Curve subdivision with arc-length control

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Abstract

In this paper we present a new non-stationary, interpolatory, curve subdivision scheme. We show that the scheme converges and the subdivision curve is continuous. Moreover, starting with the chord length parametrization of the initial polygon, we obtain a subdivision curve arc-length parametrized. A bound for the Hausdorff distance between the limit curve and the initial polygon is also obtained.

1 Introduction

The classical 4-point scheme [3],[6] is one of the earliest and most popular interpolatory curve subdivision schemes. It is a member of the Dubuc-Deslauriers family of subdivision schemes [2], where the new points lie on a polynomial interpolating consecutive vertices of the control polygon. More precisely, starting from an initial polygon $P^0 = P_i^0, i \in \mathbb{Z}$ the 4-point scheme is defined by the equations

$$P_{2i}^{j+1} = P_i^j, \qquad P_{2i+1}^{j+1} = f_i^j(t_{2i+1}^{j+1}) \tag{1}$$

where $f_i^j(t)$ is the cubic polynomial interpolating the points P_k^j at *uniform* parameter values $t_k = k/2^j$ for $k = i-1, i, i+1, i+2$, and $t_{2i+1}^{j+1} = (2i+1)/2^{j+1}$.
Several authors [7],[5] have noticed that the limit curves of the 4-point scheme fit tightly to the long edges of the initial control polygon and loosely to the short edges, see Figure 1, left. This is a result of the uniform parametrization $t_i^0 = i$ for all i: the same time is used to travel between two consecutive points P_i^0, P_{i+1}^0 of the initial polygon, regardless of their distance. In other words, the limit curve of the uniform 4-point subdivision scheme is far away from being arc-length parametrized.

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One way to address this problem is to use a non-uniform parameterization for the initial polygon

$$t_{i+1}^0 = t_i^0 + \|P_{i+1}^0 - P_i^0\|^\beta \tag{2}$$

Then, the parameter values at the step $j+1$, are computed from the parameters of the previous step as

$$t_{2i}^{j+1} = t_i^j, \qquad t_{2i+1}^{j+1} = \frac{t_i^j + t_{i+1}^j}{2} \tag{3}$$

In Figure 1 we also show the limit curves of the 4-point subdivision scheme corresponding to different values of β. Notice that the change in the parameterization of the initial polygon affects the shape of the limit curve.

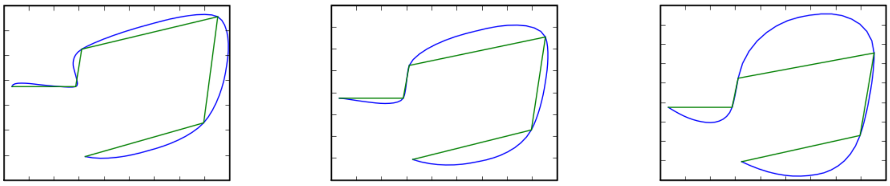

Figure 1: The limit curve of the 4-point subdivision scheme (1) with the initial parametrization (2) with $\beta = 0$ (left), $\beta = 0.5$ (middle) and $\beta = 1$ (right)

Taking this idea one step further, in [5], a reparametrization is introduced in each step, defined by the equation

$$t_0^j = 0, \qquad t_{i+1}^j = t_i^j + \|P_{i+1}^j - P_i^j\|^\beta \tag{4}$$

and then, $P_{2i}^{j+1}, P_{2i+1}^{j+1}$ are computed as in (1) and t_{2i+1}^{j+1} as in (3). The limit curve of this nonlinear scheme is smooth and when the centripetal parametrization is used, it is relatively close to the initial polygon and its shape is pleasing.

In the interpolatory scheme proposed in this paper, the new points do not necessarily lie on the cubic polynomial f in (1). Instead, we control the length of the subdivision polygon, obtaining an arc-length parametrization with respect to the chordal parametrization of the original polygon. More precisely, for all i, the length of the subdivision curve at the step j between the points P_i^0 and P_{i+1}^0 is proportional, with *the same proportionality factor* for all i, to the length of the parameter interval $t_{i+1}^0 - t_i^0 = \|P_{i+1}^0 - P_i^0\|$.
Following [9, 1], we guide the subdivision process using normal information. The process is also controlled by geometric constraints on the position of the new points, as in [8]. Similarly to [5], the proposed scheme is nonlinear and we can not study its properties through the Laurent polynomials formalism [4]. Instead, we rely on analytical and geometric arguments that are particular to this type of schemes.

2 The subdivision scheme

2.1 General definitions

Let $P^0 = \{P_i^0, i \in \mathbb{Z}\}$ be an initial polygon, where three consecutive vertices are always *noncollinear*. The equations giving the polygon at step $j+1$ can be written

$$P_{2i}^{j+1} = P_i^j \qquad P_{2i+1}^{j+1} = \frac{P_i^j + P_{i+1}^j}{2} + \rho_i^j d_i^j \tag{5}$$

where d_i^j is a normalized direction vector and $\rho_i^j > 0$ is the displacement in the direction d_i^j. We want to select d_i^j, ρ_i^j in such away that

$$\|e_{2i}^{j+1}\| + \|e_{2i+1}^{j+1}\| = \alpha^j \|e_i^j\| \tag{6}$$

where $e_i^j := P_{i+1}^j - P_i^j$ and $\alpha^j > 1$ for all j. Condition (6) means that the new point P_{2i+1}^{j+1} is on the ellipse with foci P_i^j, P_{i+1}^j, semimajor axis $\alpha^j \|e_i^j\|$, and eccentricity $1/\alpha^j$, see Figure 2 (left).

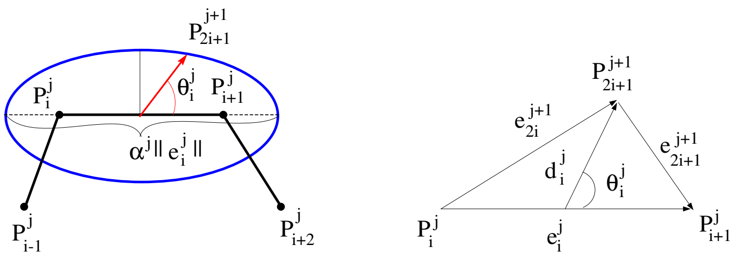

Figure 2: **Left:** The new point lies on an ellipse. **Right:** Two consecutive edges at step $j+1$.

2.2 Convergence conditions

To study the convergence of the subdivision scheme, we first define the parametric values corresponding to each point on the subdivision polygon. We keep the parameters of the even indices at level $j+1$ the same as at level j, and set the new parameter t_{2i+1}^{j+1} in the interval $[t_i^j, t_{i+1}^j]$ in such a way that

$$\frac{\|e_{2i}^{j+1}\|}{t_{2i+1}^{j+1} - t_i^j} = \frac{\|e_{2i+1}^{j+1}\|}{t_{i+1}^j - t_{2i+1}^{j+1}}$$

That is,

$$t_{2i}^{j+1} = t_i^j \qquad t_{2i+1}^{j+1} = \delta_i^j t_i^j + (1-\delta_i^j) t_{i+1}^j \quad \text{with} \quad \delta_i^j = \frac{\|e_{2i+1}^{j+1}\|}{\|e_{2i}^{j+1}\| + \|e_{2i+1}^{j+1}\|} \tag{7}$$

Theorem 1 *Consider the subdivision scheme (5) using the parametrization (7). If the new points P_{2i+1}^{j+1} are selected in such a way that, for all j, i*

$$\|e_k^{j+1}\| \leq \Gamma \|e_i^j\| \quad \textit{for} \quad k = 2i, 2i+1 \quad \textit{with} \quad \Gamma < 1 \tag{8}$$

then the subdivision scheme converges and the limit curve $c(t)$ is continuous.

