \\ 0 \end{array}\right).$$

Furthermore, for each k define the vector $w^k \in \mathbf{R}^n \times \mathbf{R}^l \times \mathbf{R}^{|A_+|} \times \mathbf{R}^{|I_1|} \times \mathbf{R}^{|I_2|}$,

$$w^k = \begin{pmatrix} \left(\frac{\partial^2 F}{\partial x^2}(\bar{\sigma},\, \bar{x})[x^k - \bar{x}]\right)^{\mathrm{T}} (\lambda^k - \bar{\lambda}) \\ + \left(\frac{\partial^2 G_{A_+}}{\partial x^2}(\bar{\sigma},\, \bar{x})[x^k - \bar{x}]\right)^{\mathrm{T}} (\mu^k - \bar{\mu})_{A_+} + \left(\frac{\partial^2 G_{I_1}}{\partial x^2}(\bar{\sigma},\, \bar{x})[x^k - \bar{x}]\right)^{\mathrm{T}} \mu^k_{I_1} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Taking into account the inequality in (3.51), and (3.46), relations (3.47)–(3.50) imply the inclusion

$$\begin{aligned} &-D\left(\mathbf{R}^n \times \mathbf{R}^l \times \mathbf{R}^{|A_+|} \times \mathbf{R}_+^{|I_1|}\right) \\ &-\{0\} \times \{0\} \times \{0\} \times \{0\} \times \mathbf{R}_-^{|I_2|} \quad \ni \quad w^k + o(\|x^k - \bar{x}\| \|(\lambda^k - \bar{\lambda},\, \mu^k - \bar{\mu})\|), \end{aligned} \tag{3.57}$$

where D is defined in (3.14), and the set in the left-hand side is a closed cone.

Note that $\{w^k/(\|x^k - \bar{x}\| \|(\lambda^k - \bar{\lambda},\, \mu^k - \bar{\mu})\|)\} \to \omega$. Dividing (3.57) by $\|x^k - \bar{x}\| \|(\lambda^k - \bar{\lambda},\, \mu^k - \bar{\mu})\|$, and passing onto the limit as $k \to \infty$, we thus obtain

$$\omega \in -D\left(\mathbf{R}^n \times \mathbf{R}^l \times \mathbf{R}^{|A_+|} \times \mathbf{R}_+^{|I_1|}\right) - \{0\} \times \{0\} \times \{0\} \times \{0\} \times \mathbf{R}_-^{|I_2|}. \tag{3.58}$$

Inclusion (3.58) means that there exist $\tilde{x} \in \mathbf{R}^n$, $\tilde{\lambda} \in \mathbf{R}^l$ and $\tilde{\mu} \in \mathbf{R}^m$ satisfying (3.25), (3.32), (3.33), (3.39) and the relations

$$\frac{\partial G_{I_1}}{\partial x}(\bar{\sigma},\, \bar{x})\tilde{x} = 0, \quad \frac{\partial G_{I_2}}{\partial x}(\bar{\sigma},\, \bar{x})\tilde{x} \leq 0,$$

$$\tilde{\mu}_{I_1} \geq 0, \quad \tilde{\mu}_{I_2} = 0.$$

These relations combined with (3.53)–(3.56) imply (3.30), (3.38), (3.34) and (3.40). Thus $(\xi,\, \eta,\, \zeta)$ satisfies (3.25)–(3.40) with some $x,\, \tilde{x} \in \mathbf{R}^n$, $\lambda,\, \tilde{\lambda} \in \mathbf{R}^l$ and $\mu,\, \tilde{\mu} \in \mathbf{R}^m$, which is a contradiction, since $\xi \neq 0$ and $(\eta,\, \zeta) \neq 0$. This completes the proof. ■

System (3.25)–(3.40) is similar to the KKT system for the MP problem

$$\begin{array}{ll} \text{minimize} & \frac{\partial \Psi}{\partial x}(\bar{\sigma},\, \bar{x},\, \bar{\lambda},\, \bar{\mu})[\xi,\, \tilde{x}] \\ \text{subject to} & \frac{\partial F}{\partial x}(\bar{\sigma},\, \bar{x})x + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(\bar{\sigma},\, \bar{x})[\xi,\, \xi] = 0, \\ & \frac{\partial G_{A_+}}{\partial x}(\bar{\sigma},\, \bar{x})x + \frac{1}{2}\frac{\partial^2 G_{A_+}}{\partial x^2}(\bar{\sigma},\, \bar{x})[\xi,\, \xi] = 0, \\ & \frac{\partial G_{A_0}}{\partial x}(\bar{\sigma},\, \bar{x})x + \frac{1}{2}\frac{\partial^2 G_{A_0}}{\partial x^2}(\bar{\sigma},\, \bar{x})[\xi,\, \xi] \leq 0, \\ & \frac{\partial F}{\partial x}(\bar{\sigma},\, \bar{x})\tilde{x} = 0, \quad \frac{\partial G_{A_+}}{\partial x}(\bar{\sigma},\, \bar{x})\tilde{x} = 0, \quad \frac{\partial G_{A_0}}{\partial x}(\bar{\sigma},\, \bar{x})\tilde{x} \leq 0 \\ & \frac{\partial F}{\partial x}(\bar{\sigma},\, \bar{x})\xi = 0, \quad \frac{\partial G_{A_+}}{\partial x}(\bar{\sigma},\, \bar{x})\xi = 0, \quad \frac{\partial G_{A_0}}{\partial x}(\bar{\sigma},\, \bar{x})\xi \leq 0 \end{array} \tag{3.59}$$

in variables $\xi,\, x,\, \tilde{x} \in \mathbf{R}^n$, though (3.40) is again stronger than the usual complementary slackness condition. In particular, if for any stationary point $(\xi,\, x)$ of problem (3.24) it holds that $\xi = 0$, and if for any stationary point $(\xi,\, x,\, \tilde{x})$ of problem (3.59) and any associated Lagrange multiplier $(\eta,\, \zeta_{A_+},\, \zeta_{A_0},\, \lambda,\, \mu_{A_+},\, \mu_{A_0},\, \tilde{\lambda},\, \tilde{\mu}_{A_+},\, \tilde{\mu}_{A_0})$ it holds that $\xi = 0$ or $(\eta,\, \zeta_{A_+},\, \zeta_{A_0}) = (0,\, 0,\, 0)$, then Theorem 3.3 is applicable.

Example 2.1 above demonstrates that estimate (3.41) in Theorem 3.3 cannot be improved.

