# LIFTING MEASURES TO INDUCING SCHEMES

YA. PESIN, S. SENTI, AND K. ZHANG

To the memory of Bill Parry

Abstract. In this paper we study the liftability property for piecewise continuous maps of compact metric spaces, which admit inducing schemes in the sense of [PS05, PS06]. We show that under some natural assumptions on the inducing schemes – which hold for many known examples – any invariant ergodic Borel probability measure of sufficiently large entropy can be lifted to the tower associated with the inducing scheme. The argument uses the construction of connected Markov extensions due to Buzzi [Buz99], his results on the liftability of measures of large entropy, and a generalization of some results by Bruin [Bru95] on relations between inducing schemes and Markov extensions. We apply our results to study the liftability problem for one-dimensional cusp maps (in particular, unimodal and multimodal maps) and for some multidimensional maps.

## 1. Introduction

In [PS05, PS06], the authors studied existence and uniqueness of equilibrium measures for a continuous map $f$ of a compact topological space $I$, which admits an inducing scheme $\{S, \tau\}$ where $S$ is a countable collection of disjoint Borel subsets of $I$ – the basic elements – and $\tau$ the integer-valued function on $S$ – the inducing time (see the next section for the definition of inducing schemes and some relevant information). More precisely, they determined a class $\mathcal{H}$ of potential functions $\varphi : I \to \mathbb{R}$ for which one can find a unique equilibrium measure $\mu_\varphi$ satisfying

$$h_{\mu_\varphi}(f) + \int_I \varphi \, d\mu_\varphi = \sup \big\{ h_\mu(f) + \int_I \varphi \, d\mu \big\}. \tag{1}$$

Here $h_\mu(f)$ is the metric entropy of the map $f$ and the supremum is taken over $f$-invariant ergodic Borel probability measures $\mu$, which are

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*liftable* with respect to the inducing scheme. If a map admits an inducing scheme its action on a subset of the phase space can be described symbolically as a tower over the full (Bernoulli) shift on a countable set of states. One can provide some conditions for the existence and uniqueness of an equilibrium measure for this shift with respect to the corresponding potential (which is the lift of the potential $\varphi$ to the tower). One then transfers these conditions into requirements on the original potential. This naturally leads to the *liftability problem*: describing all the liftable measures, i.e. those that can be expressed as the lift of invariant measures for the shift (see equation (3)). The goal of this paper is to introduce some conditions on the inducing scheme which guarantees that every $f$-invariant ergodic Borel probability measure of sufficiently large entropy, which gives positive weight to the tower, is liftable. A different point of view is to construct an inducing scheme for which a given invariant measure is liftable. This provides a symbolic description of the measure but allows only to compare this measure with invariant measures, which can be lifted to the *same* tower (since the lift operator depends on the scheme).

In Section 2 we introduce inducing schemes and state one of the main results in [PS05] on the existence and uniqueness of equilibrium measures within the class of liftable measures. Our inducing schemes are determined by Conditions (H1), (H2) and (H3). Condition (H1) introduces the induced map $F$ on the base $W$ of the tower ($W$ is the union of the basic elements). Condition (H2) states that the partition of $W$ by basic elements is generating. This allows the unique symbolic coding of every point in the base in such a way that the induced map is conjugated to the full shift on a countable set of states. Condition (H3) implies that the coding *captures* all Gibbs measures.

In Section 3 we discuss the liftability problem and some recent related results. To study liftability we follow the approach by Bruin [Bru95]: the tower associated with the inducing scheme is “embedded” into the Markov extension $(\check{I}, \check{f})$ of the system $(I, f)$ in such a way that the induced map is the first return time map to a certain subset of the Markov extension. As a result one can reduce the liftability to the inducing scheme to liftability to the Markov extension. The latter can be ensured by some results of Keller [Kel89] and Buzzi [Buz99].

In Section 4 we describe Markov extensions in the sense of Buzzi for piecewise invertible continuous maps of compact metric spaces and we state a result, which provides two conditions (called (M1) and (M2)) which guarantee liftability of measures with large entropy (see [Buz99]). Condition (M1) states that the topological entropy of the system is not concentrated merely on the image of the boundary of the invertible pieces. Condition (M2) requires the existence of a set $I_0$ of full $\mu$-measure with respect to any invariant ergodic Borel measure $\mu$ of sufficiently large entropy and such that the partition of the system into invertible pieces is generating on $I_0$ with respect to $\mu$.

The Markov extensions constructed by Buzzi have the important feature that any invariant *ergodic* measure for the system induces a measure on the Markov extension, which is also ergodic. This is crucial in our study of liftability, since in view of (1) we are only concerned with lifting ergodic invariant measures. In Section 5 we study relations between Markov extensions and inducing schemes.

To obtain liftability we need to replace Condition (H1) with a slightly stronger Condition (L). Roughly speaking it requires that the inducing time is as small as possible. In applications to one-dimensional systems, where controlling the distortion of derivatives is important, one often finds inducing schemes for which Condition (L) is replaced with the slightly different requirement ($L^+$). The principle difference between them is the way in which the action of the system on each element of the tower is extended to a small neighborhood of the element.

In Section 6 we prove our main result.

**Main Theorem 1.1.** *Let $f$ be a piecewise invertible continuous map of a compact metric space admitting an inducing scheme $\{S, \tau\}$, which satisfies Condition (L) or ($L^+$). Assume that $f$ has finite topological entropy $h(f)$ and satisfies Conditions (M1) and (M2). Then there exists $0 \leq H < h(f)$ such that any invariant ergodic Borel probability measure $\mu$ with $\mu(W) > 0$ and $h_\mu(f) > H$ is liftable.*

In the last section of the paper we describe a general approach, exploiting the notion of *nice sets*, to construct inducing schemes satisfying all the conditions of Main Theorem. Many known inducing schemes can be constructed using this approach. We also present some applications of our results to one-dimensional cusp and multimodal (in particular, unimodal) maps and to some multi-dimensional piecewise expanding maps. In some situations (e.g., one-dimensional maps) the requirement that the entropy of the liftable measure should be large can be weakened to the requirement that it is just positive. Since for many "natural" potential functions the corresponding equilibrium measures must have positive entropy, this implies that the equilibrium measure is unique within the class of *all* invariant Borel probability measures supported on the tower (see Section 3 for further discussion).

## 2. Inducing schemes

To state the liftability problem more precisely, let us introduce the notion of the inducing scheme. Let $f$ be a continuous map of a compact topological space $I$. Throughout the paper we assume that the topological entropy $h(f)$ is finite. Let $S$ be a countable collection of disjoint Borel subsets of $I$ and $\tau : S \to \mathbb{N}$ a positive integer-valued function. Set $W = \bigcup_{J\in S} J$ and consider the function $\tau : I \to \mathbb{N}$ given by

$$\tau(x) = \begin{cases} \tau(J), & x \in J \\ 0, & x \notin W. \end{cases}$$

Following [PS06] we call the pair $\{S, \tau\}$ an *inducing scheme* for $f$ if the following conditions hold:

(H1) there exists a connected open set $U_J \supset J$ such that $f^{\tau(J)}|U_J$ is a homeomorphism onto its image and $f^{\tau(J)}(J) = W$;

(H2) the partition $\mathcal{R}$ induced by the sets $J \in S$ is generating; this means that for any countable collection of elements $\{J_k\}_{k\in\mathbb{N}}$, the intersection

$$\overline{J_1} \cap \Big(\bigcap_{k\geq 2} f^{-\tau(J_1)} \circ \cdots \circ f^{-\tau(J_{k-1})}(\overline{J_k})\Big)$$

is not empty and consists of a single point; here $f^{-\tau(J)}$ denotes the inverse branch of the restriction $f^{\tau(J)}|J$ and $f^{-\tau(J)}(I) = \emptyset$ provided $I \cap f^{\tau(J)}(J) = \emptyset$.