Proof: Let $f^j(t)$ be the piecewise linear function interpolating (t_i^j, P_i^j). We will show that $\|f^j - f^{j+1}\|_\infty$ tends uniformly to 0 when $j \to \infty$. We have

$$\|f^j - f^{j+1}\|_\infty = \max_i \max_{t_i^j \leq t \leq t_{i+1}^j} \|f^j(t) - f^{j+1}(t)\| = \max_i \|f^j(t_{2i+1}^{j+1}) - f^{j+1}(t_{2i+1}^{j+1})\| \tag{9}$$

As $f^j(t)$ is linear in $[t_i^j, t_{i+1}^j]$ and t_{2i+1}^{j+1} is given by (7) we obtain,

$$f^j(t_{2i+1}^{j+1}) = \delta_i^j f^j(t_i^j) + (1-\delta_i^j) f^j(t_{i+1}^j) = \delta_i^j P_i^j + (1-\delta_i^j) P_{i+1}^j \tag{10}$$

Substituting (10) in (9), using the value of δ_i^j in (7), and $f^{j+1}(t_{2i+1}^{j+1}) = P_{2i+1}^{j+1}$, we obtain

$$\begin{aligned} \|f^j - f^{j+1}\|_\infty &= \max_i \|P_{2i+1}^{j+1} - (\delta_i^j P_i^j + (1-\delta_i^j) P_{i+1}^j)\| \\ &\leq \max_i \{\delta_i^j \|e_{2i}^{j+1}\| + (1-\delta_i^j)\|e_{2i+1}^{j+1}\|\} \leq 2 \max_i \frac{\|e_{2i+1}^{j+1}\| \|e_{2i}^{j+1}\|}{\alpha^j \|e_i^j\|} \end{aligned} \tag{11}$$

Using (6) and the arithmetic-geometric mean inequality, we get

$$2\frac{\|e_{2i+1}^{j+1}\| \|e_{2i}^{j+1}\|}{\alpha^j \|e_i^j\|} = 2\frac{\|e_{2i+1}^{j+1}\| \|e_{2i}^{j+1}\|}{\|e_{2i+1}^{j+1}\| + \|e_{2i}^{j+1}\|} \leq \frac{\|e_{2i+1}^{j+1}\| + \|e_{2i}^{j+1}\|}{2} = \frac{\alpha^j \|e_i^j\|}{2}$$

Therefore, from (11) we obtain, $\|f^j - f^{j+1}\|_\infty \leq \frac{\alpha^j}{2} \max_i \|e_i^j\|$. Using (8), we get

$$\|f^j - f^{j+1}\|_\infty \leq \Gamma(\frac{\alpha^j}{2} \max_i \|e_i^{j-1}\|) \leq \Gamma^2(\frac{\alpha^j}{2} \max_i \|e_i^{j-2}\|) \leq \cdots \leq \Gamma^j(\frac{\alpha^j}{2} \max_i \|e_i^0\|)$$

But $\Gamma < 1$, therefore (8) implies $\alpha^j \leq 2$ and passing to the limit we obtain, $\lim_{j\to\infty} \|f^j - f^{j+1}\|_\infty = 0$. The last expression means that the sequence f^j is a Cauchy sequence in the sup norm and in consequence it *converges*. Since we have proved that $f^j(t)$ converges uniformly, the limit function $c(t)$ has to be *continuous*. ■

Remark: Notice that (8) is sufficient but not necessary condition. In particular, if the hypothesis holds only after a certain step j_0, the scheme still converges to a continuous curve as we can see by applying the same proof on the polygon P^{j_0}.

2.3 Edge classification and preprocessing

Let n_i^j with $\|n_i^j\| = 1$ denote the normal vector assigned to the vertex P_i^j of the polygon P^j. Let T_i^j with $\|T_i^j\| = 1$ denote the tangent vector at P_i^j computed such that (T_i^j, n_i^j) is an anticlockwise oriented orthonormal frame. The edges of P^j are classified as *convex* or *inflection* edges according to the following definition.

Definition 1 *The edge $P_i^j P_{i+1}^j$ is called convex edge if the vectors T_i^j, T_{i+1}^j point to different halfplanes with respect to the line passing through P_i^j, P_{i+1}^j. If T_i^j and T_{i+1}^j point to the same halfplane, $P_i^j P_{i+1}^j$ is called an inflection edge. A polygon is called convex if all its edges are convex.*

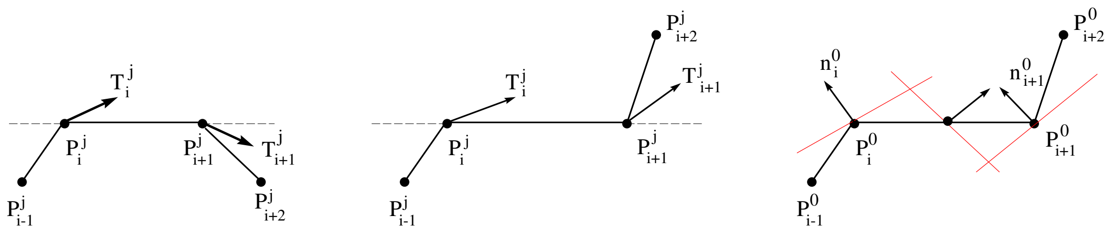

Figure 3: Edge classification. **Left:** Convex edge. **Middle:** Inflection edge. **Right:** Transforming an inflection edge in two convex edges.

The normal vector n_i^0 at a vertex P_i^0 of the initial polygon P^0 is computed as the bisector of $\angle P_{i-1}^0 P_i^0 P_{i+1}^0$ and points to the outer of the polygon, assuming that the numbering of the vertices P_i^0 introduces a clockwise orientation. Assuming that three or more consecutive points of P^0 are always noncollinear, there is no normal n_i^0 perpendicular to e_i^0, and thus, no tangent T_i^0 parallel to e_i^0. Hence, all edges of P^0 are either convex or inflection edges.
At a preprocessing step, we split every inflection edge of P^0 into two consecutive convex edges obtaining a convex polygon. If $P_i^0 P_{i+1}^0$ is an inflection edge, we subdivide it by inserting its middle-point P_m as a vertex of P^0. The normal vector n_m at P_m is computed as $n_m = \frac{n_a}{\|n_a\|}$, where

$$n_a = \lambda \frac{(e_i^0)^\perp}{\|(e_i^0)^\perp\|} + (1-\lambda) \frac{T_i^0 + T_{i+1}^0}{\|T_i^0 + T_{i+1}^0\|} \tag{12}$$

$(e_i^0)^\perp$ is the vector orthogonal to e_i^0 pointing to the same half-plane as T_i^0 and T_{i+1}^0. The parameter $\lambda \in [0, 1)$ controls the sharpness of the inflection at the point P_m.