Remark 3.2 If in Theorem 3.3 we replace system (3.25)–(3.40) by

$$\left(\frac{\partial F}{\partial x}(\bar{\sigma},\, \bar{x})\right)^{\mathrm{T}} \eta + \left(\frac{\partial G}{\partial x}(\bar{\sigma},\, \bar{x})\right)^{\mathrm{T}} \zeta = -\frac{\partial \Psi}{\partial x}(\bar{\sigma},\, \bar{x},\, \bar{\lambda},\, \bar{\mu})\tilde{x} \tag{3.60}$$

$$\frac{\partial \Psi}{\partial x}(\bar{\sigma},\, \bar{x},\, \bar{\lambda},\, \bar{\mu})\xi = -\left(\frac{\partial F}{\partial x}(\bar{\sigma},\, \bar{x})\right)^{\mathrm{T}} \lambda - \left(\frac{\partial G}{\partial x}(\bar{\sigma},\, \bar{x})\right)^{\mathrm{T}} \mu, \tag{3.61}$$

$$\frac{1}{2}\frac{\partial^2 F}{\partial x^2}(\bar{\sigma},\, \bar{x})[\xi,\, \xi] = -\frac{\partial F}{\partial x}(\bar{\sigma},\, \bar{x})x, \tag{3.62}$$

$$\frac{1}{2}\frac{\partial^2 G_{A_+}}{\partial x^2}(\bar{\sigma},\, \bar{x})[\xi,\, \xi] = -\frac{\partial G_{A_+}}{\partial x}(\bar{\sigma},\, \bar{x})x, \tag{3.63}$$

$$\frac{1}{2}\frac{\partial^2 G_{A_0}}{\partial x^2}(\bar{\sigma},\, \bar{x})[\xi,\, \xi] \leq -\frac{\partial G_{A_0}}{\partial x}(\bar{\sigma},\, \bar{x})x, \tag{3.64}$$

$$\zeta_N = 0, \tag{3.65}$$

$$\frac{\partial F}{\partial x}(\bar{\sigma},\, \bar{x})\tilde{x} = 0, \tag{3.66}$$

$$\frac{\partial G_{A_+}}{\partial x}(\bar{\sigma},\, \bar{x})\tilde{x} = 0, \tag{3.67}$$

$$\frac{\partial G_{A_0}}{\partial x}(\bar{\sigma},\, \bar{x})\tilde{x} \leq 0, \tag{3.68}$$

$$\mu_N = 0, \tag{3.69}$$

$$\frac{\partial F}{\partial x}(\bar{\sigma},\, \bar{x})\xi = 0, \tag{3.70}$$

$$\frac{\partial G_{A_+}}{\partial x}(\bar{\sigma},\, \bar{x})\xi = 0, \tag{3.71}$$

$$\frac{\partial G_{A_0}}{\partial x}(\bar{\sigma},\, \bar{x})\xi \leq 0, \tag{3.72}$$

$$\begin{aligned}(|\zeta_i| + |\mu_i|)\Bigg(&\frac{1}{2}\frac{\partial^2 G_i}{\partial x^2}(\bar{\sigma},\, \bar{x})[\xi,\, \xi] \\ &+ \left\langle \frac{\partial G_i}{\partial x}(\bar{\sigma},\, \bar{x}),\, x\right\rangle + \left\langle \frac{\partial G_i}{\partial x}(\bar{\sigma},\, \bar{x}),\, \xi\right\rangle + \left\langle \frac{\partial G_i}{\partial x}(\bar{\sigma},\, \bar{x}),\, \tilde{x}\right\rangle\Bigg) = 0,\ i \in A_0,\end{aligned} \tag{3.73}$$

then, along with the estimate (3.42), we obtain the stronger estimate for multipliers:

$$\operatorname{dist}((\lambda(\sigma),\, \mu(\sigma)),\, \mathcal{M}) = O(\|\sigma - \bar{\sigma}\|). \tag{3.74}$$

In order to prove this, one should not apply Hoffman's lemma but rather argue directly, by a contradiction, similarly to the proof of Theorem 3.3, but using the projections of (λ^k, μ^k) on $\mathcal{M}$ instead of $(\bar{\lambda}, \bar{\mu})$.

Observe that if for any (ξ, η, ζ) such that there exist $x, \tilde{x} \in \mathbf{R}^n$, $\lambda \in \mathbf{R}^l$ and $\mu \in \mathbf{R}^m$ satisfying (3.60)–(3.73) it holds that $\xi = 0$ or $(\eta, \zeta) = (0, 0)$, then the same holds for any for any (ξ, η, ζ) such that there exist $x, \tilde{x} \in \mathbf{R}^n$, $\lambda, \tilde{\lambda} \in \mathbf{R}^l$ and $\mu, \tilde{\mu} \in \mathbf{R}^m$ satisfying (3.25)–(3.40). Indeed, if (ξ, η, ζ) satisfies the latter, it evidently satisfies the former with the same x, λ and μ, and with $\tilde{x} = 0$. Note that Example 2.1 violates the former condition.

Remark 3.3 In the case of parametric KKT system with canonical perturbations (2.30), the outer estimate of limiting directions presented in Theorem 3.1 is exact. More precisely, for each tuple (d, ξ, η, ζ) satisfying (2.8)–(2.11), (2.13) and (3.1)–(3.6) with some $x \in \mathbf{R}^n$, $\lambda \in \mathbf{R}^l$ and $\mu \in \mathbf{R}^m$, and for each $t \geq 0$, system (2.30) with $\sigma = \bar{\sigma} + t^2 d$ and some $(a(t), b(t), c(t)) = o(t^2)$ has a solution of the form $(\bar{x} + t\xi + t^2 x, \bar{\lambda} + t\eta + t^2\lambda, \bar{\mu} + t\zeta + t^2\mu)$ This can be demonstrated by *explicitly* choosing the needed $a(t)$, $b(t)$ and $c(t)$. Moreover, by the same argument applied with $d = 0$, one can prove that the sufficient condition for estimate (3.23), established in Theorem 3.2, is also necessary in this case. Similarly, the first condition in Theorem 3.3 (concerned with system (2.8)–(2.11), (2.13), (3.17)–(3.22)) is necessary for the estimate (3.42) (and even so more for the estimate (3.41)). However, the second condition (concerned with system (3.25)–(3.40)) is most likely not necessary. It would have been necessary with the additional relations