We call $W$ the *inducing domain* and $\tau(x)$ the *inducing time*. Further, we introduce the *induced map* $F : W \to W$ by $F|J = f^{\tau(J)}|J$ for $J \in S$ and we set

$$X := \bigcup_{J\in S} \bigcup_{k=0}^{\tau(J)-1} f^k(J). \qquad (2)$$

Condition (H2) allows us to view the induced map $F$ as the one-sided Bernoulli shift $\sigma$ on a countable set of states $S$. More precisely, (see [PS06, PS05]) define the coding map $h : S^{\mathbb{N}} \to \cup_{J\in S} \bar{J}$ by $h : \omega = (a_0, a_1, \cdots) \mapsto x$ where $x$ is such that $x \in \overline{J}_{a_0}$ and

$$f^{\tau(J_{a_k})} \circ \cdots \circ f^{\tau(J_{a_0})}(x) \in \overline{J}_{a_{k+1}} \quad \text{for} \quad k \geq 0.$$

**Proposition 2.1.** *The map $h$ is well-defined, continuous and $W \subset h(S^{\mathbb{N}})$. It is one-to-one on $h^{-1}(W)$ and is a topological conjugacy between $\sigma|h^{-1}(W)$ and $F|W$, i.e.,*

$$h \circ \sigma|h^{-1}(W) = F \circ h|h^{-1}(W).$$

In what follows we assume that the following condition holds:

(H3) the set $S^{\mathbb{N}} \setminus h^{-1}(W)$ supports no shift invariant measures, which give positive weight to any open subset.

This condition allows one to transfer Gibbs measures for the shift via the conjugacy map to measures which give full weight to the base $W$ and are invariant under the induced map $F$. We stress that this condition will not be used in our study of liftability.

For a Borel probability measure $\nu$ on $W$ set

$$Q_\nu := \sum_{J \in S} \tau(J)\, \nu(J) = \int_W \tau(x)\, d\nu(x).$$

Define the *lifted measure* $\mathcal{L}(\nu)$ on the set $X$ (see (2)) as follows: for any Borel subset $E \subset X$,

$$\mathcal{L}(\nu)(E) := \frac{1}{Q_\nu} \sum_{J \in S} \sum_{k=0}^{\tau(J)-1} \nu(f^{-k}(E) \cap J). \tag{3}$$

We denote by $\mathcal{M}(f, I)$ the class of all $f$-invariant Borel probability measures on $I$ and by $\mathcal{M}(F, W)$ the class of all $F$-invariant Borel probability measures on $W$. Given an inducing scheme $\{S, \tau\}$, we call a measure $\mu \in \mathcal{M}(f, I)$ *liftable* if $\mu(W) > 0$ and there exists a measure $i(\mu) \in \mathcal{M}(F, W)$ such that $\mu = \mathcal{L}(i(\mu))$. We call $i(\mu)$ the *induced measure* for $\mu$ and we denote the class of all liftable measures by $\mathcal{M}_{\mathcal{L}}(f, X)$. Observe that if $\mu \in \mathcal{M}(f, I)$ is liftable with respect to an inducing scheme $\{S, \tau\}$, then $\mu(W) > 0$ (where $W$ is the inducing domain of the scheme). Also observe that different inducing schemes lead to different classes of liftable measures.

By a result in [Zwe05], if $\mu \in \mathcal{M}_{\mathcal{L}}(f, X)$ is ergodic, then the measure $i(\mu)$ is unique, ergodic, and has integrable inducing time: $Q_{i(\mu)} < \infty$.

By Proposition 2.1, liftable measures are those measures which can be expressed as lifts of shift invariant measures on the countable symbolic space. Certain important properties of the shift invariant measures can then be transferred to liftable measures, as is the case of equilibrium measures. To illustrate this let us first describe a class of potential functions $\varphi : I \to \mathbb{R}$ which admit unique equilibrium measures. Define the *induced potential function* $\bar{\varphi} : W \to \mathbb{R}$ by

$$\bar{\varphi}(x) := \sum_{k=0}^{\tau(J)-1} \varphi(f^k(x)), \quad x \in J.$$

Also, define

$$s_\varphi := \sup_{\mathcal{M}_{\mathcal{L}}(f,X)} \{h_\mu(f) + \int_X \varphi\, d\mu\}. \tag{4}$$

It is shown in [PS06] that the quantity in (4) is finite under the conditions given below. We say that the induced potential function $\bar{\varphi}$ is *locally Hölder continuous* if there exists $A > 0$ and $0 < \gamma < 1$ such that $V_n(\bar{\varphi}) \leq A\gamma^n$ for $n \geq 1$. Here $V_n(\bar{\varphi})$ is the *n-variation* defined by

$$V_n(\bar{\varphi}) := \sup_{[b_1,\dots,b_n]} \sup_{x,x' \in [b_1,\dots,b_n]} \{|\bar{\varphi}(x) - \bar{\varphi}(x')|\},$$

and the cylinder set $[b_1, \dots, b_n]$ is the subset of $J_{b_1}$ such that for every $1 \leq k \leq n-1$

$$f^{\tau(J_{b_{k-1}})} \circ \cdots \circ f^{\tau(J_{b_1})}([b_1, \dots, b_n]) \subseteq \overline{J_{b_k}}.$$

**Theorem 2.2** (see [PS06])**.** *Assume that the map $f$ admits an inducing scheme satisfying Conditions (H1)–(H3). Also assume that the potential function $\varphi$ is locally Hölder continuous and that there exists $\varepsilon > 0$ such that*

$$\sum_{J \in S} \sup_{x \in J} \exp \bar{\varphi}(x) < \infty,$$

$$\sum_{J \in S} \sup_{x \in J} \exp (\bar{\varphi} - (s_\varphi - \varepsilon)\tau(x)) < \infty.$$

*Then there exists an equilibrium measure $\mu_\varphi$ for $\varphi$ (see* (1)*). This measure is liftable and is unique among all the liftable measures.*

## 3. The liftability problem

The liftability problem has recently become a subject of intensive study. Let us stress again that the class of liftable measures $\mathcal{M}_{\mathcal{L}}(f, X)$ depends on the choice of the inducing scheme $\{S, \tau\}$ and that liftable measures are supported on $X$ (i.e., $\mu(X) = 1$; in particular, $\mu(W) > 0$). In [PZ07] an example of an inducing scheme is given for which there exist a non-liftable measure supported on $X$ as well as another (non-liftable) measure supported outside $X$; the latter is an equilibrium measure for some potential function.

We begin the study of the liftability problem by stating two general criteria that guarantee that a given measure $\mu \in \mathcal{M}(f, I)$ is liftable. Given a Borel set $A \subset X$ and $J \in S$, define

$$\epsilon(J, A) := \frac{1}{\tau(J)} \mathrm{Card}\{0 \leq k \leq \tau(J) - 1 :\ f^k(J) \cap A \neq \emptyset\},$$

where Card $E$ denotes the cardinality of the set $E$.

**Theorem 3.1** (see [PZ07]). *An $f$-invariant Borel ergodic probability measure $\mu$ with $\mu(W) > 0$ is liftable if there exists a number $N \geq 0$ and a subset $A \subset I$ such that $\mu(A) > \sup_{\tau(J)>N} \epsilon(J, A)$.*

**Theorem 3.2** (see [Zwe05]). *A measure $\mu \in \mathcal{M}(f, I)$ with $\mu(W) > 0$ and with integrable inducing time (i.e., $\tau \in L^1(I, \mu|W)$) is liftable.*

Although these two theorems give conditions under which a measure is liftable, they turn out to be difficult to check for many measures. Moreover, the study of equilibrium measures, which serves as our motivation, requires the impossible task of checking these conditions for all invariant measures. It then becomes our goal to establish sufficient conditions for the liftability of measures, which need only to be checked for reasonably few measures in the study of equilibrium measures.