Lemma 1 *$P_i^0 P_m$ and $P_m P_{i+1}^0$ are convex edges.*

Proof: As $\angle P^0_{i-1}P^0_iP^0_{i+1}$ takes values in $(0, 2\pi)$, the angle (e^0_i, n^0_i) takes values in the interval $(0, \pi)$ and the angle (e^0_i, T^0_i) takes values in $(-\pi/2, \pi/2)$. By (12), the angle (e^0_i, n_a) is also in $(-\pi/2, \pi/2)$. Thus, the angle (e^0_i, T_m) is in the interval $(-\pi, 0)$, and thus $P^0_iP_m$ and $P_mP^0_{i+1}$ are both convex.

■

At each subdivision step, the normal vector n^{j+1}_{2i+1} of the new point P^{j+1}_{2i+1} is the bisector of $\angle P^j_iP^{j+1}_{2i+1}P^j_{i+1}$, while we *keep* the normal vectors that have been computed at previous steps, i.e. $n^{j+1}_{2i} = n^j_i$. Using these normals, we preserve the convexity of the polygon by selecting the new points in a convexity preserving region.

Lemma 2 *Let P^j be a convex polygon with vertices P^j_i and tangents T^j_i. Let Q^j_i be the intersection of the line r_l passing through P^j_i with direction T^j_i and the line r_r passing through P^j_{i+1} with direction T^j_{i+1}. If the new point P^{j+1}_{2i+1} is inside the triangle $P^j_iQ^j_iP^j_{i+1}$, then P^{j+1} is also a convex polygon.*

Proof: See Figure 4 (left). The tangent lines r_l and r_r intersect. Indeed, if they were parallel and T^j_i, T^j_{i+1} were pointing to the same direction, then $P^j_iP^j_{i+1}$ would be an inflection edge. If they were parallel and T^j_i, T^j_{i+1} were pointing to opposite directions, their angle would be π, contradicting the fact that their angles with the x-axis are both in the range $(-\pi/2, \pi/2)$.
We have $T^{j+1}_{2i} = T^j_i$ and thus, it points to the outer part of $P^{j+1}_{2i}P^{j+1}_{2i+1}$. Consider the point $A^j_i := P^{j+1}_{2i+1} + e^{j+1}_{2i}$. By construction, the vector n^{j+1}_{2i+1} is the bisector of $\angle P^j_iP^{j+1}_{2i+1}P^j_{i+1}$, hence $T^{j+1}_{2i+1} := (n^{j+1}_{2i+1})^\perp$ is the bisector of $\angle A^j_iP^{j+1}_{2i+1}P^j_{i+1}$. Therefore, T^{j+1}_{2i+1} points to the inner part of $P^{j+1}_{2i}P^{j+1}_{2i+1}$. In an analogous way, we can see that T^{j+1}_{2i+1} and T^j_{i+1} point to different half planes with respect to the edge $P^{j+1}_{2i+1}, P^{j+1}_{2i+2}$.

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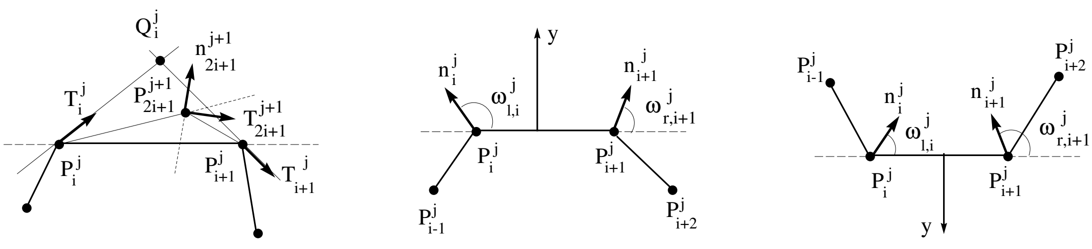

Figure 4: **Left:** Convexity preserving region. **Middle and Right:** Two convex cases. Observe the different direction of the y-positive axis.

We notice that when $P^j_iP^j_{i+1}$ is a convex edge, there are two possible configurations for n^j_i and n^j_{i+1}. In the convex case I, the angle $\omega^j_{l,i} := (e^j_i, n^j_i)$ is in $(\pi/2, \pi)$, while

the angle $\omega^j_{r,i+1} := (e^j_i, n^j_{i+1})$ is in $(0, \pi/2)$, see Figure 4 (middle). In the convex case II, the angle $\omega^j_{l,i}$ is in $(0, \pi/2)$, while the angle $\omega^j_{r,i+1}$ is in $(\pi/2, \pi)$, see Figure 4 (right).
In each case, we introduce a local coordinate system $\mathcal{S}$, where the positive x-axis is on the direction of e^j_i and the y-positive axis is perpendicular to x-axis and pointing to Q^j_i. In both cases the y-axis points to the direction of the subdivision curve. That is, outside the control polygon in case I, and inside the polygon in the case II, which corresponds to the concavities of the polygon.

2.4 Selection of the new points

In the next two Lemmas we study the ellipse with given eccentricity and foci P^j_i, P^j_{i+1}. For simplicity, we remove the indices and use the following notation on the system $\mathcal{S}$ associated with $P^j_i P^j_{i+1}$. The origin $(P^j_i + P^j_{i+1})/2$ is denoted by P_O. The vector e^j_i is denoted by e. Consequently, the coordinates of P^j_i and P^j_{i+1} are $F_l = (-\|e\|/2, 0)$ and $F_r = (\|e\|/2, 0)$ respectively.

Lemma 3 *Let $h(\alpha)$ be the ellipse with foci $F_l = (-\|e\|/2, 0)$, $F_r = (\|e\|/2, 0)$ and eccentricity $(\alpha)^{-1}$. Let P be a point on $h(\alpha)$ and let n be the normal vector of $h(\alpha)$ at P with $\|n\| = 1$. Let ω_n be the angle between e and n. If θ_P denotes the angle between e and $P - P_O$, then the following relation holds, see Figure 5 (left)*

$$\tan(\theta_P) = \left(\frac{\alpha^2 - 1}{\alpha^2}\right) \tan(\omega_n) \tag{13}$$

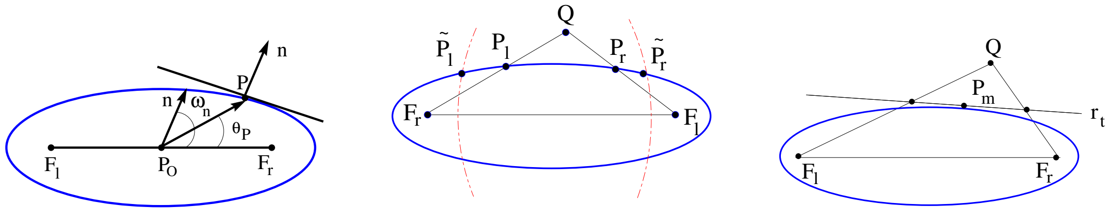

Figure 5: **Left:** The relation between the angles ω_n and θ_P. **Middle:** Selection of the new point. **Right:** Tangent at the new point.