$$\begin{aligned}
&\frac{1}{2}\frac{\partial^2\Psi}{\partial x^2}(\bar{\sigma}, \bar{x}, \bar{\lambda}, \bar{\mu})[\xi, \xi] \\
+\left(\frac{\partial^2 F}{\partial x^2}(\bar{\sigma}, \bar{x})[\xi]\right)^{\mathrm{T}}\lambda + \left(\frac{\partial^2 G}{\partial x^2}(\bar{\sigma}, \bar{x})[\xi]\right)^{\mathrm{T}}\mu \quad &= \quad -\frac{\partial\Psi}{\partial x}(\bar{\sigma}, \bar{x}, \bar{\lambda}, \bar{\mu})x \\
&\quad - \left(\frac{\partial F}{\partial x}(\bar{\sigma}, \bar{x})\right)^{\mathrm{T}}\hat{\lambda} - \left(\frac{\partial G}{\partial x}(\bar{\sigma}, \bar{x})\right)^{\mathrm{T}}\hat{\mu},
\end{aligned}$$

$$\hat{\mu}_{A_0} \geq 0, \quad \hat{\mu}_N = 0,$$

including additional auxiliary variables $\hat{\lambda} \in \mathbf{R}^l$ and $\hat{\mu} \in \mathbf{R}^m$, and with (3.40) replaced by

$$\begin{aligned}
&(\zeta_i + \mu_i + \tilde{\mu}_i + \hat{\mu})\left(\frac{1}{2}\frac{\partial^2 G_i}{\partial x^2}(\bar{\sigma}, \bar{x})[\xi, \xi]\right. \\
&\left. + \left\langle\frac{\partial G_i}{\partial x}(\bar{\sigma}, \bar{x}), x\right\rangle + \left\langle\frac{\partial G_i}{\partial x}(\bar{\sigma}, \bar{x}), \xi\right\rangle + \left\langle\frac{\partial G_i}{\partial x}(\bar{\sigma}, \bar{x}), \tilde{x}\right\rangle\right) \quad = \quad 0, \ i \in A_0.
\end{aligned}$$

In order to prove the necessity, one can fix an arbitrary $\theta \in (0, 1)$, and for each $t \geq 0$, explicitly find $(a(t), b(t), c(t)) = o(t^{2(1+\theta)})$ such that system (2.30) with $\sigma = \bar{\sigma}$ has the solution of the form $(\bar{x} + t^{1+\theta}\xi + t^{2+\theta}\tilde{x} + t^{2(1+\theta)}x, \bar{\lambda} + t\eta + t^{1+\theta}\lambda + t^{2+\theta}\tilde{\lambda} + t^{2(1+\theta)}\hat{\lambda}, \bar{\mu} + t\zeta + t^{1+\theta}\mu + t^{2+\theta}\tilde{\mu} + t^{2(1+\theta)}\hat{\mu})$.

We will now employ the notion of a critical multiplier which was originally introduced in [4] for equality-constrained optimization problems. This notion was extended to the mixed-constrained case in [7], and it was extensively used in [8, 7] in order to study the dual behavior of Newton-type methods for constrained optimization problems with nonunique multiplier associated to a solution. Furthermore, this notion was employed in [4] and in [5, Section 4] in the context of sensitivity analysis for Lagrange optimality systems, and in the rest of this paper, we extend this analysis to the mixed-constrained case.

For a given index set $I \subset A$, define the linear subspace

$$Q_I = Q_I(\bar{\sigma},\, \bar{x},\, \bar{\lambda},\, \bar{\mu}) = \left\{ x \in \ker \frac{\partial \tilde{F}_I}{\partial x}(\bar{\sigma},\, \bar{x}) \;\middle|\; \frac{\partial \Psi}{\partial x}(\bar{\sigma},\, \bar{x},\, \bar{\lambda},\, \bar{\mu})x \in \operatorname{im}\left(\frac{\partial \tilde{F}_I}{\partial x}(\bar{\sigma},\, \bar{x})\right)^{\mathrm{T}} \right\}$$

in $\mathbf{R}^n$, where we have used notation (2.40). Note that

$$x \in Q_I \quad \Longleftrightarrow \quad \exists\, (y,\, z) \in \mathbf{R}^l \times \mathbf{R}^m \text{ such that } (x,\, y,\, z_A) \in \ker D_I,$$

and if $\frac{\partial \Psi}{\partial x}(\bar{\sigma},\, \bar{x},\, \bar{\lambda},\, \bar{\mu})$ is a symmetric matrix (which automatically holds in the case of (1.2)) then

$$x \in Q_I^{\perp} \quad \Longleftrightarrow \quad (x,\, 0,\, 0) \in \operatorname{im} D_I, \tag{3.75}$$

where we have used notation (2.31).

According to [7, Definition 2.2], the multiplier $(\bar{\lambda},\, \bar{\mu})$ is refereed to as *critical* with respect to the index set I if $\bar{\mu}_{A\setminus I} = 0$ and

$$Q_I \neq \{0\}. \tag{3.76}$$

Criticality of $(\bar{\lambda},\, \bar{\mu})$ with respect to I certainly subsumes that $A \setminus I \subset A_0$, i.e., $I \supset A_+$. Evidently, if the set of multipliers which are not critical with respect to any index set I (such that $A_+ \subset I \subset A$) is nonempty then this set of multipliers is open and dense within $\mathcal{M}$. However, the next result imposes a restriction on the kind of perturbations subject to which such multipliers can be stable.

Consider arbitrary sequences $\{\sigma^k\} \subset \mathbf{R}^s \setminus \{\bar{\sigma}\}$, $\{x^k\} \subset \mathbf{R}^n$, $\{\lambda^k\} \subset \mathbf{R}^l$ and $\{\mu^k\} \subset \mathbf{R}^m$ such that $\{\sigma^k\} \to \bar{\sigma}$, $\{x^k\} \to \bar{x}$, $\{\lambda^k\} \to \bar{\lambda}$, and $\{\mu^k\} \to \bar{\mu}$, and such that for each k the point $(x^k,\, \lambda^k,\, \mu^k)$ is a solution of system (1.1) for $\sigma = \sigma^k$. The latter means that $(x^k,\, \lambda^k,\, \mu^k)$ is a solution of branch system (1.5) for some partition $(I_1,\, I_2) \in \mathcal{A}_0$, and since there is a finite number of such partitions, after passing to subsequences (if necessary) we can assume that $(I_1,\, I_2)$ does not depend on k.