For interval maps Hofbauer and Keller constructed a different type of the inducing scheme known as the Markov extension or Hofbauer-Keller tower (see [Hof79, Hof81, Kel89]). It produces a symbolic representation of the interval map via a subshift of countable type, however, the transfer matrix, defining which (symbolic) sequences are allowed, is not known a priori and can be very complicated. In [Kel89], Keller obtained some general criteria for liftability to the Markov extension for one-dimensional maps. In this case, the liftability problem consists in proving existence of a finite non-zero measure on the Markov extension, which projects (via the canonical projection $\pi$ on intervals as opposed to the operator $\mathcal{L}$ defined by (3)) to the given measure on the interval.

In [Bru95], Bruin established liftability of absolutely continuous invariant measures of positive entropy to inducing schemes satisfying some additional assumptions for piecewise continuous piecewise monotone interval maps. These assumptions allow one to "embed" the inducing scheme into the Hofbauer-Keller tower and express the induced map as the first return time map to a certain subset (in the Hofbauer-Keller tower). Using the techniques of Bruin, Pesin and Senti [PS06] showed that for any unimodal map satisfying the Collet-Eckmann condition every measure $\mu \in \mathcal{M}(f, X)$ of positive metric entropy is liftable for the particular inducing scheme whose inducing domain ranges from the smaller fixed point to its symmetric point. A similar result holds for multimodal maps.

In [Buz99], Buzzi constructed Markov extensions (i.e. a version of the Hofbauer-Keller tower) for multi-dimensional piecewise invertible maps and established liftability (in the sense of Markov extensions) for invariant measures of large entropy. In this paper we modify the approach by Bruin adjusting it to Markov extensions in the sense of

Buzzi and we establish liftability of measures of large entropy for general inducing schemes.

## 4. Markov extensions

Let $I$ be a compact metric space. A map $f : I \to I$ is said to be *piecewise invertible* if there exists a collection of open disjoint subsets $P = \{A_i \subset I\}_{i=1}^s$ satisfying:

(1) $\bigcup_{i=1}^s \overline{A}_i = I$;
(2) for each $i$ there is a connected set $U_i$ and a homeomorphism $f_{U_i} : U_i \to I$ for which $\overline{A_i} \subset U_i$ and $f_{U_i}|A_i = f|A_i$;
(3) the boundary $\partial P := \bigcup_{A_i \in P} \partial A_i$ is the singular set for $f$, i.e. for any open set $U$ for which $U \cap \partial P \neq \emptyset$, $f|U$ is not a homeomorphism.

Set

$$\partial_0 P := \partial P \text{ and } \partial_n P := \bigcup_{k=0}^{n-1} f^{-k}(\partial P),\ n \geq 1$$

and for $x \notin \partial P$ denote by $P(x)$ the element of $P$ containing $x$. Further, for $x \notin \partial_n P$ we denote by $P_n(x)$ the element of $P \vee f^{-1}P \cdots \vee f^{-n+1}P$ containing $x$.

Following [Buz99] we describe the connected Markov extension of the map $f$. This construction is slightly different from the construction of the Hofbauer-Keller tower. Set $\mathcal{D}_1 = P$ and then

$$\mathcal{D}_{n+1} := \{f(A) \cap B \neq \emptyset :\ A \in \mathcal{D}_n,\ B \in P\} \text{ and } \mathcal{D} := \bigcup_{n \geq 0} \mathcal{D}_n.$$

The *connected Markov extension* of $f$ is the pair $(\check{I}, \check{f})$ where

$$\check{I} := \{(x, D) \in I \times \mathcal{D} :\ x \in \bar{D}\}$$

is the tower and $\check{f} : \check{I} \setminus \pi^{-1}(\partial P) \to \check{I}$ is the map given by

$$\check{f}(x, D) = (f(x), E)$$

(here $E$ is the *connected* component of $f(D \cap P(x))$ containing $f(x)$ and $\pi : \check{I} \to I$ is the canonical projection, i.e., $\pi(x, \mathcal{D}) = x$). We refer to subsets of the type

$$\check{D} := \{(x, D) :\ x \in \bar{D},\ D \in \mathcal{D}\}$$

as *elements* of the Markov extension and we set

$$\check{\mathcal{D}} := \bigcup_{D \in \mathcal{D}} \check{D}.$$

Let $\mathrm{inc} : I \setminus \partial P \to \check{I}$ be the inclusion into the first level of the Markov extension, i.e., $\mathrm{inc}(x) = (x, P(x))$. For any $D \in \mathcal{D}$, we define the *level of* $D$ as $\ell(D) = \min\{n \in \mathbb{N} \colon D \in \mathcal{D}_n\}$ and, by extension, we define the *level of* $\check{D}$ as $\ell(\check{D}) = \ell(D)$ and write $\check{D} = \check{D}_\ell$.

Note that the projection $\pi : \check{I} \to I$ is countable to one on $\check{I}$, but it is injective on each $\check{D} \in \check{\mathcal{D}}$.

The Markov extension has the following properties (see [Buz99]):

(1) it is an extension of the system $(I, f)$, i.e.,
$$\pi \circ \check{f} | \check{I} \setminus \pi^{-1}(\partial P) = f \circ \pi | \check{I} \setminus \pi^{-1}(\partial P);$$
(2) $\check{\mathcal{D}}$ is a Markov partition for $(\check{I}, \check{f})$ in the sense that for any $i \in \mathbb{N}$ and any $\check{D}_a, \check{D}_b \in \check{\mathcal{D}}$, we have that $\check{f}^i(\check{D}_a) \cap \check{D}_b \neq \emptyset$ if and only if $\check{f}^i(\check{D}_a) \supseteq \check{D}_b$;

(3) for any connected set $\check{E} \subset \check{D} \in \check{\mathcal{D}}$ and any $i \in \mathbb{N}$, we have that $\check{f}^i | \check{E}$ is a homeomorphism if and only if $f^i | \pi(\check{E})$ is (see Property (3) in the definition of the piecewise invertible map);

(4) for any $\check{D} \in \check{\mathcal{D}}$ of level $n$, there exists a subset $E \subset A_i$ for some $A_i \in P$ such that $\check{f}^n$ maps $\mathrm{inc}\,(E)$ homeomorphically onto $\check{D}$.

Let $f$ be a piecewise invertible map of a compact metric space $I$ and $(\check{I}, \check{f})$ its connected Markov extension. We define $(\mathcal{I}, f_e)$ and $(\check{\mathcal{I}}, \check{f}_e)$ to be the *natural extensions* of $f$ and $\check{f}$ respectively. Recall that the natural extension $(\mathcal{I}, f_e)$ of a map $f : I \to I$ is the space of all sequences $\{x_n\}_{n \in \mathbb{Z}}$, satisfying $f(x_n) = x_{n+1}$ (i.e. orbits of $f$), along with the map $f_e$, which is the left shift. There is a natural projection $p(\{x_n\}) = x_0$ from the natural extension to the original space. If $f$ preserves a measure $\mu$, there is a unique $f_e$-invariant measure $\mu_e$ on the natural extension, which projects to $\mu$. If $\mu$ is ergodic then so is $\mu_e$ and $h_{\mu_e}(f_e) = h_\mu(f)$.

We denote by $p : \mathcal{I} \to I$ and $\check{p} : \check{\mathcal{I}} \to \check{I}$ the natural projections and by $\pi_e : \check{\mathcal{I}} \to \mathcal{I}$ the extension of the projection $\pi$ to the natural extensions. We have the following commutative diagram

$$\begin{array}{ccc} (\check{\mathcal{I}}, \check{f}_e) & \xrightarrow{\pi_e} & (\mathcal{I}, f_e) \\ \Big\downarrow \check{p} & & \Big\downarrow p \\ (\check{I}, \check{f}) & \xrightarrow{\pi} & (I, f) \end{array} \tag{5}$$

Define the $\check{f}_e$-invariant set $\check{\mathcal{I}}' \subseteq \check{\mathcal{I}}$ as

$$\check{\mathcal{I}}' := \{\{\check{x}_n\}_{n \in \mathbb{N}} \in \check{\mathcal{I}} : \text{there exists } N \geq 0 \text{ such that } \check{x}_0 = \check{f}^n(\mathrm{inc}\,(\pi(\check{x}_{-n}))) \text{ for all } n \geq N\}$$

and set $\mathcal{I}' = \pi_e(\check{\mathcal{I}}')$. It is shown in [Buz99] that $\pi_e : \check{\mathcal{I}}' \to \mathcal{I}'$ is one-to-one and bi-measurable. Let $\Delta P = f(\partial P)$.