Proof: In the system $\mathcal{S}$, the equation of $h(\alpha)$ in the standard form is

$$\left(\frac{2x}{\|e\|\alpha}\right)^2 + \left(\frac{2y}{\|e\|\sqrt{\alpha^2 - 1}}\right)^2 = 1 \tag{14}$$

If $P = (P_x, P_y)$ is a point on $h(\alpha)$, then the direction of the tangent vector to $h(\alpha)$ at P is $t_P = (-\frac{P_y}{\alpha^2 - 1}, \frac{P_x}{\alpha^2})$. The normal vector of $h(\alpha)$ at P is $n = (\cos(\omega_n), \sin(\omega_n))$ if and only if $\langle n, t_P \rangle = 0$ holds, where $\langle \cdot, \cdot \rangle$ denotes the scalar product. The equation $\langle n, t_P \rangle = 0$ is equivalent to, $\frac{P_y}{\alpha^2-1}\cos(\omega_n) = \frac{P_x}{\alpha^2}\sin(\omega_n)$. Taking into account that $\tan(\theta_P) = \frac{P_y}{P_x}$, we obtain (13).

■

In the next Lemma we compute some auxiliary points that will be used to compute the new point to be inserted between the focci of the ellipse.

Lemma 4 *Let n_l and n_r be the normal vectors associated with F_l and F_r respectively. Let ω_l, ω_r be the angles (counterclockwise) between e and n_l and e and n_r respectively. Let r_l and r_r be the tangent lines at F_l and F_r, and let Q be their intersection, as in Lemma 2. We have*

1. *There is $\alpha_Q > 1$ such that $h(\alpha)$ with $1 < \alpha < \alpha_Q$ intersects the triangle F_lQF_r.*

2. *The intersection points, with positive ordinates, between $h(\alpha)$ and r_l and $h(\alpha)$ and r_r are respectively $P_l = (\overline{x}_l, \overline{y}_l)$ and $P_r = (\overline{x}_r, \overline{y}_r)$ with*

$$\overline{x}_r = \frac{\|e\|}{2} - \overline{y}_r t_r, \quad \overline{y}_r = \frac{\|e\|(\alpha^2-1)(\alpha\sqrt{1+t_r^2}) + t_r}{2(\alpha^2+(\alpha^2-1)t_r^2)} \tag{15}$$

$$\overline{x}_l = -\frac{\|e\|}{2} - \overline{y}_l t_l, \quad \overline{y}_l = \frac{-\|e\|(\alpha^2-1)(-\alpha\sqrt{1+t_l^2}) + t_l}{2(\alpha^2+(\alpha^2-1)t_l^2)} \tag{16}$$

   with $t_l = \tan(\omega_l)$ and $t_r = \tan(\omega_r)$.

3. *The intersection point, with positive ordinates, between $h(\alpha)$ and the circle with center F_l and radius $\|e\|\Gamma$ is $\widetilde{P}_l = (\widetilde{x}_l, \widetilde{y}_l)$ with*

$$\widetilde{x}_l = \frac{\|e\|}{2}(\alpha^2 - 2\alpha\Gamma) \tag{17}$$

   The intersection point with positive ordinates, between $h(\alpha)$ and the circle with center F_r and radius $\Gamma\|e\|$ is $\widetilde{P}_r = (\widetilde{x}_r, \widetilde{y}_r)$ with $\widetilde{x}_r = -\widetilde{x}_l, \widetilde{y}_r = \widetilde{y}_l$.

4. *If $P = (x_n, y_n)$ is the point on $h(\alpha)$ with normal n, then*

$$x_n = s\frac{\|e\|\alpha^2}{2\sqrt{\alpha^2+(\alpha^2-1)\tan\omega_n^2}} \tag{18}$$

   where ω_n is the angle (counterclockwise) between e and n and $s = sign(\tan(\omega_n))$.

Proof: See Figure 5 (middle). We give the proof only for the *convex case I*, $\omega_l \in (\pi/2, \pi)$ and $\omega_r \in (0, \pi/2)$. The proof for the *convex case II* is similar.
Let $(\alpha_Q)^{-1}$ be the eccentricity of the ellipse with foci F_l and F_r passing through the point Q. To check 1) it is enough to observe that, for any $1 < \alpha < \alpha_Q$, the ellipse $h(\alpha)$ does not contain the triangle $F_l Q F_r$. Thus, there is an arc of $h(\alpha)$ contained in this triangle.
To prove 2) we write the implicit equation of the lines r_l and r_r

$$r_l : 2y \tan(\omega_l) + 2x + \|e\| = 0, \quad r_r : 2y \tan(\omega_r) + 2x - \|e\| = 0 \tag{19}$$

Solving r_r for x and substituting in (14) we obtain a quadratic equation in y. Taking into account that $\tan(\omega_r) > 0$ we compute the positive solution of this equation, which is (15). The coordinates (16) of the point P_l are computed in a similar way using r_l and (14).
To show 3) we write the equation of the circle C_l with center F_l and radius $\|e\|\Gamma$

$$C_l : (x + \|e\|/2)^2 + y^2 - \Gamma^2\|e\|^2 = 0 \tag{20}$$

Computing the resultant between (20) and (14) we obtain a biquadratic equation in y. The positive solution of this equation is $\widetilde{y}_l = \frac{\|e\|}{2}\sqrt{(\alpha^2 - 1)(1 - (\alpha - 2\Gamma))^2}$. Substituting $\widetilde{y}_l$ in (20) and solving for x we obtain $\widetilde{x}_l$ as in (17). The coordinates of the point $\widetilde{P}_r$ are obtained by symmetry.
To prove 4) observe that (13) implies $\frac{y}{x} = \left(\frac{\alpha^2-1}{\alpha^2}\right)\tan(\omega_n)$. Solving this equation for y and substituting in (14) we obtain a quadratic equation for x whose solution is given by (18).