Proposition 3.1 *Let* $(\bar{x},\, \bar{\lambda},\, \bar{\mu}) \in \mathbf{R}^n \times \mathbf{R}^l \times \mathbf{R}^m$ *be a solution of system* (1.1) *for* $\sigma = \bar{\sigma} \in \mathbf{R}^s$. *Suppose that for some partition* $(I_1,\, I_2) \in \mathcal{A}_0$ *there exist sequences* $\{\sigma^k\} \subset \mathbf{R}^s \setminus \{\bar{\sigma}\}$, $\{x^k\} \subset \mathbf{R}^n$, $\{\lambda^k\} \subset \mathbf{R}^l$ *and* $\{\mu^k\} \subset \mathbf{R}^m$ *such that* $\{\sigma^k\} \to \bar{\sigma}$, $\{x^k\} \to \bar{x}$, $\{\lambda^k\} \to \bar{\lambda}$, *and* $\{\mu^k\} \to \bar{\mu}$, *and such that for each* k *the point* $(x^k,\, \lambda^k,\, \mu^k)$ *is a solution of system* (1.5) *for* $\sigma = \sigma^k$. *Suppose further that the multiplier* $(\bar{\lambda},\, \bar{\mu})$ *is not critical with respect to the index set* $A_+ \cup I_1$.

Then

$$\|x^k - \bar{x}\| = O(\|\sigma^k - \bar{\sigma}\|), \tag{3.77}$$

and every limit point $(d,\, \xi) \in \mathbf{R}^s \times \mathbf{R}^n$ *of the sequence* $\{(\sigma^k - \bar{\sigma},\, x^k - \bar{x})/\|\sigma^k - \bar{\sigma}\|\}$ *satisfies the equalities*

$$\frac{\partial F}{\partial \sigma}(\bar{\sigma},\, \bar{x})d + \frac{\partial F}{\partial x}(\bar{\sigma},\, \bar{x})\xi = 0, \quad \frac{\partial G_{A_+\cup I_1}}{\partial \sigma}(\bar{\sigma},\, \bar{x})d + \frac{\partial G_{A_+\cup I_1}}{\partial x}(\bar{\sigma},\, \bar{x})\xi = 0. \tag{3.78}$$

Proof. Set $I = A_+ \cup I_1$, and consider the parametric system comprised by the equalities contained in (1.5):

$$\Phi(\sigma,\, x) + \left(\frac{\partial \tilde{F}_I}{\partial x}(\sigma,\, x)\right)^{\mathrm{T}} \tilde{\lambda} = 0, \quad \tilde{F}_I(\sigma,\, x) = 0, \tag{3.79}$$

with respect to $(x,\, \tilde{\lambda}) \in \mathbf{R}^n \times (\mathbf{R}^l \times \mathbf{R}^{|I|}$. Evidently, $(x^k,\, (\lambda^k,\, \mu_I^k))$ is a solution of this system for $\sigma = \sigma^k$, and for each k. Evidently, $(\bar{x},\, (\bar{\lambda},\, \bar{\mu}_I))$ is a solution of (3.79) for $\sigma = \bar{\sigma}$.

Furthermore, consider the system

$$\frac{\partial \Psi}{\partial x}(\bar{\sigma},\, \bar{x},\, \bar{\lambda},\, \bar{\mu})\xi + \left(\frac{\partial \tilde{F}_I}{\partial x}(\bar{\sigma},\, \bar{x})\right)^{\mathrm{T}} \tilde{\eta} = 0, \quad \frac{\partial \tilde{F}_I}{\partial x}(\bar{\sigma},\, \bar{x})\xi = 0, \tag{3.80}$$

with $\tilde{\eta} \in \mathbf{R}^l \times \mathbf{R}^{|I|}$, playing the same role for system (3.79) as system (2.8)–(2.13) plays for system (1.1).

Suppose that there exist $\xi \in \mathbf{R}^n \setminus \{0\}$ and $\tilde{\eta} \in \mathbf{R}^l \times \mathbf{R}^{|I|}$ such that the pair $(\xi,\, \tilde{\eta})$ is a solution of system (3.80). We then obtain that

$$\frac{\partial \Psi}{\partial x}(\bar{\sigma},\, \bar{x},\, \bar{\lambda},\, \bar{\mu})\xi \in \operatorname{im}\left(\frac{\partial \tilde{F}_I}{\partial x}(\bar{\sigma},\, \bar{x})\right)^{\mathrm{T}}, \quad \frac{\partial \tilde{F}_I}{\partial x}(\bar{\sigma},\, \bar{x})\xi = 0,$$

which contradicts (3.76). We thus proved that $\xi = 0$ holds for any solution $(\xi,\, \tilde{\eta})$ of system (3.80), and hence, Theorem 2.3 yields the estimate (3.77).

Employing (3.77) and the second equality in (3.79), we obtain

$$\begin{aligned} 0 &= \tilde{F}_I(\sigma^k,\, x^k) \\ &= \frac{\partial \tilde{F}_I}{\partial \sigma}(\bar{\sigma},\, \bar{x})(\sigma^k - \bar{\sigma}) + \frac{\partial \tilde{F}_I}{\partial x}(\bar{\sigma},\, \bar{x})(x^k - \bar{x}) + o(\|\sigma^k - \bar{\sigma}\|). \end{aligned}$$

Dividing by $\|\sigma^k - \bar{\sigma}\|$ and passing onto the limit along the appropriate subsequence gives (3.78) (recall (2.40) and the definition of I). ∎

Note that according to (3.77), a limit point $(d,\, \xi)$ of the sequence $\{(\sigma^k - \bar{\sigma},\, x^k - \bar{x})/\|\sigma^k - \bar{\sigma}\|\}$ exists, and (3.78) implies the inclusion

$$\frac{\partial \tilde{F}_{A_+ \cup I_1}}{\partial \sigma}(\bar{\sigma},\, \bar{x})d \in \operatorname{im} \frac{\partial \tilde{F}_{A_+ \cup I_1}}{\partial x}(\bar{\sigma},\, \bar{x}), \tag{3.81}$$

which is very restrictive unless

$$\operatorname{rank} \frac{\partial \tilde{F}_{A_+ \cup I_1}}{\partial x}(\bar{\sigma},\, \bar{x}) = l + |A_+| + |I_1|. \tag{3.82}$$

Thus, if (3.82) is violated for any partition $(I_1,\, I_2) \in \mathcal{A}_0$, the multiplier $(\bar{\lambda},\, \bar{\mu})$ which is noncritical with respect to $A_+ \cup I_1$ for any such partition can be stable only subject to very special perturbations.

This is precisely what happens when $A_0 = \emptyset$. Then (3.82) is just LICQ, the case we do not deal with in this work, and (3.81) turns into (2.39), a very restrictive condition, as discussed above. It is thus natural to proceed with analysis of quantitative stability of critical (with respect to $A = A_+$) strictly complementary multipliers.