**Proposition 4.1** (see [Buz99], Theorem A, Proposition 2.2)**.** *Assume that the the maps $f$ satisfies the following conditions:*

(M1) $h_{top}(\Delta P, f) < h_{top}(f)$;

(M2) *there exist a measurable subset $I_0 \subset I$ and a number $0 \le H < h_{top}(f)$ such that for any ergodic measure $\mu \in \mathcal{M}(f, I)$ with $h_\mu(f) > H$, we have $\mu(I \setminus I_0) = 0$ and $diam(P_n(x)) \to 0$ as $n \to \infty$ for $\mu$-almost every $x \in I_0$.*

*Then for any $\mu_e \in \mathcal{M}(f_e, \mathcal{I})$ with*

$$h_{\mu_e}(f_e) > \max\{H, h_{top}(\Delta P, f)\},$$

*we have $\mu_e(\mathcal{I}') = 1$. The same statement holds under the same conditions for any $\check{\mu}_e \in \mathcal{M}(\check{f}_e, \check{\mathcal{I}})$.*

One can therefore lift any measure with sufficiently large entropy to the connected Markov extension. More precisely, the following statement holds.

**Proposition 4.2.** *Assume that the map $f$ satisfies Conditions (M1) and (M2). Then for any ergodic invariant Borel probability measure $\mu$ with $h_\mu(f) > \max\{H, h_{top}(\Delta P, f)\}$,*

(1) *there exists an $\check{f}$-invariant ergodic Borel probability measure $\check{\mu}$ on the connected Markov extension $\check{I}$ with $\pi_*\check{\mu} = \mu$;*

(2) *there exists an $\check{f}_e$-invariant ergodic Borel probability measure $\check{\mu}_e$ on $\check{\mathcal{I}}$ with $\check{p}_*\check{\mu}_e = \check{\mu}$ and $\check{\mu}_e(\check{\mathcal{I}}') = 1$.*

*Proof.* Let $\mu$ be an $f$-invariant ergodic Borel probability measure with $h_\mu(f) > \max\{H, h_{top}(\Delta P, f)\}$ and $\mu_e$ the unique lift of $\mu$ to the natural extension $\mathcal{I}$. By [Buz99, Theorem A], $\pi_e$ is a measurable isomorphism between $\check{\mathcal{I}}'$ and $\mathcal{I}'$ and the measure $\check{\mu}_e := (\pi_e^{-1})_*\mu_e|\mathcal{I}'$ is well defined. Furthermore, by Proposition 4.1, $\mu_e(\mathcal{I}') = 1 = \check{\mu}_e(\check{\mathcal{I}}')$. Set $\check{\mu} = \check{p}_*\check{\mu}_e$. Since the diagram (5) commutes, we have that $\mu = \pi_*\check{\mu}$. □

## 5. Relations between Markov extensions and inducing schemes

To study the liftability problem we need to impose some stronger restrictions on the inducing schemes:

(L) *minimality*: there is a connected open set $U_J \supset J$ such that $f^{\tau(J)}|U_J$ is a homeomorphism onto its image with $f^{\tau(J)}(J) = W$ (see Condition (H1)); in addition, the inducing time is *minimal* in the following sense: for any $L \subset I$, $m \in \mathbb{N}$ and any connected open set $U_L \supset L$ such that $f^m|U_L$ is a homeomorphism with $f^m(L) = W$, we have that if $L \cap J \neq \emptyset$ for some $J \in S$ then $m \geq \tau(J)$;

In the case of one-dimensional maps one often needs bounds on the distortion of the derivative of the induced map $F$. Such bounds can be obtained using Koebe's lemma, which applies under a somewhat different assumption than (L):

($L^+$) *minimal extendibility*: there is a connected open neighborhood $W^+$ of $\overline{W}$ and for each $J \in S$ there exists a connected open neighborhood $J^+$ of $\overline{J}$ such that $f^{\tau(J)}|J^+$ is a homeomorphism onto its image, $f^{\tau(J)}(J^+) = W^+$ and $f^{\tau(J)}(J) = W$; in addition, the inducing time is *minimal extendible*, in the following sense: for any $L \subset I$, $m \in \mathbb{N}$ and any connected open neighborhood $L^+ \supset L$ such that $f^m$ is a homeomorphism of $L$ onto $W$ and of $L^+$ onto $W^+$ we have that if $L \cap J \neq \emptyset$ for some $J \in S$ then $m \geq \tau(J)$.

Conditions (L) and ($L^+$) express a kind of "minimality" of the inducing time. In particular, a refinement of a minimal (or minimal extendible) inducing scheme fails to be minimal. However, the liftability property may still hold. In fact, it does hold for any finite refinement of the inducing scheme (as the proofs below can be easily modified to work for finite refinements).

Let us stress that neither of the Conditions (L) or ($L^+$) is necessary for liftability as illustrated by the following example. Consider the map $f(x) = 2x \pmod 1$ and the inducing scheme given by $J_n = (\frac{1}{2^{n+1}}, \frac{1}{2^n})$, $n \geq 0$ and $\tau(J_n) = n + 1$. It is easy to see that the inducing scheme $\{\{J_n\}, \tau\}$ does not satisfy either of the Conditions (L) or ($L^+$), though it can be shown that every measure $\mu$ with $\mu(W) > 0$ is liftable.

Note that for any $J \in S$ and $0 \leq i \leq \tau(J)$ the map $f^i|J$ is a homeomorphism and hence it must be contained in some piecewise invertible component $A_i \in P$.

For each $J \in S$ define the map

$$\check{F}|\pi^{-1}(J) = \check{f}^{\tau(J)}|\pi^{-1}(J) \text{ and } \check{F}|\pi^{-1}(J^+) = \check{f}^{\tau(J)}|\pi^{-1}(J^+)$$

and set

$$\check{W} := \bigcup_{k\geq 0} \check{F}^k(\mathrm{inc}\,(W)) \text{ and } \check{W}^+ := \bigcup_{k\geq 0} \check{F}^k(\mathrm{inc}\,(W^+)).$$

**Theorem 5.1.** *Assume that the inducing satisfies either of the Conditions (L) or ($L^+$). Then the map $\check{F} : \check{W} \to \check{W}$ is the first return map of $(\check{I}, \check{f})$ to $\check{W}$. More precisely, for any $\check{x} \in \check{W} \cap \pi^{-1}(J)$ with some $J \in S$ we have that $\check{f}^i(\check{x}) \notin \check{W}$ for $0 < i < \tau(J)$.*

*Proof.* We will only prove the statement assuming Condition ($\mathrm{L}^+$). If the inducing scheme satisfies Condition (L) the proof goes by replacing $J^+$ with $J$, $W^+$ with $W$, and $L^+$ with $L$.

Consider an inducing scheme which satisfies Condition ($\mathrm{L}^+$). Since $\check{F}$ is a homeomorphism on any $\mathrm{inc}\,(J^+)$, for any $\check{D} \in \check{\mathcal{D}}$ with $\check{D} \cap \check{F}(\mathrm{inc}\,(J^+)) \neq \emptyset$, we have that $\check{F}(\mathrm{inc}\,(J^+)) \subset \check{D}$. Hence $\pi(\check{D}) \supset W^+$. Using induction, one can easily check that this also holds for any element $\check{D} \cap \check{W} \neq \emptyset$ of the connected Markov extension.