■

We can now describe the procedure for computing *the local coordinates* (X_m, Y_m) of the new point P^{j+1}_{2i+1}. First, we choose an α^j such that $\alpha^j < \alpha_{Q^j_i}$ for all i, where $\alpha_{Q^j_i}$ is the eccentricity of the ellipse passing through Q^j_i, see Section 2.5. From Lemma 4 we know that if $\overline{x}_l < X_m < \overline{x}_r$ and $\widetilde{x}_l < X_m < \widetilde{x}_r$, then P^{j+1}_{2i+1} is inside the triangle with vertices P^j_i, Q^j_i, P^j_{i+1} and also in the convergence region $\mathcal{I}(\Gamma)$ defined as the intersection of circles with centers P^j_i and P^j_{i+1} and radius $\Gamma\|e^j_i\|$.
Using (18), we compute the abscissae $\sigma_l, \sigma_m, \sigma_r$ of the points on $h(\alpha^j)$, such that the normals of $h(\alpha^j)$ at these points are n^j_i, $n^j_i + n^j_{i+1}$ and n^j_{i+1} respectively. Then we assign, $X_l := \max\{\overline{x}_l, \widetilde{x}_l, \sigma_l\}$, $\quad X_r := \min\{\overline{x}_r, \widetilde{x}_r, \sigma_r\}$

$$X_m := (\frac{\sigma_r - \sigma_m}{\sigma_r - \sigma_l})X_l + (\frac{\sigma_m - \sigma_l}{\sigma_r - \sigma_l})X_r \tag{21}$$

Finally, from (14) we compute *the local coordinate* Y_m of the point on the upper half of $h(\alpha^j)$ with abcisa X_m,

$$Y_m := \frac{\sqrt{(\alpha^2 - 1)(\|e\|^2\alpha^2 - 4X_m^2)}}{2\alpha} \tag{22}$$

Notice that the abscissa X_m of the new point is a convex linear combination of X_l and X_r. Hence, (X_m, Y_m) is also inside the triangle F_lQF_r and inside $\mathcal{I}(\Gamma)$. Moreover, the normal of $h(\alpha^j)$ at (X_m, Y_m) is a vector between n_i^j and n_{i+1}^j. The choice of (X_m, Y_m) respects the geometry of the configuration of the normals n_i^j, $n_i^j + n_{i+1}^j$ and n_{i+1}^j in the sense that X_l, X_m, X_r and $\sigma_l, \sigma_m, \sigma_r$ have the same cross ratio.

Lemma 5 *Let P_m be a point in the interior of the triangle F_lQF_r, and let $h(\alpha)$ be the ellipse passing through P_m. Then, the tangent of $h(\alpha)$ at P_m intersects the sides F_lQ and F_rQ of the triangle.*

Proof: See Figure 5 (right). Since P_m is on the ellipse and in the interior of the triangle F_lQF_r, the ellipse and Q are in different halfplanes with respect to the tangent line r_t to $h(\alpha)$ at P_m. Hence, the line segment F_lQ joints points located in different half planes with respect to r_t. Thus, r_t intersects the line segment F_lQ in a point between F_l and Q. The proof for F_r is similar. ■

Notice that Lemma 5 guarantees that the tangent of the ellipse at the new point (X_m, Y_m) intersects the sides $P_i^jQ_i^j$ and $Q_i^jP_{i+1}^j$ of the triangle. Therefore, in the step $j+2$ the new points $P_{4i+1}^{j+2}, P_{4i+3}^{j+2}$ are also inside the triangle $P_i^jQ_i^jP_{i+1}^j$.

2.5 Selection of the sequence α^j

Lemma 6 *With the same notation that in Lemma 4 the eccentricity of the ellipse with foci $F_l = (-\|e\|/2, 0)$, $F_r = (\|e\|/2, 0)$ passing through Q is α_Q^{-1} with*

$$\alpha_Q = \frac{\cos(\omega_r) - \cos(\omega_l)}{\sin(\omega_l - \omega_r)} \tag{23}$$

Proof: From the law of sines on the triangle F_lQF_r

$$\frac{\|F_l - Q\|}{\sin(\frac{\pi}{2} - \omega_r)} = \frac{\|F_r - Q\|}{\sin(\omega_l - \frac{\pi}{2})} = \frac{\|e\|}{\sin(\pi - \omega_l + \omega_r)} \tag{24}$$

we get

$$\alpha_Q = \frac{\|F_l - Q\| + \|F_r - Q\|}{\|F_l - F_r\|} = \frac{\sin(\omega_l - \frac{\pi}{2}) + \sin(\frac{\pi}{2} - \omega_r)}{\sin(\pi + \omega_r - \omega_l)} = \frac{\cos(\omega_r) - \cos(\omega_l)}{\sin(\omega_l - \omega_r)}$$

■

In the Append we include the procedure **Alphastep** for computing α^j. In each step j the value α^j is the minimum between a convex combination of $\alpha_Q^j := min_i \alpha_{Q_i^j}$ and the element u^j of a sequence converging to 1. Therefore, the sequence α^j obtained using this procedure tends to 1 and in all steps $\alpha^j < \alpha_{Q_i^j}$ for all i.

Theorem 2 *Given a polygon P^0 with no three consecutive collinear vertices, the subdivision scheme converges and the limit curve is continuous. Moreover, for every convex sub-polygon of P^0, the corresponding curve segment is strictly convex and all the inflection points are the midedges of the inflection edges of P^0.*

Proof: Assume that a value $1/2 < \Gamma < 1$ has been selected. By construction, the new point P_{2i+1}^{j+1}, to be inserted between P_i^j and P_{i+1}^j, is inside the triangle with vertices $P_i^j, P_{2i+1}^{j+1}, P_{i+1}^j$ and also in $\mathcal{I}(\Gamma)$. Hence, the sufficient condition of convergence (8) holds and we may conclude that the subdivision method converges and the limit curve is continuous. Moreover, after Lemma 2 we get that the sub-polygons of P^j arising from a convex sub-polygon of P^0 are also convex, hence the corresponding arc of limit curve is convex. Finally, in the preprocessing step, we split any inflection edge in two consecutive convex edges inserting its midpoint. Since the convexity is preserved in the next subdivision steps, the inflection points on the limit curve are contained in segments of the curve corresponding to inflection edges of P^0.

■

3 Properties of the limit curve

3.1 Parametrization

In this section we prove properties of the subdivision scheme when t^0 is the chord length parametrization.

Theorem 3 *Consider the subdivision scheme (5) using the parametrization (7) with (2) and $\beta = 1$. Denote by $l^j(P_0^j, P^j(t))$ the length of the subdivision curve in the step j between points P_0^j and $P^j(t)$. Then,*

$$l^j(P_0^j, P^j(t)) = (\alpha^0\alpha^1\cdots\alpha^{j-1})t \tag{25}$$

Proof: Let's assume that $t_i^{j-1} \leq t \leq t_{i+1}^{j-1}$. First we are going to show that,

$$l^j(P_{2i}^j, P^j(t)) = \alpha^{j-1}l^{j-1}(P_i^{j-1}, P^{j-1}(t)) \tag{26}$$

In fact, by linear interpolation, $P^{j-1}(t) = \left(\frac{t_{i+1}^{j-1}-t}{t_{i+1}^{j-1}-t_i^{j-1}}\right)P_i^{j-1} + \left(\frac{t-t_i^{j-1}}{t_{i+1}^{j-1}-t_i^{j-1}}\right)P_{i+1}^{j-1}$

$$\begin{aligned}\text{Thus,}\quad l^{j-1}(P_i^{j-1}, P^{j-1}(t)) &= \left\|\left(\frac{t_i^{j-1}-t}{t_{i+1}^{j-1}-t_i^{j-1}}\right)P_i^{j-1} + \left(\frac{t-t_i^{j-1}}{t_{i+1}^{j-1}-t_i^{j-1}}\right)P_{i+1}^{j-1}\right\| \\ &= \left(\frac{t-t_i^{j-1}}{t_{i+1}^{j-1}-t_i^{j-1}}\right)\|e_i^{j-1}\| \qquad (27)\end{aligned}$$