According to (1.5), in the case of strict complementarity, for $\sigma \in \mathbf{R}^s$ close enough to $\bar{\sigma}$, the solution set of (1.1) near $(\bar{x}, \bar{\lambda}, \bar{\mu})$ coincided with the solution set of a single branch system of the form

$$\begin{array}{l} \Phi(\sigma, x) + \left(\frac{\partial F}{\partial x}(\sigma, x)\right)^{\mathrm{T}} \lambda + \left(\frac{\partial G}{\partial x}(\sigma, x)\right)^{\mathrm{T}} \mu = 0, \\ F(\sigma, x) = 0, \quad G_A(\sigma, x) = 0, \quad \mu_N = 0. \end{array} \tag{3.83}$$

This is a pure system of equations, and it can be regarded as the parametric *Lagrange system.* When (1.2) holds, (3.83) is essentially the Lagrange optimality system for the equality-constrained problem

$$\begin{array}{ll} \text{minimize} & f(\sigma, x) \\ \text{subject to} & F(\sigma, x) = 0, \quad G_A(\sigma, x) = 0. \end{array}$$

Furthermore, in the strictly complementary case, relations (2.10)–(2.12) in Theorem 3.1 should be replaced by the single relation

$$\frac{\partial G_A}{\partial x}(\bar{\sigma}, \bar{x})\xi = 0, \tag{3.84}$$

while relations (3.3)–(3.5) should be replaced by the single relation

$$\frac{\partial G_A}{\partial \sigma}(\bar{\sigma}, \bar{x})d + \frac{1}{2}\frac{\partial^2 G_A}{\partial x^2}(\bar{\sigma}, \bar{x})[\xi, \xi] = -\frac{\partial G_A}{\partial x}(\bar{\sigma}, \bar{x})x, \tag{3.85}$$

The following result is obtained by deciphering the statement of [1, Theorem 4] for the specific parametric system of equations (3.83).

Theorem 3.4 *Let* $(\bar{x}, \bar{\lambda}, \bar{\mu}) \in \mathbf{R}^n \times \mathbf{R}^l \times \mathbf{R}^m$ *be a solution of system* (1.1) *for* $\sigma = \bar{\sigma} \in \mathbf{R}^s$*, and let* $A_0 = \emptyset$*. Suppose that a tuple* $(d, \xi, \eta, \zeta) \in \mathbf{R}^s \times \mathbf{R}^n \times \mathbf{R}^l \times \mathbf{R}^m$ *satisfies* (2.8)*,* (2.9)*,* (2.13)*,* (3.84) *and* (3.1)*,* (3.2)*,* (3.6)*,* (3.85) *with some* $x \in \mathbf{R}^n$*,* $\lambda \in \mathbf{R}^l$ *and* $\mu \in \mathbf{R}^m$*, and the linear system*

$$\begin{aligned} \frac{\partial \Psi}{\partial x}(\bar{\sigma}, \bar{x}, \bar{\lambda}, \bar{\mu})x^1 + \left(\frac{\partial F}{\partial x}(\bar{\sigma}, \bar{x})\right)^{\mathrm{T}} y^1 + \left(\frac{\partial G}{\partial x}(\bar{\sigma}, \bar{x})\right)^{\mathrm{T}} z^1 & \\ + \frac{\partial^2 \Psi}{\partial x^2}(\bar{\sigma}, \bar{x}, \bar{\lambda}, \bar{\mu})[\xi, x^2] & \\ + \left(\frac{\partial^2 F}{\partial x^2}(\bar{\sigma}, \bar{x})[\xi]\right)^{\mathrm{T}} y^2 + \left(\frac{\partial^2 F}{\partial x^2}(\bar{\sigma}, \bar{x})[x^2]\right)^{\mathrm{T}} \eta & \\ + \left(\frac{\partial^2 G}{\partial x^2}(\bar{\sigma}, \bar{x})[\xi]\right)^{\mathrm{T}} z^2 + \left(\frac{\partial^2 G}{\partial x^2}(\bar{\sigma}, \bar{x})[x^2]\right)^{\mathrm{T}} \zeta & = 0, \end{aligned} \tag{3.86}$$

$$\frac{\partial F}{\partial x}(\bar{\sigma},\, \bar{x})x^1 + \frac{\partial^2 F}{\partial x^2}(\bar{\sigma},\, \bar{x})[\xi,\, x^2] = 0, \tag{3.87}$$

$$\frac{\partial G_A}{\partial x}(\bar{\sigma},\, \bar{x})x^1 + \frac{\partial^2 G_A}{\partial x^2}(\bar{\sigma},\, \bar{x})[\xi,\, x^2] = 0, \tag{3.88}$$

$$z_N^1 = 0, \quad z_N^2 = 0, \tag{3.89}$$

with the additional restrictions $(x^1,\, y^1,\, z_A^1) \in (\ker D_A)^\perp$ *and* $(x^2,\, y^2,\, z_A^2) \in \ker D_A$ *has only the trivial solution.*

Then for any mapping $\rho : \mathbf{R}_+ \to \Sigma$ *such that* $\rho(t) = o(t)$*, and any* $t \geq 0$ *small enough, system* (1.1) *for* $\sigma = \bar{\sigma} + td + \rho(t)$ *has a solution of the form* $(\bar{x} + t^{1/2}\xi,\, \bar{\lambda} + t^{1/2}\eta,\, \bar{\mu} + t^{1/2}\zeta) + o(t^{1/2})$.

Employing (2.31), we can rewrite the essential part of system (3.86)–(3.89) with $x^2 = 0$ in the form

$$D_A \begin{pmatrix} x^1 \\ y^1 \\ z_A^1 \end{pmatrix} + \begin{pmatrix} \left(\frac{\partial^2 F}{\partial x^2}(\bar{\sigma},\, \bar{x})[\xi]\right)^{\mathrm{T}} y^2 + \left(\frac{\partial^2 G_A}{\partial x^2}(\bar{\sigma},\, \bar{x})[\xi]\right)^{\mathrm{T}} z_A^2 \\ 0 \\ 0 \end{pmatrix} = 0.$$