Assume, by contradiction, that there exist $\check{x} \in \check{W} \cap \pi^{-1}(J) \cap \check{D}_a$ and $0 < i < \tau(J)$ such that $\check{f}^i(\check{x}) \in \check{W} \cap \check{D}_b$. It follows from the previous observation that both $\pi(\check{D}_a) \supset W^+$ and $\pi(\check{D}_b) \supset W^+$. As $i < \tau(J)$, the map $\check{f}^i|(\pi^{-1}(J^+) \cap \check{D}_a)$ is a homeomorphism and we have that $\check{f}^i(\pi^{-1}(J^+) \cap \check{D}_a) \subset \check{D}_b$. By the Markov property of the Markov extension, $\check{f}^i(\check{D}_a) \supset \check{D}_b$ and we can take $\check{L}$ to be the unique homeomorphic pre-image of $\pi^{-1}(W^+) \cap \check{D}_b$ under $\check{f}^i$ that contains $\check{x}$.

Let $L^+ \subset I$ and $m \in \mathbb{N}$ be such that $f^m(L^+) = W^+$ and $f^m|L^+$ is a homeomorphism. By Condition ($\mathrm{L}^+$), if $L^+ \cap J \neq \emptyset$ for some $J \in S$, then $m \geq \tau(J)$. Setting $L^+ = \pi(\check{L})$ and $m = i$ we come to a contradiction. $\square$

**Theorem 5.2.** *Assume that the map $f$ satisfies Conditions (M1) and (M2) of Proposition 4.1. Let $\mu$ be an $f$-invariant ergodic Borel probability measure on $I$ with $h_\mu(f) > \max\{H, h_{top}(\Delta P, f)\}$ and $\check{\mu}$ its lift to the connected Markov extension. Let also $E \subset X$ be such that $\mu(E) > 0$ and $E \cap \partial P = \emptyset$. Then for $\check{\mu}$-almost every $\check{x} \in \check{I}$, there exists $k \in \mathbb{N}$ and $\check{y} \in inc\,(E)$ such that $\check{f}^k(\check{y}) = \check{x}$, i.e.,*

$$\check{\mu}\big(\bigcup_{k\geq 0} \check{f}^k(inc\,(E))\big) = 1.$$

*Proof.* First, define the set $\mathcal{R} \subset \check{\mathcal{X}}$ by

$$\mathcal{R} := \{\{\check{x}_n\}_{n\in\mathbb{N}} \in \check{\mathcal{I}}' : \text{ there exists } n_k \to \infty \text{ such that } \pi(\check{x}_{-n_k}) \in E,\ \check{x}_0 = \check{f}^{n_k}(\mathrm{inc}\,(\pi(\check{x}_{-n_k})))\}.$$

We claim that if $\check{\mu}_e(\mathcal{R}) = 1$ then our statement holds. Indeed, set

$$R := \{\check{x} \in \check{I} : \text{there exist } k \in \mathbb{N} \text{ and } \check{y} \in \mathrm{inc}\,(E) \text{ such that } \check{f}^k(\check{y}) = \check{x}\}.$$

We have that $\check{p}(\mathcal{R}) \subset R$ and hence by Proposition 4.2,

$$1 \geq \check{\mu}(R) \geq \check{\mu}(\check{p}(\mathcal{R})) \geq \check{\mu}_e(\mathcal{R}) = 1.$$

It follows that $\check{\mu}(R) = 1$, which implies the desired result.

We therefore are left to prove that $\check{\mu}_e(\mathcal{R}) = 1$. Note that the set $\check{\mathcal{I}}'$ has full $\check{\mu}_e$-measure and that $\check{\mu}_e(\check{p}^{-1}(\pi^{-1}(E))) = \mu(E) > 0$. Since

$\mu$ is ergodic with respect to $f$, Proposition 4.2 yields that $\check{\mu}$ is ergodic with respect to $\check{f}$. Note that the inverse map $\check{f}_e^{-1}$ is well defined on the natural extension and hence, it is ergodic with respect to $\check{\mu}_e$. By Birkhoff's ergodic theorem, for $\check{\mu}_e$-almost every $\{\check{x}_n\}_{n\in\mathbb{N}} \in \check{\mathcal{I}}'$, there exists $n_k \to \infty$ such that $\check{f}_e^{-n_k}(\{\check{x}_n\}) \in \check{p}^{-1}(\pi^{-1}(E))$. This implies that

$$\check{x}_{-n_k} = \check{p}(\check{f}_e^{-n_k}(\{\check{x}_n\})) \in \pi^{-1}(E),$$

i.e., $\pi(\check{x}_{-n_k}) \in E$. For any $\{\check{x}_n\} \in \check{\mathcal{I}}'$, we have that $\check{x}_0 = \check{f}^n(\mathrm{inc}\,(\pi(\check{x}_{-n})))$ for sufficiently large $n$. It follows that $\check{\mathcal{I}}' \subseteq \mathcal{R} \pmod{\check{\mu}_e}$ and hence,

$$1 = \check{\mu}_e(\check{\mathcal{I}}') = \check{\mu}_e(\mathcal{R}),$$

which implies the desired result. □

**Corollary 5.3.** *Assume that the map $f$ satisfies Conditions (M1) and (M2) of Proposition 4.1. Let $\mu$ be an $f$-invariant ergodic Borel probability measure on $I$ with $h_\mu(f) > \max\{H, h_{top}(\Delta P, f)\}$ and $\check{\mu}$ its lift to the connected Markov extension. If $\mu(W) > 0$ then $\check{\mu}(\check{W}) > 0$.*

*Proof.* It follows from Theorem 5.2 that

$$\check{\mu}(\bigcup_{k\geq 0} \check{f}^k(\mathrm{inc}\,(W))) = 1.$$

Since

$$\bigcup_{k\geq 0} \check{f}^k(\mathrm{inc}\,(W)) \subset \bigcup_{j\geq 0} \check{f}^{-j}(\check{W}),$$

we conclude that $\check{\mu}(\check{W}) > 0$. □

## 6. Liftability: the proof of the Main Theorem

In this section we present a proof of Main Theorem on the liftability of measures to inducing schemes. More precisely, we establish the following result.

**Theorem 6.1.** *Let $(I, P, f)$ be a piecewise invertible system and $\{S, \tau\}$ an inducing scheme satisfying Condition (L) or (L$^+$). Also assume that the map $f$ satisfies Conditions (M1) and (M2). Then any ergodic measure $\mu \in \mathcal{M}(f, I)$ supported on $X$ with $h_\mu(f) > \max\{H, h_{top}(\Delta P, f)\}$ is liftable to the inducing scheme $\{S, \tau\}$.*

*Proof.* Since $\mu$ is invariant and $X \subseteq \bigcup_{k\geq 0} f^k(W)$, the fact that $\mu(X) = 1$ implies that $\mu(W) > 0$. Then Corollary 5.3 yields that $\check{\mu}(\check{W}) > 0$.