In order to compute $l^j(P^j_{2i}, P^j(t))$ we have to take into account two cases:
Case a) $t^j_{2i} \leq t \leq t^j_{2i+1}$.
By linear interpolation, $P^j(t) = \left(\frac{t^j_{2i+1}-t}{t^j_{2i+1}-t^j_{2i}}\right)P^j_{2i} + \left(\frac{t-t^j_{2i}}{t^j_{2i+1}-t^j_{2i}}\right)P^j_{2i+1}$. Hence,

$$l^j(P^j_{2i}, P^j(t)) = \|P^j_{2i} - P^j(t)\| = \left(\frac{t - t^j_{2i}}{t^j_{2i+1} - t^j_{2i}}\right)\|e^j_{2i}\| \tag{28}$$

Now from (7) we know that, $t^j_{2i+1} - t^j_{2i} = (1-\delta^{j-1}_i)(t^{j-1}_{i+1} - t^{j-1}_i)$. Substituting in (28), taking into account that $1 - \delta^{j-1}_i = \frac{\|e^j_{2i}\|}{\alpha^{j-1}\|e^{j-1}_i\|}$ and comparing with (27) we finally get,

$$l^j(P^j_{2i}, P^j(t)) = \left(\frac{t - t^{j-1}_i}{t^{j-1}_{i+1} - t^{j-1}_i}\right)\alpha^{j-1}\|e^{j-1}_i\| = \alpha^{j-1}l^{j-1}(P^{j-1}_i, P^{j-1}(t))$$

which shows (26) for case a). *Case b)* $t^j_{2i+1} \leq t \leq t^j_{2i+2}$ is proved in a similar way. Now, observe that

$$\begin{aligned} l^j(P^j_0, P^j(t)) &= l^j(P^j_0, P^j_{2i}) + l^j(P^j_{2i}, P^j(t)) \\ &= \alpha^{j-1}(l^{j-1}(P^{j-1}_0, P^{j-1}_i) + l^{j-1}(P^{j-1}_i, P^{j-1}(t))) \\ &= \alpha^{j-1}l^{j-1}(P^{j-1}_0, P^{j-1}(t)) \end{aligned} \tag{29}$$

Hence, applying recursively (29) we get

$$l^j(P^j_0, P^j(t)) = (\alpha^{j-1}\alpha^{j-2}\ldots\alpha^0)l^0(P^0_0, P^0(t)) = (\alpha^{j-1}\alpha^{j-2}\ldots\alpha^0)t$$

■

Remark: Equation (25) means that the piecewise linear function $f^j(t)$ interpolating (t^j_i, P^j_i) is parametrized by the arc-length. Let $c(t)$ be the limit curve and assume that $\alpha = \prod_{j=0}^{\infty}\alpha^j$ is finite. Defining $L(0,t) := \lim_{j\to\infty} l^j(P^0_0, f^j(t))$ as the arc-length of the section of $c(t)$ between points $c(0) = P^0_0$ and $c(t)$ we get from (25) $L(0,t) = \alpha t$. Hence, $c(t)$ is parametrized by the arc-length.

3.2 Distance from the curve to the polygon

Lemma 7 *Any vertex P^j_k of the j-th subdivision of the edge $P^0_iP^0_{i+1}$ is inside the ellipse with foci P^0_i, P^0_{i+1} and eccentricity $(\alpha^0\alpha^1\cdots\alpha^{j-1})^{-1}$.*

Proof: We have $P^0_i = P^j_{2^ji}$ and $P^0_{i+1} = P^j_{2^j(i+1)}$. The vertices in j-th step corresponding to the edge $P^0_iP^0_{i+1}$ are P^j_k, for $k = 2^ji, ..., 2^j(i+1)$ and

$$\|P^0_i - P^j_k\| \;+\; \|P^j_k - P^0_{i+1}\| = \|P^j_{2^ji} - P^j_k\| + \|P^j_k - P^j_{2^j(i+1)}\|$$

$$\leq \sum_{l=2^j i}^{k-1} \|P_{l+1}^j - P_l^j\| + \sum_{l=k}^{2^j(i+1)-1} \|P_{l+1}^j - P_l^j\| = \sum_{l=2^j i}^{2^j(i+1)-1} \|P_{l+1}^j - P_l^j\|$$

$$= \alpha^{j-1} \sum_{l=2^{j-1}i}^{2^{j-1}(i+1)-1} \|P_{l+1}^{j-1} - P_l^{j-1}\| = \cdots = \alpha^{j-1}\alpha^{j-2}\cdots\alpha^0 \|P_{i+1}^0 - P_i^0\|$$

Hence, the sum of the distances from P_k^j to $P_l^0, l = i, i+1$ is smaller or equal to $\alpha^{j-1}\alpha^{j-2}\cdots\alpha^0$ times the distance from P_i^0 to P_{i+1}^0.

■

Using Lemma 7, we obtain an upper bound of the Hausdorff distance d_H between the segment of the limit curve $\{c(t), t \in [t_i^j, t_{i+1}^j]\}$ and the edge $P_i^0 P_{i+1}^0$.

Theorem 4 *Let $c(t)$ be the limit curve of the subdivision scheme (5). Assume that $\alpha = \prod_{j=0}^{\infty} \alpha^j$ is finite. Then*

$$d_H(\{c(t), t \in [t_i^0, t_{i+1}^0]\}, P_i^0 P_{i+1}^0) \leq \frac{\|e_i^0\|}{2}\sqrt{\alpha^2 - 1} \tag{30}$$

Proof: From Lemma 7 we know that all points P_k^j obtained at the j-th subdivision of the edge $P_i^0 P_{i+1}^0$ are contained in the ellipse with foci P_i^0, P_{i+1}^0 and eccentricity $(\alpha^0\alpha^1\cdots\alpha^{j-1})^{-1}$. In consequence, points on $c(t)$ for $t \in [t_i^0, t_{i+1}^0]$ are in the interior of the ellipse with foci P_i^0, P_{i+1}^0 and eccentricity $1/\alpha$. Observe than the length of the semiminor axis of this ellipse is $\frac{\|e_i^0\|\sqrt{\alpha^2-1}}{2}$, while the distance from each focus to the closer intersection point between the semimajor axis and the ellipse is $\frac{\|e_i^0\|(\alpha-1)}{2}$. Since $\alpha \geq 1$, the first one is bigger than the second one. Therefore, the Hausdorff distance from the section of the limit curve corresponding to the parameter interval $[t_i^0, t_{i+1}^0]$ to the edge $P_i^0 P_{i+1}^0$ is bounded above by $\frac{\|e_i^0\|\sqrt{\alpha^2-1}}{2}$.

■

4 Numerical Experiments

In this section we show curves produced from the proposed algorithm for different initial polygons. In the preprocessing step we split the inflection edges, introducing a new vertex in the middle of the edge. In all the examples we do four steps of subdivision. The sequence α^j is different in each example and was computed using the procedure described in section Append, with the value $K^j = \frac{2^j-1}{2^j} - 0.4$.

In Table 1 we show the maximum value of the cosine of the angle between the normal n_i^j and the adjacent edges $P_{i-1}^j P_i^j$ and $P_i^j P_{i+1}^j$ for $j = 1, 2, 3, 4$. Notice that as j grows the angle goes to $\pi/2$, showing that the limit curve is smooth.