Hence, if $\frac{\partial \Psi}{\partial x}(\bar{\sigma},\, \bar{x},\, \bar{\lambda},\, \bar{\mu})$ is a symmetric matrix then employing (2.40) and (3.75) we obtain that system (3.86)–(3.89) with the additional restrictions $(x^1,\, y^1,\, z_A^1) \in (\ker D_A)^\perp$ and $(x^2,\, y^2,\, z_A^2) \in \ker D_A$ having only the trivial solution implies that the system

$$\left(\frac{\partial^2 \tilde{F}_A}{\partial x^2}(\bar{\sigma},\, \bar{x})[\xi]\right)^{\mathrm{T}} \tilde{y}^2 \in Q_A^\perp, \quad \left(\frac{\partial \tilde{F}_A}{\partial x}(\bar{\sigma},\, \bar{x})\right)^{\mathrm{T}} \tilde{y}^2 = 0$$

with respect to $\tilde{y}^2 = (y^2,\, z_A^2)$ has only the trivial solution. The latter condition is equivalent to the following:

$$\operatorname{im} \frac{\partial \tilde{F}_A}{\partial x}(\bar{\sigma},\, \bar{x}) + \frac{\partial^2 \tilde{F}_A}{\partial x^2}(\bar{\sigma},\, \bar{x})[\xi,\, Q_A] = \mathbf{R}^l \times \mathbf{R}^{|A|}. \tag{3.90}$$

This condition is somewhat more restrictive than the well-known 2-*regularity* of the mapping $\tilde{F}_A(\bar{\sigma},\, \cdot)$ in the direction ξ, which is obtained from (3.90) by replacing Q_A with the generally bigger subspace $\ker \frac{\partial \tilde{F}_A}{\partial x}(\bar{\sigma},\, \bar{x})$. In particular, (3.90) subsumes the inequality

$$\operatorname{corank} \frac{\partial \tilde{F}_A}{\partial x}(\bar{\sigma},\, \bar{x}) \leq \dim Q_A,$$

and (3.90) cannot hold for a noncritical multiplier $(\bar{\lambda},\, \bar{\mu})$ unless LICQ is satisfied.

Suppose finally that $\frac{\partial \Psi}{\partial x}(\bar{\sigma},\, \bar{x},\, \bar{\lambda},\, \bar{\mu})$ is a symmetric matrix, (3.90) holds, and

$$\begin{aligned} &\frac{\partial^2 \Psi}{\partial x^2}(\bar{\sigma},\, \bar{x},\, \bar{\lambda},\, \bar{\mu})[\xi,\, x^2,\, x^2] \\ &+ \left\langle \eta,\, \frac{\partial^2 F}{\partial x^2}(\bar{\sigma},\, \bar{x})[x^2,\, x^2] \right\rangle + \left\langle \zeta_A,\, \frac{\partial^2 G_A}{\partial x^2}(\bar{\sigma},\, \bar{x})[x^2,\, x^2] \right\rangle \\ &+ 2\left\langle y,\, \frac{\partial^2 F}{\partial x^2}(\bar{\sigma},\, \bar{x})[\xi,\, x^2] \right\rangle + 2\left\langle z_A,\, \frac{\partial^2 G_A}{\partial x^2}(\bar{\sigma},\, \bar{x})[\xi,\, x^2] \right\rangle \neq 0 \end{aligned} \tag{3.91}$$

holds for all $x^2 \in Q \setminus \{0\}$ satisfying (3.87), (3.88) with some $x^1 \in \mathbf{R}^l$, and for the unique solution (y, z_A) of the system

$$\left(\frac{\partial F}{\partial x}(\bar{\sigma}, \bar{x})\right)^{\mathrm{T}} y + \left(\frac{\partial G_A}{\partial x}(\bar{\sigma}, \bar{x})\right)^{\mathrm{T}} z_A = -\frac{\partial \Psi}{\partial x}(\bar{\sigma}, \bar{x}, \bar{\lambda}, \bar{\mu})x^2$$

in $\operatorname{im} \frac{\partial \tilde{F}_A}{\partial x}(\bar{\sigma}, \bar{x}) = (\ker \frac{\partial \tilde{F}_A}{\partial x}(\bar{\sigma}, \bar{x}))^{\perp}$. It can be easily seen by a standard argument that under these assumptions, system (3.86)–(3.89) with the additional restrictions $(x^1, y^1, z^1_A) \in (\ker D_A)^{\perp}$ and $(x^2, y^2, z^2_A) \in \ker D_A$ has only the trivial solution, and hence, Theorem 3.4 is applicable.

We complete this discussion with three examples taken from [7], illustrating the results obtained above. The first example is very simple; it is the canonically parameterized version of [7, Example 3.2].

Example 3.2 Let $s = 2$, $n = 1$, $l = 0$, $m = 1$, $f(\sigma, x) = -x^2 + \sigma_1 x$, $G(\sigma, x) = x^2 - \sigma_2$. Then $\bar{x} = 0$ is a solution and a stationary point of problem (1.3) for $\sigma = \bar{\sigma} = 0$, and $\mathcal{M} = \mathbf{R}_+$ for the corresponding system (1.1) with Φ defined according to (1.2).

One can directly check that for any $\sigma \in \mathbf{R}^2$ such that $\sigma_2 > 0$ system (1.1) has a solutions of the form $(x(\sigma), \mu(\sigma)) = (\sigma_2^{1/2}, 1 - \sigma_1/(2\sigma_2^{1/2}))$ if $\sigma_1 \leq 2\sqrt{\sigma_2}$, and $(x(\sigma), \mu(\sigma)) = (-\sigma_2^{1/2}, 1 + \sigma_1/(2\sigma_2^{1/2}))$ if $\sigma_1 \leq -2\sqrt{\sigma_2}$. Moreover, if if $4\sigma_2 \geq \sigma_1^2$ then there is one more solution $(x(\sigma), \mu(\sigma)) = (\sigma_1/2, 0)$. The latter solution corresponds to the multiplier $\bar{\mu} = 0$ violating strict complementarity, which is thus Lipschitz-stable (actually just insensitive) subject to perturbations of the specified kind. If $\sigma_1 = o(\sigma_2^{1/2})$, the former pair of solutions gives in the limit (as $\sigma \to 0$) the strictly complementary critical multiplier $\bar{\mu} = 1$, which is thus stable but not Lipschitz-stable. If we take $\sigma_2 = O(\sigma_1^2)$, then we can obtain in the limit any multiplier in $\mathcal{M}$, and thus, *any* multiplier can be stable subject to some special kind of perturbations. Note that if $\sigma_2 = o(\sigma_1)$ then the limiting direction d of the normalized differences $\sigma - \bar{\sigma}$ satisfies $d_2 = 0$, and hence, (2.39) holds.