It follows from Theorem 5.1 that $(\check{W}, \check{F})$ is the first return time map of $(\check{I}, \check{f})$ to $\check{W}$. Since $\check{\mu}(\check{W}) > 0$ and $\tau(J)$ is the first return time of $\pi^{-1}(J) \cap \check{W}$ to $\check{W}$, we have that the measure $\check{\nu} = \frac{1}{\check{\mu}(\check{W})}\check{\mu}|\check{W}$ is $\check{F}$-invariant. Furthermore, for any measurable set $\check{E} \subset \check{I}$

$$(6)\quad \check{\mu}(\check{E}) = \sum_{J\in S}\sum_{k=0}^{\tau(J)-1} \check{\mu}|\check{W}(\check{f}^{-k}(\check{E}) \cap \pi^{-1}(J)) = \frac{1}{Q_{\check{\nu}}}\sum_{J\in S}\sum_{k=0}^{\tau(J)-1} \check{\nu}(\check{f}^{-k}(\check{E}) \cap \pi^{-1}(J)),$$

where by Kac's formula,

$$Q_{\check{\nu}} = \sum_{J\in S}\tau(J)\check{\nu}(\pi^{-1}(J) \cap \check{W}) = \frac{1}{\check{\mu}(\check{W})}.$$

Note that we have the following two conjugacies:

$$\pi \circ \check{f}|\check{I} \setminus \pi^{-1}(\partial P) = f \circ \pi|\check{I} \setminus \pi^{-1}(\partial P), \quad \pi \circ \check{F} = F \circ \pi.$$

It follows that $\nu := \pi_*\check{\nu}$ is an $F$-invariant Borel probability measure and

$$Q_\nu = \sum_{J\in S}\tau(J)\nu(J) = \sum_{J\in S}\tau(J)\check{\nu}(\pi^{-1}(J)) = Q_{\check{\nu}}.$$

For any $\mu$-measurable set $E$ we have

$$\begin{aligned}
\mu(E) &= \check{\mu}(\pi^{-1}(E)) \\
&= \frac{1}{Q_{\check{\nu}}}\sum_{J\in S}\sum_{k=0}^{\tau(J)-1} \check{\nu}(\check{f}^{-k}(\pi^{-1}(E)) \cap \pi^{-1}(J)) \\
&= \frac{1}{Q_\nu}\sum_{J\in S}\sum_{k=0}^{\tau(J)-1} \check{\nu}(\pi^{-1}(f^{-k}(E) \cap J)) \\
&= \frac{1}{Q_\nu}\sum_{J\in S}\sum_{k=0}^{\tau(J)-1} \nu(f^{-k}(E) \cap J) = \mathcal{L}(\nu)(E)
\end{aligned}$$

(see (3)) which is what we need. □

## 7. Applications

**7.1. A general construction of inducing schemes via nice sets.** We describe a general approach for building minimal (respectively, minimal extendible) inducing schemes, i.e. those that satisfy Condition (L) (respectively, ($L^+$)), by exploiting the notion of *nice sets*. This notion was first introduced by Martens (see [Mar94]) in the context of interval maps.

Let us write $f^n(J) \simeq V$ if $f^n|J$ is a homeomorphism with $f^n(J) = V$. An open set $V$ is said to be *nice* (for the map $f$) if

$$f^n(\partial V) \cap V = \emptyset \quad \text{for all} \quad n \geq 0$$

(here $\partial V = \overline{V} \setminus V$). In general, a given map $f$ may admit no nice sets. In the case of interval maps, however, it is easy to see that any periodic cycle contains endpoints of nice intervals. Because the pre-images of nice sets are either disjoint or nested, they are good candidates for being basic elements of minimal inducing schemes. More precisely, the collection $S$ of all first homeomorphic pre-images of a nice set, contained in the nice set, determines an inducing scheme, which satisfies Condition (L), since the preimages are homeomorphic, so the partition elements must be contained in some domain of invertibility of $f$. Let us make the above observation rigorous.

**Proposition 7.1.** *Let $V$ be a nice set for $f$ and let $J$ and $J'$ be such that $f^n(J) \simeq V \simeq f^m(J')$ with $n \leq m$. Then either*

$$\textit{int}\, J \cap \textit{int}\, J' = \emptyset \quad \textit{or} \quad J' \subset J \quad \textit{and} \quad n < m.$$

*Proof.* Assume $\partial J \cap \operatorname{int} J' \neq \emptyset$. Then $n \neq m$ and $\operatorname{int} f^n(J') \cap \partial V \neq \emptyset$. This implies that $f^{m-n}(\partial V) \subset \operatorname{int} V$ leading to a contradiction. □

Given a nice set $V$ and its open neighborhood $V^+ \supset \overline{V}$, consider the following two collections of sets

$$Q := \{J \subsetneq V \colon f^{\tau(J)}(J) \simeq V \text{ for some } \tau(J) \in \mathbb{N}\},$$
$$Q^+ := \{J \in S \colon f^{\tau(J)}(J^+) \simeq V^+ \text{ for some open } J^+ \supset J\}.$$

Further, define

$$S' := \{J \in Q \colon \tau(J) < \tau(J') \ \text{ for all } \ J' \in S \text{ with } J \cap J' \neq \emptyset\},$$
$$S'^+ := \{J \in Q^+ \colon \tau(J) < \tau(J') \ \text{ for all } \ J' \in S^+ \text{ with } J \cap J' \neq \emptyset\}.$$

We can then set

$$\mathcal{V} = \bigcup_{J \in S'} J, \quad \mathcal{V}^+ = \bigcup_{J \in S'^+} J$$

and consider the induced map $F : \mathcal{V} \to V$ defined by $F|J := f^{\tau(J)}|J$ and the $F$-invariant set

$$W := \bigcap_{k \geq 0} F^{-k}(\mathcal{V}).$$

Finally, consider the inducing schemes $\{S, \tau\}$ and $\{S^+, \tau\}$ where

$$S := \{J \cap W \colon J \in S'\}, \quad S^+ := \{J \cap W \colon J \in S'^+\}$$

and $\tau(J \cap W) = \tau(J)$ for $J \in S'$ (respectively, $J \in S'^+$).

**Theorem 7.2.** *Given a connected nice set $V$ and its connected open neighborhood $V^+ \supset \overline{V}$, the inducing scheme $\{S, \tau\}$ (respectively, $\{S^+, \tau\}$) satisfies Condition (L) (respectively, (L$^+$)) provided that $\overline{W} \supset V$.*

*Proof.* We only prove Condition (L). The proof of Condition (L$^+$) is similar. By definition, the elements $L \in S$ are homeomorphic pre-images of $W$, which are contained in elements $J \in S'$. The latter are homeomorphic pre-images of $V$, so $F(L) = f^{\tau(J)}(L) \simeq W$. Since all $L \in S$ satisfy $L \subset J$ for some $J \in S' \subseteq Q$, the inducing scheme satisfies Condition (H1) with $U_L = J$ for each $J \in S'$. In order to prove Condition (L) consider a set $L' \subset I$, an open connected set $U_{L'} \supset L'$, and $m \in \mathbb{N}$ such that $f^m|U_{L'}$ is a homeomorphism, $f^m(L') \simeq W$. Then by our assumption, there exists an open connected set $U$, $L' \subset U \subset U_{L'}$ such that $f^m(U) = V$. It follows that if $L' \cap L \neq \emptyset$ for some $L \in S$ then $U \cap V \neq \emptyset$ and $U \cap \partial V = \emptyset$. Hence, $U \in Q$. This implies that $m \geq \tau(L)$ and Condition (L) follows. $\square$

In general, the set $S$ in the previous theorem may be empty. However, in certain particular cases not only one can show that $S$ is a non-empty collection but that the set $W$ has full relative Lebesgue measure in $V$ in the sense that $\text{Leb}(V \setminus W) = \text{Leb}(V \setminus \bigcup_{J \in S'} J) = 0$ (respectively, $\text{Leb}(V \setminus \bigcup_{J \in S^+} J) = 0$ in the case we are interested in minimal extendibility, i.e. Condition (L$^+$)).

Let $\{S, \tau\}$ be an inducing scheme constructed via a nice set as described in Theroem 7.2. Assume that the set

$$\partial\mathcal{W} := \bigcup_{k=0}^{\infty} F^{-k}\Big(\bigcup_{J \in S'} \partial J\Big)$$

does not support any invariant Borel measure other than an atomic measure. Then condition (H3) is satisfied for this inducing scheme. In particular, this is true if $f$ is a one-dimensional map, as in this case the set $\partial\mathcal{W}$ is countable.