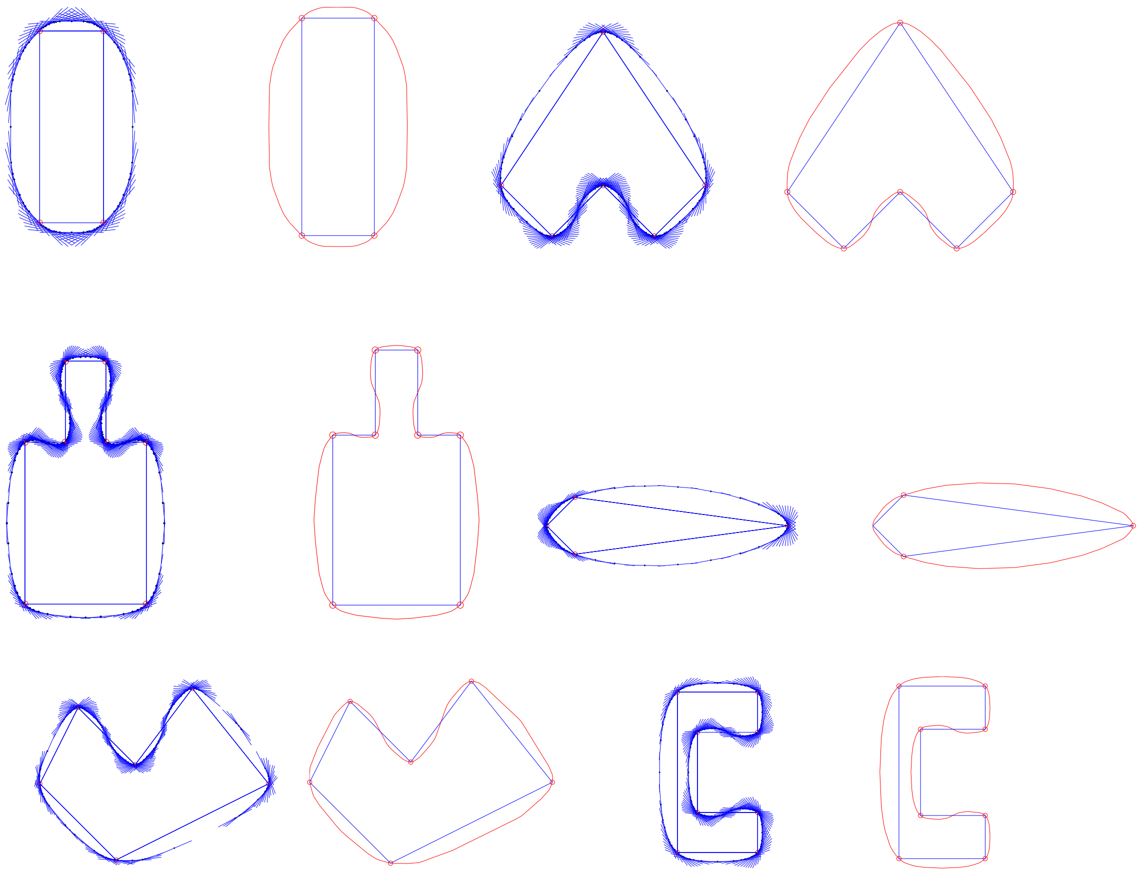

Figure 6: On the left the starting polygon and the tangent lines at points of the step 4. On the right the starting polygon and the polygonal curve after 4 steps.

Example	Γ	j=1	j=2	j=3	j=4	α
1	0.95	0.4718	0.2417	0.1331	0.0684	[1.0414 1.0270 1.0078 1.0017]
2	0.98	0.5612	0.2352	0.1198	0.0641	[1.0232 1.0165 1.0050 1.0010]
3	0.97	0.5375	0.2237	0.1310	0.0714	[1.0232 1.0143 1.0029 1.0003]
4	0.98	0.6291	0.3276	0.1633	0.0856	[1.0214 1.0163 1.0051 1.0012]
5	0.98	0.6397	0.2604	0.1853	0.0998	[1.0219 1.0135 1.0015 1.0001]
6	0.96	0.5375	0.2362	0.1145	0.0643	[1.0232 1.0158 1.0046 1.0009]

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5 Append

In this section we summarize the Algorithm *Arc-length subdivision*, which uses 2 main procedures: *Alphastep*, which computes the value of the parameter α in each step j and *Displacement*, which computes the point P_{2i+1}^{j+1} to be inserted in the step j between P_i^j and P_{i+1}^j. The auxiliary procedure *Compangles* computes the cosines and sines of the angles between the assigned normal at a vertex and the edges containing that vertex. Notation: $P^j = (P_i^j)_i, n^j = (n_i^j)_i, e^j = (e_i^j)_i$.

Algorithm Arc-length subdivision

Given an initial polygon $P^0 = \{P_i^0\}$
Preprocessing

- Compute the normal vector n_i^0 at each vertex P_i^0.
- Classify the edges of P^0 in convex or inflection edges.
- Split each inflection edge $P_i^0 P_{i+1}^0$ in two consecutive convex edges.

Main loop: Given a convex polygon P^0, normal vectors n^0, the maximum number $jmax$ of subdivision steps and a sequence $u = (u^j), u^j \in \mathbb{R}$ with $\lim_{j\to\infty} u^j = 1$,

$$
\begin{aligned}
&for\quad j = 0 : jmax - 1\\
&\quad \{c\omega_l^j, c\omega_n^j, c\omega_r^j, s\omega_l^j, s\omega_r^j, e^j\} = \textbf{Compangles}(P^j, n^j)\\
&\quad \{\alpha^j, t\omega_l^j, t\omega_n^j, t\omega_r^j\} = \textbf{Alphastep}(P^j, e^j, c\omega_l^j, c\omega_n^j, c\omega_r^j, s\omega_l^j, s\omega_r^j, u^j)\\
&\quad if\quad \alpha^j = 1\quad stop\\
&\quad else\\
&\qquad for\ \ i = 1 : length(P^j)\\
&\qquad\quad P_{2i}^{j+1} = P_i^j,\ \ n_{2i}^{j+1} = n_i^j\\
&\qquad\quad \{d_i^j, \rho_i^j\} = \textbf{Displacement}(e_i^j, t\omega_{l,i}^j, t\omega_{n,i}^j, t\omega_{r,i}^j, \alpha^j, \Gamma)\\
&\qquad\quad P_{2i+1}^{j+1} = \frac{P_i^j + P_{i+1}^j}{2} + \rho_i^j d_i^j\\
&\qquad\quad n_{2i+1}^{j+1} = \frac{(P_{2i+1}^{j+1} - P_i^j) + (P_{2i+1}^{j+1} - P_{i+1}^j)}{\|(P_{2i+1}^{j+1} - P_i^j) + (P_{2i+1}^{j+1} - P_{i+1}^j)\|}\\
&\qquad end\\
&\quad end\\
&end
\end{aligned}
$$