Consider now directional perturbations: let $\sigma = td$ with some $d \in \mathbf{R}^2$ such that $d_2 > 0$, and with $t \geq 0$ small enough. Then there is the pair of solutions $(x(t), \mu(t)) = (0, 1) + t^{1/2}(\pm d_2^{1/2}, \mp d_1/(2d_2^{1/2}))$ and the solution $(x(t), \mu(t)) = t(d_1/2, 0)$. For the former, putting $\bar{\mu} = 1$, we obtain that $(\xi, \zeta) = (\pm d_2^{1/2}, \mp d_1/(2d_2^{1/2}))$ satisfies (2.8), (2.9), (2.13), (3.84) (trivially) and (3.1), (3.2), (3.6), (3.85) (with any $x \in \mathbf{R}$ and $\mu \in \mathbf{R}$), and moreover, the specified (ξ, ζ) are the only pairs satisfying these relations. Finally, for these (ξ, ζ), one can easily verify that (3.90) is satisfied, and (3.91) holds for all $x^2 \neq 0$ (with $z_A = 0$), and hence, Theorem 3.4 is applicable.

The next example is the canonically parameterized version of [7, Example 3.3]. Note that the constraints in this example satisfy MFCQ.

Example 3.3 Let $s = 4$, $n = 2$, $l = 0$, $m = 2$, $f(\sigma, x) = -x_1 + \sigma_1 x_1 + \sigma_2 x_2$, $G(\sigma, x) = (x_1 - x_2^2 - \sigma_3,\ x_1 + x_2^2 - \sigma_4)$. Then $\bar{x} = 0$ is a solution and a stationary point of problem (1.3) for $\sigma = \bar{\sigma} = 0$, and $\mathcal{M} = \{\mu \in \mathbf{R}^2_+ \mid \mu_1 + \mu_2 = 1\}$ for the corresponding system (1.1) with Φ defined according to (1.2).

For any $\sigma \in \mathbf{R}^4$ such that $\sigma_4 > \sigma_3$ and

$$\frac{1-\sigma_1}{2} \pm \frac{\sigma_2}{2\sqrt{2}\sqrt{\sigma_4-\sigma_3}} \geq 0,$$

system (1.1) has two solutions of the form

$$\begin{aligned}(x(\sigma),\, \mu(\sigma)) \quad = \quad & \left(\left(\frac{\sigma_3+\sigma_4}{2},\, \pm\sqrt{\frac{\sigma_4-\sigma_3}{2}}\right),\, \left(\frac{1-\sigma_1}{2},\, \frac{1-\sigma_1}{2}\right)\right) \\ & \pm\left(0,\, \left(\frac{\sigma_2}{2\sqrt{2}\sqrt{\sigma_4-\sigma_3}},\, -\frac{\sigma_2}{2\sqrt{2}\sqrt{\sigma_4-\sigma_3}}\right)\right).\end{aligned}$$

Moreover, if $\sigma_4 \geq \sigma_3 + \sigma_2^2/(2(1-\sigma_1)^2)$ and $\sigma_1 < 1$ then there is one more solution

$$(x(\sigma),\, \mu(\sigma)) = \left(\left(\frac{\sigma_2^2}{4(1-\sigma_1)^2} + \sigma_3,\, \frac{\sigma_2}{2(1-\sigma_1)}\right),\, (1-\sigma_1,\, 0)\right),$$

while if $\sigma_4 \leq \sigma_3 + \sigma_2^2/(2(1-\sigma_1)^2)$ and $\sigma_1 < 1$ then there is a solution

$$(x(\sigma),\, \mu(\sigma)) = \left(\left(-\frac{\sigma_2^2}{4(1-\sigma_1)^2} + \sigma_4,\, -\frac{\sigma_2}{2(1-\sigma_1)}\right),\, (0,\, 1-\sigma_1)\right).$$

As $\sigma \to 0$, the last two solutions tend to the multipliers $\bar{\mu} = (1,\, 0)$ and $\bar{\mu} = (0,\, 1)$, respectively, both violating strict complementarity, and both these multipliers are thus Lipschitz-stable subject to perturbations of the specified kind. The limiting behavior the former pair of solutions depends on the relation between σ_2 and $\sigma_4 - \sigma_3$, but if $\sigma_2 = o((\sigma_4 - \sigma_3)^{1/2})$ then in the limit we obtain the strictly complementary critical multiplier $\bar{\mu} = (1/2,\, 1/2)$, which is thus stable but not Lipschitz-stable.

Let now $\sigma = td$ with some $d \in \mathbf{R}^4$ such that $d_4 > d_3$, and with $t \geq 0$ small enough. Then there is the pair of solutions

$$\begin{aligned}(x(t),\, \mu(t)) \quad = \quad & \left(0,\, \left(\frac{1}{2},\, \frac{1}{2}\right)\right) \\ & \pm t^{1/2}\left(\left(0,\, \sqrt{\frac{d_4-d_3}{2}}\right),\, \left(\frac{d_2}{2\sqrt{2}\sqrt{d_4-d_3}},\, -\frac{d_2}{2\sqrt{2}\sqrt{d_4-d_3}}\right)\right) \\ & + t\left(\left(\frac{d_3+d_4}{2},\, 0\right),\, \left(-\frac{d_1}{2},\, -\frac{d_1}{2}\right)\right),\end{aligned}$$

and the solution

$$(x(t),\, \mu(t)) = (0,\, (1,\, 0)) + t\left(\left(d_3,\, \frac{d_2}{2(1-td_1)}\right),\, (-d_1,\, 0)\right) + t^2\left(\left(\frac{d_2}{4(1-td_1)^2},\, 0\right),\, 0\right).$$

For the former, putting $\bar{\mu} = (1/2, 1/2)$, we obtain that $(\xi,\, \zeta)$ with

$$\xi = \left(0,\, \pm\sqrt{\frac{d_4-d_3}{2}}\right), \quad \zeta = \left(\pm\frac{d_2}{2\sqrt{2}\sqrt{d_4-d_3}},\, \mp\frac{d_2}{2\sqrt{2}\sqrt{d_4-d_3}}\right)$$

satisfies (2.8), (2.9), (2.13), (3.84) and (3.1), (3.2), (3.6), (3.85) (with any $x \in \mathbf{R}^2$ such that $x_1 = (d_3 + d_4)/2$, and any $\mu \in \mathbf{R}^2$ such that $\mu_1 + \mu_2 = -d_1$), and moreover, the specified (ξ, ζ) are the only pairs satisfying these relations. Finally, for these (ξ, ζ), (3.90) holds, and the set of $x^2 \in Q$ satisfying (3.87), (3.88) with some $x^1 \in \mathbf{R}^2$ is trivial. Hence, Theorem 3.4 is applicable.

Our last example demonstrates the case of Hölder stability of a multiplier violating strict complementarity. This is the canonically parameterized version of [7, Example 3.4] (the original source is [17, (63)]).