7.2. **One-dimensional maps.** A *cusp map* of a finite interval $I$ is a map $f : \bigcup_j I_j \to I$ of an at most countable family $\{I_j\}_j$ of disjoint open subintervals of $I$ such that

- $f$ is a $C^1$ diffeomorphism on each interval $I_j := (p_j, q_j)$, extendible to the closure $\bar{I}_j$ (the extension is denoted by $f_j$);
- the limits $\lim_{\epsilon \to 0^+} Df(p_j + \epsilon)$ and $\lim_{\epsilon \to 0^+} Df(q_j - \epsilon)$ exist and are equal to either 0 or $\pm\infty$;
- there exist constants $K_1 > K_2 > 0$ and $C > \delta > 0$ such that for every $j \in \mathbb{N}$ and every $x, x' \in \bar{I}_j$,

$$|Df_j(x) - Df_j(x')| < C|x - x'|^{\delta} \text{ if } |Df_j(x)|, |Df_j(x')| \leq K_1,$$

$$|Df_j^{-1}(x) - Df_j^{-1}(x')| < C|x - x'|^\delta \text{ if } |Df_j(x)|\,,\ |Df_j(x')| \geq K_2.$$

We call a point *singular* if it belongs to $\partial I_j$ for some $j$. Critical points of $f$ are singular.

For cusp maps one has the following result.

**Theorem 7.3.** [Dob06, Theorem 1.9.10] *Let $f$ be a cusp map with finitely many intervals of monotonicity (i.e. finite number of intervals $I_j$). Suppose $f$ has an ergodic absolutely continuous invariant probability measure $m$ with strictly positive Lyapunov exponent. Then*

(1) *$f$ possesses a nice set $V \subset I$ satisfying conditions of Theorem 7.2;*
(2) *$f$ admits inducing schemes $\{S, \tau\}$ and $\{S^+, \tau\}$ satisfying Conditions (L) and ($L^+$) respectively;*
(3) *the inducing domain $W$ has full relative Lebesgue measure in $V$ (i.e., $Leb(V \setminus W) = 0$) and $\int_I \tau\, dm < \infty$.*

In [Dob06], the fact that the inducing domain is nice (called there regularly recurrent) is not explicitly mentioned but is essentially proven.

**Remark.** *Although general Hölder continuous piecewise invertible maps, including Hölder continuous unimodal and multimodal maps, may not satisfy the boundary conditions of cusp maps, they can always be extended to a cusp map, and thus can always be viewed as the restriction of a cusp map to an invariant subinterval (see also* [Dob06, Theorem 1.9.11]*).*

*We stress again that Condition ($L^+$) serves the purpose of controlling the distortion of the derivative using Koebe's Lemma and thus control the density.*

We now establish Conditions (M1) and (M2) for piecewise invertible interval maps. Recall that a wandering interval is an interval $J$ such that the sets $f^i(J)$ are pairwise disjoint and the $\omega$-limit set of $J$ is not equal to a single periodic point.

**Proposition 7.4.** *Assume that a piecewise invertible interval map $f$ with finitely many branches has no wandering intervals or attracting periodic points on some interval $I$. Then $f|I$ satisfies Conditions (M1) and (M2) with constant $H = 0$. Furthermore, $h_{top}(\Delta P, f) = 0$.*

*Proof.* For any piecewise invertible interval map with finitely many branches (including cusp maps) the partition $P$ defined in Section 4 is finite and so the set of boundary points of $P$ is the union of the boundary points of the partition elements $A_i$. This is a finite set and as such has zero topological entropy. Thus $h_{\text{top}}(\Delta P, f) = 0$ and any map with positive topological entropy satisfies Condition (M1).

To prove Condition (M2), we set $I_1 := \cup_{j\geq 0} f^{-j}(\mathcal{C})$ where $\mathcal{C}$ denotes the set of all singular points and $I_0 = I \setminus I_1$. As $I_1$ is at most countable, an invariant measure which gives positive weight $I_1$ is either infinite or an atomic measure on a periodic point. Hence any finite ergodic measure $\mu$ with $\mu(I \setminus I_0) = \mu(I_1) > 0$ has zero entropy.

For $x \in I_0$, denote by $P_s(x)$ the maximal interval of monotonicity of $f^s$ containing the point $x$. The sets $P_s(x)$ are nested and contain $x$ so the limit $P_\infty(x) = \lim_{s\to\infty} P_s(x)$ exists. If $|P_\infty(x)| \geq \delta$ for some $\delta > 0$, then $P_\infty(x)$ contains an interval on which $f^n$ is monotone for every $n \in \mathbb{N}$. By hypothesis, there are no wandering intervals so every point of $P_\infty(x)$ is asymptotic to a periodic point (see for instance, [dMvS93, Lemma III.5.2]), contradicting the assumption that there are no attracting periodic points. Therefore $P_\infty(x) = x$ proving Condition (M2). □

**Corollary 7.5.** *Let $f$ be a Hölder continuous piecewise invertible map of a finite interval $I$ with finitely many branches. Assume that $f$ has an ergodic absolutely continuous invariant measure $m$ of positive entropy. Then*

1. *$f$ possesses a nice set $V \subset I$ satisfying conditions of Theorem 7.2;*
2. *$f|I$ satisfies Conditions (M1) and (M2) with constant $H = 0$ and $h_{top}(\Delta P, f) = 0$;*
3. *$f$ admits inducing schemes $\{S, \tau\}$ and $\{S^+, \tau\}$ satisfying Conditions (L) and ($L^+$) respectively; the inducing domain $W$ has full relative Lebesgue measure in $V$;*
4. *any ergodic $\mu \in \mathcal{M}(I, f)$ with $h_\mu(f) > 0$ and $\mu(W) > 0$ is liftable.*

*Proof.* The existence of an ergodic absolutely continuous invariant measure excludes the existence of attracting periodic orbits as well as wandering intervals, since the restriction of $f$ to the support of an ergodic measure is transitive. The statement follows from Theorem 7.3 and Proposition 7.4. □

We now consider the particular case of $S$-unimodal maps, i.e. smooth maps of the interval with one non-flat critical point at 0 and negative Schwarzian derivative (see [dMvS93] for the detailed definitions). We say that $f$ has a Cantor attractor if the $\omega$-limit set of the critical point $\omega(0)$ is a Cantor set, which coincides to the $\omega$-limit set $\omega(x)$ for almost every $x \in I$. Combining the above statements we obtain the following corollaries for $S$-unimodal maps.

**Corollary 7.6.** *Let $f$ be an $S$-unimodal map of a finite interval $I$ with a non-flat critical point. Then $f$ admits inducing schemes $\{S,\tau\}$ and $\{S^+,\tau\}$ satisfying Conditions (L) and (L$^+$) respectively and with the inducing domain $W$ of full relative Lebesgue measure in some interval $V \subset I$ if and only if there exist no Cantor attractors or attracting periodic points. In this case $f$ also satisfies Conditions (M1) and (M2) and any $\mu \in \mathcal{M}(I,f)$ with $h_\mu(f) > 0$ and $\mu(W) > 0$ is liftable. In particular, any $S$-unimodal map, satisfying the Collet-Eckmann condition, possesses an inducing scheme satisfying Condition (L$^+$), (M1) and (M2) and any non-singular (with respect to Lebesgue) invariant measure of positive entropy, which gives positive weight to the inducing domain, is liftable.*

*Proof.* Under the given hypothesis, it is well known that the unimodal maps admits an ergodic absolutely continuous invariant measure with positive Lyapunov exponent (see Ledrappier [Led81]), so Theorem 7.3 applies. The desired result follows from Theorem 6.1. □

In the more general case of multimodal maps, i.e. smooth maps of the interval (or circle) with finitely many non-flat critical points, we obtain the following result.