Procedure **Alphastep** uses three prescribed constant: $K^j \in (0,1)$, a real positive number ε close to 0 and M, a big positive number assigned to the tangent of an angle close to $\pi/2$. The input of **Alphastep** is P^j, e^j and the output of the procedure **Compangles**: $c\omega_l^j := (\cos(\omega_{l,i}^j))_i$, $c\omega_r^j := (\cos(\omega_{r,i}^j))_i$, $c\omega_n^j := (\cos(\omega_{n,i}^j))_i$, and $s\omega_l^j := (\sin(\omega_{l,i}^j))_i$, $s\omega_r^j := (\sin(\omega_{r,i}^j))_i$, where $\omega_{n,i}^j$ is the angle between $n_i^j + n_{i+1}^j$ and e_i^j.

$$
\begin{aligned}
&\{c\omega_l^j, c\omega_n^j, c\omega_r^j, s\omega_l^j, s\omega_r^j, e^j\} = \mathbf{Compangles}(P^j, n^j)\\
&\qquad for\ \ i = 1 : length(P^j) - 1\\
&\qquad\qquad e_i^j = P_{i+1}^j - P_i^j, \quad m_i^j = \frac{n_i^j + n_{i+1}^j}{\|n_i^j + n_{i+1}^j\|}\\
&\qquad\qquad c\omega_{l,i}^j = \langle n_i^j, e_i^j\rangle/\|e_i^j\|, \ \ c\omega_{r,i}^j = \langle n_{i+1}^j, e_i^j\rangle/\|e_i^j\|, \quad c\omega_{n,i}^j = \langle m_i^j, e_i^j\rangle/\|e_i^j\|\\
&\qquad\qquad s\omega_{l,i}^j = \sqrt{1-(c\omega_{l,i}^j)^2}, \ \ s\omega_{r,i}^j = \sqrt{1-(c\omega_{r,i}^j)^2}\\
&\qquad end
\end{aligned}
$$

$$
\begin{aligned}
&\{\alpha^j, t\omega_l^j, t\omega_n^j, t\omega_r^j\} = \mathbf{Alphastep}(P^j, e^j, c\omega_l^j, c\omega_n^j, c\omega_r^j, s\omega_l^j, s\omega_r^j, u^j)\\
&\qquad for\ \ i = 1 : length(P^j) - 1\\
&\qquad\quad If \quad |c\omega_{r,i}^j| > \varepsilon, \quad |c\omega_{l,i}^j| > \varepsilon\\
&\qquad\qquad t\omega_{l,i}^j = \left(\frac{s\omega_{l,i}^j}{c\omega_{l,i}^j}\right)^2, \ \ t\omega_{r,i}^j = \left(\frac{s\omega_{r,i}^j}{c\omega_{r,i}^j}\right)^2\\
&\qquad\qquad\quad \alpha_{Q_i^j} = (c\omega_{r,i}^j - c\omega_{l,i}^j)/(s\omega_{l,i}^j c\omega_{r,i}^j - c\omega_{l,i}^j s\omega_{r,i}^j)\\
&\qquad\quad else\\
&\qquad\qquad\quad t\omega_{l,i}^j = M, \ \ t\omega_{r,i}^j = M, \quad \alpha_{Q_i^j} = 1\\
&\qquad\quad end\\
&\qquad\quad If \quad |c\omega_{n,i}^j| > \varepsilon \quad t\omega_{n,i}^j = \frac{1-(c\omega_{n,i}^j)^2}{(c\omega_{n,i}^j)^2} \quad else \quad t\omega_{n,i}^j = M \quad end\\
&\qquad\quad \alpha_Q^j \quad = \min_i(\alpha_{Q_i^j}), \ \ \alpha^j = \min\{u^j, K^j\alpha_Q + (1-K^j)\}\\
&\qquad end
\end{aligned}
$$

Given $\Gamma < 1$ (Γ close to 1) and $\alpha^j > 1$, the procedure **Displacement** computes the direction of displacement d_i^j and the value ρ_i^j necessary to obtain P_{2i+1}^{j+1}. In the input we use $t\omega_{l,i}^j := \tan(\omega_{l,i}^j)^2$, $t\omega_{r,i}^j := \tan(\omega_{r,i}^j)^2$ and $t\omega_{n,i}^j := \tan(\omega_{n,i}^j)^2$.

$$
\begin{aligned}
\{d_i^j, \rho_i^j\} \quad &= \quad \textbf{Displacement}(e_i^j, t\omega_{l,i}^j, t\omega_{n,i}^j, t\omega_{r,i}^j, \alpha^j, \Gamma) \\
&\mathit{If} \quad t\omega_{n,i}^j = M \\
&\qquad d_i^j = \frac{(e_i^j)^\perp}{\|e_i^j\|}, \quad \rho_i^j = \frac{\|e_i^j\|\alpha^j\sqrt{(\alpha^j)^2-1}}{2} \\
&\mathit{else} \\
&\qquad \mathit{Compute}\ \overline{x}_l, \overline{x}_r\ \mathit{using}\ (15)-(16),\ \mathit{with}\ \alpha = \alpha^j,\ e = e_i^j,\ t_l^2 = t\omega_{l,i}^j,\ t_r^2 = t\omega_{r,i}^j \\
&\qquad \mathit{Compute}\ \widetilde{x}_l\ \mathit{using}\ (17)\ \mathit{with}\ \alpha = \alpha^j,\ e = e_i^j\ \mathit{and\ set}\ \widetilde{x}_r = -\widetilde{x}_l \\
&\qquad \mathit{Compute}\ \sigma_l, \sigma_m, \sigma_r\ \mathit{using}\ (18)\ \mathit{with}\ \alpha = \alpha^j,\ e = e_i^j\ \mathit{and} \\
&\qquad\qquad\qquad \tan(\omega_n)^2 = \{t\omega_{l,i}^j, t\omega_{n,i}^j, t\omega_{r,i}^j\}\ \mathit{respectively} \\
&\qquad X_l = \max\{\overline{x}_l, \widetilde{x}_l, \sigma_l\}, \quad X_r = \min\{\overline{x}_r, \widetilde{x}_r, \sigma_r\} \\
&\qquad \mathit{Compute}\ X_m, Y_m\ \mathit{using}\ (21)-(22)\ \mathit{with}\ \alpha = \alpha^j,\ e = e_i^j \\
&\qquad \tan(\theta_i^j) = Y_m/X_m, \quad s = sign(\tan(\theta_i^j)) \\
&\qquad \cos(\theta_i^j) = \frac{s}{\sqrt{1+\tan^2(\theta_i^j)}}, \quad \sin(\theta_i^j) = \sqrt{1-\cos^2(\theta_i^j)} \\
&\qquad d_i^j = \cos(\theta_i^j)\frac{e_i^j}{\|e_i^j\|} + \sin(\theta_i^j)\frac{(e_i^j)^\perp}{\|e_i^j\|} \\
&\qquad \rho_i^j = \frac{\|e_i^j\|}{2}(\alpha^j)^2\sqrt{\frac{(\alpha^j)^2-1}{(\alpha^j)^2-\cos^2\theta_i^j}} \\
&\mathit{end}
\end{aligned}
$$