Example 3.4 Let $s = 4$, $n = 2$, $l = 0$, $m = 2$, $f(\sigma, x) = x_1 + \sigma_1 x_1 + \sigma_2 x_2$, $G(\sigma, x) = (-x_1 - \sigma_3,\, (x_1 - 2)^2 + x_2^2 - 4 - \sigma_4)$. Then $\bar{x} = 0$ is a solution and a stationary point of problem (1.3) for $\sigma = \bar{\sigma} = 0$, and $\mathcal{M} = \{\mu \in \mathbf{R}^2 \mid \mu_1 = 1 - 4\mu_2,\ 0 \leq \mu_2 \leq 1/4\}$ for the corresponding system (1.1) with Φ defined according to (1.2).

We are concerned with the multiplier $\bar{\mu} = (1, 0)$, which is not strictly complementary, but is critical with respect to the index set $I = A = \{1, 2\}$.

For any $\sigma \in \mathbf{R}^4$ such that $\sigma_2 \neq 0$, $4 + \sigma_4 > (\sigma_3 + 2)^2$, system (1.1) has near $(\bar{x}, \bar{\mu})$ the unique solution of the form

$$\begin{aligned}(x(\sigma), \mu(\sigma)) &= \left(\left(-\sigma_3,\, \sqrt{4 + \sigma_4 - (\sigma_3 + 2)^2}\right),\, (1 + \sigma_1,\, 0)\right) \\ &\quad + \left(0,\, \left(\frac{\sigma_2(\sigma_3 + 2)}{\sqrt{4 + \sigma_4 - (\sigma_3 + 2)^2}},\, -\frac{\sigma_2}{2\sqrt{4 + \sigma_4 - (\sigma_3 + 2)^2}}\right)\right)\end{aligned}$$

if $\sigma_2 < 0$, and

$$\begin{aligned}(x(\sigma), \mu(\sigma)) &= \left(\left(-\sigma_3,\, -\sqrt{4 + \sigma_4 - (\sigma_3 + 2)^2}\right),\, (1 + \sigma_1,\, 0)\right) \\ &\quad - \left(0,\, \left(\frac{\sigma_2(\sigma_3 + 2)}{\sqrt{4 + \sigma_4 - (\sigma_3 + 2)^2}},\, -\frac{\sigma_2}{2\sqrt{4 + \sigma_4 - (\sigma_3 + 2)^2}}\right)\right)\end{aligned}$$

if $\sigma_2 > 0$. In the sequel, we deal with the former case (the latter can be considered similarly).

Let $\sigma = td$ with some $d \in \mathbf{R}^4$ such that $d_2 < 0$, $d_4 > 4d_3$, and with $t \geq 0$ small enough. Then the solution has the form

$$(x(t), \mu(t)) = (0,\, (1,\, 0)) + t^{1/2}\left((0,\, \sqrt{d_4 - 4d_3}),\, \left(\frac{2d_2}{\sqrt{d_4 - 4d_3}},\, -\frac{d_2}{2\sqrt{d_4 - 4d_3}}\right)\right) + o(t^{1/2}).$$

Thus, the multiplier $\bar{\mu} = (1, 0)$ is Hölder stable subject to perturbations of the specified class.

It is not difficult to check that the corresponding pair

$$(\xi, \zeta) = \left((0,\, \sqrt{d_4 - 4d_3}),\, \left(\frac{2d_2}{\sqrt{d_4 - 4d_3}},\, -\frac{d_2}{2\sqrt{d_4 - 4d_3}}\right)\right)$$

satisfies (2.8)–(2.11), (2.13) and (3.1)–(3.6) (with appropriate $x \in \mathbf{R}^2$ and $\mu \in \mathbf{R}^2$), and moreover, the specified (ξ, ζ) is the only pair satisfying these relations.

Let us mention that Examples 3.2–3.4 all demonstrate the use of Theorem 3.3. Moreover, they all satisfy the stronger assumption stated in Remark 3.2, and that is why estimate (3.74) is valid in these examples.

References

[1] A.V. Arutyunov and A.F. Izmailov. Abnormal equality-constrained optimization problems: sensitivity theory. *Math. Program.* 100 (2004), 485–515.

[2] J.F. Bonnans and A. Shapiro. *Perturbation analysis of optimization problems.* Springer-Verlag, New York, 2000.

[3] W.W. Hager and M.S. Gowda. Stability in the presence of degeneracy and error estimation. *Math. Program.* 85 (1999), 181–192.

[4] A.F. Izmailov. On the analytical and numerical stability of critical Lagrange multipliers. *Comput. Math. Math. Phys.* 45 (2005), 930–946.

[5] A.F. Izmailov. Sensitivity of solutions of systems of optimality conditions under the violation of constraint qualifications. *Comput. Math. Math. Phys.* 47 (2007), 533–554.

[6] A.F. Izmailov and M.V. Solodov. A note on solution sensitivity for Karush–Kuhn–Tucker systems. *Math. Meth. Oper. Res.* 61 (2005), 347–363.

[7] A.F. Izmailov and M.V. Solodov. Examples of dual behaviour of Newton-type methods on optimization problems with degenerate constraints. *Comput. Optim. Appl.* DOI 10.1007/s10589-007-9074-4.

[8] A.F. Izmailov and M.V. Solodov. On attraction of Newton-type iterates to multipliers violating second-order sufficiency conditions. *Math. Program.* DOI 10.1007/s10107-007-0158-9.

[9] D. Klatte. Upper Lipschitz behaviour of solutions to perturbed $C^{1,1}$ programs. *Math. Program.* 88 (2000), 285–311.

[10] D. Klatte and B. Kummer. Generalized Kojima functions and Lipschitz stability of critical points. *Comput. Optim. and Appl.* 13 (1999), 61–85.

[11] D. Klatte and B. Kummer. *Nonsmooth Equations in Optimization: Regularity, Calculus, Methods and Applications.* Kluwer Academic Publishers, Dordrecht, 2002.

[12] J. Kyparisis. Sensitivity analysis for nonlinear programs and variational inequalities. *Math. Oper. Res.* 15 (1990), 286–298.

[13] J. Kyparisis. Parametric variational inequalities with multivalued solution sets. *Math. Oper. Res.* 17 (1992), 341–364.

[14] A. Levy. Solution sensitivity from general principles. *SIAM J. Control Optim.* 40 (2001), 1–38.

[15] Y. Qui Y and T.L. Magnanti. Sensitivity analysis for variational inequalities. *Math. Oper. Res.* 17 (1992), 61–76.

[16] A. Shapiro A. Sensitivity analysis of generalized equations. *J. Math. Sciences* 115 (2003), 2554–2565.

[17] S.J. Wright. Constraint identification and algorithm stabilization for degenerate nonlinear programs. *Math. Program.* 95 (2003), 137–160.