**Corollary 7.7.** *Let $f$ be a multimodal map of a finite interval $I$, which has an ergodic absolutely continuous invariant measure. Then $f$ admits inducing schemes $\{S,\tau\}$ and $\{S^+,\tau\}$ satisfying Conditions (L) and (L$^+$) respectively and with inducing domain of full relative Lebesgue measure in some interval $V \subset I$. Also, $f$ satisfies Conditions (M1) and (M2) and any invariant measure of positive entropy, which gives positive weight to the inducing domain, is liftable.*

One also can obtain the corresponding theorems for rational maps of the Riemann sphere, where the notion of the Lebesgue measure is replaced by that of $t$-conformal measures.

**Theorem 7.8.** [Dob06] *Let $f$ be a rational map of the Riemann sphere and $m$ a $t$-conformal measure for $f$. Suppose that there exists an ergodic, invariant probability measure $\mu$, which is absolutely continuous with respect to $m$, and that the Lyapunov exponent of $\mu$ is strictly positive. Then $f$ possesses a nice set $V$ and consequently, admits an inducing scheme $\{S,\tau\}$ (respectively, $\{S^+,\tau\}$) with the inducing domain $W$ of full relative Lebesgue measure in $V$ and $\int \tau d\,\mu < \infty$. In particular, the inducing scheme satisfies Condition (L) (respectively, (L$^+$)).*

7.3. **A special example.** We construct a special example of a multidimensional map which illustrates some of our results.

Let $f : [b_1, b_2] \to [b_1, b_2]$ be a unimodal map with the critical point at 0 and such that $f(b_1), f(b_2) \in \{b_1, b_2\}$. Consider a family of maps $g_t : [0,1] \to [0,1]$, $t \in [0,1]$ satisfying:

(1) $g_t(0) = g_t(1) = 0$, $g_t(\frac{1}{2}) = 1$;

(2) both $g_t|(0, \frac{1}{2})$ and $g_t|(\frac{1}{2}, 1)$ are $C^{1+\alpha}$ diffeomorphisms satisfying $\left|\frac{d}{ds} g_t(s)\right| \geq a > 1$ for any $s \in (0,1) \setminus \frac{1}{2}$;

(3) $g_t(s)$ is smooth in $t$ for all $s$.

Consider the skew-product map $h : R := [b_1, b_2] \times [0,1] \to [b_1, b_2] \times [0,1]$ given by

$$h(x, y) = (f(x), g_x(y)),$$

and denote by $\pi_1$ and $\pi_2$ the projection to the first and second components respectively.

It is easy to see that $h$ is a piecewise invertible map where the partition $P$ consists of four elements

$$P = \{(b_1, 0) \times (0, \frac{1}{2}), (0, b_2) \times (0, \frac{1}{2}), (b_1, 0) \times (\frac{1}{2}, 1), (b_1, 0) \times (\frac{1}{2}, 1)\}.$$

We describe an inducing scheme for $h$. Notice that for any $k \in \mathbb{N}$, the set $h^{-k}([b_1, b_2] \times \{\frac{1}{2}\})$ consists of $2^k$ disjoint smooth curves $\{l_j^k\}_{j=1}^{2^k}$, each curve can be represented as the graph of a function from $[b_1, b_2]$ to $[0,1]$. The set $R \setminus \bigcap_{j=1}^{2^k} l_j^k$ consists of $2^k$ connected components; we denote by $\xi_k$ the collection of these components.

It follows from [PS06], [Sen03] that there exists a set $A \subset [b_1, b_2]$, a collection of intervals $Q$, and an integer-valued function $\tau : Q \to \mathbb{N}$ such that for all $J \in Q$ one has $f^{\tau(J)}(J) \simeq A$ (recall that $f^{\tau(J)}(J) \simeq A$ means that $f^{\tau(J)}$ maps $J$ homeomorphically onto $A$).

Define the collection of open sets

$$Q' := \{J \times [0,1] \cap \eta : \ J \in Q, \ \eta \in \xi_{\tau(J)}\}.$$

It follows that

$$f^{\tau(J)}(J \times [0,1] \cap \eta) \simeq A \times (0,1) \ \text{ and } \ f^{\tau(J)}(J^+ \times [0,1] \cap \eta) \simeq A^+ \times (0,1).$$

Set

$$\mathcal{W} = \bigcup_{\zeta \in Q'} \zeta \ \text{ and } \ \mathcal{H}|\zeta = h^{\tau(\zeta)}|\zeta$$

and then

$$W = \bigcup_{k \geq 0} \mathcal{H}^{-k}(\mathcal{W}) \ \text{ and } \ S = \{\zeta \cap W : \ \zeta \in Q'\}.$$

It is easy to see that $\{S, \tau\}$ is an inducing scheme.

**Lemma 7.9.** *The inducing scheme $\{S, \tau\}$ satisfies Condition (L).*

*Proof.* We observe that the inducing scheme $\{Q, \tau\}$ for $f$ satisfies Condition (L) (see [PS06], [Sen03]). Choose a number $m \in \mathbb{N}$ and a set $L$ such that

$$h^m(L) \simeq A \times (0,1),\ L \cap \zeta \neq \emptyset,\ \zeta \in Q'.$$

Assume that $\zeta = W \cap J \times (0,1) \cap \eta$ for some $J \in S$ and $\eta \in \xi_{\tau(J)}$. It follows that $f^m(\pi_1(L^+)) \simeq A^+$, $f^m(\pi_1(L)) \simeq A$, and $\pi_1(L) \cap J \neq \emptyset$. Since the inducing scheme $\{Q, \tau\}$ satisfies Condition (L) we have that $m \geq \tau(J)$, which is what we need to prove. □

**Lemma 7.10.** *The map $h : R \to R$ satisfies Conditions (M1) and (M2) with $H = \log 2$.*

*Proof.* It is easy to show that the partition of $(b_1, b_2)$ by $(b_1, 0)$ and $(0, b_2)$ is generating for the map $f$ and that the maps $g_t$ are expanding with a constant uniform in $t$. It follows that the partition $P$ for $h$ is also generating. However, since the partition element $P_n(x)$ is only well-defined outside the set $\bigcup_{k\geq 0} h^{-k}(\partial P)$, we have that $\operatorname{diam} P_n(x) \to 0$ on $R \setminus \bigcup_{k\geq 0} h^{-k}(\partial P)$. The only ergodic invariant measures supported on $\bigcup_{k\geq 0} h^{-k}(\partial P)$ are supported either on $[b_1, b_2] \times \{0\}$ or on $\{f^j(0)\}_{j=0}^{p-1} \times [0,1]$ if 0 is periodic of period $p$. The entropy of such measures is at most $\max\{\log 2, h_{top}(f)\} = \log 2$. Condition (M2) is satisfied if we set $I_0 = R \setminus \bigcup_{k\geq 0} h^{-k}(\partial P)$ and $H = \log 2 < h_{top}(h)$ (see below for the last inequality).

To check Condition (M1) note that $\Delta P = [b_1, b_2] \times \{1\} \cup f(0) \times [0,1]$. We have that $h_{top}([b_1, b_2] \times \{1\}) = h_{top}(f) \leq \log 2$. Also $h_{top}(f(0) \times [0,1)) \leq \log 2$. To see this notice that for any $\varepsilon > 0$, we can pick a number $m$ such that the horizontal diameter of $\xi_m$ is smaller than $\varepsilon$. It follows that $\{l_j^{k+m} \cap \{0\} \times [0,1]\}_{j=1}^{2^{k+m}}$ is a $(k, \varepsilon)$-spanning set and hence $h_{top}(f(0) \times [0,1)) \leq \log 2$. Since $h$ is topologically a direct product map, $h_{top}(h) = h_{top}(f) + \log 2 > \log 2$ implying Condition (M1). □

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Department of Mathematics, Pennsylvania State University, University Park, State College, PA 16802
*E-mail address*: pesin@math.psu.edu

Instituto de Matematica, Universidade Federal do Rio de Janeiro, C.P. 68 530, CEP 21945-970, R.J., Brasil
*E-mail address*: senti@impa.br

Department of Mathematics, Pennsylvania State University, University Park, State College, PA 16802
*E-mail address*: zhang@math.psu.edu